Johannes J. Duistermaat

SPRINGER
MONOGRAPHS IN MATHEMATICS

Discrete Integrable Systems

QRT Maps and Elliptic Surfaces

Springer

S*pringer* **M***onographs in* **M***athematics*

For further volumes published in this series,
www.springer.com/series/3733

Johannes J. Duistermaat

Discrete Integrable Systems

QRT Maps and Elliptic Surfaces

Springer

Johannes J. Duistermaat (deceased, March 2010)

ISSN 1439-7382
ISBN 978-1-4419-7116-6 e-ISBN 978-0-387-72923-7
DOI 10.1007/978-0-387-72923-7
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010934229

© Springer Science+Business Media, LLC 2010
All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.
The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Preface vii

1 The QRT Map 1
1.1 The Rational Formula for the QRT Map 1
1.2 Indeterminacy of the QRT Map 3
1.3 Reconstruction 4

2 The Pencil of Biquadratic Curves in $\mathbb{P}^1 \times \mathbb{P}^1$ 9
2.1 Complex Analytic Geometry 10
2.2 Complex Projective Varieties 28
2.3 Elliptic Curves 33
2.4 Biquadratic Curves 49
2.5 The QRT Mapping on a Smooth Biquadratic Curve 60
2.6 Real Points 74

3 The QRT surface 85
3.1 The surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ 86
3.2 Blowing Up 93
3.3 Blowing Up $\mathbb{P}^1 \times \mathbb{P}^1$ at the Base Points 105
3.4 The QRT Map on the QRT Surface 119

4 Cubic Curves in the Projective Plane 129
4.1 From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and Back 129
4.2 Manin Transformations 134
4.3 Manin QRT Automorphisms 138
4.4 Aronhold's Invariants 143
4.5 Pencils of Cubic Curves with Only One Base Point 148

5 The Action of the QRT Map on Homology 157
5.1 The Action of the QRT Map on Homology Classes 157
5.2 QRT Transformations of Finite Order 164

6 Elliptic Surfaces 179
6.1 Fibrations 179

6.2 The Singular Fibers . . . 187
6.3 The Weierstrass Model . . . 271
6.4 Kodaira's Classification of Elliptic Surfaces . . . 314

7 Automorphisms of Elliptic Surfaces . . . 329
7.1 The Mordell–Weil Group and the Set of Sections . . . 329
7.2 The Néron–Severi Group . . . 331
7.3 Eichler–Siegel transformations . . . 338
7.4 The Number of Periodic Fibers . . . 342
7.5 The Contributions of the Reducible Fibers . . . 346
7.6 More about the Mordell–Weil lattices . . . 351
7.7 Asymptotics of the k-Periodic Fibers . . . 355
7.8 The Mordell–Weil Group in the Weierstrass Model . . . 369

8 Elliptic Fibrations with a Real Structure . . . 377
8.1 Real Structures and Real Automorphisms . . . 377
8.2 The Real Periods near the Singular Fibers . . . 384
8.3 Hyperbolic and Elliptic I_b, $b > 0$ . . . 391
8.4 Singularities of the Real Rotation Function . . . 395
8.5 Real Pencils . . . 400

9 Rational elliptic surfaces . . . 405
9.1 Equivalent Characterizations of Rational Elliptic Surfaces . . . 405
9.2 Properties of Rational Elliptic Surfaces . . . 410

10 Symmetric QRT Maps . . . 451
10.1 The QRT Root . . . 452
10.2 Pencils of Planar Quadrics . . . 460
10.3 Poncelet . . . 467

11 Examples from the Literature . . . 475
11.1 Hesse . . . 475
11.2 The Elliptic Billiard . . . 487
11.3 The Planar Four-Bar Link . . . 512
11.4 The Lyness Map . . . 518
11.5 The McMillan Map . . . 546
11.6 Heisenberg Spin Chain Maps . . . 553
11.7 The Sine–Gordon map . . . 556
11.8 Jogia's Example . . . 572
11.9 A Non-QRT Map with the Weierstrass Data of a QRT Map . . . 577

12 Appendices . . . 591
12.1 The QRT Mapping on Singular Fibers . . . 591
12.2 Configurations of Singular Fibers in this Book . . . 608

References . . . 613

Index . . . 621

Preface

In December 2005, Theo Tuwankotta showed me a birational transformation of the plane derived from a discrete sine-Gordon equation, with the question to determine the behavior of the discrete dynamical system defined by the iterates of the map. The map belongs to the family of QRT maps, introduced in 1988 by Quispel, Roberts, and Thompson, with many examples coming from mathematical physics. Classical examples of QRT maps are the transformations in the theorem of Poncelet and the elliptic billiard. The QRT maps can be identified with the automorphisms of rational elliptic surfaces that act as translations on the smooth fibers and map a smooth section to a disjoint one. This characterization leads to very detailed information about the dynamics. For instance, it leads to an explicit formula for the number of fibers on which the transformation is periodic with a given period k, together with an asymptotic description for large k of the set of such fibers. In the real setting, it leads to a detailed qualitative description of the rotation number of the map, as a function of the real parameter which tells to which fiber the map is restricted.

The definition of the QRT maps is so simple that it can be explained to high school students. In contrast, in order to obtain their basic properties, such as their identification with automorphisms of elliptic surfaces, one needs quite a bit of algebraic and complex analytic geometry. I needed to work out many of the proofs in the literature in order to understand these, and subsequently tell the results to the discrete dynamical systems community. It is the purpose of this book to explain not only the basic facts about the QRT maps, but also the background theory of elliptic surfaces on which these are based. The completeness of the treatment hopefully will allow the reader to become familiar with any selected aspect of the story, without having to make an extensive journey through the literature.

Different categories of readers might read the book in different ways. Readers without a background in discrete dynamical systems or algebraic geometry might start with the definition of QRT maps, then read about their basic properties, take the background theory of elliptic surfaces for granted for awhile, and then browse in the chapter “Examples from the Literature” in order to see how the theory can be applied. People from Discrete Integrable Systems might first look at the examples, and then, for the proofs of the statements about these, consult the background theory

in the previous chapters. Readers with a background in algebraic geometry might be interested in Kodaira's theory of elliptic surfaces, and the rich collection of its applications presented here. Future research can be expected in the identification and analysis of other, possibly higher dimensional birational transformations preserving a fibration by elliptic curves, arising from mathematical physics or otherwise. Complex analytic families of QRT maps are such higher dimensional examples, where the problem would be to describe the bifurcations in the configurations of the singular fibers of the rational elliptic surfaces.

Prerequisites for dealing with the book are courses in differential geometry and complex analysis on a graduate level. For those not familiar with algebraic geometry, the summary in this book of the employed facts from that subject should be helpful in understanding the theory of QRT maps and elliptic surfaces.

Summary of This Book

The QRT Map

Let $p(x,\ y)$ be a polynomial function in two variables that is *biquadratic* in the sense that for each y, the polynomial $x \mapsto p(x,\ y)$ is of degree two and, for each x, the polynomial $y \mapsto p(x,\ y)$ is of degree two. If $p(x,\ y) = a(y)\,x^2 + b(y)\,x + c(y)$, then the *horizontal switch* $\iota_1 : (x,\ y) \mapsto (x',\ y)$, which switches the two points on the curve $p(x,\ y) = 0$ with the same y-coordinate, is given by $x' = -x - b(y)/a(y)$. Similarly we have the *vertical switch* ι_2, which switches the two points on the curve $p = 0$ with the same x-coordinate. The *QRT mapping* on the curve $p = 0$ is defined as the composition $\tau = \iota_2 \circ \iota_1$ of the horizontal switch and the vertical switch on $p = 0$. The horizontal and the vertical switches are *involutions*, transformations ι such that $\iota \circ \iota$ is equal to the identity, or equivalently, ι is bijective and $\iota^{-1} = \iota$.

Figure 1 shows the horizontal switch ι_1, the vertical switch ι_2, and the QRT map $\tau = \iota_2 \circ \iota_1$, acting on the Lyness curve $(x + 1)\,(y + 1)\,(x + y + a) - z\,x\,y = 0$, or equivalently, $(x + 1)\,(y + 1)\,(x + y + a)/x\,y = z$, see (11.4.2), for $a = 0.4$ and $z = 10.58$.

Remark. With this definition of the QRT map on a biquadratic curve in mind, I found it exciting to read the following at the beginning of Cayley [32]: "... a (2, 2) correspondence is such that to any given position of either point there correspond two positions of the other point... Or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form $p(x,\ y) = 0$ where p is of degree two in each of the variables ...." However, although he came this close, Cayley did not consider the QRT map of the biquadratic curve $p(x,\ y) = 0$ in [32]. ⊘

If $p^0(x,\ y)$ and $p^1(x,\ y)$ are two linearly independent biquadratic polynomials, then each nonzero linear combination $p_{(z_0,\,z_1)}(x,\ y) := z_0\,p^0(x,\ y) + z_1\,p^1(x,\ y)$

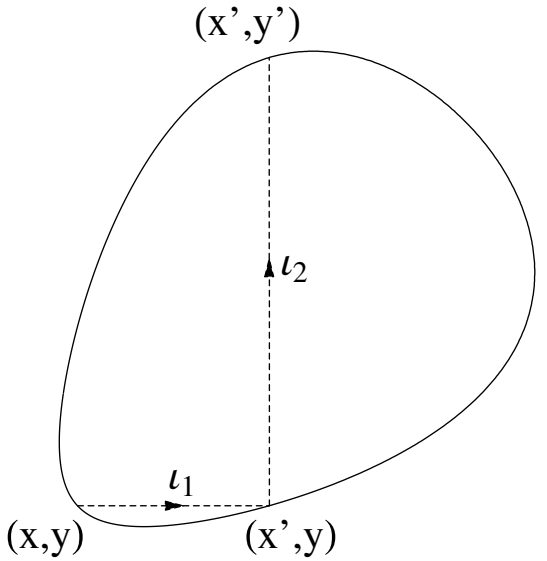


Fig. 1 QRT map on a biquadratic curve.

of p^0 and p^1 is a nonzero biquadratic poynomial. The corresponding biquadratic curve

$$z_0\, p^0(x,\ y) + z_1\, p^1(x,\ y) = 0 \tag{0.0.1}$$

in the $(x,\ y)$-plane does not change if both coefficients $z_0,\ z_1$ are multiplied by the same factor. Therefore the curves $p_{(z_0,\, z_1)}(x,\ y) = 0$ are parametrized by the projective line $\mathbb{P}^1$ of all one-dimensional linear subspaces of the $(z_0,\ z_1)$-plane. Such a one-parameter family of curves in which the parameter z appears linearly in the equations is called a *pencil* of curves.

We have that $p^0(x,\ y) = 0$ and $p^1(x,\ y) = 0$ if and only if $p_z(x,\ y) = 0$ for every z, that is, all the members of the pencil pass through the point $(x,\ y)$ if and only if two distinct members pass through it. Such a point $(x,\ y)$ is called a *base point* of the pencil. If we work over the field $\mathbb{C}$ of complex numbers, and projectively, which makes the analysis more uniform, then every pencil of biquadratic curves has eight base points, when counted with multiplicities. On the other hand, if $(x,\ y)$ is not a base point, then the set of all z such that $p_z(x,\ y) = 0$ is a one-dimensional linear subspace of the $(z_0,\ z_1)$-plane, that is, exactly one member $C = C(x,\ y)$ of the pencil passes through the point $(x,\ y)$. See Figure 3.1.1 for a case in which all eight base points are real and simple, whereas Figure 3.1.2 illustrates a base point of multiplicity two.

It is customary to write $z_0 = 1$ and $z_1 = -z$ when in the complement of the base points the equation (0.0.1) is equivalent to

$$z = p^0(x,\ y)/p^1(x,\ y). \tag{0.0.2}$$

That is, the biquadratic curve is equal to the level curve of the rational function p^0/p^1 at the level z. If we apply to $(x,\ y)$ not a base point the transformations ι_1, ι_2, and τ of the biquadratic curve $C(x,\ y)$, then we obtain birational transformations of the plane $\mathbb{P}^1 \times \mathbb{P}^1$. Here the "bi" in birational refers to the fact that both the map and its inverse are rational. These are the birational transformations of the plane that have been introduced, in this generality, by Quispel, Roberts, and Thompson in [168], [169].

The horizontal and vertical switches and the QRT map are, as birational transformations of the plane, explicitly given by the formulas (1.1.4), (1.1.5), (1.1.6). From the way these have been introduced here, it follows that ι_1, ι_2, and τ, where defined, leave each member of the pencil of biquadratic curves invariant. In Section 2.5 we prove that every smooth member of the pencil is an *elliptic curve*, on which, moreover, the QRT map acts as a *translation*. One proof uses Hamiltonian vector fields tangent to the biquadratic curves that are invariant under the QRT map, and therefore the QRT map acts as a translation in the time parameter of the solution curve of the Hamiltonian system.

The theory in this book will lead to quite detailed information about the behavior of the iterates τ^k of the QRT map τ for large k. I would like to emphasize here that although the formulas for the rational transformations τ^k in principle can be computed successively in an explicit manner, these formulas rapidly become very complicated. In fact, none of the general facts about the QRT maps has been obtained by direct inspection of the formulas for their iterates.

Figure 2 shows the orbit of a point under five iterates of the QRT map of Figure 1.

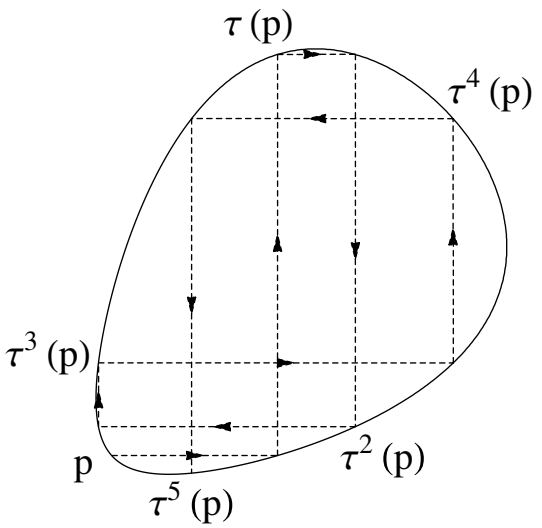


Fig. 2 Five iterates of the QRT map of Figure 1.

Because in the examples one is often interested in real QRT maps, acting as translations on real closed curves, much attention has been paid to the real rotation number, as a function of the real parameter that tells on which curve the QRT map is acting. See Section 2.6 and Chapter 8. Most of the information is obtained by viewing the real curves as the real parts of the complex curves, where it would have been difficult to understand the situation by staying exclusively in the real domain.

Very useful for computations are the formulas for the Weierstrass invariants g_2, g_3, and Δ of the biquadratic curves, and the coordinates X and Y of the image point on the Weierstrass curve under the QRT map of the point at infinity on the Weierstrass curve. In fact, X, Y, and g_2 are the basic polynomial invariants of the biquadratic polynomials that define the QRT map. See Corollary 2.4.7 and Proposition 2.5.6. For the QRT root in the case of a pencil of symmetric biquadratic curves, see Proposition 10.1.6. This leads also to an explicit determination of the inhomogeneous Picard–Fuchs equation of the QRT map, and the Beukers–Cushman monotonicity criterion for the rotation function; see Sections 2.5.3 and 2.6.3.

Singularity confinement by Blowing Up

The ambiguity at the base points of the QRT map and its invariant rational function p^0/p^1, and the phenomenon that all the invariant biquadratic curves in the pencil meet at each base point, can be removed by *blowing up* the plane at the base points. If a base point has multiplicity > 1, then in the blown-up surface a new base point will appear of multiplicity one less, which then has to be blown up again. As there are eight base points when counted with multiplicity, the process will stop after eight blowing-up transformations. We arrive at a smooth surface, a complex two-dimensional compact, connected complex analytic manifold S, on which the proper transforms of the biquadratic curves of the pencil have become disjoint, where the smooth ones are elliptic curves. The invariant rational function p^0/p^1 corresponds to an everywhere defined complex analytic mapping $\kappa : S \to \mathbb{P}^1$, of which the fibers are aforementioned proper images of the biquadratic curves, which is why κ is called an *elliptic fibration* of the surface S over the complex projective line. See Definition 6.1.7. Furthermore, the translations on the smooth fibers of S defined by the QRT map τ extend to an *automorphism* τ^S of S, an everywhere defined complex analytic diffeomorphism from S onto itself, without any ambiguities or singularities. More precisely, τ^S belongs to the group $\mathrm{Aut}(S)^+_\kappa$ of all automorphisms of S that preserve each of the fibers of κ and act as a translation on each smooth fiber, which is an elliptic curve. See Section 3.4.2.

It is a basic fact that the biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ correspond bijectively to the holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$, that is, the holomorphic sections of the anticanonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. See Section 2.1.7 for the definition of the anticanonical bundle. Therefore the two-dimensional vector space of biquadratic polynomials that define the pencil of biquadratic curves corresponds to a two-dimensional vector space of holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$,

and at each blowing up we have a corresponding two-dimensional vector space of holomorphic exterior two-vector fields on the blown-up surface. When we arrive at our surface S, the two-dimensional vector space of holomorphic exterior two-vector fields is equal to the space of *all* holomorphic exterior two-vector fields on S, where the fibers of κ are precisely the zero-sets of the nonzero holomorphic exterior two-vector fields on S. Actually, we use the two-dimensional vector spaces of holomorphic vector fields in order to find the base points in an intermediate blown up surface, which may occur over base points of higher multiplicity of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. See Section 3.3.2–3.3.4.

Automorphisms of Rational Elliptic Surfaces

Blowing up one base point b of the pencil of biquadratic curves and then blowing down the proper transforms of the horizontal and vertical axes through b, we arrive at the complex projective plane $\mathbb{P}^2$, in which the pencil of biquadratic curves corresponds to a pencil of cubic curves, with the respective blowdowns b_1 and b_2 of the horizontal and vertical axis as two of its base points. Conversely, every pencil of cubic curves in $\mathbb{P}^2$ with at least two base points arises in this way.

Every complex projective line in $\mathbb{P}^2$ intersects a cubic curve in three points. The horizontal or vertical switch on the biquadratic curve corresponds to the mapping ι_{C,b_2} or ι_{C,b_1}, which assigns to the point x on the cubic curve C the third point of intersection with C of the complex projective line through x and b_2 or b_1, respectively. Every smooth cubic curve C in $\mathbb{P}^2$ is an elliptic curve, and the composition $\tau_{C,b_1} = \iota_{C,b_1} \circ \iota_{C,b_1}$ of these two involutions, which corresponds to the QRT map on the biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$, is equal to the unique translation on C that maps b_1 to b_2. Doing this for all members of the pencil of cubic curves with the given base points b_1 and b_2, we obtain a birational transformation of the projective plane $\mathbb{P}^2$.

It follows from the theorem of Bézout that every pencil of cubic curves in $\mathbb{P}^2$ has $3 \times 3 = 9$ base points, when counted with multplicities. Blowing up the base points in the same way as for our pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, we arrive at a surface S in which the pencil of cubic curves corresponds to fibration, where the smooth fibers are elliptic curves, and the rational transformation of $\mathbb{P}^2$ corresponds to an automorphism of S that acts as a translation on each smooth fiber. These automorphisms of S, defined for each choice of two distinct base points of the given pencil of cubic curves in $\mathbb{P}^2$, were introduced by Manin [129, p. 95, 96].

The elliptic surface that is obtained by blowing up the base points of the pencil of cubic curves in $\mathbb{P}^2$ is canonically isomorphic to the surface obtained by blowing up the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ if the pencil of cubic curves in $\mathbb{P}^2$ corresponds to the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ as described above. Furthermore, the isomorphism conjugates the QRT automorphism with the Manin automorphism. In this sense the QRT automorphisms and the Manin automorphisms are the same.

In general, the elliptic surfaces that are isomorphic to the blowing up of the base points of a pencil of cubic curves in $\mathbb{P}^2$ are called *rational elliptic surfaces*. Note

that the rational elliptic surfaces obtained by blowing up the base points of a given pencil of cubic curves in $\mathbb{P}^2$ are unique up to isomorphism, but that very different pencils of cubic curves in $\mathbb{P}^2$ can give rise to isomorphic rational elliptic surfaces. See Theorem 4.3.2, Theorem 4.3.3, and Remark 9.2.28.

Each rational elliptic surface has at least one section, for instance the -1 curve that appeared at the last blowing up. If $\varphi : S \to C$ is any elliptic fibration over a curve C, then the group of all automorphisms of S that act on each smooth fiber φ as a translation will be denoted by $\mathrm{Aut}(S)^+_\varphi$. Here the plus sign refers to the convention to view the composition in a translation group as an addition. If φ has at least one section, then the group $\mathrm{Aut}(S)^+_\varphi$ acts freely and transitively on the set Σ of all sections of φ; see Lemma 7.1.1. That is, after the choice of a holomorphic section as the "zero section," the set Σ is identified with the group $\mathrm{Aut}(S)^+_\varphi$, and the set of all sections provided with such a group structure is called the *Mordell–Weil group* of S in the literature. Because we are primarily interested in the group $\mathrm{Aut}(S)^+_\varphi$, and have no canonical choice of a holomorphic section, we prefer to call $\mathrm{Aut}(S)^+_\varphi$ the Mordell–Weil group.

The identification of the QRT automorphisms of the rational elliptic surface S with the corresponding Manin automorphisms allows us to characterize the QRT automorphisms as those elements of the Mordell–Weil group of S that map some (every) section to a disjoint one. Because a theorem of Oguiso and Shioda [155, Theorem 2.5] says that the Mordell–Weil group is generated by elements that map a holomorphic section to a disjoint one, it follows that the Manin QRT automorphisms generate the Mordell–Weil group of S; see Theorem 4.3.3.

Action on Homology Classes and the Number of k-Periodic Fibers

The -1 curves that appear at each blowing up define real two-dimensional cycles in the rational elliptic surface S. The homology classes of these cycles are independent in the homology group $\mathrm{H}_2(S, \mathbb{Z})$ of S. Since $\mathrm{H}_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}^2$, it follows that $\mathrm{H}_2(S, \mathbb{Z}) \simeq \mathbb{Z}^{10}$. Each automorphism of S induces an automorphism of the group $\mathrm{H}_2(S, \mathbb{Z})$ that preserves the intersection form on $\mathrm{H}_2(S, \mathbb{Z})$. It turns out that the actions on $\mathrm{H}_2(S, \mathbb{Z})$ of the elements of the Mordell–Weil group are of a very special nature. In Section 5.1 we determine the action on $\mathrm{H}_2(S, \mathbb{Z})$ of the QRT automorphism, under the assumption that no member of the pencil of biquadratic curves contains a horizontal or a vertical axis. See Corollary 5.1.9.

Let $\varphi : S \to C$ be an elliptic fibration with at least one section E, and let $\alpha \in \mathrm{Aut}(S)^+_\varphi$. If S^{reg} denotes the set of regular points in S, then E intersects each fiber F in exactly one point $s \in F \cap S^{\mathrm{reg}}$, where the intersection is transversal. We have that $\alpha(E)$ intersects E at $s \in F$ if and only if $\alpha(s) = s$ if and only if $\alpha(f) = f$ for every $f \in F$. For this reason the topological intersection number $\nu(\alpha) = E \cdot \alpha(E)$ of the cycles E and $\alpha(E)$ is called the *number of fixed point fibers for α, counted with multiplicities*. This number is independent of the choice of the section. For any $k \in \mathbb{Z}$, the number $\nu(\alpha^k)$ is called the *number of k-periodic fibers for α, counted with multiplicities*. See Definitions 7.4.3 and 7.4.5 for more details.

Because the topological intersection number is a homology invariant, see Section 2.1.6, the numbers $\nu(\alpha^k)$ can be computed in terms of the action of α on $H_2(S, \mathbb{Z})$. In this way we obtain that the number of k-periodic fibers for the QRT automorphism, counted with multiplicities, is equal to $k^2 - 1$ if no member of the pencil of biquadratic curves contains a horizontal or a vertical axis. If at least one member of the pencil of biquadratic curves contains a horizontal or a vertical axis, then there is at least one singular fiber F of the fibration $\kappa : S \to \mathbb{P}^1$ such that the QRT automorphism permutes the irreducible components of F in a nontrivial way. See Corollary 5.1.12. In this case the behavior of the number of k-periodic fibers is more complicated, but still can be determined explicitly if one knows the action on the set of irreducible components of reducible fibers. See (4.3.2).

Periodic QRT Mappings

Let $\kappa : S \to \mathbb{P}^1$ be a rational elliptic surface, and let α be a nontrivial element of the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ of a finite order m. That is, all fibers of κ are m-periodic for α. Then the theory of Shioda [184] implies that α maps every holomorphic section of κ to a disjoint one, and therefore there is a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that S and κ are the corresponding rational elliptic surface and QRT automorphism. The order m of α can be 2, 3, 4, 5, or 6, and in Section 5.2 we give an explicit description of the Weierstrass data of the elements of the Mordell–Weil group of order m, for each $2 \leq m \leq 6$. In order to find pencils of biquadratic curves such that the QRT map is of the given order, we use the criterion of Oguiso and Shioda that α is of finite order if and only if the sum of the so-called contributions to α of all the reducible fibers is equal to two. Then a pencil is constructed with some reducible fibers, of which the irreducible components are permuted by the QRT map in the required fashion. For each of the possible orders 2, 3, 4, 5, 6, Tsuda [196, Example 3.5] provided a family of pencils for which the QRT mapping has the prescribed order, without telling how he found these families.

Elliptic Surfaces

In Chapter 6 we discuss the theory of arbitrary elliptic fibrations. After the basic definition and some preliminary observations in Section 6.1, Kodaira's classification of the possibilities for the singular fibers appears in Section 6.2.6. Kodaira's classification of elliptic surfaces in terms of their *modulus function* and *monodromy representation* is presented in Section 6.4.2. The richness of the province of the elliptic surfaces in the realm of the arbitrary surfaces is illustrated by the fact that for any meromorphic function on a compact connected complex analytic curve C there exists at least one elliptic fibration $\varphi : S \to C$ with a holomorphic section, having the given meromorphic function on C as its modulus function. In order to make the presentation more self-contained, we have included in Chapter 6 full proofs of all the basic facts about elliptic surfaces. Because of the highly nontrivial nature of many of these basic facts, Chapter 6 has grown into a small book within the book.

As for rational elliptic surfaces, the Mordell–Weil group of S is defined as the group $\mathrm{Aut}(S)^+_\varphi$ of all complex analytic diffeomorphisms α of S that act as a translation on each of the smooth fibers S_c of φ, where S_c is an elliptic curve. If $\varphi : S \to C$ has a holomorphic section, then S is projective algebraic, and it makes sense to define the Néron–Severi group $\mathrm{NS}(S)$ of S as the subgroup of $\mathrm{H}_2(S,\ \mathbb{Z})$ generated by the homology classes of algebraic curves in S. If S is a rational elliptic surface, then $\mathrm{NS}(S) = \mathrm{H}_2(S,\ \mathbb{Z})$, but for general elliptic surfaces the Néron–Severi group can be quite a bit smaller than the homology group. If $\varphi : S \to C$ has at least one section and at least one singular fiber, then the theory of Shioda [184] about the action of the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ on $\mathrm{NS}(S)$ leads to detailed information about the structure of the Mordell–Weil group. See Section 7.5. It also leads to the explicit formula (7.5.2) for the number of k-periodic fibers for $\alpha \in \mathrm{Aut}(S)^+_\varphi$, counted with multiplicities. In particular, unless α has finite order, the number of k-periodic fibers grows as k^2 as $k \to \infty$.

This illustrates that the elements of the Mordell–Weil group of an arbitrary elliptic surface share many properties with QRT transformations, the elements of the Mordell–Weil groups of rational elliptic surfaces without fixed point fibers. Since the rational elliptic surfaces form a quite small subclass of elliptic surfaces, in some sense the first nontrivial one in the hierarchy, the elements of Mordell–Weil groups of arbitrary elliptic surfaces can be viewed as a generalization of QRT maps to a large class of transformations that nevertheless have quite similar properties. However, the attractiveness of QRT maps is that many aspects of them, such as their Weierstrass normal forms, are given by explicit formulas, which I do not know of for the elements of the Mordell–Weil groups of arbitrary elliptic surfaces. Moreover, as illustrated by Chapter 11, quite a large number of birational transformations of the plane turn out to be QRT maps, whereas there are not so many explicit examples of Mordell–Weil groups of elliptic surfaces that are not QRT transformations. In this respect, further explorations might lead to future surprises. For an example in the literature of a non-QRT element of a Mordell–Weil group of an elliptic surface, see Section 11.9.1.

Asymptotic Density of the k-Periodic Fibers

The formula for the number of k-periodic fibers is based on quite nontrivial algebraic geometric or algebraic topological facts, but the corollary that, asymptotically for $k \to \infty$, it is of order k^2 also follows from the following more elementary analysis. If $p_1(c)$ and $p_2(c)$ are two basic complex periods of the Hamiltonian flow on the smooth fiber S_c in S over the point $c \in C$ of $\varphi : S \to C$, then $(r_1,\ r_2) \mapsto r_1\ p_1(c) + r_2\ p_2(c)$ defines an isomorphism of real Lie groups from the two-dimensional standard torus $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto the group of translations on S_c, where the isomorphism locally can be made to depend in a real analytic fashion on the regular value c of φ. The warning here is that the isomorphisms are determined only up to orientation-preserving automorphisms of $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, defined by elements of $\mathrm{SL}(2,\ \mathbb{Z})$, and have multivalued real analytic continuations to the set C^{reg} of all regular values of φ.

The element $\alpha \in \mathrm{Aut}(S)^+_\varphi$ then corresponds to a multivalued real analytic mapping R from C^{reg} to $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, called the *rotation map* defined by α. In the sequel we assume that the rotation map is not constant.

Let C_{α^k} denote the set of all $c \in C$ such that S_c is a k-periodic fiber for α. If $c \in C^{\mathrm{reg}}$, then $c \in C_{\alpha^k}$ if and only if $R(c) = (k_1/k + \mathbb{Z},\ k_2/k + \mathbb{Z})$ for some $k_1,\ k_2 \in \mathbb{Z}/k\,\mathbb{Z}$. It follows that, asymptotically for $k \to \infty$ and near a regular point of R, a point $c \in C^{\mathrm{reg}}$ near which R is a local diffeomorphism, the set C_{α^k} looks like a rank two lattice with distances between neighboring points of order $1/k$. Furthermore, the rescaled lattice at the point c is conformal to the period lattice of the elliptic curve S_c. The rotation map has only finitely many singular points in C^{reg}, near which points R behave like a branched covering, and where the asymptotic density of C_{α^k} is of smaller order. Near the singular values in C of φ, the density of C_{α^k} can be of higher order, but it follows from the asymptotics near every point of C that the total number $\#(C_{\alpha^k})$ of all k-periodic fibers remains of order k^2 as $k \to \infty$. The asymptotic density of C_{α^k} is described by the real-valued two-form $\mathrm{d}R_1 \wedge \mathrm{d}R_2$ on C^{reg}. This is a *single-valued* real analytic area form on C^{reg}. It is strictly positive at all points except at the finitely many singular points of the rotation mapping, where it is equal to zero. See Lemma 7.7.7, Corollary 7.7.8, and Corollary 7.7.10 for more details.

If all objects are defined over $\mathbb{R}$, then the real fibers over the real points c in C^{reg} are either empty, isomorphic to a circle $\mathbb{R}/\mathbb{Z}$, or isomorphic to two such circles. The element $\alpha \in \mathrm{Aut}(S)^+_\varphi$ acts on each circle as a rotation over a number $\rho(c) \in \mathbb{R}/\mathbb{Z}$, unless it permutes two connected components of the real fibers, when α^2 acts as a rotation on the real fiber. We assume that the *rotation function* $\rho : c \mapsto \rho(c)$ is not constant, which is equivalent to the condition that the aforementioned two-dimensional rotation map R is not constant. Asymptotically for $k \to \infty$ and near the regular points of ρ, the set of all $c \in C^{\mathrm{reg}}(\mathbb{R})$ such that $S_c(\mathbb{R})$ is a k-periodic real fiber looks like a rank-one lattice with distances between neighboring points of order $1/k$. The number of k-periodic real fibers is asymptotically of order k as $k \to \infty$. However, in contrast to the complex case, we do not have an explicit formula for the number of k-periodic real fibers.

Rational Elliptic Surfaces

Each surface that comes from a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, which is the natural domain of definition of the QRT automorphism, is a rational elliptic surface. The only rational elliptic surfaces $\kappa : S \to \mathbb{P}^1$ that are not QRT surfaces are those for which the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ is trivial, that is, those that have only one section. There are only two isomorphism classes of such exceptional rational elliptic surfaces, see Corollary 4.5.6.

In Section 9.1 we give a number of equivalent characterizations of rational elliptic surfaces, which might be useful to determine whether a given elliptic fibration is a rational elliptic surface. In Section 9.2 we collect a number of specific properties of rational elliptic surfaces. When dealing with examples, we sometimes refer to the

list of Persson [156, pp. 7–14] of all possible configurations of singular fibers in a rational elliptic surface. In the list of Oguiso and Shioda [155, pp. 84–86], singular fibers have not been distinguished from each other if the intersection diagrams of their irreducible components are the same. On the other hand, the list of Oguisa and Shioda gives more detailed information about the Mordell–Weil group in terms of the intersection form on the homology group $H_2(S, \mathbb{Z})$.

In the province of the elliptic surfaces, the rational elliptic surfaces form a village with a few hundred inhabitants. As a passer-by, I have met many of them. For a full understanding of rational elliptic surfaces one should know them all.

Examples

The interest of the whole subject is greatly enhanced by the large collection of examples that exist in the literature. One common characteristic of these examples is that each one of these exhibits some very special behavior, not at all shown by the "generic" cases. For instance, the configuration of the singular fibers in the examples is never equal to the generic one $12\ \mathrm{I}_1$, and for the pencils of symmetric biquadratic curves the configuration of the singular fibers in the examples is never equal to the generic one $3\ \mathrm{I}_2\ \ 6\ \mathrm{I}_1$. This taught me at a very early stage that I should aim at statements that are truly general, that is, not valid only under some genericity assumption.

We begin, in Section 11.1, with the fascinating pencil of cubic curves in the complex projective plane that studied by Hesse [82]. It is characterized by the property that the nine base points of the pencil are precisely the flex points of each of the smooth members of the pencil. By means of a projective linear transformation the pencil can be brought into a normal form that was Hesse's point of departure. The sections of the corresponding rational elliptic surface $\kappa : S \to \mathbb{P}^1$ correspond bijectively to the nine base points, and the Mordell–Weil group is isomorphic to $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$. Every automorphism of S is induced by a projective linear transformation, and the group of these, which has 216 elements, also was determined by Hesse.

A biquadratic polynomial $p(x, y)$ is called *symmetric* if $p(x, y) = p(y, x)$ for every $x, y \in \mathbb{C}^2$, when the corresponding curve $p(x, y) = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is invariant under the *symmetry switch* $(x, y) \mapsto (y, x)$. The great majority of the examples of QRT maps in the literature turn out to be defined by pencils of symmetric biquadratic curves. The symmetry condition implies that the QRT mapping τ can be written as $\tau = \rho \circ \rho$, where the *QRT root* ρ is defined as the composition of the horizontal switch followed by the symmetry switch. The symmetry condition implies that the QRT surface S has reducible singular fibers of which the irreducible components are permuted in a nontrivial way. In particular, the rank of the Mordell–Weil group of S is at most equal to 5, whereas for the generic rational elliptic surface the rank of the Mordell–Weil group is equal to 8. For the generic symmetric QRT surface the configuration of the singular fibers is $3\ \mathrm{I}_2\ \ 6\ \mathrm{I}_1$, in contrast to the configuration $12\ \mathrm{I}_1$ for the generic rational elliptic surface. See Section 10.1 for more details.

In Section 10.2 we show that pencils of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are in bijective correspondence with pencils of quadrics in $\mathbb{P}^2$. In Section 10.3.3 we show that this correspondence conjugates the QRT root with the *Poncelet mapping* for pairs of quadrics, where the inscribed quadric corresponds to the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ and the circumscribed quadric belongs to the pencil. Because the QRT root is a translation on each smooth fiber, we thus recover Poncelet's theorem that if a fiber contains a k-periodic point of the Poncelet mapping, then every point of the fiber is k-periodic.

In Section 11.2 we discuss the elliptic billiard. We show that the billiard map is a Poncelet mapping, and therefore a QRT root for a pencil of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, where the pencil of quadrics in $\mathbb{P}^2$ consists of the duals of the inscribed confocal quadrics for the billiard trajectories. The configuration of the singular fibers of the corresponding rational elliptic surface is $\mathrm{I}_0^*\ 3\ \mathrm{I}_2$, which illustrates that the billiard map is quite special among the general QRT roots. Another classical example is the *planar four-bar link*, the Darboux transformation of which is a QRT map on a nonsymmetric biquadratic curve. Because classically this example is not treated in the framework of a one-parameter family of elliptic curves, our discussion in Section 11.3 of the four-bar link is relatively short.

A more recent example that has obtained considerable attention in the literature is the Lyness map, which for this reason is discussed in great detail in Section 11.4. Beukers and Cushman [16], using Picard–Fuchs equations, proved the *conjecture of Zeeman* that the rotation function of the Lyness map has no stationary points in a certain interval between singular values of the fibration. Among other things, we determine the qualitative behavior of the rotation function in every such interval, including a description of the changes in the intervals when the parameter a in the Lyness map passes through its bifurcation values.

In the sections 11.5– refsGsec we discuss a number of examples of QRT roots from the mathematical physics literature, namely the KdV, the modified KdV, and the nonlinear Schrödinger maps, all of which are special McMillan maps; the Heisenberg spin chain map; and the sine–Gordon map. The last of these is treated in much detail because it is the one that introduced me to the subject of this book. Chapter 11 is concluded in Section 11.8 with a discussion of Jogia's example, and in Section 11.9 the example of Viallet, Grammaticos, and Ramani of a non-QRT map with the Weierstrass data of a QRT map.

Section 12.1 contains a list of singular fibers that appear after blowing up a singular member of a given pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, together with the action of the QRT automorphism on the set of irreducible components of the singular fiber. We have used this in some concrete examples in order to determine how the QRT automorphism permutes the irreducible components of the reducible singular fibers. For future computations it may be useful to have the complete list of Section 12.1.

What May Be New

Most of the facts in this story are known, and I have done my best to give proper references to the literature. However, there are also many results that I did not see in the literature, and might be new. Some of these results are not entirely straightforward applications of the known theory. Below follows a selection.

- The description of the holomorphic tangent vector fields to the smooth biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and the smooth cubic curves in $\mathbb{P}^2$ as Hamiltonian vector fields, in Lemma 2.4.5 and Lemma 4.1.2, respectively. See also Lemma 3.3.4.

  For a general elliptic fibration $\varphi : S \to C$, the holomorphic vector field on the complement of finitely many fibers of an elliptic fibration, which corresponds to a meromorphic section of the Lie agebra bundle $\mathfrak{f}$ over C, can be viewed as a Hamiltonian vector field as in Remark 6.2.21. The elliptic fibration can be viewed as a completely integrable Hamiltonian system; see Remark 6.2.22.
- The identification of the Eisenstein invariants of the partial discriminants of the biquadratic polynomial with the Weierstrass invariants g_2, g_3, and Δ of the period lattice of the biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$. The identification of the image point (X, Y) under the QRT transformation of the point at infinity on the Weierstrass curve with the Frobenius invariants of order two and three of the biquadratic polynomial. See Corollary 2.4.7, Proposition 2.5.6, and Remark 2.5.7. For the QRT root in the case of symmetric biquadratic polynomials, see Proposition 10.1.6.
- The identification of the Aronhold invariants of a cubic polynomial with the Weierstrass invariants of the period lattice of the cubic curve. The relation between biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and cubic curves in $\mathbb{P}^2$ leads to corresponding identities between the Eisenstein invariants of the partial discriminants of a biquadratic polynomial and the Aronhold invariants of a cubic polynomial. See Proposition 4.4.3 and Corollary 4.4.7.
- The corresponding morphisms between the various moduli spaces in Subsection 6.3.3.
- The inhomogeneous Picard–Fuchs equation = Manin homomorphism for the element of the Mordell–Weil group of an elliptic fibration, explicitly computable for the QRT map. The generalization to all QRT maps of the Beukers–Cushman criterion for monotonicity of the real period function. See Sections 2.5.3, 7.8, and 2.6.3.
- The characterization in Theorem 4.3.2 of the elements of the Mordell–Weil group of a rational elliptic surface that are QRT automorphisms or Manin automorphims for a pencil of biquadratic or cubic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, respectively, where the surface is obtained by successively blowing up the base points of the anticanonical pencils.
- The classification in Section 4.5 of the pencils of cubic curves in $\mathbb{P}^2$ with only one base point, which therefore has multiplicity 9. These are the pencils that do not have a Manin transformation. This leads to the classification in Corollary 4.5.6 of the rational elliptic surfaces with a trivial Mordell–Weil group.

- The generalization of a large part of Kodaira's theory of elliptic surfaces to the case in which it is allowed that the elliptic surfaces are not compact or that the modulus function is constant.
- The description in Lemma 6.2.38 and Table 6.2.39 of the basis of periods near a singular fiber of an elliptic fibration. This leads to an alternative proof of Kodaira's description of the monodromy around and the behavior of the modulus function near the singular fiber, as given in Table 6.2.40.
- Definition 7.4.3 of the number of k-periodic fibers for α, counted with multiplicities. The computation in (7.5.2) of this number for any element α of the Mordell–Weil group of any elliptic surface with at least one section and at least one singular fiber. The formula is in terms of the number of fixed point fibers for α, which is equal to zero for a QRT automorphism, and the way α permutes the irreducible components of reducible fibers.
- The computation in Corollary 5.1.9 of the action of the QRT automorphism on $\mathrm{H}_2(S,\ \mathbb{Z})$ when no member of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ contains a horizontal or vertical axis. The fact that in this case the number of k-periodic fibers for the QRT automorphism, counted with multiplicities, is equal to $k^2 - 1$. The characterization in Corollary 5.1.12 of the pencils of biquadratic curves for which the QRT automorphism acts on $\mathrm{H}_2(S,\ \mathbb{Z})$ as an Eichler–Siegel transformation.
- The observation in Section 5.2 that every element of finite order of the Mordell–Weil group of a rational elliptic surface is a Manin QRT automorphism. For each $2 \leq m \leq 6$, the explicit description of the Weierstrass data of the elements of the Mordell–Weil group of order m. The construction of pencils of biquadratic curves with a QRT map of order m in terms of the way the QRT map permutes the irreducible components of the reducible singular fibers.
- The generalization in Sections 6.2–6.4 of the theory of elliptic surfaces to the case that the surface need not be compact. My main motivation for this was to have the validity of a number of basic facts for local models, elliptic fibrations in a tubular neighborhood of a singular fiber. Of course this generalization does not apply to global topological statements in which the compactness of the surface is essential.
- The characterizations in Lemma 7.3.2 and Remark 7.8.5 of the elements of the Mordell–Weil group that act as Eichler–Siegel transformations on the Néron–Severi group.
- The description in Lemma 7.7.3 of the rotation map of an arbitrary element $\alpha \in \mathrm{Aut}(S)^+_\varphi$ of the Mordell–Weil group of any elliptic surface $\varphi : S \to C$. The description in Lemma 7.7.7 and Corollary 7.7.10 of the asymptotic behavior for $k \to \infty$ of the set of k-periodic fibers for α.
- The description in Chapter 8 of the behavior of the real period functions and real rotation function of an elliptic fibration with a real structure and a real element α of the Mordell–Weil group, respectively. The description in Lemma 8.1.5 and Corollary 8.4.5 of the asymptotic behavior for $k \to \infty$ of the set of real k-periodic fibers for α.
- The detailed proof of the various equivalent characterizations of rational elliptic surfaces in Theorem 9.1.3. The statement in Proposition 9.2.10 that an elliptic

surface with the same modulus function and monodromy as a rational elliptic surface is isomorphic to it.
- The various characterizations in Proposition 9.2.22 of the rational elliptic surfaces without reducible fibers. The detailed proof that their isomorphism classes are in bijective correspondence with the isomorphism classes of del Pezzo surfaces of degree one.
- The classification in Proposition 9.2.17 of the rational elliptic surfaces with a nondiscrete automorphism group.
- The analysis in Section 9.2.5 of a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that the corresponding rational elliptic surface has the generic configuration of singular fibers 12 I_1.
- The chaotic nature of the Hesse map from the complex projective line to itself, as formulated in Proposition 11.1.2, and the characterization of the Hesse surface in Proposition 11.1.6.
- The consequences in Proposition 10.1.2 of the condition that the biquadratic curves in the pencil are symmetric, or more generally, that the QRT transformation is the square of an element of the Mordell–Weil group.
- The relation in Section 10.2 between pencils of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and pencils of quadrics in $\mathbb{P}^2$. The ensuing identification in Section 10.3.3 of the QRT root defined by a pencil of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with the Poncelet mapping when the circumscribed quadrics belong to a pencil of quadrics in $\mathbb{P}^2$.
- The identification in Section 11.2 of the billiard map with a Poncelet map, and hence with a QRT root, where the pencil of circumscribed quadrics of the Poncelet mapping is dual to the confocal family of inscribed quadrics for the billiard map.
- The application in sections 11.5–11.7 and 11.4 of the theory to a number of examples from mathematical physics and to the Lyness map, respectively.
- The proof in Section 11.9 that the rational transformation of the plane, introduced by Viallet, Grammaticos, and Ramani [200, Section 2], is not birationally conjugate to a QRT map. On the other hand, since the Weierstrass data of this map are equal to the Weierstrass data of a QRT map, it behaves very much like a QRT map.
- The description in Section 12.1 of the singular fibers that appear after blowing up a singular member of a given pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, and the way the QRT automorphism permutes their irreducible components.

Acknowledgments

I thank Theo Tuwankotta, Jan Stienstra, Frans Oort, Frits Beukers, Reinout Quispel, John Roberts, Gert Heckman, Eduard Looijenga, Johan van de Leur, Erik van den Ban, Gunther Cornelissen, Heinz Hanßmann, Luuk Hoevenaars, Claude Viallet, and Alex Quintero Velez for their stimulation and expert help. Special thanks go to Jaap

Eldering, who provided the computer pictures, which in turn prompted me to develop the real aspects of QRT maps in more detail.

J.J. Duistermaat
Mathematisch Instituut, Universiteit Utrecht
January 10, 2010

Note by Johan Kolk
In March 2010 Hans Duistermaat passed away. As always he was actively engaged in research, but an aggressive illness took his life in a short period of just two weeks. By that time the manuscript of this book was completely finished. Ann Kostant, Springer, and Johan Kolk, Utrecht University, saw the book through its final editorial stages before sending it on to production.

Chapter 1
The QRT Map

1.1 The Rational Formula for the QRT Map

In Lemma 1.1.1 below we present the horizontal and vertical switches, defined in Section in terms of a pencil of biquadratic curves, as rational transformations of the plane. For this purpose it is convenient to write a biquadratic polynomial in the form

$$p(x,\ y) := \sum_{i,\ j=0}^{2} x^{2-i}\ A_{ij}\ y^{2-j} = X\ A\ Y, \tag{1.1.1}$$

where the coefficients of p are given by a 3×3 matrix $A = (A_{ij})_{i,\ j=0}^{2}$. In the shorthand notation $p(x,\ y) = X\ A\ Y$, X and Y denote the row and column vectors with coefficicients

$$Y_k := y^{2-k}, \quad X_k := x^{2-k} \quad \text{for} \quad k = 0,\ 1,\ 2. \tag{1.1.2}$$

With this notation, we have the following formulas for the QRT map.

Lemma 1.1.1 *Let A^0 and A^1 be two linearly independent 3×3 matrices, and let p^0 and p^1 be the biquadratic polynomials* (1.1.1) *with A replaced by A^0 and A^1, respectively. Let ι_1, ι_2, and τ be the horizontal switch, the vertical switch, and the QRT map defined by the pencil* (0.0.1) *of biquadratic curves. Define the vector-valued functions f and g of one variable by*

$$f(y) := (A^0\ Y) \times (A^1\ Y), \quad g(x) := (X\ A^0) \times (X\ A^1). \tag{1.1.3}$$

Then

$$\iota_1(x,\ y) = (\xi(x,\ y),\ y), \quad \text{where} \quad \xi(x,\ y) = \frac{f_0(y) - f_1(y)\,x}{f_1(y) - f_2(y)\,x}, \tag{1.1.4}$$

$$\iota_2(x,\ y) = (x,\ \eta(x,\ y)), \quad \text{where} \quad \eta(x,\ y) = \frac{g_0(x) - g_1(x)\,y}{g_1(x) - g_2(x)\,y}, \tag{1.1.5}$$

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_1,
© Springer Science+Business Media, LLC 2010

and

$$\tau(x,\ y) = \iota_2(\iota_1(x,\ y)) = (\xi(x,\ y),\ \eta(\xi(x,\ y),\ y)). \tag{1.1.6}$$

Proof. We carry out the recipe of Section for the computation of the horizontal switch. If p_z denotes the biquadratic polynomial (1.1.1) with $A = A_z := A^0 - z\,A^1$, then $p_z(x,\ y) = a_z(y)\,x^2 + b_z(y)\,x + c_z(y)$, where $a_z(y) = (A_z\,Y)_0$, $b_z(y) = (A_z\,Y)_1$, and $c_z(y) = (A_z\,Y)_2$. The horizontal switch of the point $(x,\ y)$ is equal to $(x',\ y)$, where $x' = -x - b_z(y)/a_z(y)$. If $(x,\ y)$ is not a base point, then we have to substitute in all the formulas the value of z such that $(x,\ y)$ lies on the curve $p_z(x,\ y) = 0$, that is, $z = p^0(x,\ y)/p^1(x,\ y) = (X\,A^0\,Y)/(X\,A^1\,Y)$. This leads to

$$A_z = A^0 - z\,A^1 = A^0 - \frac{X\,A^0\,Y}{X\,A^1\,Y}\,A^1 = \frac{1}{X\,A^1\,Y}\,((X\,A^1\,Y)\,A^0 - (X\,A^0\,Y)\,A^1),$$

hence $b_z(y)/a_z(y) = \beta/\alpha$, where

$$\begin{aligned}\beta &= (X\,A^1\,Y)\,(A^0\,Y)_1 - (X\,A^0\,Y)\,(A^1\,Y)_1,\\ \alpha &= (X\,A^1\,Y)\,(A^0\,Y)_0 - (X\,A^0\,Y)\,(A^1\,Y)_0.\end{aligned}$$

The coefficients of x^2, x, and 1 in β are equal to

$$\begin{aligned}(A^1\,Y)_0\,(A^0\,Y)_1 - (A^0\,Y)_0\,(A^1\,Y)_1) &= -f_2(y),\\ (A^1\,Y)_1\,(A^0\,Y)_1 - (A^0\,Y)_1\,(A^1\,Y)_1) &= 0,\end{aligned}$$

and

$$(A^1\,Y)_2\,(A^0\,Y)_1 - (A^0\,Y)_2\,(A^1\,Y)_1 = f_0(y),$$

respectively. The coefficients of x^2, x, and 1 in α are equal to

$$\begin{aligned}(A^1\,Y)_0\,(A^0\,Y)_0 - (A^0\,Y)_0\,(A^1\,Y)_0) &= 0,\\ (A^1\,Y)_1\,(A^0\,Y)_0 - (A^0\,Y)_1\,(A^1\,Y)_0) &= f_2(y),\end{aligned}$$

and

$$(A^1\,Y)_2\,(A^0\,Y)_0 - (A^0\,Y)_2\,(A^1\,Y)_0 = -f_1(y),$$

respectively. It follows that

$$\begin{aligned}x' &= -x - b_z(y)/a_z(y)\\ &= -x - \beta/\alpha\\ &= -(x^2\,f_2(y) - x\,f_1(y) - x^2\,f_2(y) + f_0(y))/(x\,f_2(y) - f_1(y))\\ &= (x\,f_1(y) - f_0(y))/(x\,f_2(y) - f_1(y)).\end{aligned}$$

The computation of the vertical switch ι_2 is analogous, and the QRT map had been defined in Section as $\tau = \iota_2 \circ \iota_1$. □

Quispel, Roberts, and Thompson *defined* the QRT map in [168] and [169] by means of the formulas (1.1.6), (1.1.4), (1.1.5). They subsequently proved that ι_1 and

ι_2 leave the biquadratic curves (0.0.1) invariant and therefore are the horizontal and vertical switches as defined in Section . Because ι_1 and ι_2, and therefore also the QRT map τ, leave each of the biquadratic curves (0.0.1) invariant, they leave the rational function

$$F(x,\ y) := q^0(x,\ y)/q^1(x,\ y) = \frac{X\ A^0\ Y}{X\ A^1\ Y} \tag{1.1.7}$$

invariant. The iterates $(\tau^k)_{k\in\mathbb{Z}}$ are called the *discrete dynamical system* generated by the QRT map, with $k \in \mathbb{Z}$ as the discrete time. It follows that F is an *integral* of the discrete dynamical system $(\tau^k)_{k\in\mathbb{Z}}$ in the sense that F is invariant under all the transformations τ^k, $k \in \mathbb{Z}$. For this reason $(\tau^k)_{k\in\mathbb{Z}}$ is called an *integrable discrete dynamical system*.

By definition, $\tau = \iota_2 \circ \iota_1$, where ι_1 and ι_2 are involutions; hence $\tau^{-1} = \iota_1 \circ \iota_2 = \iota_1 \circ \tau \circ {\iota_1}^{-1} = \iota_2 \circ \tau \circ {\iota_2}^{-1}$. This implies that for each $k \in \mathbb{Z}$, both ι_1 and ι_2 conjugate τ^k with τ^{-k}. One says in this situation that the discrete dynamical system defined by the QRT map is *time reversible*, with each of the involutions ι_1 and ι_2 of the plane acting as a time-reversing transformation.

Roberts and Quispel [175, Appendix A] observed that the area form

$$\omega := \frac{1}{X\ A^1\ Y}\ \mathrm{d}x \wedge \mathrm{d}y \tag{1.1.8}$$

is invariant under the QRT transformation τ. Here the denominator $Z\ A^1\ Y$ could have been be replaced by any of the biquadratic polynomials $z_0\ p^0(x,\ y) + z_1\ p^1(x,\ y)$. The corresponding invariance of the dual two-vector fields w such that $\omega \cdot w = 1$ follows from our Corollary 3.4.4. In summary, the QRT map is time-reversible, integrable, and area-preserving.

Finally Quispel, Roberts, and Thompson showed in [168] and [169] that quite a large number of rational transformations of the plane that occur in mathematical physics are QRT maps. Also for other examples from the literature, such as the elliptic billiard (Section 11.2) and the Lyness map (Section 11.4), it was a nontrivial discovery that these fit into the framework of QRT maps.

1.2 Indeterminacy of the QRT Map

In [168], [169] it is not mentioned whether the map acts on the real or the complex plane, although all the pictures are in the real plane. We will work in the complex plane in order to obtain more uniform results. For instance, a polynomial of degree d has d complex zeros when counted with multiplicities, whereas in the real domain d would only be an upper bound for the number of zeros. Also, complex projective algebraic manifolds often are nonempty and connected, whereas their real parts may be empty or have several connected components. The presentation of the map τ as a rational mapping from the affine plane to itself has several defects, even if we

work over the complex numbers. The same can be said about the presentation of the integral F as a rational function on the affine plane.

First, a rational function $r(x,\ y) = n(x,\ y)/d(x,\ y)$ of two variables becomes infinite at every point where $d(x,\ y) = 0$ and $n(x,\ y) \neq 0$. When this happens for one of the coordinates of $\tau(x,\ y)$, then for the iteration of τ we need to know the image of these points at infinity under the mapping τ, which is not given by the expressions in (1.1.4), (1.1.5). In our presentation of the QRT map in Chapter 3 we will resolve this by viewing both x and y as coordinates of a point on the complex projective line $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, where the infinite image point becomes a well-defined finite point in the Cartesian product $\mathbb{P}^1 \times \mathbb{P}^1$ of the two complex projective lines.

A more serious defect is that at the points $(x_0,\ y_0)$ where both the denominator and the numerator vanish, that is, where $d(x_0,\ y_0) = n(x_0,\ y_0) = 0$, the value of $r(x,\ y)$ is completely undetermined, in the sense that for any complex number c there exist points $(x,\ y)$ arbitrarily close to $(x_0,\ y_0)$ such that $d(x,\ y) \neq 0$ and $r(x,\ y) = c$. More precisely, for the given c the equation $n(x,\ y) - c\, d(x,\ y) = 0$ defines an algebraic curve C_c in the plane on which $r(x,\ y) = c$, and all the curves C_c, $c \in \mathbb{C}$, run through the same point $(x_0,\ y_0)$.

The same problem occurs for the invariant rational function (1.1.7), where the common zeros $(x_0,\ y_0)$ of the denominator and the numerator of F are the points $(x_0,\ y_0)$ that lie on every member of the pencil of biquadratic curves (0.0.1). Such a point $(x_0,\ y_0)$ is called a *base point* of the pencil (0.0.1), and we will see in Lemma 3.1.1 that, counted with multiplicities, there are eight base points in $\mathbb{P}^1 \times \mathbb{P}^1$. This implies that there is always at least one base point in $\mathbb{P}^1 \times \mathbb{P}^1$, whereas for generic pencils of biquadratic curves there are eight distinct base points. Therefore the problem of indeterminacy will always occur if one works in the complex space $\mathbb{P}^1 \times \mathbb{P}^1$, but often it will also happen at real points. In our presentation of the QRT map in Chapter 3 we will resolve the indeterminacies by passing to a complex two-dimensional manifold S on which the members of the pencil are separated from each other, basically by adding the parameter z in (0.0.2) as a tag to the variables.

1.3 Reconstruction

In this section we discuss the problem of finding explicit computational procedures to determine, for an arbitrary birational transformation τ of the plane, whether it is a QRT map and, if so, to find coefficient matrices A^0 and A^1 such that τ is equal to the QRT map defined by A^0 and A^1. We start with a reduction to a problem in fewer variables.

Because any linearly independent pair of linear combinations of A^0 and A^1 defines the same QRT map, it is the two-dimensional linear subspace spanned by A^0 and A^1 that has to be determined from τ, rather than the matrices themselves. For an n-dimensional vector space E, let $\mathrm{G}_2(E)$ denote the *Grassmann manifold* of all two-dimensional linear subspaces of E. If $L \in \mathrm{G}_2(E)$, then for each basis $e_1,\ e_2$ of L the two-vector $w = e_1 \wedge e_2$ is a nonzero element of $\Lambda^2 E$ of rank equal to two.

Let w be any element of $\Lambda^2 E$. Then w will be identified with the linear mapping $\epsilon \mapsto \mathrm{i}_\epsilon w : E^* \to E$, and the rank of w is defined as the dimension of the linear subspace $L = w(E^*)$ of E. There exist $k \in \mathbb{Z}$ such that $0 \le 2k \le n$ and linearly independent elements e_j, $1 \le j \le 2k$, in E such that $w = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2k-1} \wedge e_{2k}$. Because $\mathrm{i}_\epsilon(a \wedge b) = \epsilon(a)\, b - \epsilon(b)\, a$ for every $\epsilon \in E^*$ and $a,\ b \in E$, it follows that $w(E^*)$ is equal to the linear span of the e_j, $1 \le j \le 2k$, and rank $w = 2k$. On the other hand, the k-fold wedge product w^k of w with itself is equal to $k!\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge \cdots \wedge e_{2k-1} \wedge e_{2k} \neq 0$, whereas $w^{k+1} = 0$, and therefore k is also equal to the maximal nonnegative integer l such that $w^l \neq 0$, where $w^0 := 1$. In particular, the set $(\Lambda^2 E)_2$ of all rank-2 elements of $\Lambda^2 E$ is equal to the set of all $w \in \Lambda^2 E$ such that $w \neq 0$ and $w \wedge w = 0$.

Let $\mathrm{P}(\Lambda^2 E)$ denote the space of all one-dimensional linear subspaces of $\Lambda^2 E$, which is isomorphic to the m-dimensional projective space $\mathbb{P}^m$, with $m = (n\,(n - 1)/2) - 1$. Let $\mathrm{P}((\Lambda^2 E)_2)$ denote the set of all one-dimensional linear suspaces of $\Lambda^2 E$ that are contained in $(\Lambda^2 E)_2$. Then there is a unique mapping $\iota : \mathrm{P}((\Lambda^2 E)_2) \to \mathrm{G}_2(E)$ such that $w(E^*) = \iota(l)$ whenever $l \in (\Lambda^2 E)_2$ and $w \in l$. Furthermore, ι is an isomorphism from $\mathrm{P}((\Lambda^2 E)_2)$ onto $\mathrm{G}_2(E)$. The inverse of ι is given by the condition that $\iota(l) = L$ if and only if $e_1 \wedge e_2 \in l$ for all $e_1,\ e_2 \in L$ if and only if l is the span of $e_1 \wedge e_2$ for any basis e_1, e_2 of L.

For $n = \dim E = 4$, $\dim \Lambda^2 E = 4 \cdot 3/2 = 6$, hence $\mathrm{P}(\Lambda^2 E) \simeq \mathbb{P}^5$. Furthermore, $\dim \Lambda^4 E = 1$, and the equation $w \wedge w = 0$ identifies $\mathrm{P}((\Lambda^2 E)_2)$ with a quadric hypersurface in $\mathbb{P}^5$. It was the idea of Plücker [163] to identify the projective lines in $\mathbb{P}^3$, that is, the elements of $\mathrm{G}_2(E)$, with the elements of the quadric hypersurface in $\mathrm{P}(\Lambda^2 E)$ defined by the equation $w \wedge w = 0$. For arbitrary finite-dimensional vector spaces, the mapping $\mathbb{C}\, e_1 + \mathbb{C}\, e_2 \mapsto \mathbb{C}\, e_1 \wedge e_2 : \mathrm{G}_2(E) \to \mathrm{P}(\Lambda^2 E)$ is called the *Plücker embedding*. The linear coordinates on $\Lambda^2 E$ are called the *Plücker coordinates* on $\mathrm{G}_2(E)$, and the equation $w \wedge w = 0$ the *Plücker equation*. Because $\mathrm{G}_2(E)$ is a smooth manifold of dimension $2\,(n - 2)$, the solution set in $\Lambda^2 E \setminus \{0\}$ of the equation $w \wedge w = 0$ is a smooth manifold of dimension $2(n - 2) + 1$. It follows that if $n > 4$, there are dependencies between the $\binom{n}{4}$ quadratic equations $(w \wedge w)_j = 0$ for w.

In our situation, E is equal to the nine-dimensional vector space of all 3×3-matrices. The vector-valued functions $f(y)$ and $g(x)$ in (1.1.3) are polynomials in y and x of degree ≤ 4,

$$f_k(y) = \sum_{l=0}^{4} f_{kl}\, y^{4-l}, \quad g_k(x) = \sum_{l=0}^{4} g_{kl}\, x^{4-l}, \quad k = 0,\ 1,\ 2. \tag{1.3.1}$$

Each of the $2 \cdot 3 \cdot 5 = 30$ coefficients f_{kl}, g_{kl} depends in an antisymmetric bilinear way on $(A^0,\ A^1)$, and therefore is a linear function of the element $A^0 \wedge A^1 \in \Lambda^2 E$, the "Plücker coordinate" of the two-dimensional vector space spanned by A^0 and A^1. Let A_{ij} be the $(3i + j + 1)$th coordinate of the element $A \in E$, and let e_k, $1 \le k \le 9$, denote the standard basis of E with respect to these coordinates. Then the $e_k \wedge e_l$ with $1 \le k < l \le 9$ form a basis of $\Lambda^2 E$, where $\dim \Lambda^2 E = 36$.

The first test for τ to be a QRT map as in (1.1.6) is that its first coordinate has to be of the form $\xi(x,\ y)$ as in (1.1.4). This function determines the horizontal switch ι_1, and then the second test for τ is whether $\iota_2 = \tau \circ \iota_1$ is a vertical switch as in (1.1.5). The functions $\xi(x,\ y)$ and $\eta(x,\ y)$ in (1.1.4) and (1.1.5) determine the respective vector-valued functions function $f(y)$ and $g(x)$ up to a constant scalar multiple, and the third test for τ is whether these are polynomial functions of degree ≤ 4. The elements of $\Lambda^2 E$ that yield (1.1.3) with f and g replaced by $\alpha\, f$ and $\beta\, g$, respectively, are the

$$w = \alpha\, u + \beta\, v + \sum_{i=1}^{6} c_i\, n_i,$$

where

$$\begin{aligned}
u = {} & f_{00}\, e_4 \wedge e_7 + (f_{01}/2)\,(e_4 \wedge e_8 + e_5 \wedge e_7) + f_{02}\, e_5 \wedge e_8 \\
& + (f_{03}/2)\,(e_5 \wedge e_9 + e_6 \wedge e_8) + f_{04}\, e_6 \wedge e_9 - f_{10}\, e_1 \wedge e_7 \\
& - (f_{11}/2)\,(e_1 \wedge e_8 + e_2 \wedge e_7) - f_{12}\, e_2 \wedge e_8 - (f_{13}/2)\,(e_2 \wedge e_9 + e_3 \wedge e_8) \\
& - f_{14}\, e_3 \wedge e_9 + f_{20}\, e_1 \wedge e_4 + (f_{21}/2)\,(e_1 \wedge e_5 + e_2 \wedge e_4) \\
& + f_{22}\, e_2 \wedge e_5 + (f_{23}/2)\,(e_2 \wedge e_6 + e_3 \wedge e_5) + f_{24}\, e_3 \wedge e_6, \\
v = {} & g_{00}\, e_2 \wedge e_3 + (g_{01}/2)\,(e_2 \wedge e_6 - e_3 \wedge e_5) + g_{02}\, e_5 \wedge e_6 \\
& + (g_{03}/2)\,(e_5 \wedge e_9 - e_6 \wedge e_8) + g_{04}\, e_8 \wedge e_9 - g_{10}\, e_1 \wedge e_3 \\
& - (g_{11}/2)\,(e_1 \wedge e_6 - e_3 \wedge e_4) - g_{12}\, e_4 \wedge e_6 - (g_{13}/2)\,(e_4 \wedge e_9 - e_6 \wedge e_7) \\
& - g_{14}\, e_7 \wedge e_9 + g_{20}\, e_1 \wedge e_2 + (g_{21}/2)\,(e_1 \wedge e_5 - e_2 \wedge e_4) \\
& + g_{22}\, e_4 \wedge e_5 + (g_{23}/2)\,(e_4 \wedge e_8 - e_5 \wedge e_7) + g_{24}\, e_7 \wedge e_8,
\end{aligned}$$

$n_1 = e_4 \wedge e_9 + e_6 \wedge e_7 - 2\, e_5 \wedge e_8$, $n_2 = e_1 \wedge e_8 - e_2 \wedge e_7 - 2\, e_4 \wedge e_5$, $n_3 = e_2 \wedge e_9 + e_3 \wedge e_8 - 2\, e_5 \wedge e_6$, $n_4 = e_1 \wedge e_6 + e_3 \wedge e_4 - 2\, e_2 \wedge e_5$, $n_5 = e_3 \wedge e_7 - e_2 \wedge e_8 + e_4 \wedge e_6$, and $n_6 = e_1 \wedge e_9 - e_2 \wedge e_8 - e_4 \wedge e_6$. The fourth and final test for τ is whether there exist $\alpha \neq 0$, $\beta \neq 0$, and c_j, $1 \leq j \leq 6$, such that the rank of w is equal to two, that is, $w \wedge w = 0$. These are $\binom{9}{4} = 126$ quadratic equations for the eight unknowns α, β, and c_j, $1 \leq j \leq 6$. I have not tried to find the explicit equations and inequalities for the f_{kl} and g_{kl} such that these equations have a solution $\alpha \neq 0$ and $\beta \neq 0$. On the other hand, if w corresponds to such a solution, then $w(E^*)$ is the two-dimensional space of coefficient matrices of the sought-for biquadratic polynomials.

A complication that occurs in quite a number of the explicit examples in the literature is that the polynomials $f_0(y)$, $f_1(y)$, $f_2(y)$ in (1.1.3) have a common strictly positive degree, which is divided out in (1.1.4). That is, the horizontal switch may have degree $d_1 < 4$; see Definition 5.1.3. In this case the common factor is one of the unknowns in the problem.

There are special cases in which the reconstruction is straightforward. For instance, if τ is a QRT map, then each point of indeterminacy of ι_1 or ι_2 is a base point of the pencil of biquadratic curves. If these points of indeterminacy can be explicitly determined, then the vanishing of a biquadratic polynomial p at these points is a system of as many linear equations for p as the number of points of indeterminacy. If

this system of equations determines a two-dimensional vector space P of biquadratic polynomials, then we only need to check whether the mapping τ is equal to the QRT map defined by P.

Another case in which the reconstruction is explicit is for QRT roots of pencils of symmetric biquadratic curves; see Section 10.1.1. A case in which the reconstruction is very straightforward is for McMillan maps, see Section 11.5.

A quite different question is whether there are some general principles behind the fact that so many rational transformations of the plane that come from mathematical physics are QRT maps.

Chapter 2
The Pencil of Biquadratic Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In many of the examples of QRT maps in the literature, see Chapter 11, one is interested in the action of the iterates of the mapping in the real affine plane. However, as mentioned in Section 1.2, the results are much more uniform if the field $\mathbb{R}$ is replaced by its algebraic closure $\mathbb{C}$, and the complexified affine plane is compactified to the Cartesian product of two complex projective lines. The point is that in the complex projective setting, the full force of complex projective algebraic geometry is at our disposal. If the objects are defined over $\mathbb{R}$, then the real objects will be studied as the fixed-point sets in the complex manifolds of a complex conjugation, an involution that acts on tangent spaces as complex antilinear mappings.

A natural generalization of complex algebraic geometry is complex analytic geometry, and we will freely use the terminology of the latter. Our background reference for algebraic geometry in a complex analytic setting is the book of Griffiths and Harris [74]. More algebraically oriented, and working over more general fields than only the field of complex numbers, are Hartshorne [80] and Shafarevich [182]. In Sections 2.1 and 2.2 we introduce some of the definitions and notation that will be used throughout this book. These sections have grown into the present size in order to serve as a reasonably complete source of references. At a first reading of this book these sections may be skipped, and then consulted when referred to.

In Section 2.3 we discuss the basic properties of elliptic curves. In Section 2.4 we show that every smooth biquadratic curve is an elliptic curve, and compute, among others, the coefficients g_2 and g_3 of its Weierstrass normal form as explicit polynomial expressions in the coefficients of the biquadratic polynomial. In Section 2.5 we prove that the QRT map acts as a translation on the elliptic curve, and compute, in the Weierstrass normal form, the coordinates of the image point under the QRT map of the point at infinity, again as explicit polynomial expressions in the coefficients of the biquadratic polynomial. In Section 2.5.2 we apply this to pencils of biquadratic curves, whereas Section 2.5.3 is an exposition of the Picard–Fuchs equations for one-parameter families of Weierstrass curves, due to Bruns and Manin. Some real aspects of the above topics are discussed in Section 2.6

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_2,
© Springer Science+Business Media, LLC 2010

2.1 Complex Analytic Geometry

A *complex analytic manifold* of complex dimension n is a Hausdorff topological space M, provided with an atlas $\mathcal{A}$ of local coordinatizations χ onto open subsets of $\mathbb{C}^n$, such that for any $\chi,\ \psi \in \mathcal{A}$ in the atlas, the change of coordinates $\psi \circ \chi^{-1}$ is a complex analytic diffeomorphism, also called a biholomorphic transformation, from the open subset $\chi(U)$ of $\mathbb{C}^n$ onto the open subset $\psi(U)$ of $\mathbb{C}^n$, where U denotes the intersection of the domains of definition in M of χ and ψ. If $\mathbb{C}^n$ is identified in the usual way with $\mathbb{R}^{2n}$, then it follows that a complex analytic manifold of complex dimension n is a real analytic manifold of real dimension $2n$. Because the derivative of a complex analytic mapping is a complex linear mapping, it follows that for each $m \in M$ the tangent space $\mathrm{T}_m M$ of M at m, where M is viewed as a real analytic manifold, has a unique complex structure such that for each $\chi \in \mathcal{A}$ and m in the domain of definition of χ, the derivative at m of χ is a complex linear isomorphism from $\mathrm{T}_m M$ onto $\mathbb{C}^n$. If $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are complex analytic manifolds of respective dimensions n and p, then a mapping $f : M \to N$ is called *complex analytic* or *holomorphic* if for every $\chi \in \mathcal{A}$ and $\psi \in \mathcal{B}$, $\psi \circ f \circ \chi^{-1}$ is a complex analytic mapping from its domain of definition to $\mathbb{C}^p$.

2.1.1 Complex Analytic Sets

A subset A of M is called a *complex analytic subset of* M if for every $m \in M$ there are an open neighborhood U of m in M and a collection F of holomorphic functions $f : U \to \mathbb{C}$, such that $A \cap U$ is equal to the common zero-set of all $f \in F$. The local form of the definition is needed, because in our applications M is often compact and connected. It then follows from the maximum principle that every holomorphic function $f : M \to \mathbb{C}$ is a constant, and therefore the common zero-set of any collection of global holomorphic functions would be either M or empty. The condition that A is a complex analytic subset of M looks very weak, but it actually has very strong consequences. Below we mention some basic ones. For a more complete exposition we refer to Łojasiewicz [125, Chapters II, IV].

Because each $m \in M$ has an open neighborhood U in M such that $A \cap U$ is closed in U, A is a closed subset of M. A point $a \in A$ is called a *regular point* (of dimension k) of A if there exists an open neighborhood U of a in M such that $A \cap U$ is a complex analytic submanifold (of dimension k) of U. It follows that the set A° ($A^{\circ,k}$) of all regular points of A (of dimension k) is an open subset of A, where each $A^{\circ,k}$ is an immersed k-dimensional submanifold of M. The set of *singular points* is defined as $A^* = A \setminus A^\circ$. The complex analytic subset A of M is called *irreducible* if it is not equal to $B \cup C$, where B and C are complex analytic subsets of M, $B \neq A$ and $C \neq A$. There is a family $(A_i)_{i \in I}$, unique up to permutations of I, such that

(i) For each $i \in I$, A_i is an irreducible complex analytic subset of M.
(ii) A is equal to the union of the A_i over all $i \in I$.

(iii) For each $a \in A$ there exists an open neighborhood U of a in M such that the set of all $i \in I$ such that $A_i \cap U \neq \emptyset$ is finite.
(iv) If $i \neq j$ then A_i is not contained in A_j.

The A_i are called the *irreducible components* of A. Note that if A is compact, then (iii) implies that I is finite. The unique decomposition into irreducible components allows that for many questions the discussion can be reduced to irreducible complex analytic sets.

If A is irreducible, then there is a unique $k \in \mathbb{Z}_{\geq 0}$, called the *dimension* $\dim A$ of A, such that $A^\circ = A^{\circ,k}$. Furthermore, A^* is an analytic subset of M and $\dim B < k$ for each irreducible component B of A^*. Finally, A and A° are connected subsets of M. It follows that every compact complex analytic subset of M has only finitely many connected components.

Let A, B be complex analytic subsets of M such that A is irreducible and A is not contained in B. Then $A \setminus B$ is dense in A and an irreducible complex analytic subset of $M \setminus B$. In particular, $A \setminus B$ and $A^\circ \setminus B$ are connected. If U is an open subset of M such that $A \cap U \neq \emptyset$, then $A \cap U$ is not contained in $B \cap U$. It follows that if A and B are irreducible complex analytic subsets of M, and there exists an open subset U of M such that $A \cap U = B \cap U \neq \emptyset$, then $A = B$. In other words, an irreducible complex analytic set is determined by its germ at any of its points.

It follows immediately from the definition that the intersection of any collection of complex analytic subsets of M is a complex analytic subset of M. If A and B are analytic subsets of M, then for any $m_0 \in M$ there are open neighborhoods U and V of m_0 in M and collections F and G of holomorphic functions on U and V with common zero-sets $A \cap U$ and $B \cap V$, respectively. Let H denote the set of all product functions $f\,g$ on $W := U \cap V$, with $f \in F$ and $g \in G$. Then $(A \cup B) \cap W$ is equal to the common zero-set of H in W. This proves that $A \cup B$ is a complex analytic subset of M. In other words, the complements in M of analytic subsets of M form a topology on M, much weaker than the ordinary topology, which is the analogue of the Zariski topology in the complex analytic setting.

2.1.2 Divisors

A smooth *complex analytic surface* is a complex analytic manifold M such that $n = \dim M = 2$. A closed *complex analytic curve* in M is a one-dimensional complex analytic subset of M. In the study of QRT maps and elliptic surfaces we will quite often meet curves Z in surfaces M that locally are zero-sets of functions f_α, where it is essential to retain the vanishing orders o_i of the f_α's along the irreducible components Z_i of Z in the information. In this subsection we formalize this in the somewhat more general setting of complex analytic manifolds M of arbitrary dimension n. I learned the basic facts about divisors and line bundles from Griffiths and Harris [74, Section 1.1].

A complex analytic subset A of M is called a *hypersurface* if each of its irreducible components has dimension $n - 1$. Near each of its points a, a complex analytic

hypersurface is equal to the zero-set of a single holomorphic function f defined in an open neighborhood of a, a germ at a of a holomorphic function. Let $\mathcal{O}_a$ denote the ring of germs of holomorphic functions at a, with $\mathcal{I}_a := \{\varphi \in \mathcal{O}_a \mid \varphi(a) = 0\}$ as its unique maximal ideal. The germ $u \in \mathcal{O}_a$ is called a *unit* if it is an invertible element of $\mathcal{O}_a$, that is, $u(a) \neq 0$. If $\varphi \in \mathcal{I}_a$, then φ is called *irreducible* if one cannot write $\varphi = \psi\ \chi$ for $\psi,\ \chi \in \mathcal{I}_a$. For each nonzero element $f \in \mathcal{I}_a$ there exist finitely many irreducible $g_i \in \mathcal{I}_a$, where g_i is not a unit times g_j if $i \neq j$, and $k_i \in \mathbb{Z}_{>0}$, such that $f = {g_1}^{k_1} \cdots {g_h}^{k_h}$. The g_i are unique up to permutations and multiplications by units. The zero-sets Z_i^a near a of the g_i are called the *local components at a* of A. Each local component is contained in a unique irreducible component B of A such that $a \in B$. If $\varphi \in \mathcal{I}_a$ is irreducible with zero-set Z, then $\mathrm{d}\varphi(z) \neq 0$ for every $z \in Z^\circ$. It follows that f vanishes to order exactly k_i along $A^\circ \cap Z_i$.

Suppose that we have a covering of M with open subsets U_α and complex analytic functions $f_\alpha : U_\alpha \to \mathbb{C}$ not identically equal to zero on any connected open subset of U_α such that if $U_\alpha \cap U_\beta \neq \emptyset$, then $f_\alpha = f_{\alpha\beta}\, f_\beta$ for a holomorphic function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$. Interchanging the roles of α and β, we see that $1/f_{\alpha\beta}$ extends to a holomorphic function $f_{\beta\alpha}$ on $U_\alpha \cap U_\beta$, and it follows that $f_{\alpha\beta}$ has no zeros in $U_\alpha \cap U_\beta$. If Z_α denotes the zero-set of f_α, then $Z_\alpha \cap U_\beta = Z_\beta \cap U_\alpha$, and therefore the union Z of the Z_α's is a complex analytic subset of M. The vanishing order of f_α along $Z^\circ \cap U_\alpha$ defines a locally constant $\mathbb{Z}_{>0}$-valued function o_α on $Z^\circ \cap U_\alpha$, and it follows from $f_\alpha = f_{\alpha\beta}\, f_\beta$ for a holomorphic function $f_{\alpha\beta}$ without zeros on $U_\alpha \cap U_\beta$ that $o_\alpha = o_\beta$ on $Z^\circ \cap U_\alpha \cap U_\beta$. Because the sets $Z^\circ \cap U_\alpha$ form an open covering of Z°, it follows that the functions o_α have a unique common extension to a locally constant function $o : Z^\circ \to \mathbb{Z}_{>0}$. In particular, o is constant on every connected component of Z°. If Z_i, $i \in I$, denote the irreducible components of Z, then the sets

$$Z_i' = Z_i^\circ \setminus \bigcup_{j \neq i} Z_j$$

are the connected components of Z°, open and dense in Z_i°. It follows that for each $i \in I$ there is a constant $o_i \in \mathbb{Z}_{>0}$ such that $o = o_i$ on Z_i'. The number o_i is called the *order of vanishing of the f_α's along Z_i*.

Let conversely $Z_i, i \in I$, be a locally finite system of irreducible complex analytic hypersurfaces in M, and $o_i \in \mathbb{Z}_{>0}$. Then the union Z of the Z_i, $i \in I$, is a complex analytic hypersurface in M. Let $a \in Z$ and choose, for each local component Z_j^a at a of Z, an irreducible $g_j \in \mathcal{I}_a$ such that Z_j^a is the zero-set of g_j near a. Then $f \in \mathcal{I}_a$ vanishes to order o_i along Z_i if and only if $f = u\, {g_1}^{k_1} \cdots {g_h}^{k_h}$, where u is a unit at a and $k_j = o_i$ for every j such that $Z_j^a \subset Z_i$. If f' is another such germ, then $f' = v\, f$ for a unit v at a. It follows that there exists a system of holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that the U_α form an open covering of M and, for each $i \in I$, f_α vanishes to order o_i along $Z_i^\circ \cap U_\alpha$. For any such system, we have $f_\alpha = f_{\alpha\beta}\, f_\beta$ for a unique holomorphic function $f_{\alpha\beta}$ without zeros on $U_\alpha \cap U_\beta$. If $g_\beta : V_\beta \to \mathbb{C}$ is another such system, then we can arrange, by passing to a common refinement of the coverings U_α and V_β of M, that the coverings are the same, and we have $f_\alpha = u_\alpha\, g_\alpha$ for a unique holomorphic function u_α without zeros on U_α. In

this situation the system of the g_β's is said to be equivalent to the system of the f_α's. Therefore the mapping that assigns to each system of f_α's the system of vanishing orders o_i induces a bijection from the set of equivalence classes of f_α's onto the set of locally finite systems of distinct irreducible hypersurfaces Z_i with corresponding $o_i \in \mathbb{Z}_{>0}$, where we take the empty hypersurface if none of the f_α's have zeros. A complex analytic hypersurface Z in M with a function that assigns to each irreducible component Z_i of Z a positive integer o_i is called an *effective divisor*, where $Z = \emptyset$ is allowed. With the sum

$$(Z,\ (o_i)_{i\in I}) + (Z',\ (o_i')_{i\in I}) = (Z \cup Z',\ (o_i + o_i')_{i\in I}), \tag{2.1.1}$$

the set of divisors is a commutative semigroup. In (2.1.1), the index set I parametrizes the set of irreducible components Z_i of $Z \cup Z'$ and we take $o_i = 0$ and $o_i' = 0$ if Z_i is not an irreducible component of Z and Z', respectively. The notation

$$D = (Z,\ (o_i)_{i\in I}) = \sum_{i\in I} o_i\, Z_i, \tag{2.1.2}$$

where the Z_i such that $o_i > 0$ are the irreducible components of Z, is compatible with (2.1.1), and the number $o_i \in \mathbb{Z}_{\geq 0}$, the vanishing order of the f_α's on Z_i, is called the *multiplicity* in the divisor D of the irreducible component Z_i of Z.

A *meromorphic function* on a connected complex analytic manifold M is a function of the form $f = g/h$, where f and g are holomorphic functions on M, and h is not identically zero. If $x \in M$, $h(x) = 0$, and $g(x) \neq 0$, then one writes $f(x) = \infty$. In the following lemma the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is identified with the complex projective line $\mathbb{P}^1$, where $\infty = [0 : 1]$; see Section 2.2.

Lemma 2.1.1 *Let f be a nonzero meromorphic function on M. Then there is a subset N of M with the following properties:*

(i) *N is a complex analytic subset of M, of codimension two at any of its points.*
(ii) *There is a unique extension of f to a holomorphic mapping $M \setminus N \to \mathbb{P}^1$, which is again denoted by f.*
(iii) *The holomorphic mapping $f : M \setminus N \to \mathbb{P}^1$ is undetermined along N in the strong sense that for every $x \in N$ and every $c \in \mathbb{P}^1$ there is a sequence $x_j \in U \setminus N$ such that $x_j \to x$ and $f(x_j) \to c$ as $j \to \infty$.*

Proof. Let $a \in M$. Dividing away common factors in the germs at a of g and h, we obtain $f = G/H$ near a, where G and H are mutually prime germs of holomorphic functions at a. It follows that there is an open neighborhood U of a in M such that the common zero-set N in U of G and H is a complex analytic subset of U of codimension two. Let Z_c denote the zero-set Z_c of $G - c\,H$ and H if $c \in \mathbb{C}$ and $c = \infty$, respectively. Then Z_c is a complex analytic hypersurface in U such that $N \subset Z_c$, where $f = c$ on $Z_c \setminus N$. Because $\dim N < \dim Z_c$, N is contained in the closure of $Z_c \setminus N$. □

The set N in Lemma 2.1.1 is called the *set of indeterminacy* of the meromorphic function f. Note that $N = \emptyset$ if $\dim M \leq 1$, that is, every meromorphic function on

a complex analytic curve C extends to a holomorphic mapping $C \to \mathbb{P}^1$. If $N \neq \emptyset$, then we write $f : M \supset\!\!\to \mathbb{P}^1$ instead of $f : M \to \mathbb{P}^1$. Conversely, the *Levi extension theorem* says that if A is a complex analytic subset of M of codimension ≥ 2, and $f : M \setminus A \to \mathbb{P}^1$ is a holomorphic mapping, then f extends to a meromorphic function on M, the set of indeterminacy of which is contained in A. See Griffiths and Harris [74, p. 396, 397].

Let f be a nonzero meromorphic function on M. The prime decomposition of g and h in $\mathcal{O}_a$ leads to irreducible $g_i \in \mathcal{I}_a$, $1 \leq i \leq h$, unique up to permutations and multiplication by units, and corresponding $k_i \in \mathbb{Z}_{\neq 0}$ such that $f = {g_1}^{k_1} \cdots {g_h}^{k_h}$. Here $h = 0$ if and only if the germ at a of f extends to a unit. One says that the meromorphic function f has a *zero and pole of order* k_i *and* $-k_i$ *along* Z_i if $k_i > 0$ and $k_i < 0$, respectively. Here "along Z_i" means along the complement of the Z_j, $j \neq i$, in the nonsingular part Z_i° of the complex analytic hypersurface Z_i.

If in the above discussion of effective divisors we replace the holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ by meromorphic ones, where we retain that the $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$ are holomorphic and have no zeros, then we obtain a bijection from the set of equivalence classes of meromorphic f_α's onto the set of locally finite systems of distinct irreducible hypersurfaces Z_i with corresponding integers o_i, called *divisors*. The divisors, written as (2.1.2) with $o_i \in \mathbb{Z}$, form a commutative group $\mathrm{Div}(M)$ with respect to the sum (2.1.1). The group of divisors is the group generated by the semigroup of the effective divisors, where we write $D \geq 0$ if D is an effective divisor.

Let $\mathcal{O}^\times$, $\mathcal{M}^\times$, and $\mathcal{D}$ denote the sheaf of germs of nowhere vanishing holomorphic functions, meromorphic functions, and divisors, respectively. Here $\mathcal{O}^\times$ and $\mathcal{M}^\times$ are sheaves of commutative groups with respect to multiplication, whereas $\mathcal{D}$ is a sheaf of commutative groups with respect to addition. The group of divisors $\mathrm{Div}(M)$ is equal to the group $\mathrm{H}^0(M, \mathcal{D})$ of global sections of $\mathcal{D}$.

The homomorphism $\mathrm{Div} : \mathcal{M}^\times \to \mathcal{D}$ that assigns to any germ of a meromorphic function its divisor is surjective, with kernel equal to $\mathcal{O}^\times$. The short exact sequence of sheaves of commutative groups

$$1 \to \mathcal{O}^\times \to \mathcal{M}^\times \stackrel{\mathrm{Div}}{\to} \mathcal{D} \to 0 \tag{2.1.3}$$

induces the long exact sequence

$$1 \to \mathrm{H}^0(M, \mathcal{O}^\times) \to \mathrm{H}^0(M, \mathcal{M}^\times) \stackrel{\mathrm{Div}}{\to} \mathrm{Div}(M) \stackrel{\delta}{\to} \mathrm{H}^1(M, \mathcal{O}^\times) \to \cdots, \tag{2.1.4}$$

where $\mathrm{H}^0(M, \mathcal{O}^\times)$ is the space of all global holomorphic functions without zeros on M and $\mathrm{H}^0(M, \mathcal{M}^\times)$ is the space of all global meromorphic functions on M that do not vanish identically on any connected component of M. If M is compact and connected, then it follows from the maximum principle that $\mathrm{H}^0(M, \mathcal{O}^\times) \simeq \mathbb{C}^\times$, the multiplicative group of all nonzero complex numbers. The exactness of (2.1.4) at $\mathrm{H}^0(M, \mathcal{M}^\times)$ means that the divisor of a global meromorphic function f on M is equal to zero if and only if f is a nowhere vanishing holomorphic function on M. The exactness of (2.1.4) at $\mathrm{Div}(M)$ means that if D is a divisor on M, then $\delta(D) = 1$ if and only if D is the divisor of a global meromorphic function on M. For more

details on the *sheaf cohomology* used in other places of the book, see for instance Gunning [79, §3].

2.1.3 Holomorphic Line Bundles

In this subsection we show that every divisor D in M is equal to the divisor of a global meromorphic section s of some holomorphic line bundle L over M, where D is effective if and only if s is a global holomorphic section of L. See Lemma 2.1.2 below.

If $\pi : L \to M$ is a holomorphic complex line bundle over M, then there is a covering of M with nonempty open subsets M_α such that $\pi^{-1}(U_\alpha)$ admits a *trivialization*, a complex analytic diffeomorphism τ_α from $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{C}$ such that the restriction to $\pi^{-1}(U_\alpha)$ of π is equal to τ_α followed by the projection $(m,\ c) \mapsto m$ from $U_\alpha \times \mathbb{C}$ onto the first factor U_α.

It follows that for each $\alpha,\ \beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the *retrivialization* $\tau_\alpha \circ {\tau_\beta}^{-1}$ is a diffeomorphism of $(U_\alpha \times U_\beta) \times \mathbb{C}$ of the form

$$(m,\ c) \mapsto (m,\ f_{\alpha\beta}(m)\, c) \tag{2.1.5}$$

for a uniquely determined nowhere vanishing holomorphic complex-valued function $f_{\alpha\beta}$ on $U_{\alpha\beta}$. The 1-cocycle $f_{\alpha\beta}$ of elements of $\mathcal{O}^\times$ satisfies the *cocycle condition* $f_{\alpha\beta}\, f_{\beta\gamma} = f_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Conversely, every 1-cocycle of elements of $\mathcal{O}^\times$ leads to a holomorphic line bundle L over M, which is constructed by gluing the $U_\alpha \times \mathbb{C}$ together by means of the gluing maps (2.1.5). A 0-cocycle g is a system g_α of elements of $\mathcal{O}^\times$, defined on an open covering U_α of M. It defines a 1-cocycle $f_{\alpha\beta} := g_\alpha / g_\beta$, which is called the *coboundary* of g. The holomorphic line bundles L and L' defined by the 1-cocycles f and f' are isomorphic if and only if the 1-cocycle f/f' is a coboundary. The cohomology group $\mathrm{H}^1(M,\ \mathcal{O}^\times)$ is defined as the group of cocycles of elements of $\mathcal{O}^\times$ modulo its subgroup of coboundaries. It is a multiplicative group, called the *Picard group* $\mathrm{Pic}(M)$ of M. In other words, the above identifies the Picard group with the group of isomorphism classes $[L]$ of holomorphic line bundles L over M, where the tensor product of line bundles corresponds to the multiplication in the multiplicative group $\mathrm{Pic}(M) := \mathrm{H}^1(M,\ \mathcal{O}^\times)$.

We have the homomorphism

$$\delta : \mathrm{Div}(M) \to \mathrm{Pic}(M), \tag{2.1.6}$$

see (2.1.4), of which the kernel is equal to the group of divisors of global meromorphic functions on M, and the image is equal to the kernel of the homomorphism $\mathrm{H}^1(M,\ \mathcal{O}^\times) \to \mathrm{H}^1(M,\ \mathcal{M}^\times)$. In other words, a line bundle L is of the form $\delta(D)$ for a divisor D if and only if the 1-cocycle $f_{\alpha\beta}$ of its transition functions can be written as $f_{\alpha\beta} = g_\alpha / g_\beta$ for a 0-cocycle of meromorphic functions g_α, and if this is the case then the line bundle L is trivial if and only if D is the divisor of a global meromorphic function on M. See Griffiths and Harris [74, p. 134]. The homomor-

phism (2.1.6) leads to the habit that often the group structure in $\mathrm{Pic}(M)$ is denoted additively, which strictly speaking is not correct. Similarly one often writes "the line bundle $L = \delta(D)$" instead of "a line bundle L such that $[L] = \delta(D)$."

Let s be a global meromorphic section of L that does not vanish on any connected component of M. For each local trivialization $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, let f_α be equal to $\tau_\alpha \circ s$ followed by the projection $\pi_2 : (m,\ c) \mapsto c$ from $U_\alpha \times \mathbb{C}$ onto the second factor $\mathbb{C}$. Then f_α is a meromorphic function on U_α, and (2.1.5) shows that $f_\alpha = \pi_2 \circ \tau_\alpha \circ s = \pi_2 \circ (\tau_\alpha \circ {\tau_\beta}^{-1}) \circ \tau_\beta \circ s = f_{\alpha\beta} \circ \pi_2 \circ \tau_\beta \circ s = f_{\alpha\beta}\, f_\beta$ on $U_\alpha \cap U_\beta$. In other words, the f_α form a system of locally defined meromorphic functions as in Section 2.1.2, of which the divisor D is defined as *the divisor* $\mathrm{Div}(s)$ *of the meromorphic section* s *of* L. It follows that $[L] = \delta(\mathrm{Div}(s))$. We have the following conclusions, see Griffiths and Harris [74, p. 136].

Lemma 2.1.2 *Let M be connected and L a holomorphic line bundle over M. Then L has a nonzero meromorphic section s if and only $[L] = \delta(D)$ for some divisor D on M, where we can take $D = \mathrm{Div}(s)$. The section s of L is holomorphic if and only if the divisor D is effective. If s' is any other nonzero meromorphic section of L, then the line bundle $\delta(\mathrm{Div}(s') - D)$ is trivial, or equivalently, there exists a meromorphic function f on M such that* $\mathrm{Div}(s') = D + \mathrm{Div}(f)$.

The mapping $\mathrm{e}^{2\pi\,\mathrm{i}} : z \mapsto \mathrm{e}^{2\pi\,\mathrm{i}\,z}$ is a surjective homomorphism from the additive group $\mathbb{C}$ of all complex numbers to the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers, with kernel equal to the group $\mathbb{Z}$ of all integers. The short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\mathrm{e}^{2\pi\,\mathrm{i}}}{\to} \mathcal{O}^\times \to 1, \tag{2.1.7}$$

in which $\mathbb{Z}$, the constant sheaf of the integers, and $\mathcal{O}$, the sheaf of germs of locally defined holomorphic functions, are viewed as additive groups, induces the long exact sequence

$$\cdots \to \mathrm{H}^1(M,\ \mathcal{O}) \to \mathrm{H}^1(M,\ \mathcal{O}^\times) \overset{\mathrm{c}}{\to} \mathrm{H}^2(M,\ \mathbb{Z}) \to \mathrm{H}^2(M,\ \mathcal{O}) \to \cdots \tag{2.1.8}$$

in sheaf cohomology. The homomorphism c from the multiplicative group $\mathrm{H}^1(M,\ \mathcal{O}^\times) = \mathrm{Pic}(M)$ to the additive group $\mathrm{H}^2(M,\ \mathbb{Z})$ is the boundary operator of the long exact sequence. For each isomorphism class $[L]$ of a holomorphic line bundle L over M, the element $\mathrm{c}(L) := \mathrm{c}([L]) \in \mathrm{H}^2(M,\ \mathbb{Z})$ is called the *Chern class* of L; see Griffiths and Harris [74, p. 139]. If M is compact, D is a divisor in M, and $[L] = \delta(D)$, then the Chern class of L is equal to the image in $\mathrm{H}^2(M,\ \mathbb{Z})$ of the homology class in $\mathrm{H}_{2n-2}(M,\ \mathbb{Z})$ of D under the Poincaré duality isomorphism (2.1.17). See formula (2.1.24) below.

If L and L' are two holomorphic complex line bundles over M, then the *(tensor) product* of L and L' is the holomorphic complex line bundle $L \otimes L' = L\,L'$ over M of which the fiber over $m \in M$ is equal to $L_m \otimes L'_m = (L \otimes L')_m = (L\ L')_m =$ the tensor product of the fibers of L and L' over m. If $f_{\alpha\beta}$ and $f'_{\alpha\beta}$ are the respective transition functions of L and L', then the products $f_{\alpha\beta}\, f'_{\alpha\beta}$ are the transition functions of $L\,L'$. In other words, the induced product on the set of isomorphism classes of holomorphic

complex line bundles over M is equal to the product structure on $\mathrm{H}^1(M,\ \mathcal{O}^\times)$ induced by the product structure on $\mathcal{O}^\times$ used above. The isomorphism class of the trivial bundle $M \times \mathbb{C}$, with transition functions equal to 1, is the identity element for this product. Furthermore, if L^* denotes the *dual bundle* of L, of which the fiber over m is equal to the complex one-dimensional space ${L_m}^*$ of all linear forms $L_m \to \mathbb{C}$, then the pairing $(l,\ \lambda) \mapsto \langle l,\ \lambda\rangle := \lambda(l)$ defines an isomorphism from $L\ L^*$ onto the trivial bundle; hence L^* is canonically isomorphic to L^{-1}.

2.1.4 Pullback of Vector Bundles

Let $\lambda : L \to M$ be a holomorphic complex vector bundle over a complex analytic manifold M. Our main application will be to the case that L is a line bundle, that is, the rank of L, the dimension of the fibers of L, is equal to one. If Φ is a complex analytic mapping from a complex analytic manifold N to M, then the *pullback* Φ^*L *of* L *by means of* Φ is defined as the set

$$\Phi^*L := \{(n,\ l) \in N \times L \mid \Phi(n) = \lambda(l)\}.$$

This is a closed and smooth complex analytic submanifold of $N \times L$. Let $\pi_1 : \Phi^*L \to N$ and $\pi_2 : \Phi^*L \to L$ denote the restriction to Φ^*L of the respective projections $(n,\ l) \mapsto n$ and $(n,\ l) \mapsto l$. Then $\pi_1 : \Phi^*L \to N$ is a holomorphic complex vector bundle over N, of the same rank as L. For each $n \in N$ the restriction of π_2 to $(\Phi^*L)_n := {\pi_1}^{-1}(\{n\})$ is a complex linear isomorphism from the fiber $(\Phi^*L)_n$ of π_1 over n onto the fiber $L_{\Phi(n)}$ of λ over the point $\Phi(n) \in M$. We have the commutative diagram

$$\begin{array}{ccc} \Phi^*L & \stackrel{\pi_2}{\to} & L \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \lambda} \\ N & \stackrel{\Phi}{\to} & M \end{array}$$

in which the vertical arrows exhibit Φ^*L and L as holomorphic complex vector bundles over N and M, respectively. Phrased in a somewhat less precise way, Φ^*L is the holomorphic vector bundle over N of which the fiber over the point $n \in N$ is equal to the fiber $L_{\Phi(n)}$ of L over the point $\Phi(n) \in M$. If L is a line bundle, then the Chern homomorphism is functorial in the sense that

$$\mathrm{c}(\Phi^*L) = \Phi^*(\mathrm{c}(L), \tag{2.1.9}$$

where on the right-hand side we have used the pullback mapping $\Phi^* : \mathrm{H}^2(M,\ \mathbb{Z}) \to \mathrm{H}^2(N,\ \mathbb{Z})$ on sheaf cohomology.

Let $s : M \to L$ be a holomorphic section of $\lambda : L \to M$, that is, s is a holomorphic mapping from M to L such that $\lambda \circ s$ is equal to the identity on M. Then $\Phi^*(s) : n \mapsto (n,\ s(\Phi(n)))$ is a holomorphic section of $\pi_1 : \Phi^*L \to N$. Note that $\Phi^*(s)$ is the unique section $t : N \to \Phi^*L$ of π_1 such that $\pi_2 \circ t = s \circ \Phi$. This defines a linear mapping

$$\Phi^* : \mathrm{H}^0(M,\, \mathcal{O}(L)) \to \mathrm{H}^0(N,\, \mathcal{O}(\Phi^* L))$$

from the space of all holomorphic sections of L to the space of all holomorphic sections of $\Phi^* L$.

The following lemma is included for later reference. In the proof we use the *theorem on removable singularities of Riemann*, which states that if A is a complex analytic subset of positive codimension in a complex analytic manifold M and $f : M \setminus A \to \mathbb{C}$ is a holomorphic function such that each $a \in A$ has a neighborhood U in M such that f is bounded on $U \setminus A$, then f extends to a holomorphic function on M. See for instance Łojasiewicz [125, p. 106]. The origin of the theorem is Riemann [173, Section 12]. It follows that if φ is a continuous mapping from M to another complex analytic manifold N that is holomorphic if its restriction to $M \setminus A$ is holomorphic, then φ is holomorphic.

Lemma 2.1.3 *Let $\Phi : N \to M$ be a proper and surjective holomorphic mapping such that each nonsingular fiber of Φ is connected. Then the pullback mapping Φ^* acting on sections is a linear isomorphism from the space of all holomorphic sections of $\lambda : L \to M$ onto the space of all holomorphic sections of $\pi_1 : \Phi^* L \to N$.*

Proof. The injectivity of Φ^* follows from the surjectivity of Φ.

Let $t : N \to \Phi^* L$ be a holomorphic section of $\pi_1 : \Phi^* L \to N$. Then $t(n) = (n,\, \pi_2(t(n)))$ and $\Phi(n) = \lambda(\pi_2(n))$ for every $n \in N$. For every regular value m of Φ, the restriction of $\pi_2 \circ t$ to the fiber $N_m = \Phi^{-1}(\{m\})$ is a holomorphic mapping from N_m to the fiber L_m of L over m. Because N_m is compact and connected, it follows from the maximum principle that $(\pi_2 \circ t)|_{N_m}$ is constant, that is, there exists $s(m) \in L_m$ such that $\pi_2 \circ t(n) = s(m)$ for every $n \in N_m$. A consideration in local coordinates shows that $s(m)$ depends holomorphically on m. That is, s is a holomorphic section of L over the open subset M^{reg} of all regular values of Φ, and $t = \Phi^*(s)$ on $\Phi^{-1}(M^{\mathrm{reg}})$.

If K is a compact subset of M, then the properness of Φ implies that $\Phi^{-1}(K)$ is a compact subset of N. Because t is continuous, it follows that the restriction of t to $\Phi^{-1}(K)$ is bounded, which in combination with $t = \Phi^*(s)$ on $\Phi^{-1}(M^{\mathrm{reg}})$ implies that s is bounded on $M^{\mathrm{reg}} \cap K$. Because the complement of M^{reg} in M is a complex analytic hypersurface, it follows from Riemann's theorem on removable singularities that s has an extension to a holomorphic section of L over M, which we also denote by s. Because $t = \Phi^*(s)$ on $\Phi^{-1}(M^{\mathrm{reg}})$, $\Phi^{-1}(M^{\mathrm{reg}})$ is dense in N, and t and $\Phi^*(s)$ both are continuous, it follows that $t = \Phi^*(s)$ on N. □

2.1.5 Compact Riemann Surfaces

A *compact Riemann surface* C is a compact connected smooth complex one-dimensional complex analytic manifold. It is a real two-dimensional compact and connected smooth manifold, and the complex structure defines an orientation on it. Therefore, although from the complex point of view C is a curve, from the real point of view it is a surface, and both names are used in the literature.

The abelian differentials, that is, complex analytic complex one-forms on a compact Riemann surface C, constitute a finite-dimensional vector space, the (complex) dimension of which is called the *genus* $\mathrm{g}(C)$ of C. In the notation of Section 2.1.7, a holomorphic complex one-form on C is a global section of the sheaf Ω^1 and of the canonical bundle K_C of C, and therefore the space of all abelian differentials on C is equal to $\mathrm{H}^0(C,\, \Omega^1) = \mathrm{H}^0(C,\, \mathcal{O}(\mathrm{K}_C))$.

As a real two-dimensional surface, C is diffeomorphic to a sphere with a finite number of handles attached to it, and the number of handles is called the *topological genus* g of C. The group $\mathrm{H}_1(C,\, \mathbb{Z})$ of homology classes of 1-dimensional cycles on C is isomorphic to $\mathbb{Z}^{2g}$; see Farkas and Kra [61, I.2.5]. It is a quite remarkable fact that the complex analytic genus of C is equal to the topological genus of C in the sense that $\mathrm{g}(C) = g$, see Farkas and Kra [61, Proposition III.2.7]. If $\mathrm{g}(C) = 0$ then C is not only diffeomorphic to a sphere as a real manifold, but even complex analytic diffeomorphic to the complex projective line $\mathbb{P}^1$; see Farkas and Kra [61, III.4.9, Corollary 1].

If L and D are holomorphic complex line bundles and divisors on C, then the *degrees* of L and D are defined as

$$\deg(L) := \langle\, \mathrm{c}(L),\, \mathrm{H}(C)\rangle \quad \text{and} \quad \deg(D) := \langle 1,\, \mathrm{H}(D)\rangle, \tag{2.1.10}$$

respectively, where H is the mapping that assigns to the curve or a divsor on the curve its homology class. Here $\mathrm{c}(L) \in \mathrm{H}^2(C,\, \mathbb{Z})$, $\mathrm{H}(C) \in \mathrm{H}_2(C,\, \mathbb{Z})$, $1 \in \mathrm{H}^0(C,\, \mathbb{Z}) \simeq \mathbb{Z}$, $\mathrm{H}(D) \in \mathrm{H}_0(C,\, \mathbb{Z})$, and we have used the canonical pairing (2.1.18). Note that $\deg(D) = \sum_j m_j$ if $D = \sum_j m_j \{c_j\}$, where $c_j \in C$. The Poincaré duality (2.1.17) implies that $\mathrm{H}^2(C,\, \mathbb{Z}) \simeq \mathrm{H}_0(C, \mathbb{Z}) \simeq \mathbb{Z}$, which in turn implies that

$$\gamma \mapsto \langle \gamma,\, \mathrm{H}(C)\rangle : \mathrm{H}^2(C,\, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \tag{2.1.11}$$

is an isomorphism.

Let $c \in C$, and choose a local complex analytic coordinate z on an open neighborhood U of c in $\mathbb{C}$ such that $z(c) = 0$. Let δ_c be the holomorphic complex line bundle over C obtained by gluing the trivial bundles $(C \setminus \{c\}) \times \mathbb{C}$ and $U \times \mathbb{C}$ together along $U \setminus \{c\}$ by means of the transition function z. Then the section $s_c : C \to \delta_c$ of L_c, equal to 1 and z in the local trivializations over $C \setminus \{c\}$ and U, has only one zero, at p, where the zero is simple; hence $\mathrm{Div}(s_c) = \{c\}$, and it follows from Lemma 2.1.2 that $[\delta_c] = \delta(\{c\})$. Using the identification of Čech cohomology with singular cohomology, we have $\deg(\{c\}) = 1 = \deg(\delta_c)$. Because δ and the degrees are homomorphisms, it follows that $\deg(D) = \deg(\delta(D))$ for every divisor D on C. If L is a holomorphic line bundle over C with a meromorphic section s, then it follows from Lemma 2.1.2 that $[L] = \delta(\,\mathrm{Div}(s))$, and hence

$$\deg(L) = \deg(\,\mathrm{Div}(s)), \tag{2.1.12}$$

the number of zeros of s minus the numbers of poles of s, counted with multiplicities.

Theorem 2.1.4 *Every holomorphic line bundle L over a compact Riemann surface has at least one nonzero meromorphic section. The degree of L is equal to the number of zeros minus the number of poles of any meromorphic section of L, and also equal to the topological intersection number of any two continuous sections of L.*

Proof. Let $\mathcal{O}(L)$ denote the sheaf of germs of holomorphic sections of L. Let D be an effective divisor, a finite sum $D = \sum_j m_j c_j$ where $c_j \in C$ and $m_j \in \mathbb{Z}_{>0}$. Let $\delta \in \delta(D)$ and s a holomorphic section of δ such that $\mathrm{Div}(s) = D$. Write $L' = L\,\delta$. Then multiplication by s defines a homomorphism $\mu_s : \mathcal{O}(L) \to \mathcal{O}(L')$. The stalk at $c \in C$ of the quotient sheaf $\mathcal{S}_D := \mathcal{O}(L')/s\,\mathcal{O}(L)$ equals zero if $c \notin |D|$ and is isomorphic to $\mathbb{C}^{m_j}$ if $c = c_j$. In particular, the sheaf $\mathcal{S}_D$ is fine, which implies that $\mathrm{H}^j(C,\, \mathcal{S}_D) = 0$ for every $j > 0$. It follows that the long exact sequence induced by the short exact sequence $0 \to \mathcal{O}(L) \xrightarrow{\mu_s} \mathcal{O}(L') \to \mathcal{S}_D \to 0$ breaks off at

$$0 \to \mathrm{H}^0(C,\, \mathcal{O}(L)) \xrightarrow{\mu_s} \mathrm{H}^0(C,\, \mathcal{O}(L')) \to \mathrm{H}^0(C,\, \mathcal{S}_D) \xrightarrow{\delta} \mathrm{H}^1(C,\, \mathcal{O}(L)) \xrightarrow{\mu_s} \mathrm{H}^1(C,\, \mathcal{O}(L')) \to 0.$$

Because C is compact, it follows from (6.2.24) with $M = C$ and $p = 0$ that the cohomology groups in this sequence are finite-dimensional, whereas the higher cohomology groups are equal to zero. Because the alternating sum of the dimensions of the vector spaces in a finite exact sequence is equal to zero, and $\dim \mathrm{H}^0(C,\, \mathcal{S}_D) = \deg(D)$, we obtain, with the notation $\chi(\mathcal{O}(L)) := \dim \mathrm{H}^0(C,\, \mathcal{O}(L)) - \dim \mathrm{H}^1(C,\, \mathcal{O}(L))$ that $\chi(\mathcal{O}(L')) = \chi(\mathcal{O}(L)) + \deg(D)$. Note that $\chi(\mathcal{O}(L))$ is equal to the holomorphic Euler number defined in (6.2.30). Because $\dim \mathrm{H}^0(C,\, \mathcal{O}(L')) = \chi(\mathcal{O}(L')) + \dim \mathrm{H}^1(C,\, \mathcal{O}(L')) \geq \chi(\mathcal{O}(L'))$, we obtain that $\dim \mathrm{H}^0(C,\, \mathcal{O}(L')) \geq 1$ if $\deg(D) \geq 1 - \chi(\mathcal{O}(L))$. Then the holomorphic line bundle L' has at least one nonzero holomorphic section s', and s'/s is a nonzero meromorphic section of L, as desired.

The bundle P, of which the fiber over $c \in C$ is equal to the space of complex linear isomorphisms $\mathbb{C} \to L_c$, is a principal $\mathbb{C}^\times$-bundle over C such that L is isomorphic to the associated bundle $P \times_{\mathbb{C}^\times} \mathbb{C}$. Here $u \in \mathbb{C}^\times$ acts on $P \times \mathbb{C}$ by sending $(p,\, v)$ to $(p\,u^{-1},\, u\,v)$. The associated $\mathbb{P}^1$-bundle $P \times_{\mathbb{C}^\times} \mathbb{P}^1$, over C, where $u \in \mathbb{C}^\times$ sends $(p,\, [v_0 :\, v_1])$ to $(p\,u^{-1},\, [v_0 :\, u\,v_1])$, is identified with a compactification $\widehat{L}$ of L, where $\widehat{L_c} = L_c \cup \{\infty_c\}$ for every $c \in C$. These $\mathbb{P}^1$-bundles are examples of ruled surfaces; see Section 6.2.3.

The bundle $\widehat{L}$ has the two disjoint holomorphic sections $0_{\widehat{L}}$ and $\infty_{\widehat{L}}$, and $\deg(L) = s \cdot 0_{\widehat{L}} - s \cdot \infty_{\widehat{L}}$ for every meromorphic section s of L. Here $A \cdot B$ denotes the intersection number of the real two-dimensional cycles A and B as in Subsection 2.1.6. Using C^∞ partitions of unity, constant Hermitian structures on local trivializations of L can be pieced together to a C^∞ Hermitian structure h on L. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ cutoff function, that is, χ is compactly supported and $\chi = 1$ on a neighborhood of the origin in $\mathbb{R}$. Then

$$\rho(c) := \chi(|s(c)|)\, s(c) + (1 - \chi(|s(c)|)\, h_c(s(c),\, s(c))^{-1}\, s(c), \quad c \in C,$$

defines a C^∞ section ρ of L, where $\rho = s$ near points where $s = 0$, and $\rho = h(s,\ s)^{-1}\, s$ near points where $s = \infty$. It follows that $\rho \cdot 0_{\widehat{L}} = s \cdot 0_{\widehat{L}} - s \cdot \infty_{\widehat{L}} = \deg(L)$. Because all continuous sections of L, that is, continuous sections of $\widehat{L}$ that do not meet $\infty_{\widehat{L}}$, are homotopic, hence homologous to each other in L, and because the intersection number is a homology invariant, it follows that $\deg(L) = r \cdot 0_{\widehat{L}} = r \cdot r'$ for any two continuous sections r and r' of L. □

Remark 2.1.5. For any continuous section s of $\widehat{L}$ we have $s \cdot 0_{\widehat{L}} - s \cdot \infty_{\widehat{L}} = \rho \cdot 0_{\widehat{L}} = \deg(L)$. In particular, $\infty_{\widehat{L}} \cdot \infty_{\widehat{L}} = -\deg(L)$. Therefore, if $\deg(L) \neq 0$, then either the zero section or the section at infinity of $\widehat{L}$ is rigid in $\widehat{L}$, in the sense of Lemma 2.1.9 below. ⊘

It follows from Theorem 2.1.4 and Lemma 2.1.2 that for every holomorphic line bundle L there exist a finite sequence of points $c_j \in C$ and $m_j \in \mathbb{Z}_{\neq 0}$ such that L is isomorphic to $\prod_j {\delta_{c_j}}^{m_j}$. If L_+ is the product of the $\delta_{c_j}^{m_j}$ for the positive m_j's, and $L_- = L\, {L_+}^{-1}$, then it follows from the proof of Theorem 2.1.4 that $\chi(\mathcal{O}(L)) = \chi(\mathcal{O}(L_-)) + \deg(L_+)$ and $\chi(\mathcal{O}(L)) = \chi(\mathcal{O}(L_-)) - \deg(L_-))$ hence $\chi(\mathcal{O}(L)) = \chi(\mathcal{O}) + \deg(L_-) + \deg(L_+) = \chi(\mathcal{O}) + \deg(L)$.

The Serre duality theorem (6.2.25) with $M = C$, $n = q = 1$, and $p = 0$ implies that

$$\dim \mathrm{H}^1(C,\ \mathcal{O}(L)) = \dim \mathrm{H}^0(C,\ \mathcal{O}(\mathrm{K}_C\ L^*)), \tag{2.1.13}$$

where we note that $\Omega^1(L^*) = \mathrm{K}_C\ L^*$ if K_C denotes the canonical line bundle of C. For L equal to the trivial line bundle this number is equal to the dimension of the space of all holomorphic sections of K_C = holomorphic complex one-forms on C, and therefore equal to the genus $\mathrm{g}(C)$ of C. Therefore the equation $\chi(\mathcal{O}(L)) = \chi(\mathcal{O}) + \deg(L)$ is equivalent to the highly nontrivial and useful *Riemann–Roch formula*

$$\chi(\mathcal{O}(L)) = \dim \mathrm{H}^0(C,\ \mathcal{O}(L)) - \dim \mathrm{H}^0(C,\ \mathcal{O}(\mathrm{K}_C\ L^*)) = 1 - \mathrm{g}(C) + \deg(L). \tag{2.1.14}$$

See, for instance, Gunning [79, p. 111] or Griffiths and Harris [74, pp. 245, 246]. If $L = \mathrm{K}_C$, then $\dim \mathrm{H}^0(C,\ \mathcal{O}(L)) = \mathrm{g}(C)$ and $\dim \mathrm{H}^0(C,\ \mathcal{O}(\mathrm{K}_C\ L^*)) = 1$, and (2.1.14) implies that

$$\deg(\mathrm{K}_C) = 2\,\mathrm{g}(C) - 2. \tag{2.1.15}$$

I learned the following proposition from Griffiths and Harris [74, 3. on p. 215].

Proposition 2.1.6 *Let L be a holomorphic line bundle over a compact Riemann surface, and assume that* $\deg L > 2\ \mathrm{g}(C)$. *Let $V = \mathrm{H}^0(C,\ \mathcal{O}(L))$ denote the space of all holomorphic sections of L over C. Then for each $c \in C$ the set $\kappa(c)$ of all $v \in V$ such that $v(c) = 0$ is a complex codimension one linear subspace of V, and the mapping $\kappa : c \mapsto \kappa(c)$ defines a complex analytic diffeomorphism from C onto a nonsingular algebraic curve in* $\mathrm{P}(V^*)$.

Proof. For any $c,\ c' \in C$, write $L' = L\, {\delta_c}^{-1}\, {\delta_{c'}}^{-1}$. Then $\deg(\mathrm{K}_C\, (L')^*) = \deg(\mathrm{K}_C) - (\deg(L) - 2) = 2\ \mathrm{g}(C) - \deg(L) < 0$, where we have used (2.1.15). Therefore (2.1.13) with L replaced by L' yields that $\mathrm{H}^1(C,\ \mathcal{O}(L')) = 0$.

Because $[L] = [L'] \, \delta(\{c\} + \{c'\})$, we have the exact sequence as in the first part of the proof of Theorem 2.1.4, with L', L, and D replaced L, L', and $\{c\} + \{c'\}$, respectively. The exactness of the part $\mathrm{H}^0(C, \mathcal{O}(L)) \to \mathrm{H}^0(C, \mathcal{S}_{\{c\}+\{c'\}}) \to 0$ means that $V_{c,c'} := \{v \in V \mid v(c) = v(c') = 0\}$ is a linear subspace of V of codimension two when $V_{c,c}$ is the space of all $v \in V$ such that v vanishes of order ≥ 2 at c. It follows that $\kappa(c)$ and $\kappa(c')$ have codimension one in V and are not equal to each other when $c' \neq c$, whereas the statement that $V_{c,c}$ has codimension two means that at each $c \in C$ the tangent map of κ is not equal to zero. That is, $\kappa : C \to \mathrm{P}(V^*)$ is a complex analytic injective immersion. Because C is compact, it follows that κ is a complex analytic embedding from C onto a compact complex one-dimensional complex analytic submanifold $\kappa(C)$ of $\mathrm{P}(V^*)$. In view of Chow's theorem, $\kappa(C)$ is a nonsingular algebraic curve in $\mathrm{P}(V^*)$. □

The fact that for every holomorphic line bundle L over C there exists a divisor D on C such that $[L] = \delta(D)$ means that the homomorphism δ in the long exact sequence (2.1.4) is surjective, and therefore the mapping $\mathrm{H}^1(C, \mathcal{O}^\times) \to \mathrm{H}^1(C, \mathcal{M}^\times)$ is injective. Because divisors in C are just integral linear combinations of isolated points in C, the sheaf $\mathcal{M}^\times/\mathcal{O}^\times \simeq \mathcal{D}$ is fine, and therefore $\mathrm{H}^1(C, \mathcal{M}^\times/\mathcal{O}^\times)$ is trivial. It follows that the group $\mathrm{H}^1(C, \mathcal{M}^\times)$ is trivial as well. In general such conclusions do not hold for compact complex analytic manifolds M of complex dimension > 1.

We have seen above that $\mathrm{H}^1(C, \mathcal{O}) \simeq \mathbb{C}^g$ and $\mathrm{H}^2(C, \mathcal{O}) = 0$, where $g = \mathrm{g}(C)$. The exactness of (2.1.8) at $\mathrm{H}^2(C, \mathbb{Z})$ in combination with $\mathrm{H}^2(C, \mathcal{O}) = 0$ implies that the Chern homomorphism $\mathrm{c} : \mathrm{H}^1(C, \mathcal{O}^\times) \to \mathrm{H}^2(C, \mathbb{Z})$ is surjective. That is, for every $d \in \mathbb{Z}$ there exists a holomorphic line bundle L over C with degree equal to d. The exactness of (2.1.8) at $\mathrm{H}^1(C, \mathcal{O}^\times)$ means that the kernel of c is equal to the image of $\mathrm{H}^1(C, \mathcal{O}) \to \mathrm{H}^1(C, \mathcal{O}^\times)$. The exactness of (2.1.8) at $\mathrm{H}^1(C, \mathcal{O})$ means that the kernel of $\mathrm{H}^1(C, \mathcal{O}) \to \mathrm{H}^1(C, \mathcal{O}^\times)$ is equal to the image of $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O})$, whereas the exactness of (2.1.8) at $\mathrm{H}^1(C, \mathbb{Z})$ means that the kernel of $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O})$ is equal to the image of the boundary operator $\mathrm{H}^0(C, \mathcal{O}^\times) \to \mathrm{H}^1(C, \mathbb{Z})$. Now $\mathrm{H}^0(C, \mathcal{O}^\times) \simeq \mathbb{C}^\times$ and $\mathrm{H}^1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, and therefore the continuous homomorphism $\mathrm{H}^0(C, \mathcal{O}^\times) \to \mathrm{H}^1(C, \mathbb{Z})$ is trivial; hence $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O})$ is injective, and the kernel of c is isomorphic to the quotient of $\mathrm{H}^1(C, \mathcal{O}) \simeq \mathbb{C}^g$ by means of an additive subgroup Γ isomorphic to $\mathbb{Z}^{2g}$. Because $\ker(\mathrm{c})$ is a Hausdorff topological space, it follows that Γ is a lattice in $\mathrm{H}^1(C, \mathcal{O})$ and $\ker(\mathrm{c})$ is a complex g-dimensional torus $\mathbb{C}^g/\Gamma$. Because $\ker(\mathrm{c}) = \ker(\deg)$, see (2.1.10) and (2.1.11), it follows that the isomorphism classes of holomorphic line bundles of a given degree form a complex g-dimensional torus. See Gunning [79, pp. 134–138] for more information about $\ker(\mathrm{c}) = \ker(\deg)$.

Remark 2.1.7. If $g = 0$ when C is isomorphic to the complex projective line $\mathbb{P}^1$, then there exists for every $d \in \mathbb{Z}$ exactly one isomorphism class of holomorphic line bundles of degree d over C, the line bundle $\mathrm{O}(d)$ discussed in Section 3.2.3 and in Example 5. ⊘

2.1.6 Intersection Numbers

The *triangulation theorem of Łojasiewicz* [123] states that for every locally finite family $\mathcal{F}$ of closed real semianalytic subsets of $\mathbb{R}^N$ there exists a semianalytic stratification of $\mathbb{R}^N$ compatible with $\mathcal{F}$, of which the strata are the images of simplices of a geometric complex K in $\mathbb{R}^N$ by means of a homeomorphism h from the support $|K|$ of K to $\mathbb{R}^N$, such that all restrictions $h|_S$, $S \in K$, are real analytic isomorphisms from S onto $h(S)$. The embedding theorem of Grauert [68] implies that this also holds with $\mathbb{R}^N$ replaced by any real analytic manifold M.

Let E and F be complex n-dimensional vector spaces. If $A : E \to F$ and $B : E \to F$ are linear isomorphisms, then $A = B \circ C$ for a linear automorphism $C : E \to E$. If $E_{\mathbb{R}}$ denotes E viewed as a vector space over $\mathbb{R}$, then the determinant of the real linear transformation $C_{\mathbb{R}} : E_{\mathbb{R}} \to E_{\mathbb{R}}$ is equal to $\det C_{\mathbb{R}} = |\det C|^2 > 0$. It follows that if we choose an orientation on one complex n-dimensional vector space, viewed as a vector space over $\mathbb{R}$, then there is a unique orientation on all complex n-dimensional vector spaces such that all complex linear isomorphisms are orientation-preserving. The usual identification $\mathbb{R}^2 \xrightarrow{\sim} \mathbb{C} : (x,\ y) \mapsto x + \mathrm{i}\, y$, defines a real linear isomorphism $\mathbb{R}^{2n} \simeq (\mathbb{R}^2)^n \xrightarrow{\sim} \mathbb{C}^n$, by means of which the standard orientation of $\mathbb{R}^{2n}$ is transferred to an orientation of $\mathbb{C}^n$. In this way all finite-dimensional complex vector spaces have a canonically defined orientation. Applying this to the tangent spaces of complex analytic manifolds, we obtain a canonically defined orientation on every complex analytic manifold such that every complex analytic diffeomorphism between complex analytic manifolds is orientation-preserving.

If A is a complex k-dimensional complex analytic subset of the complex n-dimensional complex analytic manifold M, then its regular part A° is a real $2k$-dimensional oriented manifold, whereas its singular part A^* is a complex analytic subset of M of complex dimension $\leq k-1$. That is, all strata of A^* have real dimension $\leq 2k-2$. It follows that *every complex k-dimensional complex analytic set is an oriented real $2k$-dimensional cycle*.

Let M be an oriented smooth real d-dimensional manifold, and let A and B be oriented chains in M of respective real dimensions a and b, where $a + b = d$. Furthermore, let $|A| \cap |B|$ be compact, $|A| \cap |\partial B| = \emptyset$, and $|\partial A| \cap |B| = \emptyset$, where $|C|$ denotes the support of any chain C. These conditions hold if A and B are compactly supported cycles, but especially when M is not compact, it will be convenient to have the intersection number in more general situations.

For each connected component I of $|A| \cap |B|$ and each open neighborhood U of I in M disjoint from the other connected components of $|A| \cap |B|$, there exist oriented cycles A' and B' such that $A' - A = \partial\alpha$, $B' - B = \partial\beta$, where α and β are chains of respective dimensions $a+1$ and $b+1$ with supports in U, such that $A' \cap U$ and $B' \cap U$ intersect only in their smooth parts, and all the intersections are *transversal*, that is, $\mathrm{T}_{i'} A' \cap \mathrm{T}_{i'} B' = 0$ for every $i \in A' \cap B' \cap U$. The intersection number $A' \cdot_{i'} B'$ at $i' \in A' \cap B' \cap U$ is equal to $+1$ and -1 if the bases $e_1, \ldots, e_a, f_1, \ldots, f_b$ of $\mathrm{T}_{i'} M$ are positively and negatively oriented, respectively, when $e_1, \ldots, e_a$ is a positively oriented basis of $\mathrm{T}_{i'} A'$ and $f_1, \ldots, f_b$ is a positively oriented basis of $\mathrm{T}_{i'} B'$. The

intersection number $A \cdot_I B \in \mathbb{Z}$ *of* A *and* B *along* I is defined as the sum over all $i' \in A' \cap B' \cap U$ of the intersection numbers $A' \cdot_{i'} B'$, where this number does not depend on the choices of the chains α and β with support in U. The number

$$A \cdot B = \sum_I A \cdot_I B \in \mathbb{Z}, \tag{2.1.16}$$

where the sum is over the finitely many connected components I of $|A| \cap |B|$, is called the *intersection number* of A and B. We have $B \cdot A = (-1)^{a\,b}\, A \cdot B$. The intersection number of A and B is a *homological invariant*, in the sense that $A \cdot B = A \cdot B'$ if $B' = B + \partial C$, where C is a $(b+1)$-dimensional chain, $|A| \cap |C|$ is compact, and $|\partial A| \cap |C| = \emptyset$.

The intersection number has been extended by de Rham [172] to distributional differential forms, in the framework of which the proofs of identities for intersection numbers are straightforward. In this book, the de Rham isomorphism will be used in order to identify the space of all closed k-forms on M modulo exact ones with the cohomology group $\mathrm{H}^k(M, \mathbb{R}) \simeq \mathrm{H}^2(M, \mathbb{Z}) \otimes \mathbb{R}$. The presentation of Weil [208] explains, among others, the canonical identification of the Čech cohomology groups, which are sheaf cohomology groups for the sheaf of germs of constant functions, such as the group $\mathrm{H}^2(M, \mathbb{Z})$ appearing in (2.1.8), with the singular cohomology groups used in intersection theory.

If M is compact, then it has a C^∞ triangulation K. Using barycentric subdivisions, one can arrange that M has a pair of dual C^∞ triangulations K and K', where each a-dimensional simplex in K intersects exactly one $b = (d-a)$-dimensional simplex in K', with a unique intersection point, which lies in the interior of each of the simplices and at which the intersection is transversal. In this situation the mapping $A \mapsto (B \mapsto A \cdot B)$ defines an isomorphism i from the chain complex $\mathrm{C}_a(K)$ onto the cochain complex $\mathrm{C}^b(K') = \mathrm{Hom}(\mathrm{C}_b(K'), \mathbb{Z})$ such that $i \circ \partial = (-1)^{b+1} \delta \circ i$, where $\delta = \partial^*$ is the coboundary operator. Here $\mathrm{C}_a(K)$ is the free abelian group generated by the a-dimensional simplices in K. Since $\mathrm{H}_a(M, \mathbb{Z})$ and $\mathrm{H}^b(M, \mathbb{Z})$ are canonically isomorphic to $((\ker \partial) \cap \mathrm{C}_a(K))/\partial(\mathrm{C}_{a+1}(K))$ and $((\ker \delta) \cap \mathrm{C}^b(K'))/\delta(\mathrm{C}^{b-1}(K'))$, respectively, the isomorphism i induces a canonical isomorphism

$$\mathrm{pd} : \mathrm{H}_a(M, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^b(M, \mathbb{Z}), \quad a + b = d := \dim_{\mathbb{R}} M, \tag{2.1.17}$$

called the *Poincaré duality isomorphism*. The map (2.1.17) is the map D of Griffiths and Harris [74, p. 55]. The canonical pairing

$$(\gamma, c) \mapsto \langle \gamma, c \rangle : \mathrm{H}^b(M, \mathbb{Z}) \times \mathrm{H}_b(M, \mathbb{Z}) \to \mathbb{Z}, \tag{2.1.18}$$

induced by the canonical pairing between $\mathrm{C}_b(K')$ and $\mathrm{C}^b(K')$, is *unimodular* in the sense that for every homomorphism $h : \mathrm{H}^b(M, \mathbb{Z}) \to \mathbb{Z}$ there exists a $c \in \mathrm{H}_b(M, \mathbb{Z})$ such that $h(\gamma) = \langle \gamma, c \rangle$ for every $\gamma \in \mathrm{H}^b(M, \mathbb{Z})$, and $c \in \mathrm{H}_b(M, \mathbb{Z})_{\mathrm{tor}}$ if and only if $\langle \gamma, c \rangle = 0$ for every $\gamma \in \mathrm{H}^b(M, \mathbb{Z})$. A similar statement holds with $\mathrm{H}_b(M, \mathbb{Z})$ and $\mathrm{H}^b(M, \mathbb{Z})$ interchanged. Here for any commutative group H the *torsion subgroup*

H_{tor} *of* H is defined by

$$H_{\text{tor}} := \{h \in H \mid m\,h = 0 \quad \text{for some} \quad m \in \mathbb{Z}_{>0}\}. \tag{2.1.19}$$

The formula

$$A \cdot B = \mathrm{H}(A) \cdot \mathrm{H}(B) := \langle \mathrm{pd}(\mathrm{H}(A)),\ \mathrm{H}(B)\rangle, \tag{2.1.20}$$

for any oriented cycles A and B of respective dimensions a and $b = d - a$, defines a $\mathbb{Z}$-valued $\mathbb{Z}$-bilinear form on $\mathrm{H}_a(M,\ \mathbb{Z}) \times \mathrm{H}_b(M,\ \mathbb{Z})$, called the *intersection form*. Here $\mathrm{H}(C) \in \mathrm{H}_k(M,\ \mathbb{Z})$ denotes the homology class of the k-dimensional cycle C in M. The Poincaré duality isomorphism (2.1.17) implies that the intersection form is unimodular. It is the latter, somewhat weaker, statement that in Griffiths and Harris [74, p. 53] is called the Poincaré duality theorem.

If M is a complex analytic manifold of dimension n, then it is an oriented smooth real $2n$-dimensional manifold. If A and B are complex analytic subsets of M of respective complex dimensions k and l such that $k + l = n$, then A and B are oriented cycles in M of the complementary real dimensions $2k$ and $2l$. Therefore, if $A \cap B$ is compact, we have a symmetric intersection number $A \cdot B = B \cdot A$, where $A \cdot B = A \cdot B'$ if $B' = B + \partial C$ for a real $(2l + 1)$-dimensional chain C such that $A \cap |C|$ is compact. If $A \cap B \subset A^\circ \cap B^\circ$, and all intersections are transversal, then $A \cdot B = \#(A \cap B) =$ the set-theoretic number of intersection points of A and B. This number is finite, nonnegative, and equal to zero if and only if $A \cap B = \emptyset$. More generally, if i is an isolated point of $A \cap B$, then $A \cdot_i B \geq \mathrm{mult}_i(A)\ \mathrm{mult}_i(B)$. Furthermore, $A \cdot_i B = 1$ if and only if A and B are nonsingular at i and intersect transversally at i. See Griffiths and Harris [74, p. 393]. Here, for any k-dimensional analytic subset A of M, the multiplicity $\mathrm{mult}_a(A)$ of A at $a \in A$ is defined as the degree of a generic projection, in a small coordinate neighborhood in M around a, onto a small open subset of $\mathbb{C}^k$.

Remark 2.1.8. For arbitrary complex projective algebraic varieties, algebraic topological intersection theory was developed by van der Waerden [203]. He applied it to give a rigorous foundation to Schubert's calculus of enumerative geometry, one of Hilbert's problems at the 1900 Congress of Mathematicians. In [203, Appendix 1, pp. 355, and 349–352] one finds the statements that a complex projective variety can be triangulated and forms an oriented cycle of real dimension equal to twice its complex dimension, and that the topological intersection number at an isolated intersection point is equal to the algebraic multiplicity of the intersection, and hence is positive. In [203, Appendix 2] van der Waerden determined the homology groups of the complex n-dimensional projective space $\mathbb{P}^n$ as follows. If $0 \leq j \leq n$, then $\mathrm{H}_{2j}(\mathbb{P}^n,\ \mathbb{Z})$ is freely generated by the homology class of any j-dimensional complex projective subspace, whereas $\mathrm{H}_k(\mathbb{P}^n,\ \mathbb{Z}) = 0$ for all other k. This implies the *theorem of Bézout*, which states that the intersection number of two complex algebraic curves in $\mathbb{P}^2$ is equal to the product of the degrees of the curves.

Van der Waerden [203, p. 339] mentioned that earlier, Lefschetz [117] applied the topological intersection number to algebraic surfaces and correspondences on algebraic curves. In [118, p. 13] Lefschetz wrote, "As I see it at last it was my lot

to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry." ⊘

If $\lambda : L \to M$ is a holomorphic complex line bundle over M with Chern class $c(L) \in H^2(M, \mathbb{Z})$, and C a compact complex analytic curve in M, then

$$L \cdot C := 0_L \cdot 0_C = (pd^{-1} \circ c(L)) \cdot H(C) =: [L] \cdot H(C) \tag{2.1.21}$$

is called the *intersection number of L with C*. Here $pd : H_{2n-2}(M, \mathbb{Z}) \xrightarrow{\sim} H^2(M, \mathbb{Z})$ is the Poincaré duality (2.1.17), and $H(C) \in H_2(M, \mathbb{Z})$ denotes the homology class of C. Furthermore, 0_L denotes the zero section of L, and 0_C is the curve C embedded into 0_L. The zero sections 0_C and 0_L are viewed as cycles of the respective real dimensions 2 and $2n$ in the real $2(n+1)$-dimensional manifold L. The zero section of L is homotopic, hence homologous, to any real-differentiable section s of L over M, and therefore $0_L \cdot C = s \cdot 0_C$. It can be arranged that the only zeros of s in C occur in the smooth part C° of C and are simple, when $C \cdot L$ is equal to the number of zeros of $s|_{C^\circ} : C^\circ \to L$, counted with their orientations. This leads to the middle identity in (2.1.21).

If C is nonsingular and connected, an embedded Riemann surface in M, then the pullback $\iota_C{}^*(L)$ of L by means of the embedding $\iota_C : C \to M$ is a holomorphic line bundle over C. See Section 2.1.4 for the definition of pullbacks of bundles, and Section 2.1.5 for the degree of a holomorphic line bundle over a compact Riemann surface. In this case, (2.1.10), (2.1.21), and (2.1.9) imply that

$$L \cdot C = \deg(\iota_C{}^*(L)). \tag{2.1.22}$$

If D is a divisor in M, then

$$D \cdot C = \delta(D) \cdot H(C), \tag{2.1.23}$$

which is further motivation for the notation in (2.1.21). If M is compact, then

$$c(\delta(D)) = pd(H(D)), \tag{2.1.24}$$

where $pd : H_{2n-2}(M, \mathbb{Z}) \xrightarrow{\sim} H^2(M, \mathbb{Z})$ is the Poincaré duality isomorphism (2.1.17) for $a = 2n-2$ and $d = 2n$. If we combine (2.1.20) with $A = D$ and $B = C$ with (2.1.24) and (2.1.21), then we recover (2.1.23). Summarized in a somewhat less precise way, all topological objects that we have assigned to a holomorphic line bundle L are canonically isomorphic to each other.

If $M = S$ is a complex analytic *surface*, a complex analytic manifold of complex dimension 2, then any two divisors A and B in S have complex dimension 1, hence are cycles of complementary real dimensions, both equal to two, in S. Therefore, if $|A| \cap |B|$ is compact, we have a well-defined intersection number $A \cdot B$. If the curve A is compact, we have a well-defined *self-intersection number* $A \cdot A$. For negative intersection numbers we have the following strong rigidity result.

Lemma 2.1.9 *Let S be a compact complex analytic surface, A an irreducible compact complex analytic curve in S, and B an effective divisor in S. If $A \cdot B < 0$, then A is an irreducible component of B, with $A \cdot A < 0$. If $A \cdot A < 0$, and B is an irreducible complex analytic curve in M that is homologous to A, then $B = A$.*

Proof. We have $B = \sum_{i \in I} k_i B_i$, where for each $i \in I$, B_i is an irreducible complex analytic curve in S, and $k_i \in \mathbb{Z}_{>0}$. Because $0 > A \cdot B = \sum_{i \in I} k_i (A \cdot B_i)$, there exists an $i \in I$ such that $A \cdot B_i < 0$. If $A \neq B_i$, then $A \cap B_i$ is finite and $A \cdot B = \sum_{p \in A \cap B_i} A \cdot_p B_i \geq 0$ because $A \cdot_p B_i > 0$ when p is an isolated intersection point of A and B_i, where we note that $A \cdot B_i = 0$ if and only if $A \cap B_i = \emptyset$. Since $A \cdot B_i \geq 0$ contradicts $A \cdot B_i < 0$, we conclude that $A = B_i$.

If $A \cdot A < 0$ and B is homologous to A, then $A \cdot B = A \cdot A < 0$, and therefore A is an irreducible component of B. If B is irreducible, then $A = B$. □

If A is a smooth compact curve in the surface S, then the *normal bundle of A in S* is the holomorphic line bundle $\mathrm{N}(A) = \mathrm{N}_S(A)$ over A of which, for any $a \in A$, the fiber over a is equal to the one-dimensional complex vector space $\mathrm{N}(A)_a := \mathrm{T}_a S / \mathrm{T}_a A$. Let $L \in \delta(A)$ and s a holomorphic section of L such that $A = \mathrm{Div}(s)$. Then for each $a \in A$ the tangent map $\mathrm{T}_a s$ induces a linear isomorphism from $\mathrm{N}(A)_a$ onto L_a, which leads to an isomorphism $s' : \mathrm{N}(A) \xrightarrow{\sim} \iota_A{}^*(L)$. Therefore (2.1.23) and (2.1.22) with $C = D = A$ lead to the formula

$$A \cdot A = \deg(\mathrm{N}_S(A)) \tag{2.1.25}$$

for the self-intersection number.

If L and L' are holomorphic line bundles over the compact surface S, then the intersection number of L and L' is defined as

$$L \cdot L' := (\mathrm{pd}^{-1} \circ \mathrm{c}(L)) \cdot (\mathrm{pd}^{-1} \circ \mathrm{c}(L')), \tag{2.1.26}$$

where $\mathrm{c} : \mathrm{Pic}(S) \to \mathrm{H}^2(S, \mathbb{Z})$ is the Chern homomorphism (2.1.8) and $\mathrm{pd} : \mathrm{H}_2(S, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(S, \mathbb{Z})$ the Poincaré duality (2.1.17). If $[L'] = \delta(D')$ for a divisor D' in S, then (2.1.24) and (2.1.21) yield $L \cdot L' = (\mathrm{pd}^{-1} \circ \mathrm{c}(L)) \cdot D' = L \cdot D'$. If also $[L] = \delta(D)$, then (2.1.23) yields that $L \cdot L' = D \cdot D'$. See Griffiths and Harris [74, p. 470].

2.1.7 Complex Differential Forms and Multivector Fields

Let $p \in \mathbb{Z}_{\geq 0}$. If E is an n-dimensional vector space over $\mathbb{C}$, then a *complex p-form on E* is an antisymmetric complex p-linear form on E. The space of $\Lambda^p E^*$ of all complex p-forms on E is complex vector space of complex dimension $\binom{n}{p}$. If M is an n-dimensional complex manifold, then the $\Lambda^p(\mathrm{T}_m M)^*$, $m \in M$, form a holomorphic complex vector bundle $\Lambda^p \mathrm{T}^* M$ over M. A section of $\Lambda^p \mathrm{T}^* M$ is called a *complex p-form on M*. Because the vector bundle $\Lambda^p \mathrm{T}^* M$ is holomorphic,

we have a well-defined notion of holomorphic complex p-forms on M. The sheaf of germs of holomorphic complex p-forms is denoted by Ω^p.

When $p = n$ we have $\binom{n}{p} = 1$, and the holomorphic line bundle $\mathrm{K}_M := \Lambda^n \mathrm{T}^*M$ over M, of which the fiber over the point $m \in M$ is defined as

$$\mathrm{K}_{M,\,m} := \Lambda^n (\mathrm{T}_m M)^*, \tag{2.1.27}$$

is called the *canonical line bundle* of M. A complex n-form on M, where n is the complex dimension of M, is also called a *complex volume form* on M.

If $n = 1$, when M is a complex analytic curve i.e., a Riemann surface, a complex one-form on M is called an *abelian differential* on M instead of a complex volume form. If $n = 2$, when M is called a *complex analytic surface*, then a complex two-form on M is also called a *complex area form* on M, instead of a complex volume form.

The complex p-forms on M are the $(p,\,0)$-forms as defined in Section 6.2.8. Because the $(p,\,q)$-forms with $q > 0$ appear in this book only in Section 6.2.8, I have chosen to write "complex p-form" instead of "$(p,\,0)$-form."

The dual space of $\Lambda^p E^*$ is the p-fold exterior product $\Lambda^p E$, of which the elements are called the *(exterior) p-vectors* in E. The $\Lambda^p \mathrm{T}_m M$, $m \in M$, form a holomorphic vector bundle $\Lambda^p \mathrm{T}M$ over M, the sections of which are called *p-vector fields on M*. When $p = n$, the holomorphic line bundle $\Lambda^n \mathrm{T}M = \mathrm{K}_M^*$ over M is called the *anticanonical bundle* of M. In this book we will meet holomorphic exterior two-vector fields on surfaces especially when $n = p = 2$.

2.2 Complex Projective Varieties

Let V be an $(n + 1)$-dimensional vector space over $\mathbb{C}$. Then the *projective space* $\mathrm{P}(V)$ of V is defined as the set of all one-dimensional linear subspaces L of V. The mapping that assigns to any nonzero vector x in V the one-dimensional linear subspace $[x] := \mathbb{C}\,x$ of V that contains x is a surjective mapping from $V \setminus \{0\}$ onto $\mathrm{P}(V)$, and $[x] = [y]$ if and only if there is a nonzero $c \in \mathbb{C}$ such that $y = c\,x$. That is, $\mathrm{P}(V)$ can also be viewed as the orbit space $(V \setminus \{0\})/\mathbb{C}^\times$ for the action of the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers on the complement $V \setminus \{0\}$ of the origin in V.

If $V = \mathbb{C}^{n+1}$ then $\mathbb{P}^n := \mathrm{P}(\mathbb{C}^{n+1})$ is called *the n-dimensional complex projective space*. For $n = 1$ and $n = 2$, $\mathbb{P}^1$ and $\mathbb{P}^2$ are called the *complex projective line* and the *complex projective plane*, respectively. If $x = (x_0,\, x_1,\, \ldots,\, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, then the element $[x] = \mathbb{C}\,x$ of $\mathbb{P}^n$ is denoted by $[x_0 : x_1 : \cdots : x_n]$, which are called *projective coordinates* of $[x]$.

Let e_i, $0 \le i \le n$, be a basis of the $(n + 1)$-dimensional vector space V. The *affine coordinates* on $\mathrm{P}(V)$ with respect to this basis are defined as follows. Let W be the n-dimensional linear subspace of V spanned by the e_i, $1 \le i \le n$. Then every element of $\mathrm{P}(V) \setminus \mathrm{P}(W)$ is of the form $[x]$, where $x = e_0 + \sum_{i=1}^n x_i\, e_i$ for unique

$x_i \in \mathbb{C}$. The mapping $\chi_{e_1,\ldots,e_n} : [x] \mapsto (x_1, \ldots, x_n)$ is bijective from $\mathrm{P}(V) \setminus \mathrm{P}(W)$ onto $\mathbb{C}^n$, and is called the *affine coordinatization* defined by the basis $e_i, 0 \le i \le n$ of V. Because $[x_0 : x_1 : \cdots : x_n] = [1 : x_1/x_0 : \cdots : x_n/x_0]$, the affine coordinates of the point $[x_0 : x_1 : \cdots : x_n]$ are $(x_1/x_0, \ldots, x_n/x_0)$.

With the affine coordinatizations, $\mathrm{P}(V)$ is a compact n-dimensional smooth complex analytic manifold. It is of a very special algebraic nature, because the coordinate transformations are given by rational functions, more precisely, by fractional linear functions. If $e_i, 0 \le i \le n$ is any basis of V, and we define W_j as the span of the e_i with $i \ne j$, then $\mathrm{P}(V) \setminus \mathrm{P}(W_j)$ is the domain of definition for the affine coordinatization defined by the basis $e_j, e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$. Because V is equal to the union of the $\mathrm{P}(V) \setminus \mathrm{P}(W_j), 0 \le j \le n$, it follows that the n-dimensional complex projective space $\mathrm{P}(V)$ can be constructed by gluing $n+1$ copies of $\mathbb{C}^n$ together by means of the corresponding coordinate transformations.

Because the compact space $\mathrm{P}(V)$ is equal to the union of $\mathrm{P}(V) \setminus \mathrm{P}(W) \simeq \mathbb{C}^n$ and $\mathrm{P}(W)$, the n-dimensional complex projective space $\mathrm{P}(V)$ can also be viewed as a *compactification* of the noncompact n-dimensional complex vector space $\mathbb{C}^n$ by attaching *the* $(n-1)$*-dimensional complex projective space* $\mathrm{P}(W)$ *at infinity* to it. The resulting space $\mathrm{P}(V)$ is not only compact but also a smooth complex analytic manifold. Note that every $(n-1)$-dimensional complex projective subspace $\mathrm{P}(W)$ can be declared to be the subspace at infinity, where its complement in $\mathrm{P}(V)$ is isomorphic to $\mathbb{C}^n$. For example, the complex projective line $\mathbb{P}^1$ is obtained by adding one point at infinity, and one writes $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It can be constructed by means of gluing two copies of $\mathbb{C}$ together along $\mathbb{C} \setminus \{0\}$ by means of the gluing map $y \mapsto 1/y$.

If A is an invertible linear transformation in V, then it maps each one-dimensional linear subspace of V onto a one-dimensional linear subspace of V, and therefore induces a transformation $\mathrm{P}(A)$ on $\mathrm{P}(V)$. In affine coordinates it is given by the aforementioned affine coordinate transformations given by fractional linear expressions, which implies that $\mathrm{P}(A)$ is a complex analytic diffeomorphism of $\mathrm{P}(V)$. This defines a complex analytic action of the general linear group $\mathrm{GL}(V)$ of V on $\mathrm{P}(V)$ that is transitive, which shows that as a complex analytic manifold, $\mathrm{P}(V)$ is a homogeneous space. The action of $\mathrm{GL}(V)$ on $\mathrm{P}(V)$ is not faithful, because any nonzero scalar multiple of the identity in V acts trivially on $\mathrm{P}(V)$. The quotient group $\mathrm{PGL}(V) := \mathrm{GL}(V)/\mathbb{C}^\times$, which acts faithfully on $\mathrm{P}(V)$, is called the *projective linear group of* V. For the proof of the following remarkable fact, see Griffiths and Harris [74, pp. 64, 65].

Lemma 2.2.1 *Every complex analytic diffeomorphism of* $\mathrm{P}(V)$ *is equal to* $\mathrm{P}(A)$ *for some* $A \in \mathrm{GL}(V)$. *This leads to an identification of* $\mathrm{PGL}(V)$ *with the group* $\mathrm{Aut}(\mathrm{P}(V))$ *of all complex analytic diffeomorphisms of* V.

If $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a homogeneous polynomial in $n+1$ variables of degree d, then $f(\lambda x) = \lambda^d f(x)$, and it follows that if $x \ne 0$ and $f(x) = 0$, then the whole one-dimensional linear subspace $[x] := \mathbb{C}x$ spanned by x is contained in the zero-set of f, in which case we say that the point $[x] \in \mathbb{P}^n$ belongs to the zero-set of f. An *algebraic set* in $\mathbb{P}^n$ is defined as the common zero-set Z of a collection of homogeneous polynomials on $\mathbb{C}^{n+1}$. See Hartshorne [80, p. 9] and Shafarevich [182, Chapter I, §4.1]. We will further follow Hartshorne [80, p. 10] in defining a *projective*

algebraic variety as an irreducible projective algebraic set, where Griffiths and Harris [74, p. 166] use this term for any projective algebraic set. The *theorem of Chow* states that every closed complex analytic subset of $\mathbb{P}^n$ is algebraic. See Griffiths and Harris [74, p. 167]. This is an illustration of the *G.A.G.A. principle*, which roughly says that any global analytic object on a complex projective algebraic variety is algebraic, see Griffiths and Harris [74, p. 171]. The name comes from the article of Serre [180], which contains several precise formulations of the principle.

Any complex analytic manifold that is complex analytic diffeomorphic to a smooth projective algebraic variety in some complex projective space is also called a *smooth complex projective algebraic variety*. The *Kodaira embedding theorem*, see Griffiths and Harris [74, pp. 181–192], states that a compact complex analytic manifold is a smooth complex projective algebraic variety if and only if it carries a positive holomorphic line bundle.

The *Segre map*

$$([x_0 : x_1 : \cdots : x_m], [y_0 : y_1 : \cdots : y_n]) \mapsto [(w_{ij} = x_i\, y_j)_{0 \le i \le m,\, 0 \le j \le n}]$$

is an embedding of $\mathbb{P}^m \times \mathbb{P}^n$ onto the smooth algebraic variety in $\mathbb{P}^{(n+1)\,(m+1)-1}$ determined by the equations $w_{ij}\, w_{kl} = w_{kj}\, w_{il}$ for all $i,\ j,\ k,\ l$. It follows that the Cartesian product of complex projective algebraic varieties is a complex projective algebraic variety. Furthermore, a subset of a Cartesian product of projective spaces is mapped onto a projective variety if and only if it is the zero-set of polyhomogeneous polynomials. See [182, Chapter I, §5.1]. Our Lemma 2.4.1 is a variant of this for complex analytic subsets of $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric $z_0\, z_3 = z_1\, z_2$ in $\mathbb{P}^3$.

A rational function on $\mathbb{P}^n$ is a map $f : \mathbb{P}^n \supset\!\!\to \mathbb{P}^1$ such that there exist relatively prime homogeneous polynomials p and q on $\mathbb{C}^{n+1}$ of the same degree such that $f = p/q$ in the complement of the common zero-set of p and q. Dividing away common factors of p and q, it can be arranged that p and q are relatively prime. Since a rational function is meromorphic, it has the same indeterminacy as a meromorphic function on the codimension-two algebraic subset where p and q both vanish. If $V \subset \mathbb{P}^n$ is a projective algebraic variety, then a *rational function on* V is the restriction to V of a rational function $f : \mathbb{P}^n \supset\!\!\to \mathbb{P}^1$, where V is not contained in the set of indeterminacy of f. The rational functions on V form a field, called the *function field* $\mathbb{C}(V)$ *of* V.

Let V be a projective algebraic variety. The sets $V \setminus W$, where W is an algebraic subset of V and $W \neq V$, define a topology on V, called the *Zariski topology*. A function $f : V \to \mathbb{C}$ is called *regular* if every $v \in V$ has a Zariski-open neighborhood V_0 in V such that $f|_{V_0}$ is equal to the restriction to V_0 of a rational function, with set of indeterminacy disjoint from V_0. If W is another projective algebraic variety, then an *algebraic morphism* $\varphi : V \to W$ is a Zariski continuous map such that for every Zariski-open subset W_0 of W and every regular function $f : W_0 \to \mathbb{C}$ the function $f \circ \varphi : \varphi^{-1}(W_0) \to \mathbb{C}$ is regular. A *rational map* from V to W is an equivalence class of pairs (V_0, φ_{V_0}), where V_0 is nonempty Zariski-open set in V, $\varphi : V_0 \to Y$ is an algebraic morphism, and (V_0, φ_{V_0}) is equivalent to $(V_1,\ \varphi_{V_1})$ if

$\varphi_{V_0} = \varphi_{V_1}$ on $V_0 \cap V_1$. It follows that the φ_{V_0} have a common extension $\varphi : U \to W$ to the union U of all the V_0. The rational map from V to W is identified with the map $\varphi : V \supset\!\!\to W$. The rational map φ is called a *birational transformation* if there exists a rational map $\psi : W \supset\!\!\to V$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are contained in the identity in V and W, respectively. In this case the mapping $\varphi^* : g \mapsto g \circ \varphi$ is an isomorphism from the field $\mathbb{C}(W)$ of rational functions on W onto the field $\mathbb{C}(V)$ of rational functions on V. For the material in this paragraph, see Hartshorne [80, pp. 10–26], where everything is done for algebraic varieties that may be affine, quasi-affine, projective, or quasi-projective; see [80, definition on p. 15].

The following definition in the complex analytic category is due to Remmert [170].

Definition 2.2.2. Let X and Y be connected complex analytic manifolds. A *meromorphic mapping from X to Y* is defined as a mapping $f : X_0 \to Y$, where X_0 is a dense subset of X, with the following properties:

(a) The closure in $X \times Y$ of the graph $\{(x,\ f(x)) \in X \times Y \mid x \in X_0\}$ is a complex analytic subset G of $X \times Y$.
(b) If $\pi : G \to X$ denotes the restriction to G of the projection $(x,\ y) \mapsto x$, then π is a proper and surjective holomorphic mapping from G onto X.
(c) There is a closed complex analytic subspace N of codimension ≥ 2 of X such that $X_0 \subset X \setminus N$, the restriction of π to $G \setminus \pi^{-1}(N)$, is a complex analytic diffeomorphism from $G \setminus \pi^{-1}(N)$ onto $X \setminus N$, and for every $x \in N$ the fiber $\pi^{-1}(\{x\})$ of π over x is a compact connected complex analytic subset of G of dimension ≥ 1 at any of its points.

If $\psi : G \to Y$ denotes the restriction to G of the projection $(x,\ y) \to y$, then $\phi := \psi \circ (\pi|_{G \setminus \pi^{-1}(N)})^{-1}$ is a holomorphic mapping from $X \setminus N$ to Y, equal to f on X_0. Because X_0 is dense in X, ϕ is the unique continuous extension of f to $X \setminus N$. Because $X \setminus N$ is a connected subset of X, $x \mapsto (x,\ \phi(x))$ is a complex analytic diffeomorphism from $X \setminus N$ onto a complex analytic submanifold of $X \to Y$ that is dense in G, from which it follows that G is an irreducible subset of $X \times Y$ and $\dim G = \dim X$.

Let $x \in N$. Then (c) implies that $\pi^{-1}(\{x\}) = \{x\} \times F(x)$, where $F(x)$ is a compact and connected complex analytic subset of Y of dimension ≥ 1 at any of its points. It follows from (a) that for every $y \in F(x)$ there exists a sequence $x_j \in X_0$ such that $x_j \to x$ and $f(x_j) \to y$ as $j \to \infty$. Therefore, as for meromorphic functions, the complex analytic extension of f to $X \setminus N$ is undetermined along N, and the set N in (c) is called the *set of indeterminacy of the meromorphic map f*. It follows that the holomorphic mapping $\phi : X \setminus N \to Y$ is the unique maximal continuous extension of f. It is customary to denote every continuous extension of f again by f.

The transformation f is called *bimeromorphic* if f maps X_0 bijectively onto a dense subset Y_0 of Y and the inverse $f^{-1} : Y_0 \to X$ is a meromorphic mapping from Y to X. That is, if the conclusions in (a)–(c) also hold with the roles of π and ψ interchanged. For a characterization of bimeromorphic transformations between complex analytic surfaces, see Corollary 6.2.54. ⊘

Remark 2.2.3. Remmert [170, Definition 15] defined a meromorphic mapping from an irreducible and locally irreducible complex analytic space X to a locally irreducible complex analytic space Y as a map $f : X_0 \to Y$, where X_0 is a dense subset of X, with the following properties:

(i) The closure G of the graph of f is an irreducible complex analytic subset of $X \times Y$, of dimension equal to the dimension of X.
(ii) If $x \in X$ then $F(x) := \{y \in Y \mid (x, y) \in G\}$ is a nonempty compact subset of Y.
(iii) If $x \in X_0$ then $F(x) = \{f(x)\}$.

For the definition and basic properties of complex analytic spaces, see the beginning of Section 6.2.13. The complex analytic space X is locally irreducible if for every $x \in X$ the germ of X at x is irreducible. Remmert's definition is more general than Definition 2.2.2, because his complex analytic spaces are much more general than complex analytic manifolds, and the conditions (i), (ii), (iii) look much weaker than (a), (b), (c), even when X and Y are nonsingular. However, Remmert [170, pp. 368–370] proved that (i), (ii), (iii) imply (a), (b), (c) if X and Y are locally irreducible complex analytic spaces with X irreducible. An ingredient in Remmert's proof is the *proper mapping theorem* of Remmert, which says that if p is a proper complex analytic mapping from a complex analytic space A to a complex analytic space B, then $p(A)$ is a closed complex analytic subspace of B. Remmert's [170, Satz 23] is in the setting of locally irreducible spaces, where in Łojasiewicz [125, p. 290] this assumption does not appear.

Remmert's idea of using the closure of the graph of a meromorphic mapping as a complex analytic subspace of the Cartesian product leads to efficient proofs in many situations. Remmert himself indicated the following application. Let X be a compact irreducible and locally irreducible complex analytic space, and f_i, $1 \leq i \leq l$, meromorphic functions on X. Then the functions $f_1, \ldots, f_l$ are algebraically dependent if they are analytically dependent, and also if $l > \dim X$.

The proof is based on the observation of Remmert [170, Satz 34] that the mapping $f = (f_1, \ldots, f_l) : X \supset\!\!\to Y$ is meromorphic, where $Y := \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is the Cartesian product of l copies of $\mathbb{P}^1$. Therefore the closure G of the graph of f is a complex analytic subspace of $X \times Y$, and the restriction π to G of the projection $(x, y) \mapsto x$ is a proper map $G \to X$. Because X is compact, G is compact, and the restriction ψ to G of the projection $(x, y) \mapsto y$ is a proper holomorphic map from G to Y. Remmert's proper mapping theorem now implies that $\psi(G)$ is a closed complex analytic subset of Y, with $\dim \psi(G) \leq \dim G = \dim X$. Because Y is a complex projective algebraic variety, see Section 2.2, Chow's theorem yields that $\psi(G)$ is an algebraic subset of Y of dimension $\leq \dim X$. The statements follow because $(f_1(x), \ldots, f_l(x)) \in \psi(G)$ for every x in the common domain of definition of the functions $f_1, \ldots, f_l$. ⊘

I learned the following illustration of the G.A.G.A. principle from Griffiths and Harris [74, pp. 168, 490–493].

Theorem 2.2.4 *Let $V \subset \mathbb{P}^n$, $W \subset \mathbb{P}^m$ be projective algebraic varieties, where V is nonsingular. Then the following stements are equivalent:*

(i) *f is a meromorphic mapping from V to W.*
(ii) *f is a rational mapping from V to W.*
(iii) *There exist homogeneous polynomials p_j, $0 \le j \le m$, in $n+1$ variables, all of the same degree, such that $V \cap \bigcap_j (p_j = 0)$ is contained in the set of indeterminacy of f, and f is equal to the restriction to V of the mapping $x \mapsto [p_0(x) : \cdots : p_m(x)]$.*

Note that the definition of meromorphic functions is in terms of the structure of V as a complex analytic manifold, and therefore is independent of an embedding in a projective space. It follows therefore from Theorem 2.2.4 that the field of rational functions on a smooth complex projective algebraic variety is independent of the embedding of V into a projective space.

2.3 Elliptic Curves

Let C be a compact Riemann surface of genus $g(C) = 1$, and let ω be a holomorphic complex one-form on C, not identically equal to zero. It follows from (2.1.15) that the degree of K_C is equal to zero, and because ω has no poles, ω has no zeros either. As a consequence, there is a unique tangent vector field v on C such that $v \cdot \omega = 1$, v is holomorphic, and it has no zeros on C. If u is any holomorphic tangent vector field on C, then $u \cdot \omega$ is a global holomorphic function on C, and because C is compact and connected it follows from the maximum principle that $u \cdot \omega$ is equal to a constant c; hence $u = c\, v$. In other words, also the space of holomorphic vector fields on C is a complex one-dimensional vector space. Conversely, if C is a Riemann surface that carries a holomorphic tangent vector field v without zeros, then there is a unique complex one-form ω on C such that $v \cdot \omega = 1$, ω is holomorphic, and it has no zeros. Moreover, if ν is a holomorphic complex one-form on C, then $v \cdot \nu$ is a holomorphic function on C, hence equal to a constant $c = c\, v \cdot \omega$, and it follows that $\nu = c\, \omega$. We have proved the following:

Lemma 2.3.1 *For a compact Riemann surface C the following conditions are equivalent:*

(i) $g(C) = 1$.
(ii) *C carries a holomorphic complex one-form without zeros.*
(iii) *C carries a holomorphic tangent vector field without zeros.*

2.3.1 The Flow on the Curve

For any $t \in \mathbb{C}$, let the flow of v after the *complex time* t be denoted by $e^{t\,v}$. Because C is compact, this flow is globally defined, for every $t \in \mathbb{C}$ it is a complex analytic diffeomorphism of C, and $(t,\ c) \mapsto e^{t\,v}(c) : \mathbb{C} \times C \to C$ is a complex analytic action of the additive group $\mathbb{C}$ on C. Because v has no zeros, all the orbits of this action are

open, and because C is connected and different orbits are disjoint, it follows that there is only one orbit. In other words, the action of $\mathbb{C}$ on C is transitive, in the sense that for any choice of an initial point $c \in C$, the mapping $t \mapsto \mathrm{e}^{t\,v}(c) : \mathbb{C} \to C$ is surjective, and locally a complex analytic diffeomorphism. The transitivity of the action, in combination with the commutativity of $\mathbb{C}$, yields that if $t \in \mathbb{C}$ has the property that $\mathrm{e}^{t\,v}(c) = c$ for some $c \in C$, then $\mathrm{e}^{t\,v}(c) = c$ for every $c \in C$. The set P of all these t is a discrete subgroup of $\mathbb{C}$, called the *period group* of the action defined by the vector field v. For any choice of the initial point $c \in C$, the mapping $t \mapsto \mathrm{e}^{t\,v}(c)$ induces a bijective mapping $\Phi : \mathbb{C}/P \to C$, which locally is a complex analytic diffeomorphism, and therefore a complex analytic diffeomorphism from $\mathbb{C}/P$ onto C. The additive group $\mathbb{C}/P$ is identified with the group of all *translations* on C, the automorphisms of C that preserve v, or equivalently that preserve every holomorphic complex one-form on C. The translation group of C, which is a compact connected commutative complex one-dimensional complex Lie group, is equal to the identity component of the group of all automorphisms of C. Here the *identity component* G^{o} of a topological group G is the connected component of G that contains the identity element of G.

Remark 2.3.2. In this book, a compact Riemann surface C of genus one will be called an *elliptic curve*. Many authors use this name only for a pair $(C,\ c)$, where C is a genus-one Riemann surface and c is a chosen base point on C. The aforementioned bijection $\mathbb{C}/P \to C$ leads to a group structure on C, with c as the zero element of the group. However, in Section 6.1 an elliptic fibration is defined as a complex analytic fibration of a complex analytic surface over a complex analytic curve such that the generic fiber is a genus-one curve, and it would be awkward if we would not be allowed to call the generic fiber an elliptic curve. If the fiber over each point b in the base curve is provided with a base point $c = \sigma(b)$ that depends in a complex analytic way on b, then we have an elliptic fibration with a holomorphic section as in Definition 6.1.4. With the stricter definition of elliptic curve one therefore could define a fibration in elliptic curves as an elliptic fibration together with a holomorphic section. The objection to this is that in general an elliptic fibration need not have a holomorphic section. Furthermore, if there exist holomorphic sections, there can be many, and no preferred one. ⊘

Because C is compact, $\mathbb{C}/P$ is compact, which implies that the discrete subgroup P is a *full lattice* in $\mathbb{C}$, meaning that P has a $\mathbb{Z}$-basis $p_1,\ p_2$, which at the same time is an $\mathbb{R}$-basis of C. The mapping $\mathbb{R}^2 \ni (x_1,\ x_2) \mapsto \Phi(x_1\,p_1 + x_2\,p_2)$ induces a real analytic diffeomorphism from the standard real two-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ onto C, and we recover that C is diffeomorphic to a sphere with one handle. However, this identification with $(\mathbb{R}/\mathbb{Z})^2$ "forgets" the complex structure of C, and it is a more precise statement that C is isomorphic as a Riemann surface, that is, complex analytic diffeomorphic, to the complex one-dimensional torus $\mathbb{C}/P$. Because conversely, for any full lattice P in $\mathbb{C}$, $\mathbb{C}/P$ is a Riemann surface of genus one, this leads to a classification of elliptic curves in terms of period lattices.

Any (real one-dimensional) curve $\gamma : [0,\ 1] \to C \simeq \mathbb{C}/P$ lifts to a curve $\delta : [0,\ 1] \to \mathbb{C}$. If ω is the holomorphic complex one-form on C such that $v \cdot \omega = 1$,

then ω corresponds to the complex one-form $\mathrm{d}t$ on $\mathbb{C}/P$, and we have that

$$\int_\gamma \omega = \delta(1) - \delta(0). \tag{2.3.1}$$

We have $\gamma(1) = \gamma(0)$ if and only if $\delta(1) - \delta(0) \in P$, and it follows that the lattice P is equal to the set of all integrals $\int_\gamma \omega$, where γ ranges over the closed curves in C. If γ_1 and γ_2 are closed curves in C, then the complex numbers $\int_{\gamma_1} \omega$ and $\int_{\gamma_2} \omega$, called the *period integrals*, form a $\mathbb{Z}$-basis of P if and only if the homology classes of γ_1 and γ_2 generate $\mathrm{H}_1(C, \mathbb{Z})$.

Let $\Phi : C \to D$ be a complex analytic diffeomorphism from the elliptic curve C onto D. Then D also is an elliptic curve. Let u be a nonzero holomorphic vector field on D and $Q \subset \mathbb{C}$ the period lattice defined by u. Then the pushforward $\Phi_* v$ of v by Φ is a nonzero holomorphic vector field on D; hence there is a nonzero constant $\lambda \in \mathbb{C}$ such that $\Phi_* v = \lambda u$, which implies that $Q = \lambda P$. It follows that two elliptic curves are isomorphic if and only if they have the same period lattices up to multiplication by a nonzero complex number.

If $D = C = \mathbb{C}/P$ then the previous paragraph yields that $\Phi : C \to C$ is an automorphism if and only if there exists $\lambda \in \mathbb{C}$ such that $\lambda P = P$ and $T \in \mathbb{C}$ such that

$$\Phi(t + P) = \lambda t + T + P \quad \text{for every} \quad t \in \mathbb{C}. \tag{2.3.2}$$

The possible λ's are the sixth roots of unity, the fourth roots of unity, and ± 1 if the lattice P is regular hexagonal, square, or none of these two exceptional cases, respectively. This corresponds to the description of the automorphisms in Jogia, Roberts, and Vivaldi [97, Theorem 3].

For any $\Phi \in \mathrm{Aut}(C)$ the respective actions Φ_* and Φ^* of Φ on the one-dimensional space of all holomorphic vector fields and holomorphic one-forms on C are equal to multiplication by a complex number $\lambda = \Lambda(\Phi)$, where $\Phi \mapsto \lambda(\Phi)$ is a homomorphism from $\mathrm{Aut}(C)$ to the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers. The number $\lambda = \Lambda(\Phi)$ is the one appearing in (2.3.2), and it follows that the kernel of Λ is equal to the group of all translations on C. Because this group is connected and $\Lambda(\mathrm{Aut}(C))$ is finite, it follows that $\ker \Lambda = \mathrm{Aut}(C)^\circ$, the connected component of the identity element in $\mathrm{Aut}(C)$. Therefore Λ induces an isomorphism from the component group $\mathrm{Aut}(C)/\mathrm{Aut}(C)^\circ$ onto the group of sixth, fourth, and second roots of unity, isomorphic to $\mathbb{Z}/6\,\mathbb{Z} \simeq (\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$, $\mathbb{Z}/4\,\mathbb{Z}$, or $\mathbb{Z}/2\,\mathbb{Z}$, when the period lattice is regular hexagonal, or square, or none of these, respectively. In the special cases that the component group is isomorphic to $\mathbb{Z}/6\,\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$, when the period lattice is regular hexagonal or square, then the elliptic curve is called *anharmonic* or *harmonic*, respectively.

If $\Phi \in \mathrm{Aut}(C)$ and $\Lambda(\Phi) \neq 1$, then the set of fixed points of Φ in C is nonempty and finite. On the other hand, if $\Lambda(\Phi) = 1$, that is, Φ is a translation, and Φ has a fixed point in C, then Φ is equal to the identity on C. Applying this with Φ replaced by Φ^k for some $k \in \mathbb{Z}_{>0}$, it follows that if $\Lambda(\Phi) = 1$ and there exists a $c \in C$ such that $\Phi^k(c) = c$, then $\Phi^k(c) = c$ for every $c \in C$. This is called the *Poncelet porism for elliptic curves*, after Theorem 10.3.3.

Figure 2.3.1 In each case the shaded area is a fundamental domain F, a parallelogram such that the mapping $t \mapsto t + P$ is bijective from F onto $\mathbb{C}/P$, where the point t on the left and lower boundary of F is identified with the point $t + p_1$ and $t + p_2$ on the right and upper boundary of F, respectively.

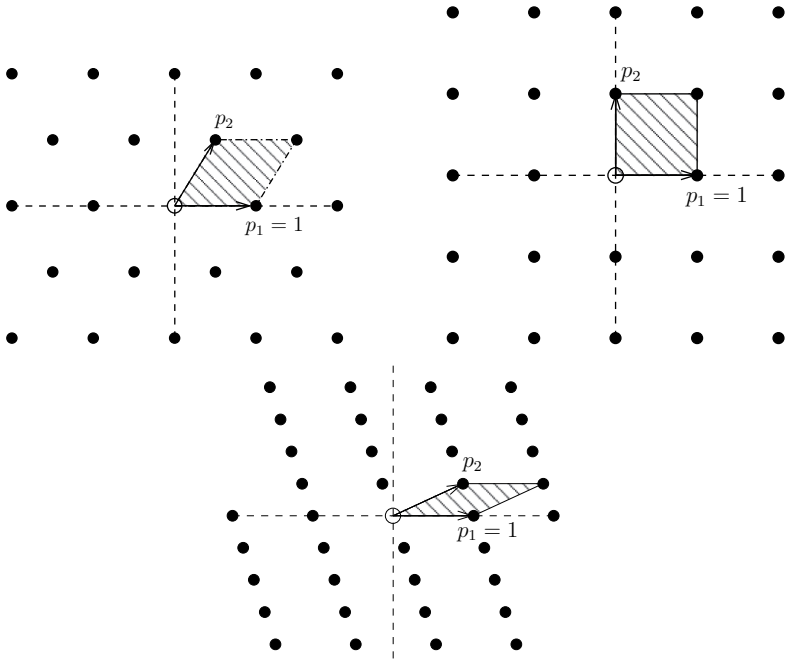


Fig. 2.3.1 Hexagonal, square, and "general" lattice.

2.3.2 The Weierstrass Normal Form

Every elliptic curve $\mathbb{C}/P$ is isomorphic to a cubic curve in the projective plane $\mathbb{P}^2$ of a very special sort, which leads to another classification of (isomorphism classes of) elliptic curves. The classical proof uses the *Weierstrass $\wp$-function* and its derivative:

$$\wp(t) := t^{-2} + \sum_{p \in P,\, p \neq 0} ((t-p)^{-2} - p^{-2}), \quad \wp'(t) = -2 \sum_{p \in P} (t-p)^{-3}. \tag{2.3.3}$$

These series converge locally uniformly in the complement of P in $\mathbb{C}$, and therefore define holomorphic functions on the complement of P in $\mathbb{C}$, which, moreover, are invariant under translations over elements of P. In other words, $\wp$ can be viewed as a meromorphic function on $\mathbb{C}/P$ with a pole of order 2 at $0 + P$. Comparing the Laurent expansions of $\wp'(t)^2$, $\wp(t)^3$, and $\wp(t)$ at $t = 0$, we obtain that the function $\phi(t) = \wp'(t)^2 - 4\wp(t)^3 + g_2\,\wp(t) + g_3$ has no pole and is equal to zero at $t = 0$ if

we take

$$g_2 = g_2(P) := 60 \sum_{p\in P,\, p\neq 0} p^{-4} \quad \text{and} \quad g_3 = g_3(P) := 140 \sum_{p\in P,\, p\neq 0} p^{-6}. \tag{2.3.4}$$

It follows that ϕ defines a global holomorphic function on $\mathbb{C}/P$, which is a constant because $\mathbb{C}/P$ is compact and connected. Because $\phi(0) = 0$, we conclude that $\phi(t) \equiv 0$. Therefore the mapping $t \mapsto [1 : x : y] = [1 : \wp(t) : \wp'(t)]$ induces a holomorphic mapping π from $\mathbb{C}/P$ to the curve W in $\mathbb{P}^2$, which in the affine coordinates $(x,\ y)$ is defined by the equation

$$y^2 = 4x^3 - g_2\, x - g_3. \tag{2.3.5}$$

Here $t = 0 + P$ is mapped to the point $[0 : 0 : 1]$ if the homogeneous equation of degree 3 for the curve W is written as

$$x_0\, {x_2}^2 - 4{x_1}^3 + g_2\, {x_0}^2\, x_1 + g_3\, {x_0}^3 = 0. \tag{2.3.6}$$

Figure 2.3.2 The left hand picture is a plot of the curves (2.3.5) in the real $(x,\ y)$-plane for $g_2 = 1$ and various values of g_3, as explained below. These are level curves of the function $y^2 - 4\,x^3 + x$, but are also members of the pencil of real cubic curves defined in (4.5.1). There are two singular curves in this one-parameter family of curves. For $g_2 = 1$, $g_3 = -1/3\sqrt{3}$ the curve (2.3.5) has a hyperbolic singular point at $x = 1/2\sqrt{3}$, $y = 0$. The other singular level curve of the function

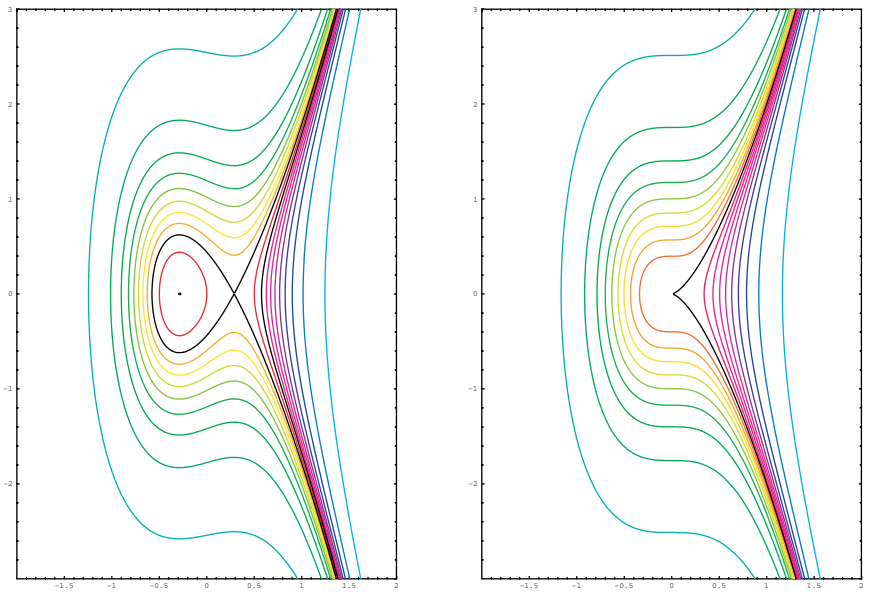


Fig. 2.3.2 Level curves of $y^2 - 4\,x^3 + x$ (left), and $y^2 - 4\,x^3$ (right).

$y^2 - 4x^3 + x$ is (2.3.5) for $g_2 = 1, g_3 = 1/3\sqrt{3}$, with an isolated elliptic singular point at $x = -1/2\sqrt{3}$, $y = 0$. This singular curve has a second connected component, which is smooth and intersects the x-axis at $x = 1/\sqrt{3}$. In each case the complex curve in the complex projective plane is a singular curve of Kodaira type I_1; see Section 6.2.6.

The right-hand picture in Figure 2.3.2 is a plot of the curves (2.3.5) in the real (x, y)-plane for $g_2 = 0$ and various values of g_3, as explained below. These are level curves of the function $y^2 - 4x^3$, and also real members of the pencil (4.5.2). There is one singular curve, the curve (2.3.5) for $g_2 = 0$, $g_3 = 0$, that has a cusp at the origin. The complex curve in the complex projective plane is a singular curve of Kodaira type II; see Section 6.2.6.

In all the pictures in this book of real pencils, we have chosen the parameters $z = -z_1/z_0$ of the members as follows. Because $[z_0 : z_1]$ runs on the real projective line $\mathbb{P}^1(\mathbb{R})$, which is isomorphic to a circle, we have tried to choose the angles $\alpha = \arctan z$ to be evenly spaced. Because the picture should at least contain the singular real curves, we start with the angles corresponding to the $[z_0 : z_1] \in \mathbb{P}^1(\mathbb{R})$ for which the curves are singular, also marking the real singular points. The angles of nonsingular points between two subsequent singular values have been chosen to be evenly spaced, where we have arranged that the spacings between the angles in the different intervals do not differ too much from each other. More elegant choices might be possible, but we found that this choice usually leads to quite reasonable pictures of the real pencils.

We will prove that the curve W is smooth and that π is a complex analytic diffeomorphism from $\mathbb{C}/P$ onto W. Note that the coordinate t on $\mathbb{C}/P$ satisfies $\mathrm{d}t = \pi^*(y^{-1}\,\mathrm{d}x)$, and therefore the fact that π is an isomorphism implies that the restriction to W of the complex one-form $y^{-1}\,\mathrm{d}x$ is a holomorphic complex one-form on W without zeros.

If we write $q(x, y) = (y^2 - 4x^3 + g_2 x + g_3)/2$, then $\pi(t) = [1 : x(t) : y(t)]$ is a solution on $W = \{q = 0\}$ of the Hamiltonian system

$$\mathrm{d}x/\mathrm{d}t = \partial q(x, y)/\partial y = y, \quad \mathrm{d}y/\mathrm{d}t = -\partial q(x, y)/\partial x = 6x^2 - g_2/2, \quad (2.3.7)$$

that is, a classical mechanical system of a particle with unit mass on the x-axis, with potential energy equal to the cubic polynomial $(-4x^3 + g_2 x + g_3)/2$. In order to understand the behavior of the solution near $t = 0$, when $\pi(0) = [0 : 0 : 1]$, we write $[1 : x : y] = [\eta : \xi : 1]$ with $\eta = 1/y$, $\xi = x/y$. In these new affine coordinates the differential equations (2.3.7) take the form

$$\mathrm{d}\eta/\mathrm{d}t = -6\xi^2 + g_2\eta^2/2, \quad -\mathrm{d}\xi/\mathrm{d}t = 1/2 + g_2\xi\eta + 3g_3\eta^2/2, \quad (2.3.8)$$

where we have used (2.3.5). This is the Hamiltonian system on $r = 0$ for the function $r(\eta, \xi) = (\eta - 4\xi^3 + g_2\eta^2\xi + g_3\eta^3)/2$. Note that $\mathrm{d}\xi/\mathrm{d}t \neq 0$ when $\eta = \xi = 0$.

The point (x, y) is a singular point of the curve $\{q = 0\}$ if and only if $q(x, y) = 0$, $\partial q(x, y)/\partial x = 0$ and $\partial q(x, y)/\partial y = 0$ if and only if (x, y) is a point on $q = 0$ where the Hamiltonian vector field H_q vanishes. Because the vector field H_q is holomorphic, we have uniqueness for the initial value problem of the Hamiltonian

system, and because at a zero $(x,\ y)$ of H_q the constant function $t \mapsto (x,\ y)$ is a solution, and our solution $t \mapsto (\wp(t),\ \wp'(t))$ is not constant, it follows that the image of π is contained in the set of smooth points of W. Note that $[0 :\ 0 :\ 1]$, the only point at infinity of W, is a smooth point of W. Because $\mathbb{C}/P$ is compact and π is continuous, $\pi(\mathbb{C}/P)$ is compact, and because $\pi'(t) \neq 0$ for every $t \in \mathbb{C}/P$ we conclude that $\pi(\mathbb{C}/P)$ is an open and closed subset of W. Because W is connected, it follows that $\pi(\mathbb{C}/P) = W$.

We write

$$\Delta := {g_2}^3 - 27\,{g_3}^2, \tag{2.3.9}$$

which is $1/16$ times the discriminant of the polynomial $f : x \mapsto 4\,x^3 - g_2\,x - g_3$ as defined in van der Waerden [204, p. 102]. Then $\Delta = 0$ if and only if the polynomial f has multiple zeros. Because $\pi(\mathbb{C}/P)$ consists of smooth points of W, it follows that W is smooth, which is equivalent to the condition that $\Delta \neq 0$.

The mapping $\pi : \mathbb{C}/P \to W$ is a holomorphic covering map without branch points, and because $\pi^{-1}(\{[0 :\ 0 :\ 1]\})$, which is the set of $t \in \mathbb{C}/P$ where $\wp(t)$ has a pole, is equal to $\{0 + P\}$, it follows that π is a complex analytic diffeomorphism from $\mathbb{C}/P$ onto W. The smooth cubic curve (2.3.6) in the complex projective plane is called the *Weierstrass normal form* of the elliptic curve $\mathbb{C}/P$.

It follows from Theorem 2.2.4 that every *elliptic function*, defined as a meromorphic function on the elliptic curve $C \simeq \mathbb{C}/P$, is equal to $t \mapsto R(\wp(t),\ \wp'(t))$ for some rational function R of two variables. See Weierstrass [207, pp. 119–121, 146]. It may be noted here that Weierstrass did not discuss the identification of an abstract elliptic curve with the cubic curve in the projective plane defined by (2.3.6). His object of study was elliptic functions, defined as meromorphic functions on $\mathbb{C}/P$, where P is a lattice in $\mathbb{C}$.

Now assume conversely that g_2 and g_3 are complex numbers such that $\Delta := {g_2}^3 - 27\,{g_3}^2 \neq 0$, that is, (2.3.6) defines a smooth curve W in the complex projective plane $\mathbb{P}^2$. Let

$$p(t) = [1 :\ x(t) :\ y(t)] = [\eta(t) :\ \xi(t) :\ 1], \quad t \in \mathbb{C},$$

be the solution curve of the Hamiltonian system (2.3.7), (2.3.8) such that $p(0) = [0 :\ 0 :\ 1]$. This implies that $p(0) \in W$, and because the Hamiltonian vector field is tangent to W, we have $p(t) \in W$ for every $t \in \mathbb{C}$. Because W is smooth, $p'(t) \neq 0$ for every $t \in \mathbb{C}$, and $p : \mathbb{C} \to W$ is a holomorphic covering map. If P denotes the period group, the set of all $t \in \mathbb{C}$ such that $p(t) = p(0)$, then p induces a complex analytic diffeomorphism from $\mathbb{C}/P$ onto W. Because W is compact, $\mathbb{C}/P$ is compact; hence P is a lattice in $\mathbb{C}$ in the sense that P has a $\mathbb{Z}$-basis that at the same time is an $\mathbb{R}$-basis of $\mathbb{C}$.

It follows from (2.3.8) and $\eta(0) = \xi(0) = 0$ that $\eta(t) = -t^3/2 + \mathrm{O}(t^6)$ and $\xi(t) = -t/2 + \mathrm{O}(t^5)$; hence $x(t) = \xi(t)/\eta(t) = t^{-2} + \mathrm{O}(t)$ as $t \to 0$. If $\wp(t)$ denotes the Weierstrass $\wp$-function for the lattice P, then also $\wp(t) = t^{-2} + \mathrm{O}(t)$ as $t \to 0$, and it follows that the function $x(t) - \wp(t)$ extends to a holomorphic function d on $\mathbb{C}/P$. Because $\mathbb{C}/P$ is compact and connected, it follows from the maximum principle that d is a constant, and because $d(0) = 0$ we conclude that

$d = 0$. That is, $x(t) \equiv \wp(t)$, which implies that $y(t) = x'(t) = \wp'(t)$. Because $[1 : \wp(t) : \wp'(t)]$ runs through the curve defined by the equation (2.3.6) with g_2 and g_3 replaced by $g_2(P)$ and $g_3(P)$, respectively, we have $y^2 = 4x^3 - g_2 x - g_3$ if and only if $y^2 = 4x^3 - g_2(P)x - g_3(P)$, and therefore $g_2(P) = g_2$ and $g_3(P) = g_3$. It follows that every smooth curve (2.3.6) is the Weierstrass normal form of an elliptic curve $\mathbb{C}/P$ for a suitable lattice P in $\mathbb{C}$.

It follows from (2.3.4) that $g_2(\lambda P) = \lambda^{-4} g_2(P)$ and $g_3(\lambda P) = \lambda^{-6} g_3(P)$. Therefore the curve W', defined by the equation (2.3.6) with g_2 and g_3 replaced by g_2' and g_3', respectively, is isomorphic to the curve W defined by the equation (2.3.6), in the sense that there is a complex analytic diffeomorphism from W onto W' if and only if ${g_2}^3/{g_3}^2 = {g_2'}^3/{g_3'}^2$. As the *modulus* of an arbitrary elliptic curve $C \simeq \mathbb{C}/P \simeq W$, the complex number that parameterizes the isomorphism classes of the elliptic curves, we take the number

$$\mathrm{J}(C) := {g_2}^3/\Delta = {g_2}^3/({g_2}^3 - 27\,{g_3}^2). \tag{2.3.10}$$

Note that if the lattice is regular hexagonal, that is, the curve is anharmonic, then $\lambda P = P$ for $\lambda = \mathrm{e}^{2\pi \mathrm{i}/6}$, and because $\lambda^{-4} \neq 1$ it follows that $g_2(P) = 0$. Similarly the case of the square lattice, when the curve is harmonic, is characterized by $g_3(P) = 0$. It follows that $\mathrm{Aut}(C)/\mathrm{Aut}(C)^{\mathrm{o}} \simeq \mathbb{Z}/6\mathbb{Z} \Leftrightarrow$ if and only if the period lattice is regular hexagonal and the curve is anharmonic $\Leftrightarrow \mathrm{J}(C) = 0$. Furthermore, $\mathrm{Aut}(C)/\mathrm{Aut}(C)^{\mathrm{o}} \simeq \mathbb{Z}/4\mathbb{Z} \Leftrightarrow$ the period lattice is square and the curve is harmonic $\Leftrightarrow \mathrm{J}(C) = 1$. We have $\mathrm{J}(C) \notin \{0,\ 1\}$ if and only if $\mathrm{Aut}(C)/\mathrm{Aut}(C)^{\mathrm{o}} \simeq \mathbb{Z}/2\mathbb{Z}$.

Remark 2.3.3. If ${g_2}^3 - 27\,{g_3}^2 = 0$ and $g_2 \neq 0$, $g_3 \neq 0$, then the curve defined by the equation (2.3.6) has a singular point, and is isomorphic to a Kodaira fiber of type I_1; see Section 6.2.6. If $g_2 = g_3 = 0$, then the curve defined by (2.3.6) has a cusp and is isomorphic to a Kodaira fiber of type II. In all other cases the curve defined by (2.3.6) is a smooth elliptic curve.

The fact that the singular Weierstrass curves are only of type I_1 or II, which precisely are the irreducible singular fibers in Kodaira's list in Section 6.2.6, is related to the fact that in the Weierstrass model of an arbitrary elliptic surface, see Theorem 6.3.6, for each reducible singular fiber all irreducible components that do not meet the given section are contracted to a point. ⊘

2.3.3 The Modular Function

The content of this subsection will be used in Sections 6.4.2 and 6.4.1.

Let C be an elliptic curve. Then the set $\mathfrak{g}$ of all holomorphic vector fields on C is a complex one-dimensional complex vector space. For any $c_0 \in C$, the mapping $v \mapsto \mathrm{e}^v(c_0)$ induces an isomorphism, a complex analytic diffeomorphism, from $\mathfrak{g}/P$ onto C, where P is the set of all $v \in \mathfrak{g}$ such that $\mathrm{e}^v(c_0) = c_0$. This period group P in $\mathfrak{g}$ is a discrete additive subgroup of $\mathfrak{g}$. Because $\mathfrak{g}/P \simeq C$ is compact, P contains a $\mathbb{Z}$-basis v_1, v_2 that at the same time is an $\mathbb{R}$-basis of $\mathbb{C}$. The $\mathbb{Z}$-basis v_1, v_2 will always be chosen such that the ordered basis v_1, v_2 is positively oriented with respect to the orientation of $\mathfrak{g}$ defined by the complex structure of $\mathfrak{g}$. Because $\dim_{\mathbb{C}} \mathfrak{g} = 1$, there is

a unique complex number q such that $v_2 = q\, v_1$, where the basis $v_1,\ v_2$ is positively oriented if and only if $\operatorname{Im} q > 0$, that is, q is a point in the *complex upper half-plane*

$$H := \{q \in \mathbb{C} \mid \operatorname{Im} q > 0\}.$$

If $v \in \mathfrak{g}$, $v \neq 0$, then $t \mapsto t\, v$ is a complex linear isomorphism from $\mathbb{C}$ onto $\mathfrak{g}$, which induces an isomorphism

$$\mathbb{C}/P_v \xrightarrow{\sim} \mathfrak{g}/P \xrightarrow{\sim} C,$$

where $P_v \subset \mathbb{C}$ is the period group defined by the vector field v, and the composition of the two isomorphisms is the isomorphism $\mathbb{C}/P_v \xrightarrow{\sim} C$ of Section 2.3.1. If p_1 and p_2 are the complex numbers such that $p_i\, v = v_1$, then p_1 and p_2 form a $\mathbb{Z}$-basis of the period group P_v, and it follows from $p_2\, v = v_2 = q\, v_1 = q\, p_1\, v$ that $p_2/p_1 = q$, the aforementioned point in the complex upper half plane H. If $v = v_1$, then $P_v = \mathbb{Z} + q\,\mathbb{Z}$, and we obtain that C is isomorphic, as an elliptic curve, to $\mathbb{C}/(\mathbb{Z} + q\,\mathbb{Z})$.

If $v \in P$, then $\gamma_v : t \mapsto \mathrm{e}^{t\, v}(c_0)$, $0 \leq t \leq 1$, is a closed real curve in C, and the map that assigns to each $v \in P$ the homology class $[\gamma_v]$ of γ_v is an isomorphism from P onto $\mathrm{H}_1(C,\ \mathbb{Z})$. If $v \in \mathfrak{g}$, $v \neq 0$, and ω is the unique holomorphic complex one-form on C such that $\omega(v) = 1$, then the basic periods p_1 and p_2 are equal to the integrals of ω over $\gamma_1 = \gamma_{v_1}$ and $\gamma_2 = \gamma_{v_2}$, and the complex number $q \in H$ is also equal to the quotient

$$q = \left(\int_{\gamma_1} \omega\right) / \left(\int_{\gamma_2} \omega\right)$$

of the period integrals,[1] where $[\gamma_1]$ and $[\gamma_2]$ form a $\mathbb{Z}$-basis of $\mathrm{H}_1(C,\ \mathbb{Z})$.

Definition 2.3.4. For every $q \in H$, let $\mathcal{J}(q)$ be the modulus of the elliptic curve $\mathbb{C}/P$ defined by (2.3.10), with $P = \mathbb{Z} + q\,\mathbb{Z}$. The complex-valued function $\mathcal{J} : q \mapsto \mathcal{J}(q)$ on the complex upper half-plane H is called the *modular function*. It follows from (2.3.10) and (2.3.4) that the function $\mathcal{J} : H \to \mathbb{C}$ is holomorphic. ⊘

In view of (2.3.4) with $P = \mathbb{Z} + q\,\mathbb{Z}$, the modular function is a holomorphic, that is, complex analytic, function on H. Note also that $\mathrm{J}(C) = \mathcal{J}(q)$ if $v_2 = q\, v_1$ for a positively oriented $\mathbb{Z}$-basis v_1, v_2 in the period subgroup P of $\mathfrak{g}$ as discussed above.

If v_1', v_2' is another positively oriented $\mathbb{Z}$-basis of P, then there exists a unique $M \in \mathrm{SL}(2,\ \mathbb{Z})$, a 2×2 matrix with integral coefficients and determinant equal to one, such that

$$v_i' = \sum_{j=1}^{2} M_i^j\, v_j, \quad i = 1,\ 2. \tag{2.3.11}$$

If we write $v_2' = q'\, v_1'$, $q' \in H$, then it follows from $v_2 = q\, v_1$ that

[1] In the literature, see for instance Kodaira [109, II, (7.3)], it seems to be more customary to work with a quotient $p_1/p_2 \in H$ of two periods p_1 and p_2 that form a negatively oriented $\mathbb{Z}$-basis of the period lattice.

$$(M_2^1 + M_2^2\, q)\, v_1 = v_2' = q'\, v_1' = q'\, (M_1^1 + M_1^2\, q)\, v_1,$$

where

$$q' = \Phi_M(q) := (M_2^1 + M_2^2\, q)/(M_1^1 + M_1^2\, q). \tag{2.3.12}$$

The mapping $\Phi_M : H \to H$ is called the *fractional linear transformation of H* defined by the element $M \in \mathrm{SL}(2, \mathbb{Z})$. The mapping $M \mapsto \Phi_M$ defines an action of $\mathrm{SL}(2, \mathbb{Z})$ on the complex upper half-plane H by means of fractional linear transformations. This action of $\mathrm{SL}(2, \mathbb{Z})$ on H is called the *modular group action*.

If C and C' are elliptic curves, we have $\mathrm{J}(C) = \mathrm{J}(C')$ if and only if the elliptic curves C and C' are isomorphic. On the other hand, $\mathcal{J}(q) = \mathcal{J}(q')$ if and only if the elliptic curves $\mathbb{C}/(\mathbb{Z} + q\,\mathbb{Z})$ and $\mathbb{C}/(Z + q'\,\mathbb{Z})$ are isomorphic if and only if there exists a complex number c such that $c\,(\mathbb{Z} + q\,\mathbb{Z}) = \mathbb{Z} + q'\,\mathbb{Z}$ if and only if c and $c\,q$ form a $\mathbb{Z}$-basis of the period group with $\mathbb{Z}$-basis 1 and q', if and only if $q' = \Phi_M(q)$ for some $M \in \mathrm{SL}(2, \mathbb{Z})$. We have proved the following result:

Proposition 2.3.5 *When q, $q' \in H$, we have $\mathcal{J}(q) = \mathcal{J}(q')$ if and only if $q' = \Phi_M(q)$ for some $M \in \mathrm{SL}(2, \mathbb{Z})$. In other words, the modular function $\mathcal{J} : H \to \mathbb{C}$ induces a bijective mapping from the orbit space for the modular group action on H onto $\mathbb{C}$.*

The functions $g_i(q) = g_i(\mathbb{C}/(\mathbb{Z}+q\,\mathbb{Z}))$ defined by (2.3.4) are holomorphic functions of $\epsilon = \mathrm{e}^{2\pi\,\mathrm{i}\,q}$, $|\epsilon| < 1$, with

$$g_2(q) = \pi^4\,(4/3 + 320\,\epsilon + \cdots),$$
$$g_3(q) = \pi^6\,(8/27 - 448\,\epsilon/3 + \cdots),$$

which implies, with the notation $\Delta = {g_2}^3 - 27\,{g_3}^2$,

$$\Delta(q) = \pi^{12}\,4096\,\epsilon + \cdots, \tag{2.3.13}$$
$$\mathcal{J}(q) = (1728\,\epsilon)^{-1} + c_0 + c_1\,\epsilon + \cdots. \tag{2.3.14}$$

Furthermore, $g_2(\mathrm{e}^{2\pi\,\mathrm{i}/6}) = 0$, $g_2(\mathrm{e}^{2\pi\,\mathrm{i}/6}) \neq 0$, whereas $g_3(\mathrm{e}^{2\pi\,\mathrm{i}/4}) \neq 0$ and $g_3(\mathrm{e}^{2\pi\,\mathrm{i}/4}) = 0$. It follows that $\mathcal{J}(\mathrm{e}^{2\pi\,\mathrm{i}/6}) = 0$ and $\mathcal{J}(\mathrm{e}^{2\pi\,\mathrm{i}/4}) = 1$. See for instance Ford [63, p. 155, 156] for the proofs of the statements about g_2 and g_3. It follows from (2.3.14) that $\mathcal{J}(q)$, as a function of ϵ, has a pole of order one at $\epsilon = 0$, with residue equal to $1/1728$. For this reason one often sees in the literature $1728\,\mathcal{J}$ instead of $\mathcal{J}$ as the modular function, because then the residue is equal to one. However, this choice leads to the appearance of the number 1728 every time the square lattice occurs, where I have followed the convention that the square lattice has modulus one.

The action of $\mathrm{SL}(2, \mathbb{Z})$ is proper, but not free. The stabilizar subgroup $\mathrm{SL}(2, \mathbb{Z})_q$ of the point $q \in H$ is equal to $\{1, -1\} \simeq \mathbb{Z}/2\,\mathbb{Z}$, except if q belongs to the orbit of $\mathrm{e}^{2\pi\,\mathrm{i}/6} = (1 + \mathrm{i}\,\sqrt{3})/2$ or of $\mathrm{e}^{2\pi\,\mathrm{i}/4} = \mathrm{i}$, where $\mathcal{J}$ has the value 0 or 1, respectively.[2] If $q = \mathrm{e}^{2\pi\,\mathrm{i}/6}$ then $\mathrm{SL}(2, \mathbb{Z})_q$ consists of the elements

[2] In the literature, see for instance Kodaira [109, II, p. 586], it seems to be more customary to use $\mathrm{e}^{2\pi\,\mathrm{i}/3} = (-1 + \mathrm{i}\,\sqrt{3})/2$ instead of $\mathrm{e}^{2\pi\,\mathrm{i}/6} = (1 + \mathrm{i}\,\sqrt{3})/2$.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},\ A^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ A^3 = -1,\ A^4 = -A,\ A^5 = -A^2,\ A^6 = 1. \tag{2.3.15}$$

Since all stabilizer subgroups in the same orbit are conjugate to each other, it follows that $\mathrm{SL}(2,\ \mathbb{Z})_q \simeq \mathbb{Z}/6\,\mathbb{Z}$ if $\mathcal{J}(q) = 0$. If $q = \mathrm{e}^{2\pi\,\mathrm{i}/4}$, then $\mathrm{SL}(2,\ \mathbb{Z})_q$ consists of the elements

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ B^2 = -1,\ B^3 = -B,\ B^4 = 1, \tag{2.3.16}$$

hence $\mathrm{SL}(2,\ \mathbb{Z})_q \simeq \mathbb{Z}/4\,\mathbb{Z}$ if $\mathcal{J}(q) = 1$. Therefore the proper and effective action of $\mathrm{PSL}(2,\ \mathbb{Z}) := \mathrm{SL}(2,\ \mathbb{Z})/\{\pm 1\}$ on H is free on $\mathcal{J}^{-1}(\mathbb{C} \setminus \{0,\ 1\})$, but not at the point q where $\mathcal{J}(q) = 0$ or $\mathcal{J}(q) = 1$. The function $\mathcal{J}$ exhibits the open subset $\mathcal{J}^{-1}(\mathbb{C} \setminus \{0,\ 1\})$ of H as a principal $\mathrm{PSL}(2,\ \mathbb{Z})$-bundle over $\mathbb{C} \setminus \{0,\ 1\}$, also called a Galois covering of $\mathbb{C} \setminus \{0,\ 1\}$ with covering group $\mathrm{PSL}(2,\ \mathbb{Z})$. For this reason $\mathrm{PSL}(2,\ \mathbb{Z}) := \mathrm{SL}(2,\ \mathbb{Z})/\{\pm 1\}$ is called the *modular group*.

If f is a nonconstant holomorphic mapping from a connected complex analytic curve U to a complex analytic curve V, then for each $u \in U$ the *degree* $\deg_u(f)$ *of* f *at* u is defined as the number of u' near u such that $f(u') = c'$, where $c' \neq c, c'$ near c. In local coordinates we have $f^{(d)}(u) \neq 0$ and $f^{(j)}(u) = 0$ for $1 \leq j \leq d-1$ if $d = \deg_u f$. Note that $\deg_u(f) = 1$ if and only if f is a complex analytic diffeomorphism from some open neighborhood of u in U onto some open neighborhood of $f(u)$ in V. We have $\deg_q(\mathcal{J}) = 3$ if $\mathcal{J}(q) = 0$, $\deg_q(\mathcal{J}) = 2$ if $\mathcal{J}(q) = 1$, and $\deg_q(\mathcal{J}) = 1$ otherwise. The mapping $\mathcal{J} : H \to \mathbb{C}$ resembles a branched covering, branching over the points $0,\ 1 \in \mathbb{C}$, and with all points of $\mathcal{J}^{-1}(\{0,\ 1\})$ as ramification points. However, since H and $\mathbb{C}$ are not compact and the mapping $\mathcal{J} : H \to \mathbb{C}$ is not proper and has infinite fibers, we are not in the situation of Section 2.3.4.

The complex upper half-plane can be viewed as the upper hemisphere in the complex projective line $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \{\infty\}$, and the modular group action on H is the restriction to this upper hemisphere of the action of $\mathrm{PSL}(2,\ \mathbb{Z})$ on $\mathbb{P}^1(\mathbb{C})$. The restriction of this action to the boundary $\mathbb{P}^1(\mathbb{R})$ is not proper, and every point in $\mathbb{P}^1(\mathbb{Q})$ has an infinite stabilizer subgroup, generated by a unipotent element. For instance, the stabilizer group of the point $\{\infty\}$ consists of the elements

$$\pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{2.3.17}$$

of $\mathrm{PSL}(2,\ \mathbb{Z})$.

2.3.4 Branched Coverings

Let C and P be compact Riemann surfaces and let $\pi : C \to P$ be a holomorphic map from C to P. For any $c \in C$ we choose local holomorphic coordinate functions z and w near c and $\pi(c)$ in C and P such that $z = 0$ and $w = 0$ correspond to the points c and $\pi(c)$, respectively. With respect to such local coordinates the mapping

π is given near c by a holomorphic function f in a neighborhood of 0 in $\mathbb{C}$ such that $f(0) = 0$. If π is not constant near c, then f has a zero at 0 of a finite order $m \in \mathbb{Z}_{>0}$, and we can choose the local coordinate z such that $f(z) = z^m$. For $p \in P \setminus \{\pi(c)\}$ close to $\pi(c)$ there are exactly m points in $\pi^{-1}(\{p\})$ close to c, approximately in a regular m-gon around c, which implies that the number $m = m_c$ does not depend on the choice of the local coordinates. We have $m_c > 1$ if and only if $\mathrm{T}_c\,\pi = 0$, where c is called a *ramification point* in C of *order* m of the mapping π. The set of all points $c \in C$ such that π is constant in a neighborhood of c is an open and closed subset of C, and because C is connected it follows that this set is empty if $\pi : C \to D$ is not constant, as we will assume in the sequel of this subsection. The set C^{sing} of all ramification points of π is discrete in C, hence finite because C is compact. It follows that the set $P^{\mathrm{sing}} = \pi(C^{\mathrm{sing}})$ of all *branch points* in P of the mapping π is finite as well. Because P is connected, the set $P^{\mathrm{reg}} = P \setminus P^{\mathrm{sing}}$ of all regular values of π is a connected open subset of P, and it follows that the number d of elements in the fiber $\pi^{-1}(\{p\})$ of π over $p \in D^{\mathrm{reg}}$ does not depend on $p \in D^{\mathrm{reg}}$. The number d is equal to the topological mapping degree of $\pi : C \to P$. For any $b \in P$ the number d is equal to the sum of the orders m_c over all $c \in \pi^{-1}(\{b\})$. In the classical literature on Riemann surfaces the nonconstant complex analytic mapping $\pi : C \to P$ is called a *d-fold branched covering* of C of P, where the word "branched" is deleted or replaced by "unbranched" if $\mathrm{T}_c\,\pi \neq 0$ for every $c \in C$.

For each $b \in B$, let n_b be the number of points in the fiber over b. Then we have the *Riemann–Hurwitz formula*

$$\mathrm{g}(C) = 1 + d\,(\mathrm{g}(D) - 1) + \sum_{b \in B} (d - n_b)/2. \tag{2.3.18}$$

See Farkas and Kra [61, I.2.7]. In particular, if $g(D) = 0$, that is, $D \simeq \mathbb{P}^1$, $d = 2$, and $\#(B) = 4$, then $n_b = 1$ for every $b \in B$, and we have $\mathrm{g}(C) = 1 + 2\,(0-1) + 4/2 = 1$. In other words, any twofold branched covering of $\mathbb{P}^1$ with four branch points is an elliptic curve. The following lemma is certainly known, but because I did not find a reference for the isomorphism ψ, I have included a proof.

Lemma 2.3.6 *Let $\pi : C \to P$ be a twofold branched covering. Then the mapping $\iota : C \to C$ that interchanges the two points in each fiber of π over a nonbranch point and fixes the ramification points in C is an involutory automorphism of C, a complex analytic diffeomorphism from C to itself such that $\iota \circ \iota$ is equal to the identity in C. If $\mathrm{g}(P) = 0$ and $\pi' : C' \to P$ is another twofold branched covering with the same branch locus as π, then there is a complex analytic diffeomorphism ψ from C onto C' such that $\pi' \circ \psi = \pi$. If $\chi : C \to C'$ is a complex analytic diffeomorphism such that $\pi' \circ \chi = \pi$, then $\chi = \psi$ or $\chi = \psi \circ \iota$.*

Proof. Let B denote the set of branch points of π. Then π is an unbranched twofold covering map from $\pi^{-1}(P \setminus B)$ onto $P \setminus B$, and it follows that for every simply connected open subset U of $P \setminus B$ the set $\pi^{-1}(U)$ has two connected components V_+, V_-, where $\pi_+ = \pi|_{V_+}$ and $\pi_- = \pi|_{V_-}$ are complex analytic diffeomorphisms from V_+ and V_- onto U, respectively. It follows that $\iota|_{V_+} = {\pi_-}^{-1} \circ \pi_+$ and $\iota|_{V_-} = {\pi_+}^{-1} \circ \pi_-$ are complex analytic diffeomorphisms from V_+ onto V_- and from V_-

onto V_+, respectively. Let $b \in B$. Then there is a unique $c \in C$ such that $\pi(c) = b$. Furthermore there are complex analytic coordinates z and w near c and b in C and P such that $z = 0$ and $w = 0$ correspond to c and b, respectively, and the mapping π takes the form $z \mapsto z^2$. Then $\iota(z) = -z$, which shows that ι is also complex analytic in a neighborhood of any ramification point. It follows that ι is an involutory automorphism of C.

The local description of π and π' implies that P has a covering by open subsets U_α, $\alpha \in A$ and complex analytic diffeomorphisms ψ_α from $\pi^{-1}(U_\alpha)$ onto $(\pi')^{-1}(U_\alpha)$ such that $\pi' \circ \psi_\alpha = \pi$ on $\pi^{-1}(U_\alpha)$. It can be arranged that $U_\alpha \cap U_\beta$ is connected when nonvoid, when there is a unique $k_{\alpha\beta} \in \mathbb{Z}/2\,\mathbb{Z}$ such that $\psi_\beta = \psi_\alpha \circ \iota^{k_{\alpha\beta}}$ on $\pi^{-1}(U_\alpha \cap U_\beta)$. If $k_{\alpha\beta} = k_\alpha - k_\beta$ for a map $\alpha \mapsto k_\alpha : A \to \mathbb{Z}/2\,\mathbb{Z}$, then $\psi_\beta \circ \iota^{k_\beta} = \psi_\alpha \circ \iota^{k_\alpha}$ on $\pi^{-1}(U_\alpha \cap U_\beta)$ yields that the ψ_α have a common extension to a complex analytic diffeomorphism ψ from C onto C' such that $\pi' \circ \psi = \pi$, and vice versa, when $\psi \circ \iota$ is the only other such diffeomorphism. The $k_{\alpha\beta}$ define a cohomology class $k \in \mathrm{H}^1(P,\ \mathbb{Z}/2\,\mathbb{Z})$, and the desired diffeomorphism ψ exists if and only if $k = 0$. For quite general topological spaces X and arbitrary commutative groups G, there is a canonical surjective homomorphism $\mathrm{H}^q(X,\ G) \to \mathrm{Hom}(\mathrm{H}_q(X,\ \mathbb{Z}),\ G)$, which is injective if $\mathrm{H}_{q-1}(X,\ \mathbb{Z})$ is torsion-free. See for instance Spanier [190, pp. 243, 241]. For $q = 1$ and connected X, when $\mathrm{H}_0(X,\ \mathbb{Z}) \simeq \mathbb{Z}$ is torsion-free and the exact sequences in the proof of l.c. can be understood explicitly, this leads to a canonical isomorphism $\mathrm{H}^1(X,\ G) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}_1(X,\ \mathbb{Z}),\ G)$. If $\mathrm{g}(P) = 0$, then $\mathrm{H}_1(P,\ \mathbb{Z}) = 0$, see Section 2.1.5, and therefore $\mathrm{H}^1(P,\ \mathbb{Z}/2\,\mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_1(P,\ \mathbb{Z}),\ \mathbb{Z}/2\,\mathbb{Z}) = 0$. □

Remark 2.3.7. Let C be a curve of genus zero. Then any involutory automorphism ι of $C \simeq \mathbb{P}^1$ not equal to the identity is induced by a complex linear transformation of $\mathbb{C}^2$ with eigenvalues ± 1, and the corresponding eigenspaces are the two fixed points in $\mathbb{P}^1$. If z is an affine coordinate on $\mathbb{P}^1$ such that the fixed points correspond to $z = 0$ and $z = \infty$, then ι corresponds to the mapping $z \mapsto -z$. The mapping that assigns to every nontrivial involutory automorphism of $C \simeq \mathbb{P}^1$ its fixed point set is bijective onto the set of all subsets of C consisting of two elements. ⊘

Remark 2.3.8. Let C be a curve of genus one. That is, $C \simeq \mathbb{C}/P$, where P is a period lattice; see Section 2.3.1. If ι is a nontrivial involutory automorphism of C, then it is of the form $\iota : z + P \mapsto -z + a + P$ for some $a + P \in \mathbb{C}/P$. We have $\iota(z + P) = z + P$ if and only if $2z \in a + P$ if and only if $z = a/2$ modulo $((1/2)\,P)/P$. Because the group $((1/2)\,P)/P$ is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$, it follows that every nontrivial involution of C has four fixed points. One of these points can be chosen freely, but then the other three fixed points are uniquely determined. ⊘

Remark 2.3.9. A twofold branched covering $\pi : C \to \mathbb{P}^1$, branching over a finite subset B of $\mathbb{P}^1$, is the same as a meromorphic function on C with mapping degree equal to two, that is, having two poles, counted with multiplicities.

For a twofold branched covering we have $d = 2$ and $n_b = 1$ for every $b \in B$, and therefore (2.3.18) implies that $\#(B) = 2\,(\,\mathrm{g}(C) + 1)$ if the curve C is a twofold branched covering of a complex projective line. The cases that $\mathrm{g}(C) = 0$, that is, $C \simeq \mathbb{P}^1$, and $\mathrm{g}(C) = 1$, that is, C is an elliptic curve, have been discussed in

Remark 2.3.7 and Remark 2.3.8, respectively. A curve C with genus > 1 that carries a meromorphic function with two poles is called a *hyperelliptic curve*. See Farkas and Kra [61, Section III.7], where a hyperelliptic curve is called a hyperelliptic surface because the book is about Riemann surfaces, that is, real two-dimensional surfaces that at the same time are complex analytic curves.

It follows from Lemma 2.3.6 that C admits an involutory automorphism with fixed-point set equal to $\pi^{-1}(B)$, where $\#(\pi^{-1}(B)) = \#(B) = 2\,(\,\mathrm{g}(C) + 1)$. Conversely, if C is a compact and connected complex analytic curve with $\mathrm{g}(C) > 1$, then Farkas and Kra [61, Proposition III.7.9] states that C is a hyperelliptic curve if and only if C admits an involutory automorphism with $2\,(\,\mathrm{g}(C) + 1)$ fixed points. Corollary 1 and 2 of [61, Proposition III.7.9] state that the fixed points of such an involutory automorphism are unique, equal to the so-called Weierstrass points of C, and that therefore the involutory automorphism of C with $2\,(\,\mathrm{g}(C) + 1)$ fixed points is unique as well. The unique involutory automorphism of C with $2\,(\,\mathrm{g}(C) + 1)$ fixed points is called the *hyperelliptic involution* of the hyperelliptic curve C. ⊘

2.3.5 The Eisenstein Invariants

Let $\mathcal{F}$ denote the space of all homogeneous polynomials

$$f = a_0\,{x_1}^4 + 4a_1\,x_0\,{x_1}^3 + 6a_2\,{x_0}^2\,{x_1}^2 + 4a_3\,{x_0}^3\,x_1 + a_4\,{x_0}^4 \tag{2.3.19}$$

of degree 4 in two variables x_0 and x_1. If $\mathcal{F}'$ denotes the set of all $f \in \mathcal{F}$ with simple zeros in $\mathbb{C}^2 \setminus \{0\}$, then the mapping that assigns to each $f \in \mathcal{F}'$ its zero-set induces a bijection between $\mathcal{F}$ modulo nonzero scalar multiples and the set of all subsets of $\mathbb{P}^1$ with four elements.

On $\mathcal{F}$ we have the action $f \mapsto f \circ L$ of the linear substitutions of variables $L \in G := \mathrm{GL}(\mathbb{C}^2)$. A polynomial I on $\mathcal{F}$ is called a *projective invariant* for this action if for every $L \in G$ there is a constant $c = c(L)$ such that $I(f \circ L) = c\,I(f)$ for every $f \in \mathcal{F}$. The following polynomials in the coefficients of (2.3.19),

$$D = a_0\,a_4 + 3{a_2}^2 - 4a_1\,a_3 \tag{2.3.20}$$

and

$$E = a_0\,{a_3}^2 + {a_1}^2\,a_4 - a_0\,a_2\,a_4 - 2a_1\,a_2\,a_3 + {a_2}^3, \tag{2.3.21}$$

satisfy $D(f \circ L) = (\det L)^4\,D(f)$ and $E(f \circ L) = (\det L)^6\,E(f)$ for every $f \in \mathcal{F}$ and $L \in \mathrm{GL}(\mathbb{C}^2)$, and therefore are projective invariants. The discriminant of f, as defined in van der Waerden [204, p. 102], is equal to 256 times $-F := D^3 - 27\,E^2$, where $F = 0$ if and only if f has multiple zeros. The oldest reference I found for the expression D, E, F is Eisenstein [58], who used these in his formulas for the zeros of the quartic polynomial f. For this reason I will call these the *Eisenstein invariants of plane quartics*, although Eisenstein did neither discuss their invariance properties

nor the relation $D^3 - 27\,E^2 + F = 0$ between D, E, and F. These were found one year later by Cayley and Boole, see [30].

Sylvester [191] proved that D and E are fundamental projective invariants for plane quartics in the sense that every polynomial projective invariant of plane quartics is a polynomial in D and E. The proof given below is inspired by Sylvester's. Analogous statements in this book, about the invariants for cubic polynomials in three variables (Sylvester, Section 4.4), the biquadratic polynomials in pairs of two variables (Frobenius, Remark 2.5.7), and the symmetric biquadratic polynomials (Frobenius, Remark 10.1.7), can perhaps be proved in a similar way.

Proof. Let I be a nonzero polynomial projective invariant, homogeneous of degree q. Then $c : L \mapsto c(L)$ is a homogeneous polynomial homomorphism from the group $G := \mathrm{GL}(\mathbb{C}^2)$ to the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers, and it follows that there exists an integer m such that $c(L) = (\det L)^m$ for every $L \in G$. The homogeneity of I of degree q yields that $m = 2\,q$, that is, $I(f \circ L) = (\det L)^{2\,q}\, I(f)$ for every $f \in \mathcal{L}$ and $L \in G$.

Let $\mathcal{S}$ denote the two-dimensional linear subspace of $\mathcal{F}$ consisting of all f in (2.3.19) such that $a_1 = a_3 = 0$ and $a_0 = a_4$. We use a_0 and a_2 as the coordinates in $\mathcal{S}$, and write the restriction of I to $\mathcal{S}$ as $I(a_0,\, a_2)$. The restrictions of D and E to $\mathcal{S}$ are given by $D = {a_0}^2 + 3\,{a_2}^3$ and $E = -{a_0}^2\,a_2 + {a_2}^3$, respectively. The set of all $f \circ L \in \mathcal{F}$ such that $f \in \mathcal{S}$ and $L \in G$ has a nonempty interior in $\mathcal{F}$, and therefore any projective invariant on $\mathcal{F}$ is uniquely determined by its restriction to $\mathcal{S}$, and it suffices to prove that the restriction $I|_{\mathcal{S}}$ to $\mathcal{S}$ of our polynomial projective invariant I is a polynomial in $D|_{\mathcal{S}}$ and $E|_{\mathcal{S}}$.

The substitution of variables $x_0 = \xi_0$, $x_1 = \mathrm{i}\,\xi_1$ maps $a_0\,({x_0}^4 + {x_1}^4) + 6\,a_2\,{x_0}^2\,{x_1}^2$ to $a_0\,({\xi_0}^4 + {\xi_1}^4) - 6\,a_2\,{\xi_0}^2\,{\xi_1}^2$; hence $I(a_0,\, -a_2) = \mathrm{i}^{2\,q}\, I(a_0,\, a_2) = (-1)^q\, I(a_0,\, a_2)$. Because of the homogeneity of degree q of I, it follows that $I(-a_0,\, a_2) = I(a_0,\, a_2)$, that is, $a_0 \mapsto I(a_0,\, a_1)$ is an even polynomial. Furthermore, the substitution of variables $x_0 = \xi_0 + \xi_1$, $x_1 = \xi_0 - \xi_1$ maps $a_0\,({x_0}^4 + {x_1}^4) + 6\,a_2\,{x_0}^2\,{x_1}^2$ to $(2\,a_0 + 6\,a_2)\,({\xi_0}^4 + {\xi_1}^4) + 6\,(2\,a_0 - 2\,a_2)\,{\xi_0}^2\,{\xi_1}^2$; hence $I(2\,a_0 + 6\,a_2,\, 2\,a_0 - 2\,a_2) = (-2)^{2\,q}\, I(a_0,\, a_2) = 4^q\, I(a_0,\, a_2)$, and the homogeneity of degree q of I implies that $I(a_0/2 + 3\,a_2/2,\, a_0/2 - a_2/2) = I(a_0,\, a_2)$. In other words, I is invariant under the linear reflections $R_1 : (a_0,\, a_2) \mapsto (-a_0,\, a_2)$ and $R_2 : (a_0,\, a_2) \mapsto (a_0/2 + 3\,a_2/2,\, a_0/2 - a_2/2)$ about the lines $a_0 = 0$ and $a_0 - 3\,a_2 = 0$, respectively. The reflections R_1 and R_2 generate a subgroup Γ of six elements of the group of linear transformations in the $(a_0,\, a_2)$-space. The element $R_3 = R_1 \circ R_2 \circ R_1 : (a_0,\, a_2) \mapsto (a_0/2 - 3\,a_2/2,\, -a_0/2 - a_2/2)$ is the third reflection element of Γ, which is about the line $a_0 + 3\,a_2 = 0$. The other two nontrivial elements of Γ, namely $R_1\,R_2$ and $R_2\,R_1$, have only the origin as fixed point. It follows that the group Γ acts freely on the complement of the union $\Sigma : a_0\,(a_0 - 3\,a_2)\,(a_0 + 3\,a_2) = 0$ of the three fixed points lines of the reflections R_1, R_2, R_3; hence each orbit of Γ in the complement of Σ has six elements.

The equations ${a_0}^2 + 3\,{a_2}^3 = D$, $-{a_0}^2\,a_2 + {a_2}^3 = E$ are equivalent to ${a_0}^2 = D - 3\,{a_2}^2$, $-4\,{a_2}^3 + D\,a_2 + E = 0$. Given D and E, the latter equation has three distinct solutions a_2, locally depending holomorphically on $(D,\, E)$, if and only if $-F := D^3 - 27\,E^2 \neq 0$. In turn, the inequality $F \neq 0$ implies that the solutions a_2

satisfy $D - 3\,a_2{}^2 \neq 0$, and for each solution a_2 there are two distinct solutions a_0 of the equation $a_0{}^2 = D - 3\,a_2{}^2$, locally depending holomorphically on $(D,\ E)$. It follows that for $(D,\ E)$ in the complement of the cusp $F = 0$, the equations $a_0{}^2 = D - 3\,a_2{}^2$ have six distinct solutions $(a_0,\ a_2)$, locally depending holomorphically on $(D,\ E)$. We also have the estimates that either $4\,|a_2|^2 \leq 1 + |D|$ and $|a_0|^2 \leq |D| + 3\,|a_2|^2 \leq (7/4)\,|D| + 3/4$, or $4\,|a_2|^2 > 1 + |D|$, when $|a_2| \leq |E|$ and $|a_0|^2 \leq |D| + |E|^3$.

It follows that Φ defines a sixfold complex analytic covering map from $\Phi^{-1}(\{F \neq 0\})$ onto $\{F \neq 0\}$. The Jacobi determinant of Φ is equal to $-2\,a_0\,(a_0 - 3\,a_2)\,(a_0 + 3\,a_2)$, of which the zero-set, the set of all singular points of Φ, is equal to aforementioned set Σ. The covering property implies that $\Phi^{-1}(\{F \neq 0\})$ is contained in the complement of Σ, on which the group Γ of six elements acts freely, and because the fibers of the covering have six elements, it follows that Γ acts transitively on the fibers of the covering. Because I is invariant under Γ, we conclude that there is a unique function $J : \{F \neq 0\} \to \mathbb{C}$ such that $I = J \circ \Phi$ on $\Phi^{-1}(\{F \neq 0\})$. The bounds on Φ^{-1} imply bounds on $J = I \circ \Phi^{-1}$, and Riemann's theorem on removable singularities implies that J extends to an entire analytic function on $\mathbb{C}^2$, where the polynomial bounds on Φ^{-1}, hence on $J = I \circ \Phi^{-1}$, imply that J is a polynomial. Because $I = J \circ \Phi$ on the nonempty open subset $\Phi^{-1}(\{F \neq 0\})$ of $\mathbb{C}^2$, we have $I = J \circ \Phi$ on $\mathbb{C}^2$. That is, I is a polynomial in $(D,\ E)$. More precisely, the homogeneity of degree q of I implies that

$$I = \sum_{j \in \mathbb{Z},\, q/3 \leq j \leq q/2} I_j\, D^{3\,j-q}\, E^{q-2\,j}$$

for unique coefficients $I_j \in \mathbb{C}$. This completes the proof of Sylvester's theorem.

It is a theorem of Chevalley [35] that if Γ is a finite group generated by linear reflections in an n-dimensional vector space V over a field K of characteristic zero, then the K-algebra J of all Γ-invariant polynomials on V is generated by n algebraically independent homogeneous elements (and the unit). Furthermore, the product of the degrees of the generators of J is equal to the order of Γ. Because Γ is a finite group generated by linear reflections in $\mathbb{C}^2$, $\deg D = 2$, $\deg E = 3$, and $\#(\Gamma) = 6$, the above proof can be viewed as an illustration of Chevalley's theorem. □

Let W denote the Weierstrass curve defined by equation (2.3.6). The mapping $p : [x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ from $\mathbb{P}^2$ to $\mathbb{P}^1$ is well defined except at the point $[0 : 0 : 1]$. However, if $x_2 = 1$ then (2.3.6) implies that $x_0 = \mathrm{O}(x_1{}^3)$ for $(x_0,\ x_1) \to (0,\ 0)$, and therefore the restriction to $W \setminus \{[0 : 0 : 1]\}$ of p extends to a complex analytic mapping $\pi : W \to \mathbb{P}^1$, where $\pi([0 : 0 : 1]) = [0 : 1]$. This mapping is a twofold branched covering with branch points at the three zeros of $4\,x_1{}^3 - g_2\,x_0{}^2\,x_1 - g_3\,x_0{}^3$ and at the point $[0 : 1]$ "at infinity." That is, the branch points are the zeros of the homogeneous polynomial

$$f = x_0\,(4\,x_1{}^3 - g_2\,x_0{}^2\,x_1 - g_3\,x_0{}^3) \tag{2.3.22}$$

of degree four in $(x_0,\ x_1)$. In the form (2.3.19) for (2.3.22), we have $a_0 = 0, a_1 = 1$, $a_2 = 0, a_3 = -g_2/4, a_4 = -g_3$, hence $D = -4\,a_1\,a_3 = g_2$, $E = {a_1}^2\,a_4 = -g_3$, and therefore the discriminant in (2.3.9) is equal to $\Delta = {g_2}^3 - 27\,{g_3}^2 = D^3 - 27\,E^2 = -F$. It follows that $J = {g_2}^3/\Delta = D^3/(D^3 - 27\,E^2)$.

If B is any subset of $\mathbb{P}^1$ with four elements, then we can arrange by means of a projective linear transformation of $\mathbb{P}^1$ that B is equal to the zero-set of (2.3.22) for a suitable choice of g_2 and g_3. It follows that the twofold branched covering of $\mathbb{P}^1$ with branch locus B is isomorphic to the curve in $\mathbb{P}^2$ defined by (2.3.6), and therefore has the same modulus. This leads to the following conclusion.

Lemma 2.3.10 *Let f be a homogeneous polynomial of degree four in two variables with simple zeros in $\mathbb{C}^2 \setminus \{0\}$, and let B be the set in $\mathbb{P}^1$, consisting of four elements, corresponding to the zero-set of f. Let D and E be the Eisenstein invariants of f and $-F = D^3 - 27\,E^2$ its discriminant. Then any twofold branched covering of $\mathbb{P}^1$ with branch locus B is an elliptic curve with modulus equal to*

$$J = \frac{D^3}{-F} = \frac{D^3}{D^3 - 27\,E^2}. \tag{2.3.23}$$

Note that the right-hand side of (2.3.23), viewed as a function $J(f)$ of the homogeneous polynomial f of degree four in two variables, is an *absolute invariant*, in the sense that $J(f \circ L) = J(f)$ for every linear substitution of variables $L \in \mathrm{GL}(\mathbb{C}^2)$. It is a rational function on the space of all homogeneous polynomials of degree four in two variables, where the numerator and the denominator are homogeneous polynomials of degree 6 in the coefficients of f.

2.4 Biquadratic Curves

Let

$$p(x,\ y) = p((x_0,\ x_1),\ (y_0,\ y_1)) = \sum_{i,\,j=0}^{2} {x_0}^i\,{x_1}^{2-i}\,A_{ij}\,{y_0}^j\,{y_1}^{2-j} \tag{2.4.1}$$

be a *biquadratic polynomial* on $\mathbb{C}^2 \times \mathbb{C}^2$, meaning that for each $y \in \mathbb{C}^2$, $x \mapsto p(x,\ y)$ is homogeneous of degree two and for each $x \in \mathbb{C}^2$, $y \mapsto p(x,\ y)$ is homogeneous of degree two. Each biquadratic polynomial can be viewed as a holomorphic section of the holomorphic line bundle $L = L^{(2,\,2)}$ over $\mathbb{P}^1 \times \mathbb{P}^1$, where the fiber $L_{([x],\,[y])}$ is equal to the one-dimensional complex vector space of all biquadratic polynomials on $[x] \times [y]$. If p is not identically zero, then the divisor $\mathrm{Div}(p)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ will be called *the zero-set of p in $\mathbb{P}^1 \times \mathbb{P}^1$*, or the *biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by p*. Here the terms "zero-set" and "curve" are a bit misleading, since the effective divisor $\mathrm{Div}(p)$ includes the information on the multiplicities of the irreducible components of the curve Z where $p = 0$, equal to the vanishing orders of p along the irreducible

components. If p has simple zeros along each irreducible component of Z, then $\mathrm{Div}(p)$ can be identified with Z.

The following lemma can be viewed as a Chow theorem and a Bézout theorem for $\mathbb{P}^1 \times \mathbb{P}^1$. It is a complex analytic version for $\mathbb{P}^1 \times \mathbb{P}^1$ of the statement that any complex projective subvariety of a Cartesian product of projective spaces is the zero-set of polyhomogeneous polynomials; see [182, Chapter I, §5.1]. For the definition of the intersection number $D \cdot D'$ of the divisors D and D', see Section 2.1.6.

Lemma 2.4.1 *Let D be an effective divisor in $\mathbb{P}^1 \times \mathbb{P}^1$. Then there is a nonzero bihomogeneous polynomial function s on $\mathbb{C}^2 \times \mathbb{C}^2$, unique up to multiplication by a nonzero scalar factor, such that $D = \mathrm{Div}(s)$. If the bihomogeneity degree of s is equal to $(d_1,\ d_2)$ and s' is a nonzero bihomogeneous polynomial of bidegree $(d_1',\ d_2')$, then $\mathrm{Div}(s) \cdot \mathrm{Div}(s') = d_1\, d_2' + d_2\, d_1'$. If $d_1 > 0$ and $d_2 > 0$, then the support $|D|$ of D is connected.*

Proof. If A is a compact Riemann surface, then $\mathrm{H}^0(A,\ \mathbb{Z}) \simeq \mathbb{Z}$, $\mathrm{H}^1(A,\ \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ with g equal to the genus of A, $\mathrm{H}^2(A,\ \mathbb{Z}) \simeq \mathbb{Z}$, and $\mathrm{H}^k(A,\ \mathbb{Z}) = 0$ for $k > 2$ because $\dim_{\mathbb{R}} A = 2$. Let B be another compact Riemann surface. Because the cohomology groups $\mathrm{H}^k(A,\ \mathbb{Z})$ and $\mathrm{H}^k(B,\ \mathbb{Z})$ have no torsion, the *Künneth formula* implies that

$$\mathrm{H}^k(A \times B,\ \mathbb{Z}) \simeq \bigoplus_{j=0}^{k} \mathrm{H}^j(A,\ \mathbb{Z}) \otimes \mathrm{H}^{k-j}(B,\ \mathbb{Z}). \tag{2.4.2}$$

See for instance Spanier [190, p. 249], which I found easier to understand than the original articles of Künneth [116]. If $A = \mathbb{P}^1$ then $\mathrm{H}^1(A,\ \mathbb{Z}) = \{0\}$, and the isomorphism in (2.4.2), as explicited in Spanier, takes the form

$$\mathrm{H}^2(\mathbb{P}^1 \times B,\ \mathbb{Z}) = {\pi_1}^*(\mathrm{H}^2(\mathbb{P}^1,\ \mathbb{Z})) \oplus {\pi_2}^*(\mathrm{H}^2(B,\ \mathbb{Z})), \tag{2.4.3}$$

where $\pi_1 : \mathbb{P}^1 \times B \to \mathbb{P}^1 : (a,\ b) \to a$ and $\pi_2 : \mathbb{P}^1 \times B \to B : (a,\ b) \to b$ denote the projections from $\mathbb{P}^1 \times B$ onto the first and second factor, respectively. By Poincaré duality (2.1.17), (2.4.3) in turn implies that

$$\mathrm{H}_2(\mathbb{P}^1 \times B,\ \mathbb{Z}) = \mathbb{Z}\,[\mathbb{P}^1 \times \{b\}] \oplus \mathbb{Z}\,[\{a\} \times B] \tag{2.4.4}$$

for any $(a,\ b) \in \mathbb{P}^1 \times B$. In the proof of Lemma 2.4.1 we will use (2.4.3) only for $B = \mathbb{P}^1$, where we have included (2.4.4) for later reference.

It follows from Lemma 2.1.2 that there exist a holomorphic line bundle L over $\mathbb{P}^1 \times \mathbb{P}^1$ and a nonzero holomorphic section s of L such that $D = \mathrm{Div}(s)$. Recall the line bundle $\mathrm{O}(1)$ over $\mathbb{P}^1$ of degree one, the fiber of which over $[x] \in \mathbb{P}^1$ is equal to the space $[x]^*$ of all linear forms on the one-dimensional vector space $[x]$. Because $\mathrm{H}^2(\mathbb{P}^1,\ \mathbb{Z})$ is generated by the Chern class of $\mathrm{O}(1)$, it follows in view of (2.4.3) with $B = \mathbb{P}^1$ that there are $d_1,\ d_2 \in \mathbb{Z}$ such that the Chern class of L is equal to the Chern class of

$$\mathrm{O}(d_1,\ d_2) := ({\pi_1}^*(\mathrm{O}(1)))^{d_1}\ ({\pi_2}^*(\mathrm{O}(1)))^{d_2}.$$

See Section 2.1.4 for the definition of pullbacks of line bundles. Because $\mathrm{H}^1(\mathbb{P}^1,\,\mathbb{Z}) = \{0\}$, the Künneth formula (2.4.2) with $k = 1$ implies that $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1,\,\mathbb{Z}) = \{0\}$; hence $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1,\,\mathbb{Z}) = \{0\}$, $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1,\,\mathbb{C}) \simeq \mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1,\,\mathbb{Z}) \otimes \mathbb{C} = \{0\}$, and therefore $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1,\,\mathcal{O}) = \{0\}$ in view of (6.2.35), because $\mathbb{P}^1 \times \mathbb{P}^1$ is Kähler. It follows that the Chern homomorphism c in the exact sequence (2.1.8) is injective, and we conclude that $L \simeq \mathrm{O}(d_1,\,d_2)$.

We recognize $\mathrm{O}(d_1,\,d_2)$ as the line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ such that the fiber over $([x],\,[y])$ is equal to the space of bihomogeneous functions of bidegree $(d_1,\,d_2)$ on $([x] \setminus \{0\}) \times ([y] \setminus \{0\})$. Let u_1 and u_2 be nonzero holomorphic sections of $\mathrm{O}(1)$, with only one zero at a point $a_i \in \mathbb{P}^1$, which moreover is simple. Then $v = (u_1 \circ \pi_1)^{d_1}\,(u_2 \circ \pi_2)^{d_2}$ is a nonzero meromorphic section of $\mathrm{O}(d_1,\,d_2)$, and $\mathrm{Div}(v) = d_1\,(\{a_1\} \times \mathbb{P}^1) + d_2\,(\mathbb{P}^1 \times \{a_2\})$. It follows that D is homologous to $\mathrm{Div}(v)$, and therefore $D \cdot H = d_1$ and $D \cdot V = d_2$ for each horizontal and vertical axis H and V, respectively. Because we had assumed that D is effective, we have $d_1 \geq 0$ and $d_2 \geq 0$.

The holomorphic section s of $\mathrm{O}(d_1,\,d_2)$ is identified with a bihomogeneous holomorphic function of bidegree $(d_1,\,d_2)$ on $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$. Because the complement $(\{0\} \times \mathbb{C}^2) \cup (\mathbb{C}^2 \times \{0\})$ of $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ in $\mathbb{C}^2 \times \mathbb{C}^2$ has codimension two, Hartog's lemma yields that s extends to a holomorphic function on $\mathbb{C}^2 \times \mathbb{C}^2$. Using the bihomogeneity of s, we obtain that s is equal to the sum of the monomials of bidegree $(d_1,\,d_2)$ in the Taylor expansion of s at $(0,\,0)$; hence s is a bihomogeneous *polynomial* function of bidegree $(d_1,\,d_2)$.

This proves the first statement in the lemma. Because $\mathrm{Div}(s)$ and $\mathrm{Div}(s')$ are homologous to $d_1\,V + d_2\,H$ and $d_1'\,V' + d_2'\,H'$, respectively, where $V,\ V'$ and $H,\ H'$ are any vertical and horizontal axes, respectively, we have $\mathrm{Div}(s) \cdot \mathrm{Div}(s') = (d_1\,V + d_2\,H) \cdot (d_1'\,V' + d_2'\,H') = d_1\,d_1'\,V \cdot V' + d_1\,d_2'\,V \cdot H' + d_2\,d_1'\,H \cdot V' + d_2\,d_2'\,H \cdot H' = d_1\,d_2' + d_2\,d_1'$.

Each connected component of $|D|$ is an analytic set. Therefore, if $|D|$ is not connected, then $D = E + E'$ for some nonzero effective divisors E and E' with disjoint supports; hence $E \cdot E' = 0$. If $(e_1,\,e_2)$ and $(e_1',\,e_2')$ are the respective bidegrees of E and E', then $(e_1,\,e_2) \neq (0,\,0)$, $(e_1',\,e_2') \neq (0,\,0)$, and $0 = E \cdot E' = e_1\,e_2' + e_2\,e_1'$ implies that $e_1\,e_2' = 0$ and $e_2\,e_1' = 0$. It follows that either $e_1 = e_1' = 0$ or $e_2 = e_2' = 0$; hence $d_1 = e_1 + e_1' = 0$ or $d_2 = e_2 + e_2' = 0$. □

Corollary 2.4.2 *The biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are the effective divisors D in $\mathbb{P}^1 \times \mathbb{P}^1$ such that the intersection number of D with each horizontal axis and with each vertical axis is equal to two. Each biquadratic curve is connected. If D is reducible, and the D_i are the irreducible components of D with the respective multiplicities $m_i \in \mathbb{Z}_{>0}$, then each D_i is the zero-set of a bihomogeneous polynomial of bidegree $(d_{i1},\,d_{i2}) \neq (0,\,0)$, where $\sum_i d_{i1} = \sum_i d_{i2} = 2$.*

2.4.1 The Eisenstein Invariants of the Partial Discriminants

For each $(x_0,\,x_1) \in \mathbb{C}^2$, the discriminant of $y \mapsto p(x,\,y)$ is equal to

$$\Delta_2(p)(x_0,\, x_1) := \left[\sum_{i=0}^{2} {x_0}^i\, {x_1}^{2-i}\, A_{i1}\right]^2 - 4\left[\sum_{i=0}^{2} {x_0}^i\, {x_1}^{2-i}\, A_{i0}\right]\left[\sum_{i=0}^{2} {x_0}^i\, {x_1}^{2-i}\, A_{i2}\right], \tag{2.4.5}$$

which is a homogeneous polynomial of degree 4 in $x = (x_0,\, x_1)$. The polynomial (2.4.5) will be called the *partial discriminant of p with respect to the second variable*. Similarly the discriminant $\Delta_1(p)(y_0,\, y_1)$ of $x \mapsto p(x,\, y)$ is called the *partial discriminant of p with respect to the first variable*.

If $\Delta_2(p)(x_0,\, x_1) \neq 0$, then $(x_0,\, x_1) \neq (0,\, 0)$ and there are exactly two points $[y_0 :\, y_1] \in \mathbb{P}^1$ such that $c := ([x_0 :\, x_1],\, [y_0 :\, y_1]) \in C = \mathrm{Z}_p$, the total derivatives of $y \mapsto p(x,\, y)$ at $y = (y_0,\, y_1)$ are nonzero, and it follows that C is smooth at these points c. If $\Delta_2(p)(x_0,\, x_1) = 0$ and $\mathrm{d}\Delta_2(p) \neq 0$ at $(x_0,\, x_1)$, then $(x_0,\, x_1) \neq (0,\, 0)$ and there is exactly one $[y_0 :\, y_1] \in \mathbb{P}^1$ such that $c := ([x_0 :\, x_1],\, [y_0 :\, y_1]) \in C$, but this time the total derivative of $x \mapsto p(x,\, y)$ is nonzero, and C is again smooth at the point c. Also, if $[x_0' :\, x_1'] \neq [x_0 :\, x_1]$ converges to $[x_0 :\, x_1]$, then $\Delta_2(p)(x_0',\, x_1') \neq 0$, and the two points $([x_0' :\, x_1'],\, [y_0' :\, y_1'])$ on C converge to $([x_0 :\, x_1],\, [y_0 :\, y_1])$. The homogeneous polynomial $\Delta_2(p)$ of degree 4 has only simple zeros in $\mathbb{C}^2 \setminus \{0\}$ if and only if its discriminant $-F = D^3 - 27\,E^2$ from Section 2.3.4 is not equal to zero. Therefore, if $F \neq 0$, then C is smooth, whereas Corollary 2.4.2 implies that C is connected. The projection

$$C \ni ([x_0 :\, x_1],\, [y_0 :\, y_1]) \mapsto [x_0 : x_1] \in \mathbb{P}^1$$

exhibits C as a twofold branched covering of $\mathbb{P}^1$, with branch points at the four simple zeros $[x_0 :\, x_1]$ of f. In fact, the coming together of the branches over the branch points yields another proof that C is connected. It therefore follows from (2.3.18) with $D = \mathbb{P}^1$, and hence $\mathrm{g}(D) = 0$, and $d = 2$ that $\mathrm{g}(C) = 1 - 2 + 4/2 = 1$, that is, C is an elliptic curve.

On the other hand, if $F = 0$, then either $\Delta_2(p) \equiv 0$, in which case all zeros of p have multiplicity ≥ 2 and C is not smooth in the sense that $\mathrm{d}p \neq 0$ at all zeros of p in $(\mathbb{C}^2 \setminus \{0\})^2$, or the projection from C onto the first factor is not a covering because C contains a vertical axis, in which case C is not smooth either, or the projection from C onto the first factor is a twofold branched covering, but with multiple branch points, in which case C again is not smooth.

In view of Lemma 2.3.10 we have proved the following:

Proposition 2.4.3 *Let $p = p(x,\, y)$ be a biquadratic polynomial of which the zero-set defines a biquadratic curve C in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $k = 1$ or $k = 2$. Let $D_k := D(\Delta_k(p))$, $E_k := E(\Delta_k(p))$, and $-F_k := -F(\Delta_k(p)) = {D_k}^3 - 27\,{E_k}^2$ denote the Eisenstein invariants, see Section* 2.3.4*, of the partial discriminant of p with respect to the kth variable. Then C is smooth if and only if $F_i \neq 0$. Furthermore, if C is smooth, then C is an elliptic curve, with modulus J given by* (2.3.23) *with $D = D_k$, $E = E_k$, $F = F_k$. As a function of p, J is a rational function on the*

space of all biquadratic polynomials, with numerator and denominator each equal to a polynomial of degree 12 *on the space of all biquadratic polynomials.*

The Eisenstein invariants D_k, E_k, and $-F_k = {D_k}^3 - 27\,{E_k}^2$ in Proposition 2.4.3 are homogeneous polynomials in the coefficients A_{ij} of the biquadratic polynomials (2.4.1) of degrees 4, 6, and 12, respectively. We would like to emphasize that although the general formulas for these polynomials are quite formidable, they are completely explicit. In examples in which not too many of the coefficients A_{ij} are nonzero, the formulas for D_k, E_k, and F_k become accordingly more manageable.

Remark 2.4.4. Let $\mathcal{B}$ denote the nine-dimensional vector space of all biquadratic polynomials p, and let G denote the eight-dimensional group $\mathrm{GL}(2,\,\mathbb{C}) \times \mathrm{GL}(2,\,\mathbb{C})$. If $p \in \mathcal{B}$ and $L = (L_1,\, L_2) \in G$, then $\Delta_2(p \circ L) = (\det L_2)^2\, \Delta_2(p) \circ L_1$, with a similar equation for $\Delta_1(p \circ L)$, and therefore

$$D(\Delta_k(p \circ L)) = (\det L_1)^4\, (\det L_2)^4\, D(\Delta_k(p)) = (\det L)^4\, D(\Delta_k(p)) \quad (2.4.6)$$

$$E(\Delta_k(p \circ L)) = (\det L_1)^6\, (\det L_2)^6\, E(\Delta_k(p)) = (\det L)^6\, E(\Delta_k(p)) \quad (2.4.7)$$

Here we have used that D and E are homogeneous of degree two and three and are projective invariants of order 4 and 6, respectively. In other words, $p \mapsto D(\Delta_k(p))$ and $p \mapsto E(\Delta_k(p))$ are projective invariants of order 4 and 6, respectively. For a discussion of the general projective invariants for the action of G on $\mathcal{B}$, see Remark 2.5.7. ⊘

2.4.2 The Vector Field on the Biquadratic Curve

In the following lemma we identify, in affine coordinates, the holomorphic vector field without zeros on the biquadratic curve $p = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with the restriction to $p = 0$ of the Hamiltonian vector field defined by the function p. This leads to another proof that the smooth solution curves C of (2.5.3) are elliptic curves. It also leads to the identification in Corollary 2.4.7 of the Eisenstein invariants of the partial discriminants of p with the Weierstrass invariant (2.3.4) of the period lattice of the vector field v.

Lemma 2.4.5 *Let p be a bihomogeneous polynomial of bidegree* $(2,\, 2)$ *on $\mathbb{C}^2 \times \mathbb{C}^2$ that corresponds to the polynomial $q(x,\, y) := p((1,\, x),\, (1,\, y))$ in the two affine coordinates $(x,\, y)$. Let C be the biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the zero-set of p and let C^{reg} denote the smooth part of C. Then the restriction to the smooth part of the curve $q = 0$ in $\mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ of the Hamiltonian vector field*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial q(x,\, y)}{\partial y}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial q(x,\, y)}{\partial x},$$

defined by the function q, extends to a holomorphic vector field v without any zeros on C^{reg}. In the other affine coordinates this vector field is also Hamiltonian.

Let $\Delta_2(x)$ and $\Delta_1(y)$ denote the discriminant of the polynomial $y \mapsto q(x,\ y)$ and $x \mapsto q(x,\ y)$, respectively. The time needed to run from $(x^0,\ y^0)$ to $(x^1,\ y^1)$ on the curve $q(x,\ y) = 0$ is equal to

$$\pm \int_{x^0}^{x^1} \Delta_2(x)^{-1/2}\, \mathrm{d}x = t^1 - t^0 = \pm \int_{y^0}^{y^1} \Delta_1(y)^{-1/2}\, \mathrm{d}y, \tag{2.4.8}$$

where in the first and the last integrals we have parameterized the curve $q(x,\ y) = 0$ by the coordinate x and y, respectively.

Proof. Because the curve $q = 0$ is smooth at $(x,\ y)$ if and only if $\mathrm{d}q$ is nonzero at $(x,\ y)$ if and only if the Hamiltonian vector field H_q defined by q is nonzero at $(x,\ y)$, and because $\mathrm{d}q\ \mathrm{H}_q = 0$, it follows that v is a holomorphic tangent vector field without zeros on the smooth part of $q = 0$.

If we make the substitution of variables $x = 1/\xi$, and write

$$q(x,\ y) = p((1,\ x),\ (1,\ y)) = x^2\, p(1/x,\ 1),\ (1,\ y)) = \xi^{-2}\, r(\xi,\ y),$$

then

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -x^{-2}\, \frac{\mathrm{d}x}{\mathrm{d}t} = -x^{-2}\, \frac{\partial q(x,\ y)}{\partial y} = -x^{-2}\, \xi^{-2}\, \frac{\partial r(\xi,\ y)}{\partial y} = -\frac{\partial r(\xi,\ y)}{\partial y},$$

whereas at any point where $q(x,\ y) = 0$, we have

$$\frac{\partial r(\xi,\ y)}{\partial \xi} = \xi^2\, \frac{\partial q(x,\ y)}{\partial x}\, \frac{\mathrm{d}x}{\mathrm{d}\xi} = -\frac{\partial q(x,\ y)}{\partial x} = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

In other words, in the $(\xi,\ y)$-chart the vector field is equal to the restriction to $r(\xi,\ y) = 0$ of the Hamiltonian vector field defined by the function $-r(\xi,\ y)$, which is a holomorphic tangent vector field to the smooth part of $r(\xi,\ y) = 0$ without zeros. Similar computations in the other two charts of $\mathbb{P}^1 \times \mathbb{P}^1$ lead to the desired conclusions.

For the proof of (2.4.8), we write $q(x,\ y) = a(x)\, y^2 + b(x)\, y + c(x)$, where $a(y), b(y), c(y)$ are polynomials of degree ≤ 2 in y. Then $q(x,\ y) = 0$ is equivalent to $y = (-b(x) \pm \Delta_2(x)^{1/2}/2\, a(x)$, where $\Delta_2(x) = b(x)^2 - 4\, a(x)\, c(x)$ is the discriminant of the polynomial $y \mapsto q(x,\ y)$, which is a polynomial in x of degree ≤ 4. Therefore

$$\mathrm{d}x/\,\mathrm{d}t = \partial q(x,\ y)/\partial y = 2\, a(x)\, y + b(x) = \pm\, \Delta_2(x)^{1/2}$$

on $q(x,\ y) = 0$, and similarly $\mathrm{d}y/\,\mathrm{d}t = -\partial q(x,\ y)/\partial x = \mp\, \Delta_1(y)^{1/2}$. □

Figure 2.4.1 shows the Hamiltonian vector field on the real part of a biquadratic curve $p = 0$. Here $p = 0$ is the Lyness curve (11.4.2) with $a = 0.4$ and $z = 10.58$, the same as in Figure 1, 2. A nearby level curve $p = -0.05$ has been added to the picture in order to illustrate that the Hamiltonian vector field is smaller where the gradient of p is smaller, that is, the nearby level curve is farther away.

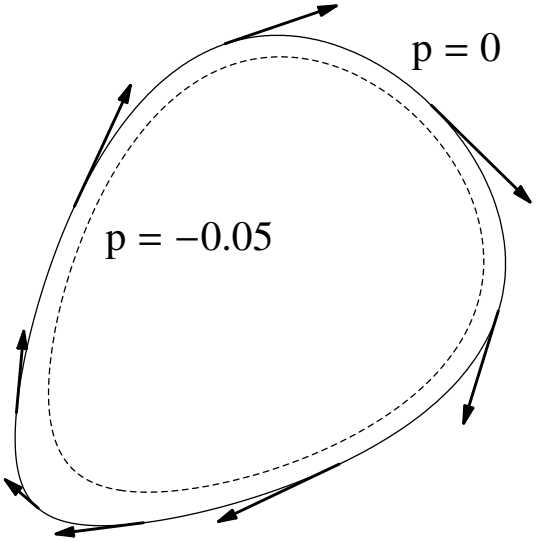


Fig. 2.4.1 The Hamiltonian vector field on the curve $p = 0$.

Remark 2.4.6. Lemma 2.4.5 also follows from Lemma 3.3.4 and Lemma 3.3.3, where the biquadratic polynomials are identified with the holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$. ⊘

Corollary 2.4.7 *In the situation of Lemma 2.4.5, assume that the curve C is smooth, which is equivalent to* $F(\Delta_k) \neq 0$. *Let P be the period lattice of v, the holomorphic vector field without zeros on C. Then* $D(\Delta_1) = g_2(P) = D(\Delta_2)$ *and* $E(\Delta_1) = -g_3(P) = E(\Delta_2)$.

Proof. In order to indicate the dependence on the polynomial p in the notation, we write $q = q(p)$, $\Delta_1 = \Delta_1(p)$, and $P = P(p)$. A linear substitution of variables $x_1 = a\,\xi_1 + b\,\xi_0$, $x_0 = c\,\xi_1 + d\,\xi_0$, corresponds to the fractional linear substitution of variables $x = \lambda(\xi) = (a\,\xi + b)/(c\,\xi + d)$ in the affine coordinate. If L denotes the linear transformation that maps $(\xi_0,\ \xi_1)$ to $(x_0,\ x_1)$, then $q(p \circ (L \times 1))(\xi,\ y) = (c\,\xi + d)^2\, q^p(\lambda(\xi),\ y)$, whereas on the other hand we have $\mathrm{d}x/\,\mathrm{d}t = (a\,d - b\,c)\,(c\,\xi + d)^{-2}\ \mathrm{d}\xi/\,\mathrm{d}t$. If $\mathrm{d}\xi/\,\mathrm{d}t = \partial q(p \circ (L \times 1))(\xi,\ y)/\partial y$, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (a\,d - b\,c)\,\frac{\partial q(p)(\lambda(\xi),\ y)}{\partial y},$$

and it follows that the mapping $(\xi,\ y) \mapsto (\lambda(\xi),\ y)$ intertwines the Hamiltonian vector field defined by the function $q(p \circ (L \times 1))$ with $\det L$ times the Hamiltonian vector field defined by the function $q(p)$. Because period lattices of intertwined vector fields are the same, it follows that $P(p \circ (L \times 1)) = (\det L)^{-1}\, P(p)$, and therefore (2.3.4) implies that $g_2(P(p \circ (L \times 1))) = (\det L)^4\, g_2(P(p))$ and $g_3(P(p \circ (L \times 1))) = (\det L)^6\, g_3(P(p))$.

The first equation in (2.3.7) in combination with (2.3.5) yields

$$\mathrm{d}t = y^{-1}\,\mathrm{d}x = \pm f(1,\,x)^{-1/2}\,\mathrm{d}x,$$

with $f(x_0,\,x_1)$ as in (2.3.22). A comparison with the first identity in (2.4.8) shows that the period lattice $P(p)$ is equal to the period lattice of the Hamiltonian vector field (2.3.7) on the Weierstrass curve (2.3.5) if $\Delta_2(p) = f$. It therefore follows from the observation after (2.3.22) that $g_2(P(p)) = g_2 = D(\Delta_2(p))$ and $g_3(P(p)) = g_3 = -E(\Delta_2(p))$. Using the invariance properties of the Eisentein invariants D and E, we have, for every linear substitution of variables L,

$$\begin{aligned} g_2(P(p \circ (L \times 1))) &= (\det L)^4\, g_2(P(p)) = (\det L)^4\, D(\Delta_2(p)) \\ &= D(\Delta_2(p) \circ L) = D(\Delta_2(p \circ (L \times 1))), \end{aligned} \tag{2.4.9}$$

and

$$\begin{aligned} g_3(P(p \circ (L \times 1))) &= (\det L)^6\, g_2(P(p)) = (\det L)^6\, E(\Delta_2(p)) \\ &= E(\Delta_2(p) \circ L) = E(\Delta_2(p \circ (L \times 1))). \end{aligned} \tag{2.4.10}$$

Let $\mathcal{P}$ denote the space of all homogeneous polynomials of degree four in two variables, and $\mathcal{W}$ the set of all $f \in \mathcal{P}$ of the form (2.3.22), with ${g_2}^3 - 27\,{g_3}^2 \neq 0$. Then the set $\mathcal{P}'$ of all homogeneous polynomials of degree four in two variables that can be transformed to an element of $\mathcal{W}$ by means of a linear substitution of variables is a nonempty open subset of $\mathcal{P}$.

It follows that if $\mathcal{B}$ denotes the space of all bihomogeneous polynomials of bidegree $(2,\,2)$ on $\mathbb{C}^2 \times \mathbb{C}^2$, the set $\mathcal{B}'$ of all $p' \in \mathcal{B}$ such that $\Delta_2(p') \in \mathcal{P}'$ is a nonempty open subset of $\mathcal{B}$. We have $p' \in B'$ if and only if there exist a linear transformation L and $f \in \mathcal{W}$ such that $\Delta_2(p') = f \circ L$. Because $\Delta_2(p \circ (L \times 1)) = \Delta_2(p) \circ L$, this means that $\Delta_2(p) = f \in \mathcal{W}$ if $p := p' \circ (L^{-1} \times 1)$, which is equivalent to $p' = p \circ (L \times 1)$. It now follows from (2.4.9) and (2.4.10) that $g_2(P(p')) = D(\Delta_2(p'))$ and $g_3(P(p')) = -E(\Delta_2(p'))$ for every $p' \in \mathcal{B}'$.

Let $\mathcal{B}^{\mathrm{reg}}$ denote the set of all $p \in \mathcal{B}$ such that the equation $p = 0$ defines a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, that is, $D(\Delta_2(p))^3 - 27\,E(\Delta_2(p))^2 \neq 0$. Then $\mathcal{B}^{\mathrm{reg}}$ is a connected open subset of $\mathcal{B}$, and it follows from the theorem on the unique continuation of complex analytic functions that the complex analytic functions $p \mapsto g_2(P(p)) - D(\Delta_2(p))$ and $p \mapsto g_3(P(p)) + E(\Delta_2(p))$ that are equal to zero on the nonempty open subset $\mathcal{B}'$ of $\mathcal{B}^{\mathrm{reg}}$ are equal to zero on $\mathcal{B}^{\mathrm{reg}}$. Based on the second identity in (2.4.8), a similar argument shows that $g_2(P(p)) = D(\Delta_1(p))$ and $g_3(P(p)) = -E(\Delta_1(p))$ for every $p \in \mathcal{B}^{\mathrm{reg}}$. □

Remark 2.4.8. The mapping that assigns to each biquadratic polynomial p the period lattice $P(p)$ of the Hamiltonian vector field in Lemma 2.4.5 is highly transcendental. Also the mapping which assigns to any lattice P the Weierstrass invariants $g_2(P)$ and $g_3(P)$ in (2.3.4) is transcendental. It is quite remarkable that the compositions $p \mapsto g_2(P(p))$ and $p \mapsto g_3(P(p))$ are equal to the *polynomial* mappings $D \circ \Delta_k$ and $-E \circ \Delta_k$, respectively. ⊘

Remark 2.4.9. The identities $D \circ \Delta_1 = D \circ \Delta_2$ and $E \circ \Delta_1 = E \circ \Delta_2$ were proved by Frobenius [64, §2]. The consequence that the discriminants of Δ_1 and Δ_2 are equal was observed by Pettigrew [157], and this observation led me to Corollary 2.4.7.

Previously Cayley [32] had observed that by means of a linear substitution in (y_0, y_1) the function Δ_1 can be made equal to Δ_2, resulting in a biquadratic polynomial p that is symmetric in the sense that $p(x, y) = p(y, x)$ for all (x, y). Frobenius [64, §3, 4] made this statement more precise by also investigating in which degenerate cases this is not true. See also Remark 10.3.7, about Cayley [32]. ⊘

Remark 2.4.10. According to Section 2.3.1, the time of the flow of the Hamiltonian vector field v in Lemma 2.4.5 is the natural coordinate function on C, which leads to the identification of the curve C with $\mathbb{C}/P$. Here P is the period lattice of the vector field v on C, an additive subgroup of $\mathbb{C}$ that is generated by two complex numbers that are linearly independent over $\mathbb{R}$. The numerical integration of the vector field v, or equivalently the integrals in (2.4.8), is a straightforward matter, and this leads to a straightforward numerical computation of two generators of the period lattice P. If an explicit translation T in C is given, such as the restriction to C of the QRT map, see Section 2.5, then also the position of the translation vector relative to the period lattice P, which is the time needed for the solution curve to run from a to $T(a)$, where a is any point on C, can be numerically computed in a straightforward manner. ⊘

Remark 2.4.11. In (2.4.8) the time parameter is obtained as an *incomplete elliptic integral*, where the periods are the *complete elliptic integrals* obtained by integrating over closed loops in $q(x, y) = 0$; see Remark 2.5.5.

Also note that the differential form

$$\pm \Delta_2(x)^{-1/2}\, \mathrm{d}x = \pm \Delta_1(y)^{-1/2}\, \mathrm{d}y \tag{2.4.11}$$

on C, which is integrated in (2.4.8), is the Poincaré residue mentioned in Remark 3.3.5. The identity between the left- and right-hand sides in (2.4.11) is a differential equation along the biquadratic curve introduced by Cayley [32, p. 15]. ⊘

Remark 2.4.12. It follows from Section 2.6 that if the coefficients A^0_{ij}, A^1_{ij} in (2.5.3) are real numbers, and for real $z = -z_1/z_0$ the curve C of solutions of (2.5.3) has real points, then the real part of C has either one or two connected components, and on each component the real time flow of v is periodic, with the same period on both components if the real part of C has two connected components. ⊘

2.4.3 $\wp$ and $\wp'$ on a Smooth Biquadratic Curve

Let p be a biquadratic polynomial as in (2.4.1), and assume that the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p = 0$ is smooth. Assume that we have a given point

on the biquadratic curve $p = 0$, which we arrange by means of projective linear transformations in each of the variables to be the point $o = ([1 : 0], [1 : 0])$. That is, in the notation of (2.4.1) we assume that $A_{22} = 0$. In the sequel of this subsection, we will use the affine coordinates $x_0 = 1$, $x_1 = x$, $y_0 = 1$, $y_1 = y$, and write $p(x, y) = p((1, x), (1, y))$. Note that $p(0, 0) = 0$.

Let t be the complex time parameter, modulo the period lattice P, of the solution curve ϕ of the Hamiltonian system in Lemma 2.4.5, such that $\phi(0) = o$. Then the mapping $\mathbb{C} \ni t \mapsto \phi(t)$ induces a complex analytic diffeomorphism Φ from $\mathbb{C}/P$ onto C_o such that $\Phi(0 + P) = o$.

Let $\wp(t)$ denote the Weierstrass $\wp$-function defined by the period lattice P as in (2.3.3). It follows from Section 2.3.2 that the mapping $t \mapsto [1 : \wp(t) : \wp'(t)]$ induces a complex analytic diffeomorphism π from $\mathbb{C}/P$ onto the smooth Weierstrass curve W in the complex projective plane defined by the equation (2.3.6). Therefore $\Psi := \pi \circ \Phi$ is a complex analytic diffeomorphism from C onto W, where $\Psi(o) = [0 : 0 : 1]$, the point at infinity on W. Because both C and W are projective algebraic curves, the GAGA principle tells us that Ψ is equal to the restiction to C of a rational mapping from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. Encouraged by this principle, it turns out that the simplest attempt to write $\wp(t)$ and $\wp'(t)$ as rational functions of $(x(t), y(t))$ already is successful.

Lemma 2.4.13 *Let C be the biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p = 0$, where p is as in* (2.4.1) *such that $A_{22} = 0$. Assume that C is smooth, hence an elliptic curve. Let t be the complex time parameter, modulo the period lattice P, of the solution curve $([1 : x(t)], [1 : y(t)])$ of the Hamiltonian system in Lemma* 2.4.5*, such that $([1 : x(0)], [1 : y(0)]) = ([1 : 0], [1 : 0])$. Let $\wp(t)$ be the Weierstrass function of the period lattice P defined as in* (2.3.3)*. Then $\wp(t) = \mathcal{P}(x(t), y(t))$ and $\mathrm{d}\wp(t)/\mathrm{d}t = \mathcal{P}'(x(t), y(t))$, where $\mathcal{P}(x, y)$ and $\mathcal{P}'(x, y)$ are the rational functions of two variables defined by*

$$\begin{aligned}\mathcal{P}(x, y) := &-(A_{02}\,x + A_{12})\,(A_{20}\,y + A_{21})/x\,y \\ &+ ({A_{11}}^2 - 4\,A_{10}\,A_{12} - 4\,A_{01}\,A_{21} + 8\,A_{02}\,A_{20})/12,\end{aligned}$$

and $\mathcal{P}'(x, y) = \mathcal{Q}(x, y)/x^2\,y^2$, respectively, where

$$\begin{aligned}\mathcal{Q}(x, y) := &-{A_{12}}^2\,A_{21}\,x - 3\,A_{02}\,A_{12}\,A_{21}\,x^2 - 2\,{A_{02}}^2\,A_{21}\,x^3 + A_{12}\,{A_{21}}^2\,y \\ &- (A_{02}\,A_{11} + A_{01}\,A_{12})\,A_{21}\,x^2\,y - 2\,A_{01}\,A_{02}\,A_{21}\,x^3\,y \\ &+ 3A_{12}\,A_{20}\,A_{21}\,y^2 + (A_{11}\,A_{20} + A_{10}\,A_{21})\,A_{12}\,x\,y^2 \\ &+ (A_{01}\,A_{12}\,A_{20} - A_{02}\,A_{10}\,A_{21})\,x^2\,y^2 - 2\,A_{00}\,A_{02}\,A_{21}\,x^3\,y^2 \\ &+ 2A_{12}\,{A_{20}}^2\,y^3 + 2\,A_{10}\,A_{12}\,A_{20}\,x\,y^3 + 2\,A_{00}\,A_{12}\,A_{20}\,x^2\,y^3.\end{aligned}$$

It follows that $\Psi([1 : x], [1 : y]) = [1 : \mathcal{P}(x, y) : \mathcal{P}'(x, y)]$ whenever $([1 : x], [1 : y]) \in C$.

Proof. The computations are based on the observation that for every meromorphic function $f(x, y)$ of two variables, we have $\mathrm{d}f(x(t), y(t))/\mathrm{d}t = (\mathrm{H}_p\ f)(x(t), y(t))$,

where H_p is the Hamiltonian vector field defined by the function p,

$$\mathrm{H}_p := \frac{\partial p(x,\ y)}{\partial y}\,\frac{\partial}{\partial x} - \frac{\partial p(x,\ y)}{\partial x}\,\frac{\partial}{\partial y},$$

viewed as a first order linear partial differential operator with polynomial coefficients. It follows that $\mathrm{d}^k\, f(x(t),\ y(t))/\,\mathrm{d}t^k = ((\mathrm{H}_p)^k\, f)(x(t),\ y(t))$, which allows us to compute, with the help of a formula manipulation program, the Taylor expansion of $t \mapsto f(x(t),\ y(t))$ at $t = 0$ up to any desired order.

We first assume that $A_{12} \neq 0$ and $A_{21} \neq 0$, that is, at the base point $([1 :\ 0], [1 :\ 0])$ the biquadratic curve C is not tangent to the horizontal axis or the vertical axis through that point. Then the restriction a to C of the function $1/x\,y$ has a pole of order two at $(0,\ 0)$. Because $p(x,\ 0) = (A_{02}\,x + A_{12})\,x$ and $p(0,\ y) = (A_{20}\,y + A_{21})\,y$, the function a has also a pole of order one at the points $(x,\ 0)$ and $(0,\ y)$ on C where $A_{02}\,x + A_{12} = 0$ and $A_{20}\,y + A_{21} = 0$, respectively. It follows that the restriction to C of the function

$$b(x,\ y) := (A_{02}\,x + A_{12})\,(A_{20}\,y + A_{21})/x\,y$$

has a pole of order two at $(0,\ 0)$ and no other poles. Using the Taylor expansions at $t = 0$ of $t \mapsto x(t)$ and $t \mapsto y(t)$ of order three, we obtain that $b(x(t),\ y(t)) = -1/t^2 + c + \mathrm{O}(|t|)$ as $t \to 0$, where

$$c := ({A_{11}}^2 - 4\,A_{10}\,A_{12} - 4\,A_{01}\,A_{21} + 8\,A_{02}\,A_{20})/12.$$

Therefore, if we define $\mathcal{P}(x,\ y) = -b(x,\ y) + c$ as in the lemma, then $t \mapsto f(t) := \mathcal{P}(x(t),\ y(t))$ is a meromorphic function of $t \in \mathbb{C}/P$, with a pole of order two at $0 + P$, no other poles, and the coefficients of the terms of order -1 and 0 of the Laurent expansion of $f(t)$ at $t = 0$ equal to zero. Because the Weierstrass $\wp$-function has the same properties, it follows that $r(t) := f(t) - \wp(t)$ is a holomorphic function of $t \in \mathbb{C}/P$ such that $r(0) = 0$. Because $\mathbb{C}/P$ is compact and connected, it follows from the maximum principle that $r(t)$ is a constant, and hence $r(t) \equiv r(0) = 0$, that is, $f(t) \equiv \wp(t)$, or equivalently, $\wp(t) \equiv \mathcal{P}(x(t),\ y(t))$.

The formula for $\mathcal{P}' := \mathrm{H}_p\ \mathcal{P}$ has been obtained by means of a formula manipulation program, and then $\wp'(t) \equiv \mathcal{P}'(x(t),\ y(t))$. The formulas for arbitrary smooth biquadratic curves, where it is allowed that $A_{12} = 0$ or $A_{21} = 0$, follow from the observation that $\wp(t)$ and $\mathrm{d}\wp(t)/\,\mathrm{d}t$ depend continuously on the coefficients of p.

□

Note that the rational functions $\mathcal{P}$ and $\mathcal{P}'$ of $(x,\ y)$ that appear in Lemma 2.4.13 depend in a homogeneous polynomial way on the coefficients A_{ij} of the biquadratic polynomial p, with homogeneity degrees two and three for $\mathcal{P}$ and $\mathcal{P}'$, respectively. In view of the fact that a priori $\wp(t)$ is defined only for biquadratic curves p such that the curve $p = 0$ is smooth, it is quite remarkable that $\wp(t) = \mathcal{P}(x(t), y(t))$ and $\mathrm{d}\wp(t)/\,\mathrm{d}t = \mathcal{P}'(x(t),\ y(t))$, where $x\,y\,\mathcal{P}(x,\ y)$ and $x^2\,y^2\,\mathcal{P}'(x,\ y)$ are polynomials in $(x,\ y)$ of which all the coefficients depend in a polynomial fashion on the coefficients of p. That is, these coefficients also have well-defined values when the curve

$p = 0$ is not smooth and even when the special point ([1 : 0], [1 : 0]) is a singular point of the curve $p = 0$.

2.5 The QRT Mapping on a Smooth Biquadratic Curve

The description of the automorphism groups of elliptic curves at the end of Subsection 2.3.1 leads to the following characterization of involutions of elliptic curves.

Lemma 2.5.1 *Let C be an elliptic curve, and let Φ be an* involution *of C, an automorphism of C such that $\Phi \circ \Phi$ is equal to the identity on C. Then we have the following alternatives.*

(i) *The involution Φ is a* translational involution *on C, that is, $\Phi_*(v) = v$ for every holomorphic vector field v on C, and $\Phi^*(\omega) = \omega$ for every holomorphic complex one-form ω on C. If Φ has a fixed point on C, then Φ is equal to the identity on C.*

(ii) *Φ is an* inversion *on C, that is, $\Phi_*(v) = -v$ for every holomorphic vector field v on C, and $\Phi^*(\omega) = -\omega$ for every holomorphic complex one-form ω on C. Every inversion on C is an involution, and has four fixed points on C.*

Proposition 2.5.2 *On each smooth solution curve C of (2.5.3) that is an elliptic curve, the horizontal switch ι_1 and the vertical switch ι_2 are inversions on C, and the QRT mapping $\tau = \iota_2 \circ \iota_1$ is a translation on C.*

Proof. The horizontal switch ι_1 in (1.1.4) defines an involution of C, with as its fixed points the four points $(x,\ y) \in C$ where x is a zero of the discriminant (2.4.5) of p with respect to y. Therefore Lemma 2.5.1 implies that ι_1 is an inversion on C. The proof that ι_2 is an inversion on C is similar. It follows that for every holomorphic vector field v on C we have $\tau_*(v) = (\iota_2 \circ \iota_1)_*(v) = (\iota_2)_*((\iota_1)_*(v)) = (\iota_2)_*(-v) = v$; hence τ is a translation on C. □

Remark 2.5.3. In affine coordinates the curve is given by an equation of the form $p(x,\ y) = 0$, where $x \mapsto p(x,\ y)$ is a polynomial of degree two for each y and $y \mapsto p(x,\ y)$ is a polynomial of degree two for each x. We take for our nonzero holomorphic vector field on the curve $p(x,\ y) = 0$ the Hamiltonian vector field in Lemma 2.4.5.

Writing $p(x,\ ,\ y) = a(y)\,x^2 + b(y)\,x + c(y)$, the horizontal switch ι_1 is the mapping $(x,\ y) \mapsto (x',\ y)$, where $p(x,\ y) = 0$, $p(x',\ y) = 0$, and $x' \neq x$. That is, $x' = -x - b(y)/a(y)$. Now the vertical part $\dot{y} = -\partial p(x,\ y)/\partial x$ of the Hamiltonian vector field is equal to $2\,a(y)\,x + b(y)$, which is equal to $2\,a(y)\,(-x - b(y)/a(y)) + b(y) = -2\,a(y)\,x - b(y)$, at the image point. Because the tangent space of the curve at every point is one-dimensional and the Hamiltonian vector field is tangent to the curve, this confirms that the Hamiltonian vector field is mapped to its opposite by the horizontal switch. ⊘

If $c \in C$ and $n \in \mathbb{Z}$, then c is called a *periodic point of* τ *with period* n if $\tau^n(c) = c$. The following corollary could be called the *Poncelet porism for the QRT map*.

Corollary 2.5.4 *Let C be a smooth solution curve of* (2.5.3) *and let $n \in \mathbb{Z}$. If there exists a periodic point of period n on C, then every point of C is a periodic point of period n of τ, that is, τ^n is equal to the identity on C.*

Proof. It follows from Proposition 2.5.2 that τ is a translation on C, and therefore the statement follows from the Poncelet porism for elliptic curves at the end of Section 2.3.1. □

Remark 2.5.5. The complex number T such that $\tau : t + P \mapsto t + T + P$ is equal, modulo P, to $\int_\gamma \omega$, where $\gamma : [0,\ 1] \to C$ is any path in C such that $\gamma(1) = \tau(\gamma(0))$. This may be used in order to obtain numerical approximations of T, for instance by evaluating complete elliptic integrals, see also Remark 2.4.11. ⊘

2.5.1 The Frobenius Invariants

Let p be a biquadratic polynomial as in (2.4.1) such that the biquadratic curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p = 0$ is smooth. Let $g_2(p) := D(\Delta_k(p))$ and $g_3(p) := -E(\Delta_k(p))$ be the homogeneous polynomials of the respective degrees 4 and 6 in the coefficients A_{ij} of p, as obtained in Corollary 2.4.7. Let W denote the Weierstrass curve in $\mathbb{P}^2$ defined by the equation (2.3.6), with the coefficients $g_2 = g_2(p)$ and $g_3 = g_3(p)$. Then the vector field on C defined by p as in Lemma 2.4.5 has the same period lattice P as the vector field on W defined by q in Section 2.3.2, which leads to an identification of the group of translations on C with the group of translations on W, both groups being isomorphic to $\mathbb{C}/P$.

Proposition 2.5.6 *Under this identification, the QRT transformation on C corresponds to the translation on W that maps the point* $[0 :\ 0 :\ 1]$ *at infinity on W to the point* $[1 :\ X :\ Y]$ *on W, where*

$$X = X(p) := ({A_{11}}^2 - 4\,A_{10}\,A_{12} - 4\,A_{01}\,A_{21} + 8\,A_{02}\,A_{20} + 8\,A_{00}\,A_{22})/12 \quad (2.5.1)$$

and

$$Y = Y(p) := -\det A. \quad (2.5.2)$$

If C is not smooth, but $C^{\mathrm{reg}} \neq \emptyset$, and C_0 is a connected component of C^{reg} not contained in a horizontal or vertical axis, and invariant under the QRT map, then this conclusion remains valid with C and W replaced by C_0 and W^{reg}, respectively.

Proof. Assume that $A_{22} = 0$, that is, $([1 :\ 0],\ [1 :\ 0]) \in C$. Then (2.5.1) and (2.5.2) with $A_{22} = 0$ follow from the last conclusion in Lemma 2.4.13, with $([1 :\ x],\ [1 :\ y]) := \tau([1 :\ 0],\ [1 :\ 0])$, where $x = -A_{12}/A_{20}$ and

$$y = -(A_{01}\,x^2 + A_{11}\,x + A_{21})/(A_{00}\,x^2 + A_{10}\,x + A_{20}).$$

In the general case, we have $([1 : r],\ [1 : 0]) \in C$ if and only if $A_{02}\, r^2 + A_{12}\, r + A_{22} = 0$; hence

$$r = (-A_{12} \pm ({A_{12}}^2 - 4\, A_{02}\, A_{22})^{1/2})/2\, A_{02}.$$

The substitution of variables $x_1 \mapsto r\, x_0 + x_1$ leads to a biquadratic polynomial $\widetilde{p}$ with

$$\widetilde{p}((1,\ 0),\ (1,\ 0)) = 0,$$

and X and Y for $\widetilde{p}$ turn out to be given by (2.5.1) and (2.5.2), respectively. I admit that I used a formula manipulation program in order to arrive at (2.5.1) and (2.5.2), and also to check that $Y(p)^2 - 4\, X(p)^3 + g_2(p)\, X(p) + g_3(p) = 0$.

For the last statement we observe that the assumptions imply that the mapping Ψ defined before Lemma 2.4.13, with $o \in C_0$, is a complex analytic diffeomorphism from C_0 onto W^{reg}. Perturbing the coefficients such that C is smooth, the conclusion follows by a continuity argument. □

Figure 4.2.2 shows the geometric construction of the translation on the Weierstrass curve that maps the point at infinity to the point $b_2 = (X,\ Y)$ on the Weierstrass curve.

It is quite remarkable that X and Y are homogeneous *polynomials* of the respective degrees two and three in the coefficients A_{ij} of the biquadratic polynomial p. It follows that these numbers also have finite limits if p approximates a biquadratic polynomial p_0 for which the curve $p_0 = 0$ is not smooth.

Remark 2.5.7. Let $\mathcal{B} := \mathrm{H}_{2,\,2}(\mathbb{C}^2 \times \mathbb{C}^2)$ denote the nine-dimensional vector space of all biquadratic polynomials p. Let G denote the eight-dimensional group consisting of all linear transformations $L : (x,\ y) \mapsto (L_1(x),\ L_2(y))$ or $L : (x,\ y) \mapsto (L_1(y),\ L_2(x))$, with $L_1,\ L_2 \in \mathrm{GL}(2,\ \mathbb{C})$. Then the respective polynomials $X = X(p)$ and $Y = Y(p)$ in (2.5.1) and (2.5.2) are projective invariants of order two and three for the representation of G in $\mathcal{B}$, in the sense that $X(p \circ L) = (\det L)^2\, X(p)$ and $Y(p \circ L) = (\det L)^3\, Y(p)$. On the other hand, it follows from (2.4.6), (2.4.7), and Corollary 2.4.7 that $g_2(p) := D(\Delta_k(p))$ and $g_3(p) := -E(\Delta_k(p))$ are projective invariants of order 4 and 6, respectively. Between the projective invariants X, Y, g_2, and g_3 we have the relation $Y^2 - 4\, X^3 + g_2\, X + g_3 = 0$, which is an equation between homogeneous polynomials of degree 6 on $\mathcal{B}$.

Frobenius [64, p. 127] found three projective invariants s, t, and u of respective degrees 2, 3, and 4 for biquadratic poynomials, and proved [64, Theorem VII on p. 181] that every polynomial projective invariant of biquadratic polynomials is a polynomial in s, t, and u. A straightforward comparison shows that $s = X/4$ and $t = -Y/8$, whereas [64, (18) on p.131] implies that $3\, s^2 + u = g_2/16$, where we note that the quartic $R(x)$ in [64, bottom of p. 133] is equal to $\Delta_2(x)/4$, and therefore $g_2(R) = g_2(\Delta_2)/16$. It follows that X, Y, and g_2 are basic polynomial projective invariants of biquadratic polynomials as well. For this reason I propose to call X, Y, and g_2 the *Frobenius invariants of biquadratic polynomials*. Our Weierstrass equation $Y^2 - 4\, X^3 + g_2\, X + g_3 = 0$ is equivalent to the Weierstrass equation $t^2 - 4\, s^3 + g_2(R)\, s + g_3(R)$ of Frobenius [64, (19) on p. 131]. ⊘

Remark 2.5.8. The Mathematica function

```
WeierstrassHalfPeriods[{a,b}]
```

returns, for any complex numerical values of $a = g_2$ and $b = g_3$, two complex numbers h_1 and h_2 such that $p_1 = 2\,h_1$ and $p_2 = 2\,h_2$ form a $\mathbb{Z}$-basis of the period lattice P of the Weierstrass curve (2.3.6), the additive group of all periods of the solutions of the Hamiltonian system (2.3.7). If $g_2 = D(\Delta_k)$ and $g_3 = -E(\Delta_k)$ as in Corollary 2.4.7, then P is equal to the period lattice of the biquadratic curve $p = 0$ defined by the biquadratic polynomial p, as in Lemma 2.4.5. That is, P is equal to the period group of the flow on $p = 0$ of the Hamiltonian vector field v defined by the Hamiltonian function p.

Let x and y be the complex numerical values of the coordinates of a point on the Weierstrass curve (2.3.5). Then, for $a = g_2$ and $b = g_3$, the Mathematica function

```
InverseWeierstrassP[{x,y},{a,b}]
```

returns a complex number t such that $\wp(t) = x$, $\wp'(t) = y$, and $t = \kappa_1\, p_1 + \kappa_2\, p_2$ with $-1/2 \le \kappa_1 \le 1/2$ and $-1/2 \le \kappa_2 \le 1/2$. If $x = X$ and $y = Y$, with X and Y as in Proposition 2.5.6, then t is equal to the complex time needed for a solution curve of the Hamiltonian system on $p = 0$, as in Lemma 2.4.5, to run from any given initial point on $p = 0$ to its image point under the QRT map. In other words, if we identify $\mathbb{C}/P$ with the translation group on the biquadratic curve $p = 0$ by means of the flow of v, then $t + P$ corresponds to the QRT map on the curve $p = 0$.

Other program packages may have functions that are equivalent to the two Mathematica functions mentioned above. In this way Corollary 2.4.7 and Proposition 2.5.6 allow a straightforward numerical computation of the period lattice P and the QRT map as an element of $\mathbb{C}/P$, for any biquadratic polynomial p. As is to be expected, the results will be singular if the curve $p = 0$ is singular, that is, if $\Delta := {g_2}^3 - 27\,{g_3}^2 = 0$. ⊘

Proposition 2.5.6 has the following amusing consequence.

Corollary 2.5.9 *Let p be a nonzero biquadratic polynomial as in* (2.4.1)*, and assume that the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p = 0$ is irreducible. Let ι_1, ι_2, and $\tau = \iota_2 \circ \iota_1$ be the horizontal switch, the vertical switch, and the QRT transformation on C, respectively. Then the following statements are equivalent:*

(i) *ι_1 and ι_2 commute.*
(ii) *τ is an involution on C, that is, $\tau \circ \tau$ is equal to the identity on C.*
(iii) *$Y(p) = 0$.*
(iv) *The matrix A is singular in the sense that* $\det A = 0$.

Proof. The assumption that C is irreducible implies that C^{reg} is connected and not contained in a horizontal or vertical axis. It follows that ι_1 and ι_2 are well-defined involutions on C^{reg}, and the last statement in Proposition 2.5.6 holds with $C_0 = C^{\mathrm{reg}}$.

We have $\tau^{-1} = {\iota_1}^{-1} \circ {\iota_2}^{-1} = \iota_1 \circ \iota_2$, which shows that (i) $\Leftrightarrow$ (ii).

On the other hand, identifying the point at infinity on the Weierstrass curve W with the origin in $\mathbb{C}/P$, we have that the inversion corresponds to the involution $(x,\ y) \mapsto$

$(x, -y)$ on the curve W defined in affine coordinates by $y^2 - 4x^3 + g_2 x + g_3 = 0$. Because on W the QRT mapping is the translation from the point at infinity to (X, Y), we have $\tau \circ \tau = 1$ if and only if $\tau = \tau^{-1}$ if and only if $(X, Y) = (X, -Y)$ if and only if $Y = 0$, and therefore (ii) $\Leftrightarrow$ (iii). Finally (iii) $\Leftrightarrow$ (iv) follows from (2.5.2). □

2.5.2 Conclusions for pencils of biquadratic curves

We now apply Propositions 2.4.3 and 2.5.6 to the pencil of curves (0.0.1) that define the QRT map. We will view the variables x, y, and z in the equation (0.0.2) as affine coordinates $x = x_1/x_0$, $y = y_1/y_0$, and $z = -z_1/z_0$, each on a copy of the projective line $\mathbb{P}^1$. This leads to the equation

$$\begin{aligned} p_{(z_0, z_1)}(x_0, x_1, y_0, y_1) &:= z_0\, p^0((x_0, x_1), (y_0, y_1)) + z_1\, p^1((x_0, x_1), (y_0, y_1)) \\ &= 0, \end{aligned} \tag{2.5.3}$$

where, for $k = 1, 2$,

$$p^k((x_0, x_1), (y_0, y_1)) := \sum_{i,\, j=0}^{2} {x_0}^i\, {x_1}^{2-1}\, A^k_{ij}\, {y_0}^j\, {y_1}^{2-j}. \tag{2.5.4}$$

Note that the left-hand side in (2.5.3) is equal to the right hand side in (2.5.4) with A^k replaced by the matrix $z_0 A^0 + z_1 A^1$.

Let $z = (z_0, z_1) \in \mathbb{C} \setminus \{(0, 0)\}$ be given. Then $p = p_z$ is a bihomogeneous polynomial of bidegree $(2, 2)$ on $\mathbb{C}^2 \times \mathbb{C}^2$. It follows that its zero-set defines a biquadratic curve $C = \mathrm{Z}_{p_z}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, of course under the condition that p_z is not identically zero, that is, if the matrix $A = z_0 A^0 + z_1 A^1$ is not equal to zero. This will be the case for every $(z_0, z_1) \neq (0, 0)$ if and only if the matrices A^0 and A^1 are linearly independent, which we assume from from now on.

Note that Z_{p_z} is equal to the level curve of the rational function p^0/p^1 at the level $z = -z_1/z_0$. Furthermore, $p^1(x, y)/p^0(x, y)$ is equal to the rational function $F(x, y)$ in (1.1.7) if we use the homogeneous versions $X_i = {x_0}^i\, {x_1}^{2-i}$, $i = 0, 1, 2$, and $Y_j = {y_0}^j\, {y_1}^{2-j}$, $j = 0, 1, 2$ of the vectors $X \in \mathbb{C}^3$ and $Y \in \mathbb{C}^3$, respectively.

If in (2.5.3), (z_0, z_1) is replaced by a nonzero multiple of (z_0, z_1), then we get the same curve C in $\mathbb{P}^1 \times \mathbb{P}^1$. That is, the curve C depends only on the point $[z] = [z_0 : z_1] \in \mathbb{P}^1$ in the complex projective line, and the family of curves $C = C_{[z]} := \mathrm{Z}_{p_z}$, $[z] \in \mathbb{P}^1$, is called the *pencil of biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$ *defined by the matrices* A^0_{ij} *and* A^1_{ij}.

Because the coefficient matrix $A = z_0 A^0 + z_1 A^1$ of $p = p_z$ depends in a linear way on $z = (z_0, z_1)$, Proposition 2.4.3 and Corollary 2.4.7 lead to the following conclusions.

Corollary 2.5.10 *For each* $(z_0, z_1) \in \mathbb{C}$, *write* $g_2(z) := D(\Delta_k(p_z))$, $g_3(z) := -E(\Delta_k(p_z))$, *and* $\Delta(z) := g_2(z)^3 - 27\, g_3(z)^2$; *see Proposition* 2.5.6. *Let* $C_{[z]}$ *and*

$W_{[z]}$ denote biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and the Weierstrass curve in $\mathbb{P}^2$ defined by the equation $p_z = 0$ and (2.3.6) *with $g_2 = g_2(z)$, $g_3 = g_3(z)$, respectively.*

Then $g_2(z)$, $g_3(z)$, and $\Delta(z)$ are homogeneous polynomials in $z = (z_0, z_1)$ of degree 4, 6, *and* 12, *respectively. The curve $C_{[z]}$ is smooth if and only if $\Delta(z) \neq 0$ if and only if the curve $W_{[z]}$ is smooth. Moreover, if this is the case, then $W_{[z]}$ is isomorphic to $C_{[z]}$.*

If $J([z])$ denotes the modulus J of $C_{[z]}$ as defined in (2.3.23) *when $C_{[z]}$ is smooth, then $J([z]) = g_2(z)^3/\Delta(z)$. It follows that the function $[z] \mapsto J([z])$ extends to a rational mapping $J : \mathbb{P}^1 \to \mathbb{P}^1$, with mapping degree $j \leq 12$. Counted with multiplicities, each isomorphism class of elliptic curves is represented in our pencil of biquadratic curves by j members, that is, when $j > 0$. If $j = 0$ then J is constant and all smooth fibers are isomorphic to each other as complex analytic curves.*

Note that the degree j of J is < 12 if and only if the polynomials $g_2(z)$ and $g_3(z)$ in z have common factors. For a formula for the degree j of J in terms of the Kodaira types of the singular fibers in the rational elliptic surface S defined by the pencil of biquadratic curves, see (6.2.48). For a discussion of the rational elliptic surfaces with $j = 0$ and $j = 1$, see Remarks 9.2.8 and 6.3.13, respectively.

Remark 2.5.11. Because every Weierstrass curve is irreducible, and there exist biquadratic curves with more than one irreducible component, it can happen that $C_{[z]}$ is not isomorphic to $W_{[z]}$.

For the description of the set of all the pairs $g_2(z)$ and $g_3(z)$ of homogeneous polynomials in $z = (z_0, z_1)$ of respective degrees 4 and 6 that can occur for pencils of biquadratic curves with at least one smooth member, see Corollary 3.3.14. ⊘

Assume that the pencil of biquadratic curves defined by (2.5.3) has at least one smooth member. It then follows from Lemma 3.1.1 that, counted with multiplicities, the pencil has eight base points. This implies that, disregarding multiplicities, the pencil has at least one base point. Lemma 2.4.13 therefore leads to the following conclusions.

Corollary 2.5.12 *Assume that the pencil of biquadratic curves defined by* (2.5.3) *has at least one smooth member, and has b as a base point. For each $[z] \in \mathbb{P}^1$ such that $C_{[z]}$ is smooth, let v_z and v'_z be the vector fields on $C_{[z]}$ and $W_{[z]}$ defined by $p = p_z$ and $q = q_z$ as in Lemma* 2.4.5 *and in Section* 2.3.2 *with $g_2 = g_2(z)$ and $g_3 = g_3(z)$, respectively. Let Ψ_z denote the isomorphism from $C_{[z]}$ onto $D_{[z]}$ that intertwines v_z with v'_z and maps b to $[1 : 0 : 0]$, the point at infinity on $W_{[z]}$. Then the Ψ_z are defined by rational transformations $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, depending in a rational way on z. These rational transformations are given by the formulas in Lemma* 2.4.13 *when $b = ([1 : 0], [1 : 0])$.*

Proof. Write $b = ([x_0 : x_1], [y_0 : y_1])$. Then we can transform $[x_0 : x_1]$ to $[1 : 0]$ and $[y_0 : y_1]$ to $[1 : 0]$, each by means of an explicit linear transformation in $\mathbb{C}^2$. This leads to an explicit reduction to the situation that $([1 : 0], [1 : 0])$ is a base point of the pencil, that is, $A_{22} = z_0 A^0_{22} + z_1 A^1_{22} = 0$ for all $(z_0, z_1) \in \mathbb{C}^2$. In this situation the conclusions follow from Lemma 2.4.13. □

In general, there is no algebraic formula for any of the base points in terms of the coefficients A^0_{ij} and A^1_{ij} of the biquadratic polynomials p^0 and p^1, respectively.

Proposition 2.5.6 leads to the following conclusions, for which no knowledge of a base point is needed.

Corollary 2.5.13 *Assume that the pencil of biquadratic curves defined by* (2.5.3) *has at least one smooth member. Write* $X(z) := X$ *and* $Y(z) = Y$ *as in* (2.5.1) *and* (2.5.2)*, respectively, with* $A_{ij} = z_0\, A^0_{ij} + z_1\, A^1_{ij}$*. Then* $X(z)$ *and* $Y(z)$ *are homogeneous polynomials in* $z = (z_0,\, z_1)$ *of degree* 2 *and* 3*, respectively.*

For each $[z] \in \mathbb{P}^1$ *such that* $C_{[z]}$ *is smooth, let* v_z *and* v'_z *denote the vector field on* $C_{[z]}$ *and* $W_{[z]}$ *as in Corollary* 2.5.12*. If* τ_z *denotes the restriction to* $C_{[z]}$ *of the QRT transformation of the pencil of biquadratic curves, and* $t \in \mathbb{C}$ *is such that* $\tau_z = \mathrm{e}^{t\, v_z}$*, then* $\mathrm{e}^{t\, v'_z}$ *is the translation on* $W_{[z]}$ *that maps the point* $[0 : 0 : 1]$ *at infinity on* $W_{[z]}$ *to the point* $[1 : X(z) : Y(z)]$*.*

If the pencil of biquadratic curves has at least one smooth member, then the set of $z \in \mathbb{C}^2 \setminus \{0\}$ such that $C_{[z]}$ is smooth is dense in $\mathbb{C}^2$, and it follows from Corollary 2.5.13 that

$$Y(z)^2 - 4\, X(z)^3 + g_2(z)\, X(z) + g_3(z) = 0 \tag{2.5.5}$$

for every $z = (z_0,\, z_1) \in \mathbb{C}^2$. Because $g_2(z)$, $g_3(z)$, $X(z)$, $Y(z)$ are homogeneous of degree 4, 6, 2, 3, respectively, the left-hand side of (2.5.5) is a homogeneous polynomial of degree 6 in $z = (z_0,\, z_1)$. If, for given $g_(z)$ and $g_3(z)$, we view (2.5.5) as an equation for the unknown homogeneous polynomials $X(z)$ and $Y(z)$ of degree 2 and 3, respectively, the equation (2.5.5) is equivalent to $6 + 1 = 7$ nonlinear polynomial equations in the $(2 + 1) + (3 + 1) = 7$ unknown coefficients of $X(z)$ and $Y(z)$. Since I see no obvious way of solving this system of seven polynomial equations in seven unknowns, I find it quite remarkable to have the explicit solutions given by $X(z) =$ (2.5.1) and $Y(z) =$ (2.5.2), with $A_{ij} = z_0\, A^0_{ij} + z_1\, A^1_{ij}$.

Remark 2.5.14. On a singular solution curve of equation (2.5.3), the QRT map may not be everywhere defined. However, if we pass to the surface $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ that is obtained by blowing up, eight times, base points of the anticanonical pencils, starting with the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, then there is a unique $\tau^S \in \mathrm{Aut}(S)^+_\kappa$, called the QRT automorphism, such that $\pi \circ \tau^S(s) = \tau \circ \pi(s)$ whenever the right-hand side is defined; see Corollaries 3.4.2 and 3.4.4. Here the singular fibers of the elliptic fibration $\kappa : S \to \mathbb{P}^1$ of Corollary 3.3.10 correspond to the singular solution curves of equation (2.5.3). While the QRT map on a biquadratic curve $C_{[z]}$ sometimes is not defined if the curve $C_{[z]}$ is singular, the restriction of τ^S to the corresponding singular fiber $S_{[z]}$ over $[z] \in \mathbb{P}^1$ of the elliptic fibration $\kappa : S \to \mathbb{P}^1$ is always an automorphism of $S_{[z]}$ such as described in Section 6.3.6.

An isomorphic construction of the QRT surface S is given by the Weierstrass model, see Section 3.3.6, which in turn refers to the Weierstrass model of arbitrary elliptic surfaces in Section 6.3. For the determination of the Kodaira types of the singular fibers in terms of the orders of the zeros of g_2 and g_3 at the zeros of Δ, see Corollary 3.3.13. For a classification of triples $(X,\, Y,\, g_2)$ that can occur for QRT maps, see Corollary 3.3.14. In Example 2, the Weierstrass model has been used to

prove that one can arrange that the $[z] \in \mathbb{P}^1$ such that the biquadratic curve $C_{[z]}$ is singular form a regular 12-gon in $\mathbb{P}^1$. Here each singular curve has only one singular point, which is a normal crossing, and the corresponding singular fiber $S_{[z]}$ in the elliptic surface S is of Kodaira type I_1. ⊘

2.5.3 Picard–Fuchs Equations

Let P be a lattice in $\mathbb{C}$ with the corresponding Weierstrass curve (2.3.5). The periods $p \in P$ are the integrals over loops in the Weierstrass curve of the one-form $y^{-1}\,\mathrm{d}x$, as explained in Section 2.3.2. Let g_2 and g_3 depend in a holomorphic way on a parameter $z \in U$, where U is some open subset of $\mathbb{C}$.

If γ_z denotes a closed path in the Weierstrass curve (2.3.5) with $g_2 = g_2(z)$, $g_3 = g_3(z)$, then the integrals

$$p = p(z) := \int_{\gamma_z} \frac{\mathrm{d}x}{y}, \qquad \eta = \eta(z) := \int_{\gamma_z} \frac{x\,\mathrm{d}x}{y} \tag{2.5.6}$$

over γ_z locally also depend holomorphically on z. Note that on the projective curve (2.3.6) corresponding to (2.3.5) the complex one-form $\omega = y^{-1}\,\mathrm{d}x$ is globally holomorphic without zeros, whereas $y^{-1}\,x\,\mathrm{d}x$ has a simple pole at the point $[0:0:1]$ at infinity on (2.3.6). The closed path γ_z is chosen such that it avoids the pole of $y^{-1}\,x\,\mathrm{d}x$.

It had been observed by Bruns [26, pp. 237, 238] that the functions $p(z)$ and $\eta(z)$ satisfy a homogeneous system of first-order linear differential equations

$$p' = -Q\,p - P\,\eta, \qquad \eta' = R\,p + Q\,\eta, \tag{2.5.7}$$

where the variable coefficients $P = P(z)$, $Q = Q(z)$, and $R = R(z)$ are explicitly given by the formulas

$$P = 3\,\Gamma/2\,\Delta, \quad Q = \Delta'/12\,\Delta, \quad R = g_2\,\Gamma/8\,\Delta, \tag{2.5.8}$$

where the prime denotes differentiation with respect to the variable z,

$$\Gamma := -3\,g_3\,g_2' + 2\,g_2\,g_3', \tag{2.5.9}$$

and $\Delta := {g_2}^3 - 27\,{g_3}^2$ is the discriminant as given in (2.3.9). Because $\Delta(z) = 0$ if and only if the Weierstrass curve with $g_2 = g_2(z)$ and $g_3 = g_3(z)$ is singular, the linear first-order system of differential equations for p and η is singular only at points where $\Delta = 0$, that is, where the Weierstrass curve is singular. It is therefore assumed that $\Delta(z)$ is not identically equal to zero.

From (2.5.7) we obtain

$$\begin{aligned} p'' &= -Q'\,p - Q\,p' - P'\,\eta - P\,\eta' = -Q'\,p - Q\,p' - P'\,\eta - P\,(R\,p + Q\,\eta) \\ &= -Q\,p' - (Q' + P\,R)\,p - (P' + P\,Q)\,\eta \\ &= -Q\,p' - (Q' + P\,R)\,p + (P' + P\,Q)\,(p' + Q\,p)/P, \end{aligned}$$

which leads to the homogeneous linear second-order differential equation $\mathcal{L}\, p = 0$ for p, where

$$\mathcal{L} := \mathrm{d}^2 / \mathrm{d}z^2 + c_1(z)\ \mathrm{d} / \mathrm{d}z + c_2(z), \tag{2.5.10}$$

$$c_1(z) := -P'(z)/P(z), \tag{2.5.11}$$

$$c_2(z) := Q'(z) + P(z)\, R(z) - Q(z)\, P'(z)/P(z) - Q(z)^2. \tag{2.5.12}$$

Here we have assumed that $P(z)$ is not identically equal to zero, that is, $\Gamma = -3\, g_3\, g_2' + 2\, g_2\, g_3'$ is not identically equal to zero. Because the derivative of the modulus function $J = {g_2}^3/\Delta$ equals

$$J' = 27\, {g_2}^2\, g_3\, \Gamma/\Delta^2, \tag{2.5.13}$$

and $g_2 = 0$, $g_3 = 0$ correspond to $J = 0$, $J = 1$, respectively, we have $\Gamma \equiv 0$ if and only if the modulus function $J(z)$ is a constant. For the case that J is constant we refer to Remark 2.5.21.

Lemma 2.5.15 *Every solution p of $\mathcal{L}\, p$ is a linear combination of period functions.*

Let $\widetilde{\mathcal{L}}$ be a differential operator as in (2.5.10) with $c_j(z)$ replaced by $\widetilde{c}_j(z)$, $j = 1, 2$. If $\widetilde{\mathcal{L}}\, p = 0$ for every period function p, then $\widetilde{\mathcal{L}} = \mathcal{L}$, that is, $\widetilde{c}_j = c_j$ for $j = 1, 2$.

Proof. Let $p_1(z)$ and $p_2(z)$ be two locally defined period functions generating the period lattice $P(z)$, where we can arrange that $q(z) = p_2(z)/p_1(z)$ has positive imaginary part. We have $J(z) = \mathcal{J}(q(z))$, see Section 2.3.3, hence $J'(z) = 0$ when $q'(z) = 0$, and the assumption that J is not a constant implies that $p_2(z)$ is not a constant multiple of $p_1(z)$. In other words, the functions p_1 and p_2 are linearly independent over $\mathbb{C}$. Because locally the solution space of the equation $\mathcal{L}\, p = 0$ is a complex two-dimensional vector space, this proves the first statement in the lemma.

If $\widetilde{\mathcal{L}}$ annihilates the period functions, then for $k = 1, 2$,

$$0 = \widetilde{\mathcal{L}}\, p_k - \mathcal{L}\, p_k = (\widetilde{c}_1 - c_1)\, p_k' + (\widetilde{c}_2 - c_2)\, p_k.$$

Because the solution space of a nontrivial linear differential equation of order ≤ 1 is of dimension ≤ 1, the linear independence over $\mathbb{C}$ of p_1 and p_2 implies that $\widetilde{c}_1 - c_1 = 0$ and $\widetilde{c}_2 - c_2 = 0$. □

It follows from (2.5.11) that the singularities of $c_1(z)$ are poles of order ≤ 1. On the other hand, (2.5.12) implies that

$$c_2 = \frac{1}{144}\, \frac{J'}{J}\, \frac{(J-1)'}{J-1} + \frac{1}{12}\, \frac{\Delta'}{\Delta} \left(c_1 + \frac{\Delta''}{\Delta'} - \frac{13}{12}\, \frac{\Delta'}{\Delta} \right), \tag{2.5.14}$$

a formula that I learned from Schmickler-Hirzebruch [176, p. 136], and therefore the singularities of $c_2(z)$ are poles of order ≤ 2. As a consequence, the linear second-order ordinary differential equation $\mathcal{L}\, p = 0$ has only *regular singular points*, which in turn implies that near any singular point z_0 every solution is a linear combination

of functions of the form $(z-z_0)^\alpha$ times a holomorphic function or $(z-z_0)^\alpha \log(z-z_0)$ times a holomorphic function. See, for instance, Coddington and Levinson [37, Chapter 4, 5].

Remark 2.5.16. The linear system of first-order ordinary differential equations has a singular point *of the first kind* at z_0 if its coefficients have at most a simple pole at z_0, that is, if the function Γ/Δ has at most a simple pole at z_0. The formula

$$\Gamma/\Delta = (3\, g_2'/g_2 - \Delta'/\Delta)\, g_2/27\, g_3$$

shows that Γ/Δ has at most a simple pole at z_0 if the order of vanishing of g_2 at z_0 is at least equal to the order of vanishing of g_3 at z_0. According to Table 6.3.2, this happens for instance if the singular fiber is of Kodaira type I_b, $b \geq 1$, or II or IV. However, if for instance the singular fiber at $z = z_0$ is of Kodaira type I_b^*, $b \geq 1$, then Γ/Δ has a pole of order two at ζ. ⊘

Let $(X(z),\ Y(z))$ be a point on the Weierstrass curve (2.3.5) with $g_2 = g_2(z)$, $g_3 = g_3(z)$, that depends holomorphically on $z \in U$. Then the integral

$$T(z) = \int_\infty^{X(z)} y^{-1}\, \mathrm{d}x \tag{2.5.15}$$

represents the point in $\mathbb{C}/P$ corresponding to $(X(z),\ Y(z))$ under the isomorphism from $\mathbb{C}/P$ with the Weierstrass curve as described in Section 2.3.2. Note that here the period lattice $P = P_z$ depends in a holomorphic way on z in the complement of the zeros of $\Delta(z)$. Although $T(z)$ is determined only modulo periods, the function $\mu := \mathcal{L}\, T$ is uniquely determined, because $\mathcal{L}\, p = 0$ for every period function p. The following computation, which I learned from Beukers and Cushman [16, (14)], leads to an explicit formula for $\mu(z) := (\mathcal{L}\, T)(z)$, as well as to a proof of (2.5.10), (2.5.11), (2.5.12).

Let $f = f_z(x)$ be a polynomial of degree 3 in x whose coefficients depend holomorphically on z. With $y = f^{1/2}$, we have

$$\begin{aligned}
\frac{\partial y^{-1}}{\partial z} &= -\frac{1}{2}\, f^{-3/2}\, \frac{\partial f}{\partial z}, \\
\frac{\partial^2 y^{-1}}{\partial z^2} &= \frac{3}{4}\, f^{-5/2} \left(\frac{\partial f}{\partial z}\right)^2 - \frac{1}{2}\, f^{-3/2}\, \frac{\partial^2 f}{\partial z^2},
\end{aligned}$$

and therefore $\mathcal{L}(y^{-1}) = y^{-5}\, g$, with

$$g := \frac{3}{4} \left(\frac{\partial f}{\partial z}\right)^2 - \frac{1}{2}\, f\, \frac{\partial^2 f}{\partial z^2} - c_1\, \frac{1}{2}\, f\, \frac{\partial f}{\partial z} + c_2\, f^2,$$

which is a polynomial in x of degree 6.

If h is a polynomial in x of degree 4, then

$$\frac{\partial (y^{-3}\, h)}{\partial x} = f^{-3/2}\, \frac{\partial h}{\partial x} - \frac{3}{2}\, f^{-5/2}\, \frac{\partial f}{\partial x}\, h = y^{-5}\, i,$$

where

$$i := f\,\frac{\partial h}{\partial x} - \frac{3}{2}\,\frac{\partial f}{\partial x}\,h$$

is also a polynomial in x of degree 6. The coefficients of g depend in a linear inhomogeneous way on c_1 and c_2, whereas the coefficients of i depend in a linear homogeneous way on the five coefficients of h. Because a polynomial of degree 6 has seven coefficients, the equation $g = i$ is equivalent to a linear inhomogeneous system of seven equations for the seven unknowns c_1, c_2, and the five coefficients of h.

Solving this system of equations for $f = 4\,x^3 - g_2(z)\,x - g_3(z)$ with the help of a formula manipulation program, we arrive at $\mathcal{L}(y^{-1}) = \partial(y^{-3}\,h)/\partial x$, where L is as in (2.5.10) with c_1 and c_2 as in (2.5.11) and (2.5.12), respectively, and $h = h_0 + h_1\,x + h_2\,x^2 + h_3\,x^3 + h_4\,x^4$, with

$$\begin{aligned}
h_0 &= \Gamma^2/8\,\Delta,\\
h_1 &= (-15\,{g_2}\,{g_3}^2\,(g_2')^3 + 18\,{g_2}^2\,g_3\,(g_2')^2\,g_3' - 4\,{g_2}^3\,g_2'\,(g_3')^2 - 24\,g_2\,g_3\,(g_3')^3\\
&\quad +4\,{g_2}^3\,g_3\,g_3'\,g_2'' - 108\,{g_3}^3\,g_3'\,g_2'' - 4\,{g_2}^3\,g_3\,g_2'\,g_3'' + 108\,{g_3}^3\,g_2'\,g_3'')/4\,\Gamma\,\Delta,\\
h_2 &= (-15\,{g_2}^2\,g_3\,(g_2')^3 + 14\,{g_2}^3\,(g_2')^2\,g_3' - 216\,{g_3}^2\,(g_2')^2\,g_3' + 324\,g_2\,g_3\,g_2'\,(g_3')^2\\
&\quad -168\,{g_2}^2\,(g_3')^3 + 8\,{g_2}^4\,g_3'\,g_2'' - 216\,g_2\,{g_3}^2\,g_3'\,g_2'' - 8\,{g_2}^4\,g_2'\,g_3''\\
&\quad +216\,g_2\,{g_3}^2\,g_2'\,g_3'')/8\,\Gamma\,\Delta,\\
h_3 &= 0,\\
h_4 &= 21\,g_2\,g_3\,(g_2')^3 - 18\,{g_2}^2\,(g_2')^2\,g_3' - 108\,g_3\,g_2'\,(g_3')^2 + 120\,g_2\,(g_3')^3 - 8\,{g_2}^3\,g_3'\,g_2''\\
&\quad +216\,{g_3}^2\,g_3'\,g_2'' + 8\,{g_2}^3\,g_2'\,g_3'' - 216\,{g_3}^2\,g_2'\,g_3'')/2\,\Gamma\,\Delta,
\end{aligned}$$

and we recall that $\Gamma = -3\,g_3\,g_2' + 2\,g_2\,g_3'$, $\Delta = {g_2}^3 - 27\,{g_3}^2$. The main point of this display is to illustrate the explicit nature of the formulas.

Because

$$T'(z) = X'(z)/Y(z) + \int_\infty^{X(z)} \frac{\partial y^{-1}}{\partial z}\;\mathrm{d}x$$

and

$$T''(z) = (X'/Y)'(z) + \frac{1}{2}\,X'(z)\,(g_2'(z)\,X(z) + g_3'(z))/Y(z)^3 + \int_\infty^{X(z)} \frac{\partial^2 y^{-1}}{\partial z^2}\;\mathrm{d}x,$$

where we have used that $f^{-3/2} = y^{-3}$ and $\partial f/\partial z = -g_2'\,x - g_3'$, we arrive at the formula $(\mathcal{L}\,T)(z) = \mu(z)$, with

$$\begin{aligned}
\mu(z) &= (X'/Y)'(z) + \frac{1}{2}\,X'(z)\,(g_2'(z)\,X(z) + g_3'(z))/Y(z)^3 + c_1(z)\,X'(z)/Y(z)\\
&\quad +h_z(X(z))/Y(z)^3. \qquad (2.5.16)
\end{aligned}$$

Here we have indicated the dependence on z of the coefficients of the polynomial $h(x)$ by writing $h(x) = h_z(x)$, where in (2.5.16) we subsequently have substituted $x = X(z)$. We also recall the formula (2.5.11) for $c_1(z)$.

The formula (2.5.16) for $\mu(z)$ suggests that $\mu(z)$ has a singular behavior at a point z_0 where $Y(z_0) = 0$, but $\Gamma(z_0)\,\Delta(z_0) \neq 0$, that is, the operator $\mathcal{L}$ is regular. However, this is an artifact of the way in which we have computed μ, because it follows from $\mu = \mathcal{L}\,T$, the fact that T is holomorphic when $\Delta \neq 0$, and the fact that the coefficients c_1 and c_2 of $\mathcal{L}$ are holomorphic when $\Gamma \neq 0$ and $\Delta \neq 0$ that μ is holomorphic where $\Gamma \neq 0$ and $\Delta \neq 0$. More precisely, we have the following lemma.

Lemma 2.5.17 *The function $\mu(z) = (\mathcal{L}\,T)(z)$ can be singular only at points $z = z_0$ where $\Delta(z_0) = 0$ or $\Gamma(z_0) = 0$. If $\Delta(z_0) \neq 0$ and $\Gamma(z_0) = 0$, then $\mu(z)$ has a pole of order ≤ 1 at $z = z_0$.*

Let $\Delta(z_0) = 0$, meaning that the Weierstrass curve (2.3.5) *with $g_2 = g_2(z_0)$ and $g_3 = g_3(z_0)$ is singular. Assume, moreover, that $T(z)$ is holomorphic in a neighborhood of $z = z_0$. Then $\mu(z)$ has a pole of order ≤ 2 at $z = z_0$. If the homogeneous equation $\mathcal{L}\,p = 0$ has a solution $p(z)$ that is holomorphic in a neighborhood of $z = z_0$ and satisfies $p(z_0) \neq 0$, then $\mu(z)$ has a pole of order ≤ 1 at $z = z_0$.*

Proof. Assume that $\Delta(z_0) \neq 0$. Then the Weierstrass curve (2.3.5) with $g_2 = g_2(z_0)$ and $g_3 = g_3(z_0)$ is smooth, and $T(z)$ is holomorphic in a neighborhood of $z = z_0$. It follows from (2.5.8) and $\Delta(z_0) \neq 0$ that the functions $P(z)$, $Q(z)$, and $R(z)$ are holomorphic in a neighborhood of $z = z_0$. If $\Gamma(z_0) \neq 0$, then also $P'(z)/P(z)$ is holomorphic in a neighborhood of $z = z_0$, and it follows that $\mu(z) = (\mathcal{L}\,T)(z)$ is holomorphic in a neighborhood of $z = z_0$. On the other hand, if $\Gamma(z_0) = 0$, then $P'(z)/P(z)$ has a simple pole at $z = z_0$. It follows therefore from (2.5.11) and (2.5.11) that the coefficients $c_1(z)$ and $c_2(z)$ of $\mathcal{L}$ have at most simple poles at $z = z_0$, and therefore $\mu(z) = (\mathcal{L}\,T)(z)$ has a pole of order ≤ 1 at $z = z_0$.

The conclusions in the case $\Delta(\zeta) = 0$ follow from the fact that $c_1(z)$ and $c_2(z)$ have poles of order ≤ 1 and ≤ 2 at $z = z_0$, respectively. If $p(z)$ is holomorphic in a neighborhood of $z = z_0$, $p(z_0) \neq 0$, and $\mathcal{L}\,p = 0$, then $c_2 = -(p'' + c_1\,p')/p$ shows that also $c_2(z)$ has a pole of order ≤ 1 at $z = z_0$, and therefore $\mu(z) = (\mathcal{L}\,T)(z)$ has a pole of order ≤ 1 at $z = z_0$. □

We now apply the above to QRT transformations. Assume that the pencil of biquadratic curves defined by (2.5.3) has at least one smooth member and nonconstant modulus function $J : \mathbb{P}^1 \to \mathbb{P}^1$. We use the affine coordinate $z \in \mathbb{C}$ such that $z_0 = 1$, $z_1 = -z$. That is, we write $p_z = p^0 - z\,p^1$ such that the biquadratic curve $p_z = 0$ is the level curve at the level z of p^0/p^1, the rational function that is invariant under the QRT transformation. With this substitution, the invariants $g_2 = g_2(z)$ and $g_3 = g_3(z)$ are polynomials in $z \in \mathbb{C}$ of degree ≤ 4 and ≤ 6, respectively. Let $\mathcal{L}$ be the second-order linear ordinary differential operator (2.5.10) with the coefficients $c_1(z)$, $c_2(z)$ defined by (2.5.11), (2.5.12). Here P, Q, R are defined by (2.5.8), with $\Gamma = -3\,g_3\,g_2' + 2\,g_2\,g_3'$ and $\Delta = {g_2}^3 - 27\,{g_3}^2$. Note that the degrees of Γ and Δ are ≤ 8 and ≤ 12, respectively.

Let $P = P_z$ denote the period lattice of vector field v_z on the biquadratic curve $C_{[1:-z]}$, defined by the biquadratic polynomial $p_z = p^0 - z\,p^1$ as in Lemma 2.4.5. Then P_z is generated by periods $p_1(z)$, $p_2(z)$ that are multivalued holomorphic functions on the complement of the zero-set of Δ. Our first conclusion is the following:

Proposition 2.5.18 *The basic periods $p_1(z)$ and $p_2(z)$ satisfy the homogeneous linear second-order ordinary differential equation $\mathcal{L}\, p = 0$.*

Remark 2.5.19. In Table 6.2.39, the behavior of the basic periods $p_j(z)$ near a singular fiber is described in terms of the Kodaira type of the singular fiber, where a local coordinate z is used such that $z = 0$ corresponds to the singular fiber. On the other hand, the Laurent expansion of the coefficients $c_1(z)$ and $c_2(z)$ near a singular point of the differential operator $\mathcal{L}$ leads to quite detailed information about the behavior of the solutions $p(z)$ of $\mathcal{L}\, p = 0$ near the singular point, which must be compatible with Table 6.2.39 if we take $p = p_j$. In particular, this leads to relations between the Laurent expansions of the coefficients $c_1(z)$, $c_2(z)$ of $\mathcal{L}$ and the Kodaira types of the singular fibers. ⊘

Our second conclusion deals with the QRT transformation τ on the biquadratic curve $C_{[1:-z]}$.

Proposition 2.5.20 *If $\Delta(z) \neq 0$, let $T(z) \in \mathbb{C}/P_z$ be such that the QRT transformation τ on the biquadratic curve $C_{[1:-z]}$ is equal to the flow of the vector field v_z after the time $T(z)$. This defines a multivalued holomorphic function $T(z)$ of z, which modulo periods is holomorphic and nonzero everywhere. The function $\mu(z) := (\mathcal{L}\, T)(z)$ is a rational function of z that can be computed explicitly, where $\mu \equiv 0$ if and only if the QRT map τ is of finite order.*

If μ is not identically equal to zero, then its poles satisfy the conclusions in Lemma 2.5.17. *Furthermore, the degree of the numerator of μ is equal to the degree of the denominator of μ minus $5 + m$. Here m is the integer ≥ -2 such that $\widehat{\mu}(\zeta)$ is of order ζ^m as $\zeta \to 0$, where $\zeta = 1/z$ and $\widehat{\mu}(\zeta)$ replaces $\mu(z)$ when $p^0 - z\, p^1$ is replaced by $\zeta\, p^0 - p^1$.*

Proof. With $p = p_z = p^0 - z\, p^1$, the invariants $X(z) := X(p_z)$ and $Y(z) := Y(p_z)$ of p in Proposition 2.5.6 are polynomials in z of degree ≤ 2 and ≤ 3, respectively, which moreover satisfy the Weierstrass equation (2.5.5). Furthermore, $T(z)$ is equal to the incomplete elliptic integral (2.5.15). Therefore the statement about T follows from Lemma 7.4.1, which in turn implies that μ satisfies the conclusions in Lemma 2.5.17.

That $\mu \equiv 0$ if and only if τ is of finite order follows from Proposition 7.8.8 for $\alpha = \tau$. The last statement follows from Remark 7.8.6. □

Remark 2.5.21. If J is a constant, then $P \equiv 0$, and the periods satisfy the homogeneous linear first-order differential equation $\mathcal{L}_1\, p := p' + Q\, p = 0$, which in view of the equation $Q = \Delta'/12\,\Delta$ in (2.5.8) implies that p is a constant multiple of Δ^{-12}. In this case T satisfies the inhomogeneous linear first-order differential equation $\mathcal{L}_1\, T = \mu_1$, where μ_1 is a meromorphic function with at most simple poles at the zeros of Δ. In the case of the QRT mapping, μ_1 is a rational function with simple poles at the zeros of Δ, and the degree of the numerator of μ_1 is equal to the degree of the denominator of μ_1 minus 3 if the curve $p^1 = 0$ is not singular, and minus 2 if the curve $p^1 = 0$ is singular. The proof of Proposition 7.8.8 again yields that $\mu_1 = 0$ if and only if the QRT transformation has finite order.

For rational elliptic surfaces the possible configurations of singular fibers in the case of constant J are listed in Remark 9.2.8. It follows from Theorem 4.3.3 that the Mordell–Weil group is generated by QRT automorphisms, and if these all have finite order, then the Mordell–Weil group is finite. Because in quite a number of cases in Remark 9.2.8 the Mordell–Weil group is infinite, it follows that there exist QRT mappings with a constant modulus J that are of infinite order, that is, for which $\mu_1 \neq 0$. ⊘

Remark 2.5.22. I learned the reference to Bruns [26] from Schmickler–Hirzebruch [176, p. 135]. Bruns [26, p. 238] also observed that with the substitutions

$$p = \Pi\, {g_2}\, {g_3}^{-5/6}/3, \qquad \eta = \mathrm{H}\, {g_3}^{1/6},$$

and with all functions viewed as functions of $g := {g_2}^3/27\,{g_3}^2$, one obtains the following "universal" hypergeometric differential equations for Π and H:

$$0 = g\,(g-1)\,\mathrm{d}^2\,\Pi/\,\mathrm{d}g^2 + (7\,g/3 - 4/3)\,\mathrm{d}\Pi/\,\mathrm{d}g + (55/144)\,\Pi,$$
$$0 = g\,(g-1)\,\mathrm{d}^2\,\mathrm{H}\,/\,\mathrm{d}g^2 + (4\,g/3 - 1/3)\,\mathrm{dH}\,/\,\mathrm{d}g - (5/144)\,\mathrm{H}\,.$$

Note that $J = {g_2}^3/({g^2}^3 - 27\,{g_3}^2) = g/(g-1)$.

Klein [106, p. 113] considered $\widetilde{\Pi} := \Delta^{12}\,p$ as a function of J and gave in [106, p. 124] for this function the universal hypergeometric differential equation

$$J\,(1-J)\,\mathrm{d}^2\,\widetilde{\Pi}/\,\mathrm{d}J^2 + (2/3 - 7\,J/6)\,\mathrm{d}\widetilde{\Pi}/\,\mathrm{d}J - (1/144)\,\widetilde{\Pi} = 0,$$

followed two pages later by a reference to the 1875 publication of Bruns [26]. Here it may be noted that if we give g_2 and g_3 the weights 4 and 6, respectively, then multiplying p and η by any factors that are expressions in g_2 and g_3 of weight equal to one will lead to universal differential equations for the resulting products, if viewed as a function of J. The equation of Klein implies that c_1, and therefore also c_2, see (2.5.14), can be expressed in terms of the discriminant Δ and the modulus function J. Indeed, it follows from $J = {g_2}^3/\Delta$, $J - 1 = 27\,{g_3}^2/\Delta$, and (2.5.13) that $J'/J = 3\,g_2'/g_2 - \Delta'/\Delta$, $(J-1)'/(J-1) = 2\,g_3'/g_3 - \Delta'/\Delta$, and $J''/J' = 2\,g_2'/g_2 + g_3'/g_3 + \Gamma'/\,\Gamma - 2\,\Delta'/\Delta$, and therefore

$$c_1 = -\frac{\Gamma'}{\Gamma} + \frac{\Delta'}{\Delta} = \frac{2}{3}\,\frac{J'}{J} + \frac{1}{2}\,\frac{(J-1)'}{J-1} - \frac{J''}{J'} + \frac{1}{6}\,\frac{\Delta'}{\Delta}. \tag{2.5.17}$$

The relation with the hypergeometric differential equation with respect to the variable J is another explanation why equation (2.5.10) is singular not only when $\Delta(z) = 0$, when $J(z) = \infty$, $J(z) = 0$, or $J(z) = 1$, but also when $\mathrm{d}J(z)/\,\mathrm{d}z = 0$, that is, when $\Gamma(z) = 0$.

Manin [128, pp. 191–192] wrote; "The generalization of the Gauss differential equations to periods of abelian integrals on curves of arbitrary genus was obtained by Picard and Fuchs (with a reference to Picard and Simart [158]).... The first section of Chap. 1 is devoted to the construction and theory of such relations, *which we propose to call Picard–Fuchs equations*." This name has caught on, and is nowadays

generally used. Shortly after this, Grothendieck [77] proposed the name *Gauss–Manin connection*, which is used more or less synonymously in the modern literature.

Manin [128, p. 191] considered not only periods, that is, integrals over closed paths, but also integrals $T(z)$ over intervals as in (2.5.15), where $\mu(z) = (\mathcal{L}\,T)(z)$ is no longer zero as for the periods, but equal to an explicitly computable function of the parameter z. For this reason I propose to call the function $\mu(z)$, when $T(z)$ corresponds to the QRT map, *Manin's function of the QRT map*. If the QRT map is replaced by an arbitrary element α of the Mordell–Weil group of an arbitrary elliptic surface, then the mapping which assigns to α the function μ is Manin's homomorphism as discussed in Proposition 7.8.8 and Remark 7.8.9. ⊘

2.6 Real Points

Let $\mathbb{P}^1(\mathbb{R})$ denote the real projective line. If the coefficients A^0_{ij}, A^1_{ij} in (2.5.3) are real numbers, then for given $(z_0,\ z_1) \in \mathbb{C}^2$ with $[z_0 : z_1] \in \mathbb{P}^1(\mathbb{R})$, the real solutions of (2.5.3) are real curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. These real curves are the fixed-point sets in the complex elliptic curves $C_{[z]}$ of the complex conjugation γ, where γ leaves $C_{[z]}$ invariant if $[z_1 : z_0] \in \mathbb{P}^1(\mathbb{R})$. We first discuss the real biquadratic curves from the perspective of arbitrary elliptic curves with a complex conjugation.

2.6.1 Real Points of an Elliptic Curve

The holomorphic vector field v in Section 2.3 can be used to obtain a quite detailed description of the set $C(\mathbb{R})$ of all real points of an arbitrary elliptic curve C. We will show that $C(\mathbb{R})$ is either empty, or it consists of one circle, or it consists of two parallel circles in the torus, with "one halfway the other." Any translation of the complex elliptic curve that leaves the set of real points invariant is a rotation in the circle in the second case. In the third case it is either a rotation in each of the two circles, or it is a translation in the torus that interchanges the two circles. In the last case, applying the complex translation twice, we obtain a rotation in each of the curves, over the same angle.

We now turn to the proof of these statements, at the same time explaining the above statements in more detail. Let C be any complex elliptic curve and let $\gamma : C \to C$ be a *complex conjugation* in C in the sense that γ is an antiholomorphic mapping from C to C such that $\gamma \circ \gamma = 1$. The antiholomorphy of γ means that γ is differentiable as a transformation of the real two-dimensional surface C and that for any $c \in C$ and $v \in \mathrm{T}_c C$, we have $\mathrm{T}_c\gamma(\mathrm{i}\ v) = -\mathrm{i}\ \mathrm{T}_c\gamma(v)$. The equation $\gamma \circ \gamma = 1$ implies that γ is a diffeomorphism with inverse equal to γ. Here $\mathrm{T}_c C$ denotes the *tangent space* of C at c, and $\mathrm{T}_c\gamma$ is the *tangent mapping* of γ at c, which is the linear mapping from $\mathrm{T}_c C$ to $\mathrm{T}_{\gamma(c)} C$, which in local coordinates is represented by the Jacobi matrix of the first-order partial derivatives of γ at the point c.

For any holomorphic vector field v on C, the complex conjugate vector field $\gamma_* v$ of v, defined by $(\gamma_* v)(\gamma(c)) = (\mathrm{T}_c\gamma)(v(c))$ for every $c \in C$, is a holomorphic vector

field on C that satisfies $\gamma_*(\mathrm{i}\,v) = -\mathrm{i}\,\gamma_* v$. That is, γ_* is a complex conjugation, a complex antilinear transformation of the one-dimensional complex vector space V of all holomorphic vector fields on C, with square equal to the identity. It follows that there is a nonzero $v \in V$ such that $\gamma_* v = v$. In the sequel, v will be such a holomorphic vector field on C. We have $\gamma_*(t\,v) = \overline{t}\,v$ for every $t \in \mathbb{C}$.

Let c be any choice of an initial point in C. Then there exists an $s \in \mathbb{C}$ such that $\gamma(c) = \mathrm{e}^{s\,v}(c)$. For any $t \in \mathbb{C}$ we have

$$\gamma(\mathrm{e}^{t\,v}(c)) = \mathrm{e}^{\overline{t}\,v}(\gamma(c)) = \mathrm{e}^{\overline{t}\,v}(\mathrm{e}^{s\,v}(c)) = \mathrm{e}^{(\overline{t}+s)\,v}(c). \tag{2.6.1}$$

Note that if $t \in P$, then it follows from (2.6.1) that $\gamma(c) = \mathrm{e}^{\overline{t}\,v}(\gamma(c))$; hence $\overline{t} \in P$. Equation (2.6.1) means that the isomorphism from $\mathbb{C}/P$ onto C, induced by the mapping $t \mapsto \mathrm{e}^{t\,v}(c)$, intertwines the transformation

$$\widehat{\gamma} : t + P \mapsto \overline{t} + s + P : \mathbb{C}/P \to \mathbb{C}/P \tag{2.6.2}$$

in $\mathbb{C}/P$ with the complex conjugation γ in C. Note that

$$\widehat{\gamma}(\widehat{\gamma}(t + P)) = \widehat{\gamma}(\overline{t} + s + P) = \overline{\overline{t} + s} + s + P = t + \overline{s} + s + P,$$

and therefore the condition that $\gamma^2 = 1$ is equivalent to the condition that $2\,\mathrm{Re}\,s = \overline{s} + s \in P$.

The point $\mathrm{e}^{t\,v}(c)$ is a real point of C, that is, a fixed point of γ, if and only if $t + P$ is a fixed point of $\widehat{\gamma}$, if and only if $\overline{t} + s + P = t + P$, if and only if $2\,\mathrm{i}\,\mathrm{Im}\,t - s = t - \overline{t} - s \in P$. It follows that C has no real points at all if and only if the vertical line $\mathrm{Re}\,s + \mathrm{i}\,\mathbb{R}$ in the complex plane does not meet any points of P. Note that this implies that $\mathrm{Re}\,s \notin P$, whereas we had $2\,\mathrm{Re}\,s \in P$.

Now assume that C has real points, one of which we take as our initial point c. That is, $\gamma(c) = c$ and $s = 0$ in (2.6.2). In this situation $\mathrm{e}^{t\,v}(c)$ is a real point of C if and only if $2\,\mathrm{i}\,\mathrm{Im}\,t \in P$. Furthermore, P is invariant under complex conjugation, which implies that for every $p \in P$ we have that $2\,\mathrm{Re}\,p = p + \overline{p} \in P$. Because P contains an $\mathbb{R}$-basis of $\mathbb{C}$, P is not contained in the imaginary axis, and there is a unique smallest strictly positive real number p_1 such that $p_1 \in P$. If q denotes the minimum of $\mathrm{Im}\,p$ where $p \in P$ and $\mathrm{Im}\,p > 0$, then there is a unique $p = p_2 \in P$ such that $\mathrm{Im}\,p = q$ and $0 \le \mathrm{Re}\,p < p_1$. The complex numbers p_1 and p_2 form a $\mathbb{Z}$-basis of P.

Because $p_2 \in P$, we have $2\,\mathrm{Re}\,p_2 \in P$. On the other hand, $0 \le 2\,\mathrm{Re}\,p_2 < 2p_1$, and because the only elements $p \in P$ such that $p \in \mathbb{R}$ and $0 \le p < 2p_1$ are $p = 0$ and $p = p_1$, we have that either $\mathrm{Re}\,p_2 = 0$ or $\mathrm{Re}\,p_2 = p_1/2$.

Let $\mathrm{Re}\,p_2 = 0$. Then $p_2 = \mathrm{i}\,q$, and $\mathrm{e}^{t\,v}(c)$ is a real point of C if and only if there are integers n_1 and n_2 such that $2\,\mathrm{i}\,\mathrm{Im}\,t = n_1\,p_1 + n_2\,p_2 = n_1\,p_1 + \mathrm{i}\,n_2\,q$, that is, $\mathrm{Im}\,t$ is an integral multiple of $q/2$. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, then we obtain that $C(\mathbb{R})$ consists of the two parallel circles $(\mathbb{R}/\mathbb{Z}\,p_1) \times \{0 + \mathbb{Z}\,q\}$ and $(\mathbb{R}/\mathbb{Z}\,p_1) \times \{b/2 + \mathbb{Z}\,q\}$ in the torus $(\mathbb{R}/\mathbb{Z}\,p_1) \times (\mathbb{R}/\mathbb{Z}\,q)$.

Finally, let $\mathrm{Re}\,p_2 = p_1/2$ when $p_2 = (p_1/2) + \mathrm{i}\,q$. In this case $\mathrm{e}^{t\,v}(c)$ is a real point of C if and only if there are integers n_1 and n_2 such that $2\,\mathrm{i}\,\mathrm{Im}\,t =$

$n_1\, p_1 + n_2\, p_2 = n_1\, p_1 + n_2\, ((p_1/2) + \mathrm{i}\, q)$. That is, $n_2 = -2\, n_1$ is even, and $\operatorname{Im} t$ is an integral multiple of q. Therefore $C(\mathbb{R})$ is equal to the circle $(\mathbb{R}/\mathbb{Z}\, p_1) \times \{0 + \mathbb{Z}\, q\}$ in the torus $(\mathbb{R}/\mathbb{Z}\, p_1) \times (\mathbb{R}/\mathbb{Z}\, q)$.

Figure 2.6.1 shows the cases $\operatorname{Re} p_2 = 0$ and $\operatorname{Re} p_2 = p_1/2$, respectively.

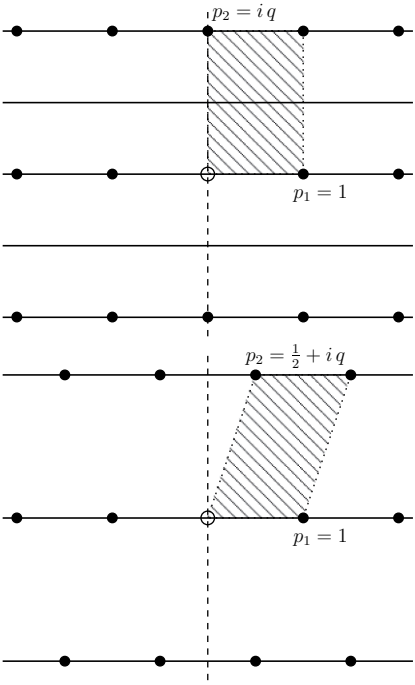


Fig. 2.6.1 The real locus $C(\mathbb{R})$. Two circles (left), or one circle (right).

Remark 2.6.1. Assume that C has real points. Then P is invariant under complex conjugation, and it follows from (2.3.3) and (2.3.4) that $\overline{\wp(t)} = \wp(\bar{t})$ for each $t \in \mathbb{C}$ and $g_2,\ g_3 \in \mathbb{R}$, respectively. Because $\overline{\wp(t)} = \wp(\bar{t}$ for each $t \in \mathbb{C}$ implies that $\overline{\wp'(t)} = \wp'(\bar{t})$ for each $t \in \mathbb{C}$, the mapping $t \mapsto (x,\ y) = (\wp(t),\ \wp'(t))$ intertwines the complex conjugation on $\mathbb{C}/P$ with the complex conjugation $(x,\ y) \mapsto (\overline{x},\ \overline{y})$ on the curve (2.3.5), or equivalently with the complex conjugation $[x_0 : x_1 : x_2] \mapsto [\overline{x_0} : \overline{x_1} : \overline{x_2}]$ on the cubic curve (2.3.6) in $\mathbb{P}^2$. Because the mapping $t \mapsto (x,\ y) = (\wp(t),\ \wp'(t))$ induces an isomorphism from $\mathbb{C}/P$ onto the cubic curve (2.3.6) in $\mathbb{P}^2$, it follows that $C(\mathbb{R})$ corresponds to the set of all $[x_0 : x_1 : x_2]$ such that $(x_0,\ x_1,\ x_2)$ is a nonzero real vector that satisfies the equation (2.3.6).

It follows that $C(\mathbb{R})$ has one or two connected components if and only if ${g_2}^3 - 27\, {g_3}^2 < 0$ or ${g_2}^3 - 27\, {g_3}^2 > 0$, respectively. In view of (2.3.10) we have $J \leq 1$ and $g_2 < 0$ when $J = 1$ if ${g_2}^3 - 27\, {g_3}^2 < 0$, whereas $J \geq 1$ and $g_2 > 0$ when $J = 1$ if ${g_2}^3 - 27\, {g_3}^2 > 0$. In other words, if $J < 1$ or $J > 1$, then $C(\mathbb{R})$ has one or

two components, respectively. If $J = 1$, when $g_3 = 0$, then both cases are possible, depending on the sign of g_2, namely $C(\mathbb{R})$ has one or two connected components if $g_2 < 0$ or $g_2 > 0$, respectively. The explanation in terms of the period lattice P is that $J = 1$, when the elliptic curve is harmonic, corresponds to a square lattice P. Then $C(\mathbb{R})$ has one connected component if the real points of P are on the diagonal of the square, whereas $C(\mathbb{R})$ has two connected components if the real points of P are on a side of the square. ⊘

Remark 2.6.2. Assume that $C(\mathbb{R})$ has no real points. It follows from (2.6.1) that

$$\gamma(\mathrm{e}^{t\,v}(c)) = \mathrm{e}^{(\overline{t}+s-t)\,v}(\mathrm{e}^{t\,v}(c)),$$

where $\overline{t} + s - t = \operatorname{Re} s$ if $\operatorname{Im} t = \operatorname{Im} s/2$. That is, we can arrange that $s \in \mathbb{R}$, where $s \notin P$ but $2\,s \in P$. Rescaling P by a real factor, we can arrange that $P \cap \mathbb{R} = \mathbb{Z}$, and we have $s \in (1/2) + \mathbb{Z}$. The complex conjugation in C corresponds to the mapping

$$t + P \mapsto \overline{t} + s + P = \overline{t} + 1/2 + P. \tag{2.6.3}$$

It follows from $P \cap \mathbb{R} = \mathbb{Z}$ that $P = \mathbb{Z} + \mathbb{Z}\,q$, where $q \in \mathbb{C}$ is uniquely determined by the conditions that $\operatorname{Im} q > 0$ and $0 \le \operatorname{Re} q < 1$. We had concluded from (2.6.1) that $\overline{P} = P$, which means that $\overline{q} = k_1 + k_2\,q$ for some $k_1,\ k_2 \in \mathbb{Z}$ if and only if $\overline{q} = k_1 - q$ for some $k_1 \in \mathbb{Z}$ if and only if $\operatorname{Re} q = 0$ or $\operatorname{Re} q = 1/2$.

It follows from (2.3.4) and (2.3.10) that $g_2,\ g_3 \in \mathbb{R}$ and $J \in \mathbb{R}$. As in Remark 2.6.1, if $J < 1$ or $J > 1$, then $\operatorname{Re} q = 1/2$ or $\operatorname{Re} q = 0$, respectively. If $J = 1$, when $g_3 = 0$, then $g_2 < 0$ or $g_2 > 0$ if $\operatorname{Re} q = 1/2$ or $\operatorname{Re} q = 0$, respectively.

In view of (2.6.3), the complex conjugation on the elliptic curve without fixed points corresponds to the "ordinary" complex conjugation $[x_0 : x_1 : x_2] \mapsto [\overline{x_0} : \overline{x_1} : \overline{x_2}]$ in the cubic curve (2.3.6) in $\mathbb{P}^2$, followed by an involution that is a translation over a nonzero real element of order two. Because the ordinary conjugation and the translation commute, the composition is an involution as well. The elliptic curve has no real points with respect to a complex conjugation without fixed points. ⊘

2.6.2 Real Biquadratic Curves

Let $p(x,\ y)$ be a biquadratic polynomial with real coefficients, where we assume that the biquadratic curve C defined by the equation $p(x,\ y) = 0$ is smooth, hence an elliptic curve. For a real $x \in \mathbb{P}^1(\mathbb{R})$, the equation $p(x,\ y) = 0$ has a real solution $y \in \mathbb{P}(\mathbb{R})$ if and only if $\Delta_2(x) \ge 0$, where $\Delta_2(x)$ denotes the discriminant of the quadratic polynomial $y \mapsto p(x,\ y)$. The assumption that C is smooth means that all the zeros of Δ_2 in $\mathbb{P}^1$ are simple, and we have four of these because Δ_2 is a homogeneous polynomial of degree four in $(x_0,\ x_1)$. Because the coefficients of Δ_2 are real, the nonreal zeros of Δ_2 appear in complex conjugate pairs, and therefore Δ_2 has 0, 2, or 4 real zeros in $\mathbb{P}^1(\mathbb{R})$, each of which simple. Similar considerations hold for the partial discriminant $\Delta_1(y)$ of $x \mapsto p(x,\ y)$. This leads to the following

cases, where we recall that $\mathbb{P}^1(\mathbb{R})$ can be identified with a circle, in the form of the real axis with one point at infinity added, with $+\infty$ attached to $-\infty$. We write $\pi_1 : ([x], [y]) \mapsto [x]$ and $\pi_2 : ([x], [y]) \mapsto [y]$ for the projection from $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ onto the first and second factors, respectively.

Lemma 2.6.3 *Let p be a biquadratic polynomial with real coefficients such that $\Delta = F(\Delta_1) = F(\Delta_2) \neq 0$. We then have the following cases:*

(a) *$\Delta_2(x) < 0$ for every $x \in \mathbb{R}^2 \setminus \{0\}$, $\Delta_1(y) < 0$ for every $y \in \mathbb{R}^2 \setminus \{0\}$, and C has no real points. We have $\Delta > 0$.*

(b) *Δ_2 has two zeros in $\mathbb{P}^1(\mathbb{R})$ and Δ_1 has two zeros in $\mathbb{P}^1(\mathbb{R})$, where all zeros are simple. $C(\mathbb{R})$ is connected, isomorphic to a circle. The restrictions of π_1 and π_2 to $C(\mathbb{R})$ are twofold branched coverings of $C(\mathbb{R})$ over the closed intervals I_1 and I_2 in $\mathbb{P}^1(\mathbb{R})$, where $\Delta_2 \geq 0$ and $\Delta_1 \geq 0$, branching over the endpoints of I_1 and I_2, which are the two zeros of Δ_2 and Δ_1, respectively. We have $\Delta < 0$.*

(c) *$\Delta_2(x) > 0$ for every $x \in \mathbb{R}^2 \setminus \{0\}$ and Δ_1 has four zeros in $\mathbb{P}^1$, each of which is simple. $C(\mathbb{R})$ has two connected components $C(\mathbb{R})^{\pm}$, each isomorphic to a circle. The restriction to $C(\mathbb{R})^{\pm}$ of π_1 is an isomorphism from $C(\mathbb{R})^{\pm}$ onto $\mathbb{P}^1(\mathbb{R})$. The set of points in the second factor $\mathbb{P}^1(\mathbb{R})$ where $\Delta_1 \geq 0$ is the union of two disjoint closed intervals $I_2^{\pm}$, and the restriction to $C(\mathbb{R})^{\pm}$ of π_2 is a twofold branched covering of $C(\mathbb{R})^{\pm}$ over $I_2^{\pm}$, branching over the endpoints of $I_2^{\pm}$, where $\Delta_1 = 0$. The horizontal and vertical switches ι_1 and ι_2 preserve and interchange the connected components of $C(\mathbb{R})$, respectively, and the QRT transformation $\tau = \iota_2 \circ \iota_1$ interchanges them. We have $\Delta > 0$.*

(d) *As in (c), but with Δ_2, π_1, and ι_1 interchanged with Δ_1, π_2, and ι_2, respectively.*

(e) *$\Delta_2(x) > 0$ for every $x \in \mathbb{R}^2 \setminus \{0\}$ and $\Delta_1(y) > 0$ for every $y \in \mathbb{R}^2 \setminus \{0\}$. $C(\mathbb{R})$ has two connected components $C(\mathbb{R})^{\pm}$, each isomorphic to a circle. The restrictions to $C(\mathbb{R})^{\pm}$ of the projections π_1 and π_2 are isomorphisms from $C(\mathbb{R})^{\pm}$ onto $\mathbb{P}^1(\mathbb{R})$. The horizontal and vertical switches ι_1 and ι_2 both interchange the connected components of $C(\mathbb{R})$, and the QRT transformation preserves them. We have $\Delta > 0$.*

(f) *Both Δ_2 and Δ_1 have four zeros in $\mathbb{P}^1$, each simple. $C(\mathbb{R})$ has two connected components $C(\mathbb{R})^{\pm}$, each isomorphic to a circle. The restrictions to $C(\mathbb{R})^{\pm}$ of the projections π_1 and π_2 are twofold branched coverings over the closed intervals $I_1^{\pm}$ and $I_2^{\pm}$ in $\mathbb{P}^1(\mathbb{R})$ where $\Delta_2 \geq 0$ and $\Delta_1 \geq 0$, respectively. The horizontal and vertical switches ι_1 and ι_2 both preserve the connected components of $C(\mathbb{R})$, and the QRT transformation preserves them as well. We have $\Delta > 0$.*

Proof. If $\Delta_2(x) < 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$, or $\Delta_1(y) < 0$ for all $y \in \mathbb{R}^2 \setminus \{0\}$, then $C(\mathbb{R}) = \emptyset$. Moreover, the zeros of $\Delta_2(x)$ appear as four nonreal complex conjugate pairs a, $\overline{a}$, b, $\overline{b}$, and the discriminant of Δ_2 is equal to

$$(a - \overline{a})^2 (a - b)^2 (a - \overline{b})^2 (\overline{a} - b)^2 (\overline{a} - \overline{b})^2 (b - \overline{b})^2.$$

The first and sixth factors are negative, whereas the product of the second and fifth factors is positive, and the product of the third and fourth factors is positive. Therefore $\Delta > 0$. The signs of Δ in the other cases are determined in a similar manner.

Assume that $\Delta_2(x)$ has two zeros in $\mathbb{P}^1(\mathbb{R})$. Let I be the closed interval in $\mathbb{P}^1(\mathbb{R})$ where $\Delta_2 \geq 0$, where the endpoints of I are the zeros of Δ_2. Then $\pi_1|_{C(\mathbb{R})}$ is a twofold branched covering of $C(\mathbb{R})$ over I, where the two branches come together over the endpoints. It follows that $C(\mathbb{R})$ is connected.

Assume now that $\Delta_2(x) > 0$ for every $x \in \mathbb{R}^2 \setminus \{0\}$. First assume that $C(\mathbb{R})$ is connected. According to Section 2.6.1, the period lattice is of the form $P = \mathbb{Z}\, p_1 + \mathbb{Z}\, p_2$ with $p_1 \in \mathbb{R}_{>0}$ and $p_2 = p_1/2 + \mathrm{i}\, q$, $q \in \mathbb{R}_{>0}$. Choosing the base point on $C(\mathbb{R})$, we have that under the identification of C with $\mathbb{C}/P$, the real locus $C(\mathbb{R})$ corresponds to $(\mathbb{R}/\mathbb{Z}\, p_1) \times \{\mathbb{Z}\, p_2\}$.

The restriction to C of π_1 exhibits the complex curve C as a twofold branched covering of C over the complex projective line $\mathbb{P}^1$, where the ramification points in C are the points $b = ([x], [y]) \in C$ such that $\Delta_2(x) = 0$. The involution ι that interchanges the two points in each fiber of $\pi_1|_C$, see Lemma 2.3.6, has these ramification points as its fixed points. According to Remark 2.3.8, the set of ramification points in $\mathbb{C}/P$ is of the form $\zeta + \mathbb{Z}\, p_1/2 + \mathbb{Z}\, p_2/2$ for some $\zeta \in \mathbb{C}$.

Because the polynomial Δ_2 has real coefficients, its zeros appear in complex conjugate pairs, which implies that $\overline{\zeta} + P$ is a ramification point. It follows that there are integers k_1 and k_2 such that $\overline{\zeta} = \zeta + k_1\, p_1/2 + k_2\, p_2/2$. Inspection of the real and imaginary parts yields $k_2 = -2\, k_1$ and $0 = 2\ \mathrm{Im}\, \zeta + k_2\, q/2$, and therefore $\mathrm{Im}\, \zeta = k_1\, q/2$. However, this implies that the ramification point $\zeta - k_1\, p_2/2 + P$ is real, in contradiction to the assumption that Δ_2 has no real zeros. This contradiction shows that $C(\mathbb{R})$ has two connected components $C(\mathbb{R})^{\pm}$, and that $\pi_1|_{C(\mathbb{R})^{\pm}}$ is an isomorphism from $C(\mathbb{R})^{\pm}$ onto $\mathbb{P}^1(\mathbb{R})$.

Finally, if Δ_2 has four zeros in $\mathbb{P}^1(\mathbb{R})$, then the set of points in $\mathbb{P}^1(\mathbb{R})$ where $\Delta_1 \geq 0$ is the union of two disjoint closed intervals $I_1^{\pm}$, over each of which π_1 exhibits $C(\mathbb{R})$ as a twofold branched covering, where the branches come together over the endpoints of the interval. It follows that $C(\mathbb{R})$ has two connected components ${\pi_1}^{-1}(I_1^{\pm}) \cap C(\mathbb{R})$, each isomorphic to a circle.

It follows that $C(\mathbb{R}) = \emptyset$ if and only if we are in case (a), and $C(\mathbb{R})$ has one connected component if and only if we are in case (b). This completes the classification of Lemma 2.6.3. □

Remark 2.6.4. If p is symmetric in the sense that $p(x,\ y) \equiv p(y,\ x)$, then $\Delta_2 = \Delta_1$, and the cases (c) and (d) do not occur. ⊘

Remark 2.6.5. If all the coefficients A_{ij} of p in (2.4.1) are real, then the Weierstrass coefficients $g_2 = D(\Delta_k(p))$ and $g_3 = -E(\Delta_k(p))$ in Corollary 2.4.7 are real, as well as the invariants $X = X(p)$ and $Y = Y(p)$ in Proposition 2.5.6.

The Weierstrass curve W defined by (2.3.6) always has the point at infinity $[0 : 0 : 1]$ as a real point. This is a smooth point of the real curve $W(\mathbb{R}) := W \cap \mathbb{P}^2(\mathbb{R})$, which implies that the affine part $W \cap \mathbb{R}^2$ of $W(\mathbb{R})$ always has an unbounded connected component.

The affine part is given by the equation $y^2 = f(x) := 4\, x^3 - g_2\, x - g_3$. The stationary points of f are $x = \pm(g_2/12)^{1/2}$, which are real if and only if $g_2 \geq 0$. At the real stationary points f takes the values $-(\pm\, 3^{-3/2}\, {g_2}^{-3/2} + g_3)$, and it follows that $\{x \in \mathbb{R} \mid f(x) \geq 0\}$ consists of two intervals if and only if $|g_3| < 3^{-3/2}\, {g_2}^{3/2}$,

or equivalently $\Delta = {g_2}^3 - 27\,{g_3}^2 > 0$. Therefore, if W is smooth, that is, $\Delta \neq 0$, then $W(\mathbb{R})$ has one or two connected components if and only if $\Delta < 0$ or $\Delta > 0$, respectively. Each connected component of the real projective curve $W(\mathbb{R})$ is isomorphic to a circle. If $\Delta > 0$, one of the connected components is bounded in the affine plane $\mathbb{R}^2$, where the bounded component lies to the left of the unbounded one in the $(x,\ y)$-plane.

If $\Delta > 0$ and $(g_2,\ g_3)$ approaches a point $(g_2^0,\ g_3^0)$ on the curve $\Delta = 0$ with $g_3^0 > 0$, then the bounded component shrinks to an isolated point of $W(\mathbb{R})$, located at $x = -(g_2^0/12)^{1/2}$, $y = 0$, where the function $y^2 - f(x)$ has a stationary point with a positive definite second-order derivative matrix. When crossing the curve $\Delta = 0$ to $\Delta < 0$, the bounded component disappears, while the unbounded component survives. This is called the *elliptic bifurcation*.

If $\Delta > 0$ and $(g_2,\ g_3)$ approaches a point $(g_2^0,\ g_3^0)$ on the curve $\Delta = 0$ with $g_3^0 < 0$, then the bounded component gets attached to the unbounded component at the point $x = -(g_2^0/12)^{1/2}$, $y = 0$, where the function $y^2 - f(x)$ has a stationary point with an indefinite nondegenerate second-order derivative matrix. In other words, the curve $W(\mathbb{R})$ has a real normal crossing at the singular point. When crossing the curve $\Delta = 0$ to $\Delta < 0$, the smooth curve $W(\mathbb{R})$ is connected, with a bulge at the place of the previous bounded component. This is called the *hyperbolic bifurcation*.

In an elliptic surface with a real structure, the elliptic and hyperbolic bifurcations correspond to singular fibers of Kodaira type I_b, $b \geq 1$, that are respectively elliptic and hyperbolic with respect to the real structure. See Section 8.3. Let $p_1(z)$ denote the real period, in its dependence on the real parameter z that describes on which curve we are, and let $z = z_0$ correspond to the singular curve when $\Delta(z_0) = 0$. In the elliptic case, $p_1(z)$ extends to a real analytic function of z on a neighborhood of $z = z_0$, which we can also see by writing $p_1(z)$ as the integral of the complex one-form $y^{-1}\,\mathrm{d}x$ over the connected component of $W(\mathbb{R})$ that passes through the point at infinity, which in the elliptic case remains smooth and depends in a real analytic fashion on z for all z in a neighborhood of $z = z_0$. In the hyperbolic case, $p_1(z)$ has logarithmic behavior, meaning that there is a constant $C > 0$ such that $p_1(z)$ is asymptotically equal to $C\,\log(1/|z - z_0|)$ as $z \to z_0$. In all other cases, when $g_2(z_0) = g_3(z_0) = 0$, the Kodaira type of the singular fiber is not equal to I_b, $b \geq 1$. See for instance Table 6.3.2.

Returning to our biquadratic curve C defined by the equation $p = 0$, suppose that we have $b \in C(\mathbb{R})$. By a suitable real translation, we can arrange that $b = ([1 : 0],\ [1 : 0])$, where the coefficients of the new biquadratic polynomial again are real. The isomorphism from C onto the Weierstrass curve D of Lemma 2.4.13 is real in the sense that it maps real points to real points, and therefore it defines an injective real analytic mapping from $C(\mathbb{R})$ to $W(\mathbb{R})$. The restriction of this mapping to a connected component of $C(\mathbb{R})$ is an isomorphism from that connected component onto a corresponding connected component of $W(\mathbb{R})$.

However, in general it is not true that the real biquadratic curve $C(\mathbb{R})$ is isomorphic to the real Weierstrass curve $W(\mathbb{R})$, because it can happen that $C(\mathbb{R}) = \emptyset$, whereas $W(\mathbb{R})$ always has a nonempty component passing through the point at infinity. It follows from Lemma 2.6.3 that if $\Delta < 0$, then the real biquadratic curve is isomorphic

to the real Weierstrass curve, and both are connected. If $\Delta > 0$ when the real Weierstrass curve has two connected components, then the real biquadratic curve is empty, or isomorphic to the real Weierstrass curve. ⊘

For the numerical computation of the period p, see Remark 2.5.5, where the reference to Remark 2.4.11 takes the following form. Using the affine coordinate $x \in \mathbb{R}$ on $\mathbb{P}^1(\mathbb{R})$ such that $x_0 = 1$, $x_1 = x$, the primitive positive real period $p = p_1$ is given in the cases (c) and (e) by the complete elliptic integral

$$p = \int_{-\infty}^{\infty} \Delta_2(x)^{-1/2}\, \mathrm{d}x. \tag{2.6.4}$$

Note that the integral converges, because $\Delta_2(x)^{-1/2} = \mathrm{O}(|x|^{-2})$ for $x \to \pm\infty$.

In the cases (b), (d) and (f), assume that the intervals I_1 and $I_1^{\pm}$ are of the form $[a,\ b]$ for $a,\ b \in \mathbb{R}$, $a < b$, where we leave it to the reader to formulate the results if the interval on the real projective line $\mathbb{P}^1(\mathbb{R})$ contains the point at infinity. Then $\Delta_2(x) > 0$ for $a < x < b$, whereas $\Delta_2(x)$ has simple zeros at a and b. In this case the primitive positive real period $p = p_1$ is equal to the complete elliptic integral

$$p = 2 \int_a^b \Delta_2(x)^{-1/2}\, \mathrm{d}x. \tag{2.6.5}$$

Note that the integrand is singular at the endpoints a and b, where $\Delta_2(x) = 0$, but the integral converges because the zeros of $\Delta_2(x)$ at the endpoints are simple.

Remark 2.6.6. In cases (b), (d) and (f), we have two disjoint intervals I_1^- and I_1^+ on which $\Delta_1 \geq 0$. It follows from the description in Section 2.6.1 of the connected components of $C(\mathbb{R})$ as subsets of $\mathbb{C}/P$ that the integrals of the complex one-form $\mathrm{d}t$ over the two connected components $C(\mathbb{R})^{\pm}$ of $C(\mathbb{R})$ are both equal to the primitive positive real period p_1. It follows that the integral of $\Delta_2(x)^{-1/2}$ over I_1^- is equal to the integral of $\Delta_2(x)^{-1/2}$ over I_1^+. ⊘

For a rotation $t + \mathbb{Z}\, p \mapsto t + T + \mathbb{Z}\, p$ in the circle $\mathbb{R}/p\,\mathbb{Z}$, the *rotation number* is the number

$$\rho := T/p \in \mathbb{R}/\mathbb{Z}. \tag{2.6.6}$$

If the points $(x^0,\ y^0)$ and $(x^1,\ y^1)$ in the $(x,\ y)$-plane corresponding to $0 + \mathbb{Z}\, p$ and $T + \mathbb{Z}\, p$ are given explicitly, and lie on the same connected component of $C(\mathbb{R})$, then it has been indicated in Remarks 2.5.5 and 2.4.11 how the number T can be approximated numerically and written as an incomplete elliptic integral, respectively. Combined with the computation of the period p, this leads to a numerical computation of the rotation number T/p.

2.6.3 The Beukers–Cushman Monotonicity Criterion

Let p^0 and p^1 be biquadratic polynomials with real coefficients. For $z \in \mathbb{R}$ such that the biquadratic curve $C_{[1:-z]}$ defined by the equation $p_z := p^0 - z\,p^1 = 0$ is smooth and has real points, we have the primitive positive real period $p = p(z)$, as in Section 2.6.2. Note that the discriminant $\Delta(z)$ of $x \mapsto (\Delta_k(p_z))(x)$ has real coefficients. On every open real z-interval I between zeros of Δ, the structure of the real locus $C_{[1:-z]}(\mathbb{R})$ does not change, and $p(z)$ is a real analytic function of $z \in I$, which moreover satisfies the homogeneous Picard–Fuchs equation $\mathcal{L}\,p = 0$, where $\mathcal{L}$ is defined by (2.5.10). The respective coefficients $c_1(z)$ in (2.5.11) and $c_2(z)$ in (2.5.12) of $\mathcal{L}$ are rational functions of z that are explicitly computable in terms of the Weierstrass data $g_2(z)$ and $g_3(z)$ of the biquadratic polynomials; see Corollary 2.5.10.

Assume that for $z \in I$, the QRT transformation τ on $C_{[1:-z]}(\mathbb{R})$ preserves the connected components of $C_{[1:-z]}(\mathbb{R})$ if the latter has two connected components. If τ interchanges the two connected components, see Lemma 2.6.3, then $\tau^2 = \tau \circ \tau$ preserves the connected components, and we adapt the following analysis to τ^2 instead of τ.

Let $T(z)$ be the real number such that the QRT transformation on $C_{[1:-z]}$ is equal to the flow after time $T(z)$ of the vector field v_z, as at the end of Section 2.5.3. By adding a suitable integral multiple of $p(z)$, we can arrange that $0 < T(z) < p(z)$. Then the rotation number $\rho(z) = T(z)/p(z)$ depends in a real analytic way on $z \in I$ and takes values between 0 and 1. It is interesting to study the behavior of the *rotation function* $\rho : I \to]0,\ 1[$. For instance, the set of all $z \in I$ such that the restriction of the QRT transformation to $C_{[1:-z]}$ is periodic with period k is equal to the set of all $z \in I$ such that there exists an integer l, $0 < l < k$, such that $\rho(z) = l/k$.

Beukers and Cushman [16] proved the monotonicity of the rotation function ρ on one of the intervals I for the Lyness map, see Section 11.4, by means of the inhomogeneous Picard–Fuchs equation $\mathcal{L}T = \mu$ for the function $T = T(z)$. Their method, described in the next paragraph, can be used to analyze the monotonicity behavior of the rotation function on every interval I such that one knows the sign and pole behavior on I of $\mu(z)$ and $P(z)$. For each pair p^0, p^1 of biquadratic polynomials the rational functions $\mu(z)$ and $P(z)$ can be computed explicitly, see Proposition 2.5.20 and formula (2.6.8), but in general the determination of their sign and pole behavior on the given interval I may be not such a straightforward matter.

The derivative $\rho'(z)$ of the rotation function has the same sign as the function

$$\sigma(z) := T'(z)\,p(z) - T(z)\,p'(z) = \rho'(z)\,p(z)^2. \tag{2.6.7}$$

The equations $\mathcal{L}\,T = \mu$, and $\mathcal{L}\,p = 0$, imply that $\sigma' = T''\,p - T\,p'' = (-c_1\,T' - c_2\,T + \mu)\,p - T\,(-c_1\,p' - c_2\,p) = -c_1\,\sigma + \mu\,p = (P'/P)\,\sigma + \mu\,p$, where we have substituted (2.5.11) in the last identity. This implies the equation

$$(\sigma/P)' = \mu\,p/P, \tag{2.6.8}$$

where $P = 3\,\Gamma/2\,\Delta$ and $\Gamma = -3\,g_3\,g_2' + 2\,g_2\,g_3'$; see (2.5.8) and (2.5.9). If Γ has no zeros in I, then P has no zeros in I, hence has a constant sign in I, and the derivative of $\sigma/|P|$ has the same sign as μ in I. If μ has no zeros of odd order in I, then, disregarding the finitely many zeros of μ, the sign of μ, equal to the sign of $(\sigma/|P|)'$, is constant on I. In particular, $f = \sigma/|P|$ is strictly increasing on I if μ is nonnegative on I. If $z_0 \in I$ and $\rho'(z_0) = 0$, then $f(z_0) = 0$, and $f(z)$ converges to an element of $[-\infty,\ 0[$ and $]0,\ \infty]$ if z approaches the left or right endpoint of I, respectively. It follows that $\rho'(z)$ has no zeros on I, and has the same sign as μ on I if the function $\sigma(z)/P(z)$ converges to zero if z approaches one of the endpoints of I. Note that the same conclusion holds if the functions $\rho'(z)$ and $\mu(z)$ have the same sign for z near the left right endpoint of I, or opposite signs for z near the right endpoint of I.

According to Table 6.3.2, we have a singular fiber over $z = \zeta$ of type I_b, $b > 0$, if and only if Δ has a zero of order b, and g_2 and g_3 are not equal both to zero at ζ. It follows that

$$\Gamma = (3\,\Delta\,g_2' - \Delta'\,g_2)/27\,g_3 = (2\,\Delta\,g_3' - \Delta'\,g_3)/{g_2}^2 \tag{2.6.9}$$

has a zero of order $b-1$ at ζ, and therefore $P = 3\,\Gamma/2\,\Delta$ has a pole of order one at ζ. Because for an elliptic fiber of type I_b both $p(z)$ and $T(z)$ extend to real analytic functions on a neighborhood of ζ, see the proof of Lemma 8.4.1, the function $\sigma(z)$ in (2.6.7) extends to a real analytic function on a neighborhood of ζ, and we conclude that $\sigma(z)/P(z)$ converges to zero as $z \to \zeta$ in the case of an elliptic singular fiber of type I_b with $b > 0$.

Beukers and Cushman [16] considered the Lyness map for $a > 0$ when there is a z_+ such that $\Delta(z_+) = 0$ and Δ, μ, and p have no zeros on $]z_+,\ \infty[$; see Table 11.4.5. The singular fiber over z_+ is of type I_1 and elliptic; see Table 11.4.3. Furthermore, μ is negative on $]z_+,\ \infty[$ when $a > 1$ and positive on $]z_+,\ \infty[$ when $0 < a < 1$. This completes the proof of Beukers and Cushman of Conjecture 1 of Zeeman [214], stating that the rotation map of the Lyness map on the interval $]z_+,\ \infty[$ is strictly decreasing when $a > 1$ and increasing when $0 < a < 1$. If $a = 1$, the Lyness map is periodic of period 5 and the rotation function is constant. In Section 11.4 we determine the monotonicity behavior of the rotation function of the Lyness map on all regular intervals and for all values of a, and we also apply the Beukers–Cushman criterion to some other QRT maps in Chapter 11.

Remark 2.6.7. The elliptic singular fibers are not the only ones for which $\sigma(z)/P(z) \to 0$ as $z \to \zeta$. If the singular fiber is hyperbolic of type I_b, $b > 0$, then $P(z)$ still has a simple pole at $z = \zeta$, but now we are in case (2) of Lemma 8.2.1. It follows that there is an $a \in \mathbb{R}_{>0}$ such that $p(z) = \mathrm{O}(\log(1/|z-\zeta|))$ and $p'(z) \sim -a/|z-\zeta|$ as $z \to \zeta$, and therefore $\sigma(z) \sim a\,f(0)/|z-\zeta|$ as $z \to \zeta$. It follows that $\sigma(z)/P(z) \to 0$ as $z \to \zeta$ if $f(0) = 0$, whereas $\sigma(z)/|P(z)|$ converges to a nonzero real number with the same sign as $f(0)$ when $f(0) \neq 0$.

For the other types, (2.6.9) and Table 6.3.2 lead to the following conclusions about the degree of the function $P(z)$ at $z = \zeta$. For type I_0^* we have $\deg P \geq -1$. For type I_b^* with $b > 0$ we have $\deg P = -2$. For the types II, III, and IV we have

$\deg P \geq -1$, with equality if and only if $\deg g_2 = 1$, $\deg g_3 = 2$, and $\deg g_2 = 2$, respectively. For types II*, III*, and IV*, we have $\deg P \geq -2$, with equality if and only if $\deg g_2 = 4$, $\deg g_3 = 5$, and $\deg g_2 = 3$, respectively. Note that for all types other than I_b and I_b^*, equality occurs if and only if the degrees of g_2 and g_3 at $z = \zeta$ are the minimal ones for the given types.

It follows from Table 6.2.39 that for all the types other than I_b there is an $\alpha \in \mathbb{Q}$, $-1 < \alpha < 0$, such that $p(z)$ and $p'(z)$ are $\mathrm{O}(|z - \zeta|^{\alpha-1})$ as $z \to \zeta$, and hence $\sigma(z) = \mathrm{O}(|z - \zeta|^{\alpha-1})$ as $z \to \zeta$, and therefore $\sigma(z)/P(z) \to 0$ as $z \to \zeta$ if $\deg P = -2$. This occurs if and only if we have the type I_b^*, $b > 0$, II* with $\deg g_2 = 4$, III* with $\deg g_3 = 5$, or IV* with $\deg g_2 = 3$. ⊘

The following lemma implies that there is an upper bound for the number of oscillations of the rotation function, where the same upper bound holds for all QRT maps. What is the lowest one?

Lemma 2.6.8 *For the QRT map, there are at most* 32 *zeros of the derivative* ρ' *of the rotation function in any open interval* I *such that* Δ *has no zeros in* I.

Proof. Let n denote the number of zeros of ρ' in I. Let I_j, $1 \leq j \leq k$, denote the successive open subintervals of I that are separated by $k - 1 = c$ zeros of Γ. If $a, b \in I_j$, $a < b$, $\rho'(a) = \rho'(b) = 0$, then $\sigma(a)/(P(a) = \sigma(b)/P(b) = 0$, and it follows from the variational principle that $(\sigma/P)'$ has a zero in $]a, b[$, where (2.6.8) implies that this is a zero of μ. Therefore the number of zeros of ρ' in I_j is ≤ 1 plus the number of zeros of μ in I_j. It follows that $n \leq k$ plus the number m of zeros of μ in $I + k - 1$, that is, $n \leq m + 2c + 1$. Because the degrees of Γ and Δ are ≤ 8 and ≤ 12, respectively, it follows from Proposition 2.5.20 that μ has at most 15 zeros, whereas $c \leq 8$. Therefore $n \leq 15 + 16 + 1 = 32$. □

Chapter 3
The QRT surface

In Section 2.5 the QRT mapping τ is described as a translation on each solution curve $C = C_{[z_0:\, z_1]}$ of (2.5.3) when $C_{[z_0,\, z_1]}$ is a smooth, hence elliptic, curve in $\mathbb{P}^1 \times \mathbb{P}^1$. This already gives quite detailed information on the QRT mapping. For instance, $C \simeq \mathbb{C}/P$ for a lattice $P = P_{[z_0:\, z_1]}$ in $\mathbb{C}$, and $\tau|_C : t + P \mapsto t + \mu + P$ for some $\mu = \mu([z_0 : z_1]) \in \mathbb{C}$. Therefore the nth iterate τ^n of τ acts on C by translation over $n\,\mu + P$, and as a consequence, we had the Poncelet porism, Corollary 2.5.4.

However, the main point of the QRT map in Chapter 1 is that *these transformations of the curves $C_{[z_0:\, z_1]}$, $[z_0 : z_1] \in \mathbb{P}^1$, range together to a birational transformation of the plane*, given by the formulas (1.1.4), (1.1.5), (1.1.6). If $([x_0 : x_1], [y_0 : y_1])$ is the point in $\mathbb{P}^1 \times \mathbb{P}^1$ with the affine coordinates $x = x_1/x_0$, $y = y_1/y_0$, one substitutes $X = ({x_1}^2,\, x_0\, x_1,\, {x_0}^2)$ and $Y = ({y_1}^2,\, y_0\, y_1,\, {y_0}^2)$ in (1.1.3), and then $x_0' = f_1(y)\, x_0 - f_2(y)\, x_1$, $x_1' = f_0(y)\, x_0 - f_1(y)\, x_1$, and $y_0' = g_1(x')\, y_0 - g_2(x')\, y_1$, $y_1' = g_0(x')\, y_0 - g_1(x')\, y_1$.

However, in $\mathbb{P}^1 \times \mathbb{P}^1$ we had the problem that the curves $C_{[z_0:\, z_1]}$, $[z_0 : z_1] \in \mathbb{P}^1$, are not disjoint from each other, since these all meet at the base points. Moreover, the rational transformation τ is not determined at points where both the numerator and the denominator of some of their defining rational functions are equal to zero. These problems will be resolved by replacing the surface $\mathbb{P}^1 \times \mathbb{P}^1$ by a modified surface S, a connected, compact, complex analytic complex two-dimensional manifold, on which the curves $C_{[z_0:\, z_1]}$, $[z_0 : z_1] \in \mathbb{P}^1$, define a fibration, a partition of S into disjoint curves, and on which the QRT map is an everywhere defined complex analytic diffeomorphism of S.

We begin with a definition and investigation in Subsection 3.1 of an algebraic subvariety S^p of $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ that is the disjoint union of the curves $C_{[z_0:\, z_1]}$, $[z_0 : z_1] \in \mathbb{P}^1$. When S^p is smooth, as is often the case, it has all the desired properties, and we can take $S = S^p$. We will investigate the smoothness of S^p in Section 3.1.3.

However, there are also examples occurring in the literature for which S^p is not smooth, and in these cases S^p has to be further "desingularized" by means of a succession of blowing-up procedures. After the introduction in Section 3.2.1 of the technique of blowing up, we will show in Section 3.3 that successively blowing up

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_3,
© Springer Science+Business Media, LLC 2010

base points (eight times) leads to a smooth surface S with the desired properties. Moreover, S is a minimal resolution of singularities of the singular subvariety S^p of $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$; see Section 3.3.5.

3.1 The surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$

3.1.1 Definition of the Surface

Let $(z_0,\ z_1) \neq (0,\ 0)$ and let C be the subset of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p_{(z_0,\, z_1)} = 0$, as in (2.5.3), (2.5.4). The algebraic closedness of $\mathbb{C}$ implies that C is not empty. The linear independence of A^0 and A^1, which we assume throughout this book, implies that $A = z_0 A^0 + z_1 A^1 \neq 0$, and therefore the biquadratic polynomial $p = p_{(z_0,\, z_1)} = z_0\, p^0 + z_1\, p^1$ is not identically equal to zero. Therefore C is a curve, a nonempty one-dimensional algebraic subset of $\mathbb{P}^1 \times \mathbb{P}^1$. The curve C does not change if we multiply z_0 and z_1 by a common factor, and therefore it depends only on the element $[z_0 : z_1]$ of the complex projective line $\mathbb{P}^1$. Therefore we write $C = C_{[z_0 : z_1]}$. When the divisor $\mathrm{Div}(p)$, see the beginning of Section 2.4, has irreducible components of multiplicity > 1, then the algebraic set C is replaced by the divisor $\mathrm{Div}(p)$. The family of divisors $C_{[z_0 : z_1]}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, parametrized by $[z_0 : z_1] \in \mathbb{P}^1$, is called a *pencil of biquadratic curves* in $\mathbb{P}^1 \times \mathbb{P}^1$.

For each $u \in \mathbb{P}^1 \times \mathbb{P}^1$, let Z_u denote the set of all $(z_0,\ z_1) \in \mathbb{C}^2$ such that $p_{(z_0,\, z_1)} = 0$ at u. The fact that the polynomial p in (2.5.3) is linear in $(z_0,\ z_1)$ implies that Z_u is a linear subspace of $\mathbb{C}^2$, where $\dim Z_u \geq 1$ because the condition that $p_{(z_0,\, z_1)} = 0$ at u corresponds to only one linear equation in $(z_0,\ z_1)$. There are only two cases:

(i) $\dim Z_u = 1$, when u is contained in exactly one member of the pencil, the curve C_{Z_u}.
(ii) $Z_u = \mathbb{C}^2$, when u is contained in every member of the pencil.

Note that (ii) occurs if and only if both p^0 and p^1 vanish at u, in which case u is called a *base point of the pencil*. Let B denote the set of all the base points of the pencil. Because B is the intersection of any two distinct members of the pencil, the base locus B is an algebraic set in $\mathbb{P}^1 \times \mathbb{P}^1$, where $\dim B \leq 1$, because $\dim C \leq 1$ for every member C of the pencil.

If we view the polynomial p in (2.5.3) as a polynomial in all the variables $(z_0,\ z_1)$ and $((x_0,\ x_1),\ (y_0,\ y_1))$, then it is trihomogeneous of tridegree $(1,\ (2,\ 2))$. That is, for each $(z_0,\ z_1)$ it is a biquadratic polynomial in $((x_0,\ x_1),\ (y_0,\ y_1))$, whereas for each $((x_0,\ x_1),\ (y_0,\ y_1))$ it is homogeneous of degree one (= linear) in $(z_0,\ z_1)$. It follows that the equation $p = 0$ defines an algebraic surface $S = S^p$ in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$, which therefore is a projective algebraic set, see the end of Section 2.2. Note that the p in (2.5.3) are the general trihomogeneous polynomial functions of tridegree $(1,\ (2,\ 2))$ on $\mathbb{C}^2 \times (\mathbb{C}^2 \times \mathbb{C}^2)$.

Let π_1 denote the restriction to S of the projection $([z_0 : z_1], ([x_0 : x_1], [y_0 : y_1])) \mapsto [z_0 : z_1]$ from $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ onto the first component $\mathbb{P}^1$. Then π_1 is an algebraic map from S to $\mathbb{P}^1$. For each $[z_0 : z_1] \in \mathbb{P}^1$, the fiber $S_{[z_0:z_1]} := (\pi_1)^{-1}(\{[z_0 : z_1]\})$ is of the form $\{[z_0 : z_1]\} \times C_{[z_0:z_1]}$. It is in this way that S^p is viewed as the disjoint union of the curves $C_{[z_0:z_1]}$, $[z_0 : z_1] \in \mathbb{P}^1$, "obtained by adding $[z_0 : z_1] \in \mathbb{P}^1$ as a tag."

Let π_2 denote the restriction to S of the projection $([z_0 : z_1], ([x_0 : x_1], [y_0 : y_1])) \mapsto ([x_0 : x_1], [y_0 : y_1])$ from $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ onto the second component $\mathbb{P}^1 \times \mathbb{P}^1$. We recall that the rational function p^0/p^1 on $\mathbb{P}^1 \times \mathbb{P}^1$ is not defined in the set B of all base points, but that in $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus B$ the curve $C_{[z_0:z_1]}$ is equal to the level curve p^0/p^1 at the level $z = -z_1/z_0$. Write $\iota(z) = [1 : -z]$. It follows that the rational function $\iota \circ (p^0/p^1) \circ \pi_2$ on S, which is not defined at $(\pi^2)^{-1}(B)$, extends to the everywhere defined algebraic morphism $\pi_1 : S \to \mathbb{P}^1$. In this way the construction of $S = S^p$ removes the indeterminacies of the rational function p^0/p^1. Also note that the domain of definition S of π_1 is compact, in contrast to the noncompact affine plane $\mathbb{C} \times \mathbb{C}$, which was the domain of definition of the rational function $F = p^0/p^1$ in (1.1.7). In order to avoid confusion with the notation $z = -z_1/z_0$, which is used throughout this book, we will not abbreviate $[z_0 : z_1]$ by z, but by ζ.

It follows from the description of case (i) that the restriction to $S \setminus (\pi_1)^{-1}(B)$ of the surjective proper algebraic morphism $\pi_2 : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is an isomorphism from $S \setminus (\pi_1)^{-1}(B)$ onto $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus B$, with inverse equal to the algebraic morphism $u \mapsto (Z_u, u)$. On the other hand, for each $b \in B$ the fiber $(\pi_2)^{-1}(\{b\})$ of π_2 over b is equal to $\mathbb{P}^1 \times \{b\}$, which is a complex projective line. It follows that π_2 is a *modification* of $\mathbb{P}^1 \times \mathbb{P}^1$ in B. For the definition of modifications in the complex analytic category, see Section 6.2.13.

Let b be a base point. Then any two members C_ζ and $C_{\zeta'}$ of the pencil of biquadratic curves intersect each other at b. If C_ζ and $C_{\zeta'}$ are smooth, then the intersection number $C_\zeta \cdot_b C_{\zeta'}$ of C_ζ and $C_{\zeta'}$ at b does not depend on the choice of the smooth members of the pencil, and is called the *multiplicity* i_b *of the base point* b, where the letter i reminds us of the intersection number of C_ζ and $C_{\zeta'}$ at the point b.

Lemma 3.1.1 *Assume that p^0 and p^1 have no nonconstant common factors and that the pencil $\mathfrak{B}$ of biquadratic curves $z_0\, p^0 + z_1\, p^1 = 0$ has at least one smooth member. Then, counted with multiplicities, the pencil $\mathfrak{B}$ has eight base points in $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let D and D' be two distinct smooth members of $\mathfrak{B}$. Because D and D' are biquadratic, it follows from Corollary 2.4.2 that $D \cdot D' = 2 \cdot 2 + 2 \cdot 2 = 8$. Because D and D' are smooth, hence irreducible, $D \cdot D'$ is equal to the sum over all intersection points b of the intersection numbers $D \cdot_b D'$. The intersection points b of D and D' are the base points of $\mathfrak{B}$ with $D \cdot_b D'$ as their multiplicities. □

Figures 3.1.1 and 3.1.2 Figure 3.1.1 shows the pencil of real sine–Gordon curves, defined by the biquadratic polynomials (11.7.6) for $s = 1$ and $\vartheta = -3$, when there are eight real base points. The affine coordinates have been chosen to show all the

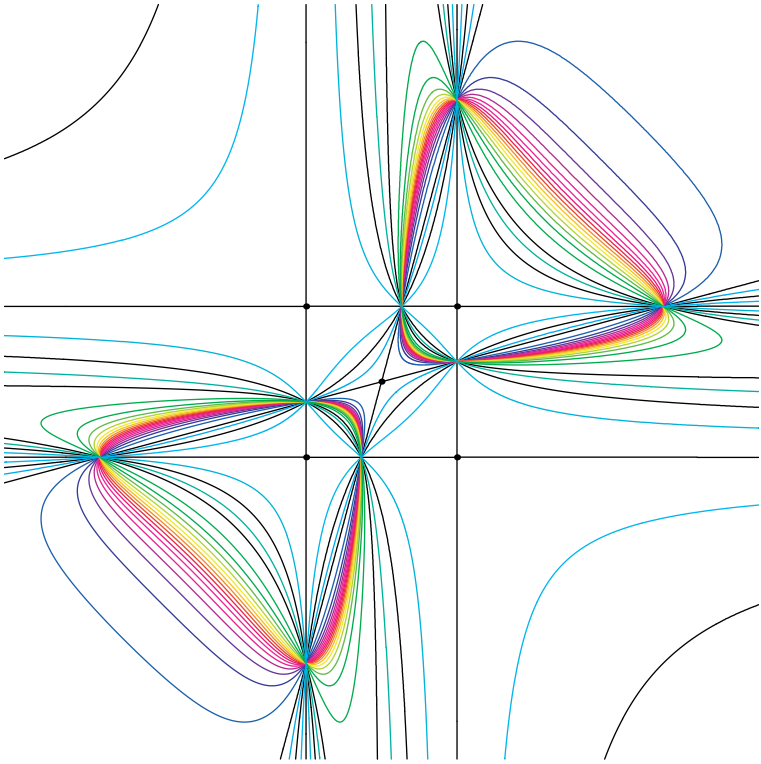


Fig. 3.1.1 Pencil of real sine–Gordon curves, with eight real base points.

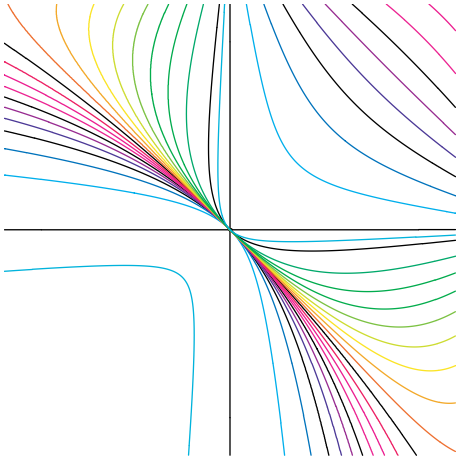


Fig. 3.1.2 The double base point of the Lyness pencil. See also Figure 3.3.1.

eight base points, of which two lie on the horizontal axis $y = \infty$ at infinity and two lie on the vertical axis $x = \infty$ at infinity.

Figure 3.1.2 shows the pencil of real Lyness curves defined by (11.4.3) for $a = 0.3$, near the point $([0 : 1], [0 : 1])$, which is a base point of the pencil of multiplicity two. The second-order contact at this base point between the curves of the pencil is clearly visible.

For the choice of the members of the real pencils that appear in the pictures, see the text under the heading "Figure 2.3.2," starting with "In all pictures in this book of real pencils …."

Remark 3.1.2. The eight base points in $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be chosen in arbitrary positions. Let M denote the space of all 3×3 matrices, and let W be the two-dimensional linear subspace of M spanned by the matrices A^0 and A^1 that appear in (1.1.3). Assume for simplicity that all the base points are simple, and let $(x^{(i)},\ y^{(i)})$, $1 \le i \le 8$, be an enumeration of the base points. The base points are characterized by the condition that $X^{(i)} \cdot A\, Y^{(i)} = 0$ for every $A \in W$; see (2.5.3). For each i, the equation $X^{(i)} \cdot A\, Y^{(i)} = 0$ is one linear equation for $A \in M$, equivalent to the condition that the matrix A is orthogonal to the rank-one matrix $X^{(i)} \otimes Y^{(i)}$. Therefore the set of all $A \in M$ such that $X^{(i)} \cdot A\, Y^{(i)} = 0$ for all $1 \le i \le 8$ is equal to the orthogonal complement O of the linear span of the matrices $X^{(i)} \otimes Y^{(i)}$. Because $\dim M = 9$ and $W \subset O$, where $\dim W = 2$, it follows that the matrices $X^{(i)} \otimes Y^{(i)}$, $1 \le i \le 8$, are linearly dependent. Because M is spanned by the matrices $X \otimes Y$ with $X = ({x_1}^2,\ x_0 x_1,\ {x_0}^2)$, $Y = ({y_1}^2,\ y_0 y_1,\ {y_0}^2)$, this leads to the conclusion that the points $(x^{(i)},\ y^{(i)})$, $1 \le i \le 8$, must satisfy an algebraic relation in order that they can be the base points of a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. ⊘

The following proposition shows in a way other than that in Remark 3.1.2 that the eight base points in $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be chosen arbitrarily. The proof of Proposition 3.1.3 is analogous to the proof of the more classical proposition 4.1.4.

Proposition 3.1.3 *Let C be a smooth biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and let D be a biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ not equal to C. If E is a biquadratic curve that passes through seven of the eight intersection points of D with C, where multiplicities are allowed, then E passes through the eighth intersection point of D with C. Furthermore, in this case E belongs to the pencil of biquadratic curves that contains C and D.*

Proof. Let r and s be biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ such that D and E are given by the zero-sets of r and s, respectively. Then s/r is a meromorphic function, bihomogeneous of degree zero, and therefore can be viewed as a meromorphic function on $\mathbb{P}^1 \times \mathbb{P}^1$. The restriction φ of s/r to C is a meromorphic function on C with divisor equal to $\sum_i P_i - \sum_i Q_i = \sum_i (P_i - Q_i)$, where the P_i, and the Q_i, for $1 \le i \le 8$, denote the intersection points of E and D with C, where multiplicities are allowed. Because C is an elliptic curve, it carries a nonzero holomorphic complex one-form ω, unique up to a constant factor, such that for each $O \in C$, the mapping that assigns to each $P \in C$ the integral of ω over a curve from O to P defines an isomorphism from C onto $\mathbb{C}/P$, where P is the period lattice. See Section 2.3.1.

Because $\sum_i (P_i - Q_i)$ is equal to the divisor of a rational function on C, it follows from Abel's theorem as formulated in Griffiths and Harris [74, p. 235], that the sum over all i of the integrals of ω over curves in C running from Q_i to P_i belongs to the period lattice P. In other words, the sum over all i of the P_i is equal to the sum over all i of the Q_i, viewed as an identity in the additive group $\mathbb{C}/P \simeq C$. If $P_i = Q_i$ for all $1 \leq i \leq 7$, then this implies that $P_8 = Q_8$.

Now assume that $P_i = Q_i$ for all $1 \leq i \leq 8$. Then φ is a holomorphic function on C, and because C is compact and connected, it follows from the maximum principle that φ is equal to a constant $c \in \mathbb{C}$. The equation $s/r = c$ on C is equivalent to $s - r\,c = 0$ on the zero-set of q. Because q has only simple zeros in $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$, it follows that the rational function $(s - c\,r)/q$ is holomorphic on $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$, which moreover is bihomogeneous of degree zero, and therefore defines a holomorphic function ψ on $\mathbb{P}^1 \times \mathbb{P}^1$. Because $\mathbb{P}^1 \times \mathbb{P}^1$ is compact and connected, it follows that ψ is equal to a constant $d \in \mathbb{C}$, and the conclusion is that $s = c\,r + d\,q$. In other words, E belongs to the pencil of biquadratic curves that contains C and D. □

3.1.2 Common Factors

In this subsection we give a short discussion of the case in which p^0 and p^1 have a nonconstant common factor.

Lemma 3.1.4 *The following conditions are equivalent:*

(i) *The base locus B has more than eight elements.*
(ii) *The biquadratic polynomials p^0 and p^1 have a nonconstant common factor q, bihomogeneous of bidegree (d_1, d_2) with $0 \leq d_1 \leq 2$, $0 \leq d_2 \leq 2$ and not $0 < d_1 + d_2 < 4$.*
(iii) *The curves $p^0 = 0$ and $p^1 = 0$ have a curve D in common.*
(iv) $\dim B = 1$.

If any of these conditions hold, then every member of the pencil of biquadratic curves is singular.

Proof. i) ⇒ (ii) follows from Lemma 3.1.1, where any common factor a of p^0 and p^1 is bihomogeneous in view of Lemma 2.4.1, applied to the curve $a = 0$. (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) are obvious.

Assume that p^0 and p^1 have a common nonconstant factor a. Then for each $[z_0 : z_1] \in \mathbb{P}^1$ the curve $C = C_{[z_0:z_1]}$ defined by the equation $z_0\,p^0 + z_1\,p^1 = 0$ has the fixed curve $a = 0$ as a component. It follows that $C_{[z_0:z_1]}$ is reducible. Because a reducible curve is singular at the intersection points of any two of its irreducible factors, it follows that $C_{[z_0:z_1]}$ is singular. □

Because the space of bihomogeneous polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ of bidegree (d_1, d_2) has dimension $(d_1 + 1)(d_2 + 1)$, the dimension of the space of pairs (p^0, p^1) such that $p^0 = a\,q^0$, $p^1 = a\,q^1$, where a has bidegree (d_1, d_2), is equal to

$$(d_1 + 1)\,(d_2 + 1) + 2\,(2 - d_1 + 1)\,(2 - d_2 + 1) - 1,$$

which is equal to 13, 13, 8, 11, 8, 9, 9, if $(d_1,\ d_2)$ is equal to $(1,\ 0)$, $(0,\ 1)$, $(2,\ 0)$, $(1,\ 1)$, $(0,\ 2)$, $(2,\ 1)$, $(1,\ 2)$, respectively. Therefore the set of pairs $(p^0,\ p^1)$ such that $\dim B = 1$ has codimension 5 in the 18-dimensional space of all pairs of biquadratic polynomials.

In none of the cases is there a QRT-like birational transformation of the plane.

Example 1. If both matrices A^0 and A^1 are antisymmetric, then

$$p_z = p = (x_1\,y_0 - x_0\,y_1)\,(A_{01}\,x_1\,y_1 - A_{20}\,(x_1\,y_0 + x_0\,y_1) + A_{12}\,x_0\,y_0),$$

where $A_{ij} = z_0\,A^0_{ij} + z_1\,A^1_{ij}$. Therefore these represent a six-dimensional linear subspace of the 11-dimensional (nonlinear) manifold of all the pairs $p^0,\ p^1$ of bidegree $(2,\ 2)$ that have a common factor of bidegree $(1,\ 1)$.

Each level curve is the union of the symmetry axis $x = y$ for the switch $x \leftrightarrow y$ and a pencil of bilinear curves. The first factor ι_1 in the QRT map, the horizontal switch, sends every point $(x,\ y)$ with $x \neq y$ to $(y,\ y)$. Therefore the QRT map is far from a birational transformation, because it decreases the dimension of the space.⊘

3.1.3 Smoothness of the Surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$

The point $(x,\ y) \in \mathbb{P}^1 \times \mathbb{P}^1$ is not a base point of the pencil of biquadratic curves if and only if the left-hand side of (2.5.3) is a nonzero linear form in $(z_0,\ z_1)$, which means that in affine coordinates the derivative of the left-hand side with respect to z is not equal to zero. If $\zeta \in \mathbb{P}^1$ is the unique solution of (2.5.3) for the given $(x,\ y) \in \mathbb{P}^1 \times \mathbb{P}^1$, then this condition in turn is equivalent to the statement that the surface S is smooth at the point $s = (\zeta,\ x,\ y)$ and the tangent space $\mathrm{T}_s S$ of S at s is not horizontal, in the sense that it does not contain the tangent space $\mathrm{T}_s(\{(x,\ y)\} \times \mathbb{P}^1) = \ker \mathrm{T}_s \pi$ at s of the fiber over $(x,\ y)$ of the projection $\pi_2 : (\zeta,\ x,\ y) \mapsto (x,\ y)$.

Therefore the only possible singular points of S are the points $(\zeta,\ x,\ y) \in S$ such that $(x,\ y)$ is a base point of the pencil of biquadratic curves.

Lemma 3.1.5 *Let $(x,\ y)$ be a base point of the pencil of biquadratic curves, that is, $\mathbb{P}^1 \times \{(x,\ y)\} \subset S$. Then the following conditions are equivalent:*

(i) *For every $\zeta \in \mathbb{P}^1$, $(\zeta,\ x,\ y)$ is a nonsingular point of S.*
(ii) *For every $\zeta \in \mathbb{P}^1$, $(x,\ y)$ is a nonsingular point of the curve C_ζ.*
(iii) *In affine coordinates, the 2×2 matrix*

$$\begin{pmatrix} \frac{\partial p^0(x,\,y)}{\partial x} & \frac{\partial p^0(x,\,y)}{\partial y} \\ \\ \frac{\partial p^1(x,\,y)}{\partial x} & \frac{\partial p^1(x,\,y)}{\partial y} \end{pmatrix}$$

is nonsingular.

(iv) *There are two points* ζ_0 *and* ζ_1 *in* $\mathbb{P}^1$ *such that* $(x,\ y)$ *is a smooth point of* C_{ζ_0} *and of* C_{ζ_1}*, and these curves intersect transversally at* $(x,\ y)$ *in the sense that* $\mathrm{T}_{(x,\,y)}C_{\zeta_0} \neq \mathrm{T}_{(x,\,y)}C_{\zeta_1}$.

(v) *Condition (ii) holds, and the mapping* $\zeta \mapsto \mathrm{T}_{(x,\,y)}C_\zeta$ *is an isomorphism from the complex projective line* $\mathbb{P}^1$ *onto the complex projective line* $\mathrm{P}(\mathrm{T}_{(x,\,y)}(\mathbb{P}^1 \times \mathbb{P}^1))$ *of all complex one-dimensional linear subspaces of the complex two-dimensional vector space* $\mathrm{T}_{(x,\,y)}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Proof. In affine coordinates we have that $(\zeta,\ x,\ y)$ is a smooth point of S if and only if $p(\zeta,\ x,\ y) = 0$ and not all the partial derivatives $\partial p(\zeta,\ x,\ y)/\partial\zeta, \partial p(\zeta,\ x,\ y)/\partial x$ and $\partial p(\zeta,\ x,\ y)/\partial y$ are equal to zero. Because the assumption that $(x,\ y)$ is a base point means that $p(\zeta,\ x,\ y) = 0$ for all ζ, hence $\partial p(\zeta,\ x,\ y)/\partial\zeta = 0$, this condition is equivalent to $(\partial p(\zeta,\ x,\ y)/\partial x,\ \partial p(\zeta,\ x,\ y)/\partial y) \neq (0,\ 0)$, which is equivalent to the condition that $(x,\ y)$ is a smooth point of C_ζ. This proves the equivalence between (i) and (ii).

With the notation (2.5.4), we have $p = z_0\, p^0 + z_1\, p^1$, whose derivative with respect to the $(x,\ y)$-variables is equal to $z_0\ \mathrm{d}p^0 + z_1\ \mathrm{d}p^1$. This derivative is nonzero for every $(z_0,\ z_1) \neq (0,\ 0)$ if and only if (iv) holds. This proves the equivalence between (ii) and (iii).

If $(x,\ y)$ is a smooth point of C_ζ, where $\zeta = [z_0 :\ z_1]$, that is, $z_0\ \mathrm{d}p^0 + z_1\ \mathrm{d}p^1 \neq (0,\ 0)$, then $\mathrm{T}_{(x,\,y)}C_\zeta$ is equal to the kernel of $z_0\ \mathrm{d}p^0 + z_1\ \mathrm{d}p^1$. If the common kernel for two different vectors $(z_0,\ z_1)$ is equal to zero, then (iii) holds. Conversely, if (iii) holds then for any two linearly independent vectors $(z_0,\ z_1)$ the common kernel is equal to zero, which means that the holomorphic mapping $\zeta \mapsto \mathrm{T}_{(x,\,y)}C_\zeta$ from $\mathbb{P}^1$ to $\mathrm{P}(\mathrm{T}_{(x,\,y)}(\mathbb{P}^1 \times \mathbb{P}^1))$ is injective, hence an isomorphism. This proves the implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), whereas (v) $\Rightarrow$ (ii) is obvious. □

We will call $(x,\ y)$ a *simple base point* of the pencil of biquadratic curves, if any of the conditions (i)–(v) in Lemma 3.1.5 holds.

Corollary 3.1.6 *The following statements are equivalent:*

(i) *The surface* S *in* $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ *defined by equation* (2.5.3) *is smooth.*

(ii) *For any base point* $(x,\ y)$ *of the pencil of biquadratic curves, any member of the pencil is smooth at* $(x,\ y)$.

(iii) *There are eight distinct base points* $(x,\ y)$ *in* $\mathbb{P}^1 \times \mathbb{P}^1$ *of the pencil of biquadratic curves.*

If the surface S in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ is smooth, then the restriction π_1 to S of the projection $(\zeta,\ (x,\ y)) \mapsto \zeta$ is a complex analytic mapping from S to $\mathbb{P}^1$, whose fibers are mapped to the biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ by means of the restriction π_2 to S of the projection $(\zeta,\ (x,\ y)) \mapsto (x,\ y)$. The mapping $\pi_1 : S \to \mathbb{P}^1$ is an elliptic fibration as defined in Section 6.1 if and only if the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ has a smooth member. Furthermore, it follows in this case from Corollary 3.4.2 that there is a complex analytic diffeomorphism τ^S from S to itself such that $\pi \circ \tau^S = \tau \circ \pi$ on the preimage under π in S of the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ at which the QRT mapping τ is well defined.

3.2 Blowing Up

3.2.1 Blowing Up a Surface at a Point

If the surface in Section 3.1 is not smooth, which often is the case in the explicit examples in the literature, then a resolution of singularities is needed in order to obtain a smooth surface. This will be done by means of the technique of blowing up, which we explain in this section. Actually, we will not use the surface in Section 3.1 as our point of departure, but start by blowing up the base points in $\mathbb{P}^1 \times \mathbb{P}^1$, repeating this procedure if in the new surface we still have base points of the proper transform of the pencil of curves until we arrive at a nonsingular surface in which the pencil of curves no longer has any base points. The relationship between this surface and the surface in Section 3.1 will be discussed in Ssction 3.3.5.

A continuous mapping f from a topological space A to a topological space B is called *proper* if for every compact subset K of B the preimage $f^{-1}(K)$ is a compact subset of A. For the definition of complex analytic spaces, see Section 6.2.13.

Lemma 3.2.1 *Let X and Y be complex analytic spaces and $f : X \to Y$ a proper complex analytic map. Assume that there is a dense open subset X_0 of X such that the restriction of f to X_0 is an open mapping from X_0 to Y. Furthermore, let Y be locally irreducible. Then the number of connected components of $f^{-1}(\{y\})$ is a lower semicontinuous function of $y \in Y$. Every fiber of f is connected if for a dense set of $y \in Y$ the fiber of f over y is connected.*

Proof. Because f is proper, each fiber of f is a compact complex analytic space and therefore has finitely many connected components. Let $y_0 \in Y$ and let C_i, $1 \leq i \leq n$, be the connected components of the fiber of f over y_0. There exist mutually disjoint open neighborhoods U_i of C_i in X, let U be the union of the U_i.

The properness of f implies that if K is a compact neighborhood of y_0 in Y, then $f^{-1}(K)$ is a compact subset of X, and therefore $f^{-1}(K) \setminus U$ is a compact subset of X. Since the intersection over all such K of the sets $f^{-1}(K) \setminus U$ is equal to $f^{-1}(\{y_0\}) \setminus U = \emptyset$, there exists a compact neighborhood K of y in Y such that $f^{-1}(K) \setminus U = \emptyset$, that is, $f^{-1}(K) \subset U$. Because Y is locally irreducible, there exists an irreducible open neighborhood Y_0 of y_0 in Y such that $Y_0 \subset K$; hence $f^{-1}(Y_0) \subset f^{-1}(K) \subset U$.

Write $V_i = f^{-1}(Y_0) \cap U_i$ when $C_i \subset f^{-1}(\{y_0\}) \cap U_i \subset V_i$ and the open subsets V_i of X are mutually disjoint. The restriction to V_i of f is a proper complex analytic mapping from V_i to Y_0, and it follows from Remmert's proper mapping theorem that $f(V_i)$ is a closed complex analytic subspace of Y_0. Because X_0 is dense in X, $X_0 \cap V_i \neq \emptyset$, and because $f|_{X_0}$ is an open mapping it follows that $f(X_0 \cap V_i)$ is a nonempty open subset of Y_0 that contains $f(V_i)$. Because Y_0 is irreducible, it follows that $f(V_i) = Y_0$. That is, for every $y \in Y$ and every $1 \leq i \leq n$ we have $f^{-1}(\{y\}) \cap V_i \neq \emptyset$, which implies that $f^{-1}(\{y\})$ has at least n connected components. □

If V is a two-dimensional vector space, then the *tautological line bundle* of the complex projective line $\mathrm{P}(V)$ is the line bundle τ of which the fiber over $l \in \mathrm{P}(V)$ is equal to the one-dimensional complex linear subspace l of V. For any nonzero

linear form ξ on V, the mapping $l \mapsto \xi|_l$ defines a holomorphic section of the dual bundle τ^* with a single simple zero at $l = \ker \xi$. It follows that $\deg(\tau^*) = 1$, and therefore $\deg(\tau) = -1$.

Theorem 3.2.2 *Let S be a complex two-dimensional complex analytic manifold, and let b be a point of S. Then there exist R and π with the following properties:*

(i) *R is a complex two-dimensional complex analytic manifold, and π is a proper complex analytic mapping from R to S.*
(ii) *If we write $E := f^{-1}(\{b\})$, then $R \setminus E$ is dense in E and the restriction to $R \setminus E$ of π is a complex analytic diffeomorphism from $R \setminus E$ onto $S \setminus \{b\}$. It follows from* (i) *and Lemma* 3.2.1 *that E is a compact connected complex analytic subset of R.*
(iii) *π is not a complex analytic diffeomorphism from R onto S, and E is irreducible.*

If R' and π' satisfy (i), (ii), *and* (iii) *with R and π replaced by R' and π', respectively, then there exists a unique continuous map $\Phi : R \to R'$ such that $\pi' \circ \Phi = \pi$, and Φ is a complex analytic diffeomorphism from R onto R'.*

Assume that (i), (ii), *and* (iii) *hold. Then the curve E is nonsingular, for every $e \in E$ the tangent mapping $\mathrm{T}_e\pi$ of π at e has rank one, and the mapping $\lambda : e \mapsto \mathrm{T}_e\pi(\mathrm{T}_e R)$ is a complex analytic diffeomorphism from E onto $\mathrm{P}(\mathrm{T}_b S)$. The restriction of $\mathrm{T}\pi$ to $\mathrm{T}_E R$ induces an isomorphism of the normal bundle $\mathrm{N}_R(E)$ of E in R onto the tautological line bundle of $\mathrm{P}(\mathrm{T}_b S)$. It follows that E is an embedded complex projective line in R with self-intersection number*

$$E \cdot E = \deg(\mathrm{N}_R(E)) = -1. \tag{3.2.1}$$

Proof. (Existence) Let U be an open neighborhood of the origin in $\mathbb{C}^2$. Let

$$I_U := \{(u, l) \in U \times \mathbb{P}^1 \mid u \in l\}$$

denote the *incidence relation* in $U \times \mathbb{P}^1$. Then $((x, y), [\xi : \eta]) \in I$ if and only if $x\eta - y\xi = 0$. Near a point where $\xi \neq 0$ we can take $\xi = 1$ and η as the affine coordinate in $\mathbb{P}^1$, and the incidence equation is equivalent to

$$y = x\eta. \tag{3.2.2}$$

This exhibits, near such points, I as a two-dimensional complex analytic submanifold of $U \times \mathbb{P}^1$, parametrized by (x, η). Near a point where $\eta \neq 0$ we can take $\eta = 1$ and ξ as the affine coordinate in $\mathbb{P}^1$, and the incidence equation is equivalent to

$$x = y\xi. \tag{3.2.3}$$

This exhibits, near such points, I as a two-dimensional complex analytic submanifold of $U \times \mathbb{P}^1$, parametrized by (y, ξ). The conclusion is that the whole of I_U is a closed two-dimensional complex analytic submanifold of the three-dimensional complex analytic manifold $U \times \mathbb{P}^1$. The projection $(u, l) \mapsto u$ is a proper complex analytic map from $U \times \mathbb{P}^1$ to U, and therefore its restriction π_U to I_U is a proper complex

analytic map from I_U to U. We have $E_U := {\pi_U}^{-1}(\{0\}) = \{0\} \times \mathbb{P}^1$, and the restriction of π_U to $I_U \setminus E_U$ is bijective from $I_U \setminus E_U$ onto $U \setminus \{0\}$, with inverse equal to the complex analytic mapping $u \mapsto (u, [u])$. It follows from (3.2.2) and (3.2.3) that the tangent map of π is given by $(\delta x, \delta\eta) \mapsto (\delta x, \eta\,\delta x + x\,\delta\eta)$ and $(\delta y, \delta\xi) \mapsto (\xi\,\delta y + y\,\delta\xi, \delta y)$, which has rank equal to 2, unless $x = 0$, hence $y = x\,\eta = 0$ and $y = 0$, and hence $x = y\,\xi = 0$, respectively, when the rank is equal to 1. We conclude that (i), (ii), and (iii) hold with S, p, R, π, and E replaced by U, 0, I_U, π_U, and E_U, respectively.

There exists a local coordinatization, a complex analytic diffeomorphism χ from an open neighborhood S_0 of b in S onto an open neighborhood U of the origin in $\mathbb{C}^2$, such that $\chi(b) = 0$. Let S^χ denote the union of the surface I_U with $S \setminus \{b\}$, where each point $(u, l) \in I_U \setminus E_U$ is identified with the point $\chi^{-1}(u) \in S_0 \setminus \{b\}$. Because $\chi^{-1} \circ \pi_U$ is a complex analytic diffeomorphism from $I_U \setminus E_U$ onto $S \setminus \{b\}$, the set S^χ has a unique structure of a complex analytic manifold such that $S \setminus \{b\}$ and I_U are open subsets of it. One says that S^χ is *the complex analytic manifold obtained from S and I_U by gluing $S_0 \setminus \{b\}$ and $I_U \setminus E_U$ together by means of the gluing map* $\chi^{-1} \circ \pi_U$. In this gluing process, the mappings $\chi^{-1} \circ \pi_U : I_U \to S$ and the identity in $S \setminus \{b\}$ are glued together to a complex analytic mapping $\pi^\chi : S^\chi \to S$ that has the properties (i), (ii), (iii), because $\pi_U : I_U \to U$ has these properties.

(Uniqueness) We follow Hopf [90, §3, No. 1, 2]. Suppose that $\pi : R \to S$ satisfies (i) and (ii). Let $\chi = (s_1, s_2)$ be a holomorphic system of local coordinates on an open neighborhood S_0 of b in S such that $\chi(b) = 0$, when $U := \chi(S_0)$ is an open neighborhood of the origin in $\mathbb{C}^2$. By shrinking S_0 if necessary, we can arrange that U is an open ball in $\mathbb{C}^2$ with center at the origin. Then the zero-set Z_j of $r_j := s_j \circ \pi$ is a complex analytic subset of $\pi^{-1}(S_0)$ that contains E. The restriction to $Z_j \setminus E$ of the holomorphic function r_k, where $k \neq j$, is a complex analytic diffeomorphism from $Z_j \setminus E$ onto $D \setminus \{0\}$, where D is an open disk in $\mathbb{C}$ with center at the origin. It follows that the closure Z'_j of $Z_j \setminus E$ in $R_0 := \pi^{-1}(S_0)$ is an irreducible component of Z_j such that $Z'_j \cap E$ is finite, and $r_k|_{Z'_j}$ is a proper complex analytic mapping from Z'_j onto D, of which the restriction to $Z'_j \setminus E = Z_j \setminus E$ is complex analytic diffeomorphism from $Z'_j \setminus E$ onto $D \setminus \{0\}$. Lemma 3.2.1 implies that the fiber $Z'_j \cap E$ over 0 consists of a single point e_j. Using the properness of $r_k|_{Z'_j}$ it follows that the bijective mapping $r_k|_{Z'_j}$ has a continuous inverse $\gamma : D \to Z'_j$. Because γ is a complex analytic as a mapping from $D \setminus \{0\}$ to R_0, with a continuous extension to D such that $\gamma(0) = e_j$, it follows from Riemann's theorem on removable singularities that γ is complex analytic as a mapping from D to R_0, with $Z'_j = \gamma(D)$. Because $r_k \circ \gamma$ is the identity on D, the chain rule of differentiation implies that $\mathrm{T}_{e_j} r_k \circ \mathrm{T}_0\gamma = 1$; hence $\mathrm{T}_0\gamma \neq 0$, the curve $Z'_j = \gamma(D)$ is nonsingular at $\gamma(0) = e_j$, and $\mathrm{T}_{e_j} r_k \neq 0$ on the tangent space $\mathrm{T}_{e_j} Z'_j$ of Z'_j at e_j. The definition of Z_j implies that $r_j = 0$ on $Z_j \supset Z'_j$; hence $\mathrm{T}_{e_j} r_j = 0$ on $\mathrm{T}_{e_j} Z'_j$. If $e_j = e_k =: e$, then $\mathrm{T}_e r_2 \neq 0$ on $\mathrm{T}_e Z'_1$, $\mathrm{T}_e r_1 = 0$ on $\mathrm{T}_e Z'_1$, $\mathrm{T}_e r_1 \neq 0$ on $\mathrm{T}_e Z'_2$, and $\mathrm{T}_e r_2 = 0$ on $\mathrm{T}_e Z'_2$; hence the linear forms $\mathrm{T}_e r_1$ and $\mathrm{T}_e r_2$ are linearly independent. Because $\mathrm{T}_e r_j = \mathrm{T}_b s_j \circ \mathrm{T}_e\pi$ it follows that $\mathrm{T}_e\pi$ is bijective, and the inverse mapping theorem implies that e is an isolated point of E. Because

E is connected, we conclude that $E = \{e\}$ and $\pi : R \to S$ is a complex analytic diffeomorphism.

Assume that π is not a diffeomorphism. Applying a linear substitution of variables to (s_1, s_2), we conclude from the previous paragraph that for each $[a : b] \in \mathbb{P}^1$ the closure $Z'_{[a:b]}$ of the complement in R_0 of the zero-set of $a\,r_1 + b\,r_2$ is a holomorphically embedded curve in R_0 that intersects E at a unique point $e_{[a:b]}$, where $e_{[a:b]} \neq e_{[c:d]}$ if $a\,d - b\,c \neq 0$. Because the connected complex analytic subset E of R has at least two elements, we have $\dim_{\mathbb{C}} E > 0$, whereas $\dim_{\mathbb{C}} E < 2$ because $R \setminus E$ is dense in R. Therefore E is a curve. Let $e \in E$. If t is the greatest common divisor of the germs at e of r_1 and r_2, then $r_1 = t\,g_1$, $r_2 = t\,g_2$, and the germs g_1 and g_2 at e of holomorphic functions are mutually prime. Because E is the common zero-set of r_1 and r_2, we have $t \neq 0$ and $(g_1, g_2) \neq (0, 0)$ in the complement of E, and therefore $[r_1 : r_2] = [g_1 : g_2]$ is a holomorphic mapping from the complement of E near e to $\mathbb{P}^1$. The germ at e of E is equal to the zero-set of a germ at e of a holomorphic function f. Let f_i denote the finitely many irreducible factors of f. If $a, b, c, d \in \mathbb{C}$, $a\,d - b\,c \neq 0$, and f_i is a factor of both $a\,g_1 + b\,g_2$ and $c\,g_1 + d\,g_2$, then f_i is a factor of both g_1 and g_2, a contradiction. It follows that for every i the set P_i of all $[a : b] \in \mathbb{P}^1$ such that f_i is a factor of $a\,g_1 + b\,g_2$ contains at most one element, the union over all i of the P_i is a finite subset of $\mathbb{P}^1$, and its complement contains at least two elements. That is, there exist $a, b, c, d \in \mathbb{C}$ such that $a\,d - b\,c \neq 0$, and for every i the germ f_i is a factor neither of $a\,g_1 + b\,g_2$ nor of $c\,g_1 + d\,g_2$. In other words, neither of the zero-sets of $a\,g_1 + b\,g_2$ and $c\,g_1 + d\,g_2$ contains an irreducible component of the germ of E at e. If $g_1(e) = g_2(e) = 0$, then both zero-sets are germs at e of curves passing through e and contained in the respective zero-sets of $a\,r_1 + b\,r_2$ and $c\,r_1 + d\,r_2$; hence $e_{[a:b]} = e = e_{[c:d]}$, a contradiction. Therefore $(g_1(e), g_2(e)) \neq (0, 0)$. It follows that $[r_1 : r_2] = [g_1 : g_2]$ has an extension to a holomorphic mapping from an open neighborhood of e in R to $\mathbb{P}^1$. Because this holds for every $e \in E$, the holomorphic mapping $[r_1 : r_2] : R_0 \setminus E \to \mathbb{P}^1$ has an extension to a holomorphic mapping from R_0 to $\mathbb{P}^1$. In the notation of the existence proof this means that the complex analytic diffeomorphism ${\pi_U}^{-1} \circ \pi : R_0 \setminus E \to I_U \setminus (\{0\} \times \mathbb{P}^1)$ has an extension to a holomorphic mapping $\psi : R_0 \to I_U$, where the identity $\pi_U \circ \psi = \pi$ on the dense open subset $R_0 \setminus E$ of R_0 extends by continuity to the identity $\pi_U \circ \psi = \pi$ on R_0. If K is a compact subset of I_U, then $\pi_U(K)$ is a compact subset of U; hence the properness of π implies that $\psi^{-1}(K) = (\pi_U \circ \psi)^{-1}(\pi_U(K)) = (\chi \circ \pi)^{-1}(\pi_U(K))$ is a compact subset of R_0. This proves that the mapping ψ is proper. Summarizing, there is a unique continuous mapping $\psi : R_0 \to I_U$ such that $\pi_U \circ \psi = \pi$, the mapping ψ is complex analytic and proper, and the restriction of ψ to $R_0 \setminus E$ is a complex analytic diffeomorphism from $R_0 \setminus E$ onto $I_U \setminus (\{0\} \times \mathbb{P}^1)$.

Finally, assume that (i), (ii), and (iii) hold, when the curve E is irreducible. Because $I_U \setminus (\{0\} \times \mathbb{P}^1)$ is dense in I_U, the properness of ψ implies that $\psi : R_0 \to I_U$ is surjective, and hence $\psi(E) = \{0\} \times \mathbb{P}^1$, where $\psi^{-1}(\{0\} \times \mathbb{P}^1) = E$. Each fiber of ψ is either a point or a curve, and because E is irreducible, each fiber of ψ over a point of $\{0\} \times \mathbb{P}^1$ is either a point or equal to E. Since the occurence of the latter would imply that $\psi(E)$ is a point, the conclusion is that each fiber of ψ over a point of $\{0\} \times \mathbb{P}^1$ is a

point. Because ψ is injective on $R_0 \setminus E$, it follows that $\psi : R_0 \to I_U$ is bijective. Since a bijective proper mapping has a continuous inverse, $\psi^{-1} : I_U \to R_0$ is continuous, its restriction to the dense open subset $I_U \setminus (\{0\} \times \mathbb{P}^1)$ is complex analytic, and it follows from Riemann's theorem on removable singularities that $\psi^{-1} : I_U \to R_0$ is complex analytic, that is, $\psi : R_0 \to I_U$ is a complex analytic diffeomorphism. The diffeomorphisms $\psi : R_0 \to I_U$ and $\pi|_{R\setminus E} \to S \setminus \{b\}$ glue together to a diffeomorphism $\Phi : S \to S^\chi$ such that $\pi^\chi \circ \Phi = \pi$, where $\pi^\chi : S^\chi \to S$ is the model obtained in the existence proof.

(Coordinate invariant properties) It suffices to prove the properties with $\pi : R \to S$ replaced by $\pi_U : I_U \to U$, where $E = \{0\} \times \mathbb{P}^1$. Let $l \in \mathbb{P}^1$ and write $e = (0, l)$. Because $\pi_U(\{0\} \times \mathbb{P}^1) = \{0\}$, the tangent map at e of π_U is equal to zero on $\mathrm{T}_e E$ and equal to the identity on $l \times \{0\} \subset \mathrm{T}_e E$, and therefore induces a linear isomorphism from $\mathrm{T}_e I_U / \mathrm{T}_e E$ onto $\mathrm{T}_0 U$, where $\mathrm{T}_e I_U / \mathrm{T}_e E$ is the fiber over e of the normal bundle of E in I_U. This proves that λ is a complex analytic diffeomorphism from E onto $\mathrm{P}(\mathrm{T}_b S)$ and that $\mathrm{T}\pi$ induces an isomorphism from $\mathrm{N}_R(E)$ onto the tautological bundle of $\mathrm{P}(\mathrm{T}_b S)$. The first identity in (3.2.1) follows from (2.1.25). □

The mapping $\pi : R \to S$ in Theorem 3.2.2 is called *the blowing up of S at the point b*, where the definite article "the" refers to the uniqueness modulo complex analytic diffeomorphisms Φ. Other names are *σ-process, dilatation, monoidal, quadratic, or quadric transformation*. The modern name "blowing up," the classical name "dilatation," and the name "σ-process" introduced by Hopf [90] all refer to the replacement of b by a projective line. Here the letter σ refers to the real two-dimensional sphere equal to the complex projective line. The adjectives "monoidal," "quadratic," and "quadric" refer to the substitutions of variables (3.2.2) and (3.2.3) that relate the local coordinates (x, y) near b in S to the respective local coordinates (x, η) and (y, ξ) near E in R. The fiber $E := \pi^{-1}(\{b\})$ of $\pi : R \to S$ over the point b is called the *exceptional fiber*. If in Theorem 3.2.2 the condition that E is irreducible is dropped, then Lemma 6.2.50 implies that π is a composition of n blowups, where n is the number of irreducible components of E.

Figure 3.2.1 illustrates a real version of the blowing-up procedure by means of the *helicoid*, a ruled surface described by a straight horizontal line that rotates at a constant angular speed around a fixed vertical axis, where at the same time the intersection point with the vertical axis moves up at a constant speed. The vertical coordinate is counted modulo the height at which the horizontal line has made a half turn when its projection onto the horizontal plane has returned to its starting position, but with reversed orientation.

The real locus of R corresponds to the helicoid, where the vertical coordinate is counted modulo the height at which the horizontal line has made a half turn. This real surface is isomorphic to the *Möbius strip*. The projection π corresponds to the vertical projection of the helicoid onto the horizontal plane, and the fiber E over the blowing-up point corresponds to the vertical axis in the helicoid modulo the aforementioned height, which is isomorphic to the real projective line $\mathbb{P}^1(\mathbb{R}) \simeq$ a circle.

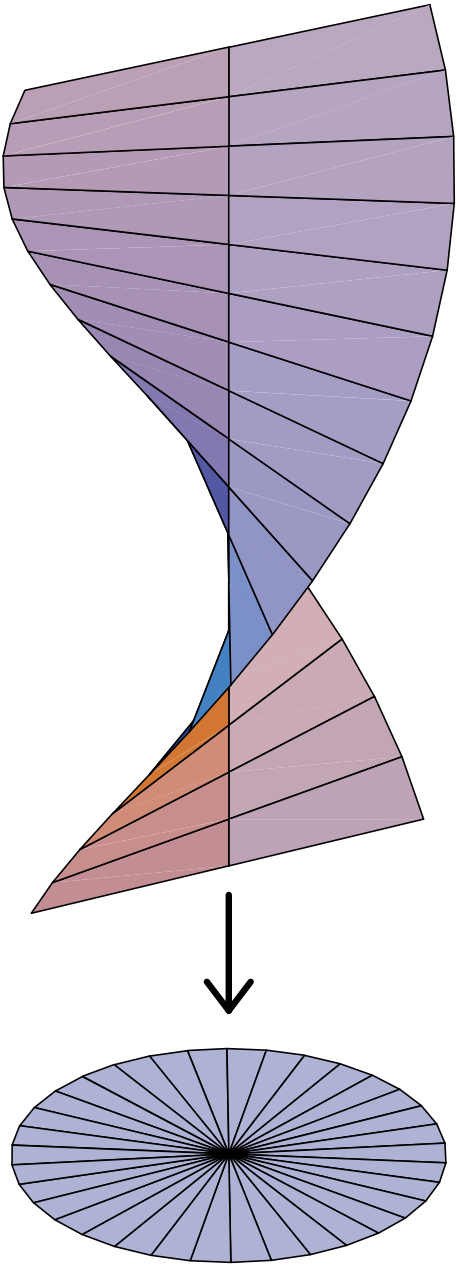


Fig. 3.2.1 Real blowing up: a Möbius strip.

Lemma 3.2.3 *Let $\pi : R \to S$ be a blowing up. Then R is a complex projective algebraic surface if and only if S is a complex projective algebraic surface.*

For a proof, see Griffiths and Harris [74, p. 192].

Let C be any complex analytic curve in a complex analytic surface S. The curve C is called a *rational curve* if it is smooth and complex analytic diffeomorphic to $\mathbb{P}^1$. The curve C is called an *exceptional curve of the first kind* or a -1 *curve* if it is rational and satisfies $C \cdot C = -1$. If $\pi : R \to S$ is a blowing up of S at the point $b \in S$, then Theorem 3.2.2 implies that $\pi^{-1}(\{b\})$ is a -1 curve in R. The following converse is called the *Castelnuovo–Enriques criterion*.

Theorem 3.2.4 *Let C be a -1 curve in R. Then there is a smooth surface S and a blowing up $\pi : R \to S$ at a point $b \in S$ such that $C = \pi^{-1}(\{b\})$. If R is projective algebraic, then S is complex analytic diffeomorphic to a projective algebraic surface.*

See Griffiths and Harris [74, p. 476] or Iskovskikh and Shafarevich [94, §6.3, Theorem 5], in the category of projective algebraic surfaces. In the category of complex analytic surfaces, Theorem 3.2.4 follows from Grauert [70, §4, No. 8]. For another proof, see Kodaira [109, II, Appendix on pp. 624–626].

The assumption that C is a -1 curve means that $C \simeq \mathbb{P}^1$ and the degree of the conormal bundle $\mathrm{N}(C)^*$ of C in R is equal to one, which in turn is equivalent to the statement that the space of all holomorphic sections over C of $\mathrm{N}(C)^*$ is complex two-dimensional. Theorem 3.2.4 is equivalent to the statement that every for every holomorphic section α over C of $\mathrm{N}(C)^*$ there are an open neighborhood U of C in R and a holomorphic function $f : U \to \mathbb{C}$ such that $f|_C = 0$ and $(\mathrm{d}f)|_C = \alpha$. Here, for each $c \in C$, the linear form α_c on $\mathrm{T}_c R/\mathrm{T}_c C$ is identified with the linear form $v \mapsto \alpha_c(v + \mathrm{T}_c C)$ on $\mathrm{T}_c R$ that vanishes on $\mathrm{T}_c C$. If f_1 and f_2 are holomorphic functions on an open neighborhood U of C in R such that $f_1 = f_2 = 0$ on C and the sections $\mathrm{d}f_1|_C$ and $\mathrm{d}f_2|_C$ of $\mathrm{N}(C)^*$ are linearly independent, then, shrinking U if necessary, the holomorphic mapping $(f_1,\ f_2) : U \to \mathbb{C}^2$ is a model of π over an open neighborhood S_0 of b in S, where $(f_1,\ f_2)$ form a holomorphic local coordinate system on S_0 that is equal to zero at the point b.

The following lemma describes how blowing up changes the topology of a surface.

Lemma 3.2.5 *In the situation of Theorem* 3.2.2*, if $r_0 \in R \setminus E$ and $s_0 = \pi(r_0)$, then π_* is an isomorphism from the fundamental group $\pi_1(R,\ r_0)$ of R onto the fundamental group $\pi_1(S,\ s_0)$ of S. In particular, if S is simply connected, then R is simply connected.*

If $\iota_{R\setminus E}$ is the identity $R \setminus E \to R$, $\psi = (\pi|_{R\setminus E})^{-1}$, and $i_{S\setminus\{b\}}$ is the identity $S \setminus \{b\} \to S$, then the homomorphisms $(i_{S\setminus\{b\}})_* : \mathrm{H}_2(S \setminus \{b\},\ \mathbb{Z}) \to \mathrm{H}_2(S,\ \mathbb{Z})$ and $\psi_* : \mathrm{H}_2(S \setminus \{b\},\ \mathbb{Z}) \to \mathrm{H}_2(R \setminus E,\ \mathbb{Z})$ are isomorphisms, the homomorphism $(\iota_{R\setminus E})_* : \mathrm{H}_2(R \setminus E,\ \mathbb{Z}) \to \mathrm{H}_2(R,\ \mathbb{Z})$ is injective, and

$$\mathrm{H}_2(R,\ \mathbb{Z}) = (\iota_{R\setminus E})_* \circ \psi_* \circ (i_{S\setminus\{b\}})_*{}^{-1}(\mathrm{H}_2(S,\ \mathbb{Z})) \oplus \mathbb{Z}\ \mathrm{H}(E). \tag{3.2.4}$$

For each $j,\ k \in \mathbb{Z}_{\geq 0}$, pullback by means of π defines a linear isomorphism π^* from the space of all holomorphic complex j-forms on S onto the space of all holomorphic

complex j-forms on R, and from the space of all holomorphic sections of $\mathrm{K}_S{}^k$ onto the space of all holomorphic sections of $\mathrm{K}_R{}^k$.

Proof. Because E has real codimension 2 in R, any loop β in R based at r_0 is homotopic by a small deformation to such a loop β' in R that misses E when the loop $\gamma = \pi \circ \beta'$ in S misses b. If γ is contractible in S to s_0, then because $\{b\}$ has real codimension 4 in S, the homotopy in S, which has real dimension 2, can be deformed to one in $S \setminus \{b\}$, and its image under ψ is a contraction of β' in $R \setminus E$ to the point r_0. This shows that the homomorphism π_* between the fundamental groups is injective. Because any loop γ in S is homotopic to a loop γ' that misses b, $\psi \circ \gamma'$ is a loop in R such that γ' is the image under π. This shows that π_* is surjective.

For the proof of (3.2.4), let U be a small open ball U around b in S when $U \setminus \{b\}$ is contractible to a three-dimensional sphere S^3, and $V = \pi^{-1}(U)$ is a tubular neighborhood of E in R that is contractible to $E \simeq \mathbb{P}^1 \simeq \mathrm{S}^2$. Because R is equal to the union of the open subsets $R \setminus E$ and V, whose intersection is equal to $V \setminus E$, we have the *Mayer–Vietoris exact sequence*

$$\cdots \to \mathrm{H}_2(V \setminus E, \mathbb{Z}) \to \mathrm{H}_2(R \setminus E, \mathbb{Z}) \oplus \mathrm{H}_2(V, \mathbb{Z}) \to \mathrm{H}_2(R, \mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_1(V \setminus E, \mathbb{Z}) \to \cdots,$$

see for instance Bott and Tu [22, p. 188]. Because π is a diffeomorphism from $V \setminus E$ onto $U \setminus \{b\}$, we have $\mathrm{H}_2(V \setminus E, \mathbb{Z}) \simeq \mathrm{H}_2(U \setminus \{b\}) \simeq \mathrm{H}_2(\mathrm{S}^3, \mathbb{Z}) = 0$. Furthermore, π is a diffeomorphism from $R \setminus E$ onto $S \setminus \{b\}$, and because $\{b\}$ has real codimension 4 in S we have that the injection from $S \setminus \{b\}$ into S induces an isomorphism from $\mathrm{H}_2(S \setminus \{b\}, \mathbb{Z})$ onto $\mathrm{H}_2(S, \mathbb{Z})$, whereas $\mathrm{H}_2(V, \mathbb{Z}) \simeq \mathrm{H}_2(E, \mathbb{Z}) = \mathbb{Z}\,\mathrm{H}(E)$. Finally $\mathrm{H}_1(V \setminus E, \mathbb{Z}) \simeq \mathrm{H}_1(U \setminus \{b\}, \mathbb{Z}) \simeq \mathrm{H}_1(\mathrm{S}^3, \mathbb{Z}) = 0$, and the displayed part of the Mayer–Vietoris sequence becomes $0 \to \mathrm{H}_2(R \setminus E, \mathbb{Z}) \oplus \mathbb{Z}\,\mathrm{H}(E) \to \mathrm{H}_2(R, \mathbb{Z}) \to 0$.

Because the mapping π is holomorphic, pullback by π defines linear mappings $\pi^* : \mathrm{H}^0(S, \Omega^j) \to \mathrm{H}^0(R, \Omega^j)$ and $\pi^* : \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S{}^k)) \to \mathrm{H}^0(R, \mathcal{O}(\mathrm{K}_R{}^k))$.

(Here it is essential that $k \geq 0$; see Proposition 3.3.7 for $k = -1$.) These mappings are injective because if $\pi^*\theta = 0$, then the fact that $\pi|_{R\setminus E}$ is a complex analytic diffeomorphism from $R \setminus E$ onto $S \setminus \{b\}$ implies that $\theta = 0$ on $S \setminus \{b\}$; hence $\theta = 0$, because $S \setminus \{b\}$ is dense in S and θ is continuous. Furthermore, if μ is a holomorphic section over R, then $((\pi|_{R\setminus E})^{-1})^*\mu$ is a holomorphic section over $S \setminus \{b\}$, which in view of Hartogs's theorem, see Griffiths and Harris [74, p. 7], extends to a holomorphic section θ over S. Because $\mu = \pi^*\theta$ on $R \setminus E$ and $R \setminus E$ is dense in R, it follows by continuity that $\mu = \pi^*\theta$. This proves that π^* is surjective. □

3.2.2 Total and Proper Transforms of Curves

Let $\pi : R \to S$ be a blowing up of S at a point $b \in S$, and let C be a curve in S. If $b \notin C$, then the preimage of C in R under π is isomorphic to C, but if

$b \in C$, then the preimage contains the -1 curve E as an additional component. If $f = 0$ is the local equation for the curve C in S, then $\pi * (f) = f \circ \pi = 0$ is the local equation in R for the preimage under π of C. For this reason the preimage under π of C in R is denoted by $\pi^*(C)$ and is called the *total transform* of C in the blown up manifold R. The *proper transform* $\pi'(C)$ of C in R is defined as the closure in R of $\pi^{-1}(C \setminus \{b\})$. If $(a_0,\ a_1)$ are local coordinates near b such that b has coordinates $(0,\ 0)$, and $f(a_0,\ a_1) = 0$ is the local defining equation for C, then we write $\mathrm{ord}_b(C) = m$ if the function f has a zero of order m at the origin. In terms of (3.2.2), we have $(\pi^*(f))(a_1,\ \alpha_1) = f((-\alpha_1\, a_1,\ a_1)) = {a_1}^m\, g(a_1,\ \alpha_1)$ where $\alpha_1 \mapsto g(0,\ \alpha_1)$ does not vanish identically, and the strict transform $\pi'(C)$ is defined by the equation $g = 0$, where $g = f/{a_1}^m$. We have $m = 1$ if and only if C is smooth at b.

Lemma 3.2.6 *If the curve C is smooth at b, then the proper transform $\pi'(C)$ of C intersects E at a unique point e, $\pi'(C)$ is smooth at e, and the restriction to $\pi'(C)$ of π defines a complex analytic diffeomorphism from $\pi'(C)$ onto C.*

It follows furthermore that the total transform $\pi^*(C)$ is equal to the union of the strict transform $\pi'(C)$ and the -1 curve E counted with multiplicity m. In terms of divisors, see Section 2.1.2, this is written as

$$\pi^*(C) = \pi'(C) + \mathrm{ord}_b(C)\, E. \tag{3.2.5}$$

If C' is another curve in S, then the intersection numbers of the total transforms in R satisfy

$$\pi^*(C) \cdot \pi^*(C') = C \cdot C', \tag{3.2.6}$$

because π is a proper mapping of degree one. We also have $\pi^*(C) \cdot E = \pi^*(C) \cdot \pi^*(\{b\}) = C \cdot \{b\} = 0$, and therefore

$$\pi'(C) \cdot E = \mathrm{ord}_b(C), \tag{3.2.7}$$

in view of (3.2.5) and (3.2.1). Combining (3.2.6) with (3.2.7) and (3.2.1), we obtain

$$\pi'(C) \cdot \pi'(C') = C \cdot C' - \mathrm{ord}_b(C)\ \mathrm{ord}_b(C'). \tag{3.2.8}$$

I found these statements in Iskovskikh and Shafarevich [94, Section 6.1]. For intersection numbers, see Section 2.1.6.

3.2.3 The Line Bundle $\pi_m^* L$

Let L be a holomorphic line bundle over S and let $m \in \mathbb{Z}_{>0}$. The pullback π^*L of L by π, defined at the beginning of Section 6.2.7, is a holomorphic line bundle over R such that the fiber $(\pi^*L)_r$ of π^*L over the point $r \in R$ is canonically isomorphic to the fiber L_s of L over the point $s = \pi(r)$. For each open subset U of S, the mapping $\lambda \mapsto \lambda \circ \pi$ is a linear isomorphism from the space of all holomorphic sections

of L over U onto the space of all holomorphic sections of π^*L over $\pi^{-1}(U)$; see Lemma 2.1.3.

If L_E is a holomorphic line bundle over R such that $[L_E] = \delta(E)$, then the holomorphic line bundle $\pi_m^*L := (\pi^*L) \otimes {L_E}^{-m}$ over R satisfies

$$[\pi_m^*L] = [\pi^*L]\,\delta(E)^{-m}. \tag{3.2.9}$$

This equation determines π_m^*L uniquely up to isomorphisms of holomorphic line bundles over R.

According to Lemma 2.1.2, the line bundle L_E has a holomorphic section s_E such that $\mathrm{Div}(s_E) = E$. That is, the zero-set of s_E is equal to E, along which s_E has simple zeros. Multiplication by ${s_E}^m$ defines a holomorphic map from π_m^*L to π^*L that for each $r \in R \setminus E$ is a linear isomorphism from $(\pi_m^*L)_r$ onto $(\pi^*L)_r = L_{\pi(r)}$, and for each $e \in E$ is equal to zero on $(\pi_m^*L)_r$. If we write $\iota_m(u) = u\, s_E(u)^m \in L_s$ if $u \in (\pi_m^*L)_r$ and $s = \pi(r)$, then ι_m is a holomorphic map from π_m^*L to L with the following properties:

(i) For each $r \in R \setminus E$, the restriction of ι_m to the fiber of π_m^*L over r is a linear isomorphism from the fiber of π_m^*L over r onto the fiber of L over $\pi(r)$.
(ii) For each open subset U of S, the equation $\iota_m \circ f_m = f \circ \pi$ defines an isomorphism $f_m \mapsto f$, also denoted by ι_k, from the space of all holomorphic sections of π_m^*L over $\pi^{-1}(U)$ onto the space of all holomorphic sections of L over U that vanish to order $\geq m$ at b.

Recall the isomorphism $\lambda : E \to \mathrm{P}(\mathrm{T}_b S)$ in Theorem 3.2.2. For each $l \in \mathrm{P}(\mathrm{T}_b\, S)$, let $\mathrm{O}(m)_l \otimes L_b$ denote the one-dimensional space of all homogeneous L_b-valued polynomials of degree m on the one-dimensional linear subspace l of $\mathrm{T}_b S$. See Example 5 for the notation $\mathrm{O}(m)$. The $\mathrm{O}(m)_l \otimes L_b$, $l \in \mathrm{P}(\mathrm{T}_b\, S)$, form a holomorphic line bundle $\mathrm{O}(m) \otimes L_b$ over $\mathrm{P}(\mathrm{T}_b\, S)$. The restriction to E of the line bundle π_m^*L is canonically isomorphic to $\lambda^*(\,\mathrm{O}(m) \otimes L_b)$. If f is a local holomorphic section of L that has a zero of order $\geq m$ at b, and f_m is the holomorphic section of π_m^*L such that $\iota_m \circ f_m = f \circ \pi$, then for each $e \in E$, $f_m(e)$ is equal to the restriction to $\lambda(e)$ of the Taylor expansion of order m of f at the point b.

3.2.4 Proper Transforms of Pencils

Now assume that L is a holomorphic line bundle over a compact complex analytic surface S, F is a two-dimensional vector space of global holomorphic sections of L, and $\mathfrak{P}$ is the pencil of divisors in S defined by F as is Section 2.1.3. We also assume that the set B of all base points of $\mathfrak{P}$, the common zeros of the elements of F, is zero-dimensional, hence finite. Finally, b is one of the base point of $\mathfrak{P}$, and $\pi : R \to S$ is the blowing up of S at b.

If $\mathrm{ord}_b(f)$ denotes the order of the zero of f at b, let $m = m_b(F) := \min_{f \in F} \mathrm{ord}_b(f)$. Then $m \in \mathbb{Z}_{>0}$ and $\mathrm{ord}_b(f) \geq m$ for every $f \in F$. According to Section 3.2.3, the equation $\iota_m \circ f_m = f \circ \pi$ defines a linear isomorphism $\pi' : f \mapsto f_m$ from F onto a two-dimensional vector space $\pi'(F)$ of global holo-

morphic sections of $\pi_m^* L$. Note that π' is equal to the restriction to F of the inverse of the mapping ι_m in Section 3.2.3. If $f \in F$ and $e \in E$, then $\pi'(f)_e$ is equal to the restriction to $\lambda(e) \in \mathrm{P}(\mathrm{T}_b\, S)$ of the Taylor expansion of order m of f at the base point b. We denote the isomorphism from $\mathrm{P}(F)$ onto $\mathrm{P}(\pi'(F))$ induced by the linear isomorphism $\pi' : F \to \pi'(F)$ also by π'. The pencil $\pi'(\mathfrak{P}) = \mathrm{P}(\pi'(F))$ of divisors in R defined by $\pi'(F)$ is called the *proper transform of the pencil* $\mathfrak{P}$; see Griffiths and Harris [74, p. 476].

Lemma 3.2.7 *With the above notation, we have the following conclusions:*

(i) *Let $f \in F$ and $f \neq 0$. Then $Z_{[\pi'(f)]} = \pi'(Z_{[f]}) + (\operatorname{ord}_b(f) - m)\, E$. In particular, $Z_{[\pi'(f)]} = \pi'(Z_{[f]})$ if and only if $\operatorname{ord}_b(f) \leq \operatorname{ord}_b(g)$ for every nonzero $g \in F$.*

(ii) *The multiplicity $\mathrm{i}_b^{\mathfrak{P}}$ of the base point b of the pencil $\mathfrak{P}$ is $\geq m \cdot \max_{f \in F,\, f \neq 0} \operatorname{ord}_b(f)$. The sum of the multiplicities of the base points on E of the pencil $\pi'(\mathfrak{P})$ is equal to $\mathrm{i}_b^{\mathfrak{P}} - m^2$, where $\pi'(\mathfrak{P})$ has no base points in E if and only if $\mathrm{i}_b^{\mathfrak{P}} = m^2$.*

(iii) *$e \in E$ is a base point of $\pi'(\mathfrak{P})$ if and only if for every (for two linearly independent) $f \in F$ the restriction to e of the Taylor expansion of order k of f at b is equal to zero.*

(iv) *If f^0 and f^1 form a basis of F, and $g^0,\ g^1 \in \pi'(F)$ are such that $f^0 = \iota_m(g^0)$ and $f^1 = \iota_m(g^1)$, then $(f^0/f^1) \circ \pi = g^0/g^1$ on $R \setminus E$.*

Proof. (i) It follows from (3.2.5) and (3.2.9) that $\pi'(Z_{[f]}) = \pi^*(Z_{[f]}) - \operatorname{ord}_b(f)\, E$ and $\pi^*(Z_{[f]}) = Z_{[\pi^*(f)]} = Z_{[\pi'(f)]} + m\, E$.

(ii) If $C = Z_{[f]}$, $C' = Z_{[f']}$, $\operatorname{ord}_b(f) = \operatorname{ord}_b(f') = m$, and $[f] \neq [f']$, then it follows from (i) and (3.2.8) that $Z_{[\pi'(f)]} \cdot Z_{[\pi'(f')]} = \pi'(Z_{[f]}) \cdot \pi'(Z_{[f']}) = Z_{[f]} \cdot Z_{[f']} - m^2$. This proves the second statement in (ii). On the other hand, if $\operatorname{ord}_b(f) > k$ and $\operatorname{ord}_b(f') = k$, then it follows from (i) that $i_b^{\mathfrak{P}} - m^2 = Z_{[\pi'(f)]} \cdot Z_{[\pi'(f')]} \geq (\operatorname{ord}_b(f) - m)\, E \cdot Z_{[\pi'(f')]} = (\operatorname{ord}_b(f) - m)\, m$, which proves the first statement in (ii).

(iii) It follows from the definition of the map $\pi' = {\iota_m}^{-1}|_F$ that for each $f \in F$ and $e \in E$, $\pi'(f)_e = 0$ if and only if the restriction to $\lambda(e)$ of the Taylor expansion of order m of f at b is equal to zero. The point e is a base point of $\mathfrak{P}'$ if and only if this happens for every $f \in F$ if and only if this happens for two linearly independent elements f of F.

(iv) This is because over $R \setminus E$ the line bundles $\pi_m^* L$ and $\pi^* L$ are canonically isomorphic. □

It follows from Lemma 3.2.7 (ii) that the number of base points of $\pi'(\mathfrak{P})$, counted with multiplicities, is $m^2 > 1$ less than the number of base points of $\mathfrak{P}$, counted with multiplicities. Therefore, if we continue blowing up base points and passing to the proper transforms of the pencils, we will arrive at a modification $\pi : \widehat{S} \to S$ of S at the set B of all base points of $\mathfrak{P}$, and an iterated proper transform $\widehat{\mathfrak{P}}$ of $\mathfrak{P}$ which is a pencil of curves in $\widehat{S}$ without base points. It follows that for any basis f^0, f^1 of F, the meromorphic function $(f^0/f^1) \circ \pi$, which a priori is well defined only on $\widehat{S} \setminus \pi^{-1}(B)$, extends to a surjective holomorphic mapping $\varphi : \widehat{S} \to \mathbb{P}^1$.

The proof of Lemma 6.1.1, with S replaced by $\widehat{S}$, shows that the set $\varphi(\widehat{S}^{\mathrm{sing}})$ of singular values of φ is finite, where $\widehat{S}^{\mathrm{sing}} := \{\widehat{s} \in \widehat{S} \mid \mathrm{d}\varphi(\widehat{s}) = 0\}$. For every nonsingular value $z \in \mathbb{P}^1$ of φ the level set $\varphi^{-1}(\{z\})$ is a smooth curve in $\widehat{S}$. That is, in this strong sense the generic level curve of φ is smooth, even if every member of the pencil $\mathfrak{P}$ is a singular curve in S.

In Lemma 3.2.7 the case $m = 1$ is of special interest.

Lemma 3.2.8 *Assume that $m = 1$, that is, at least one member of $\mathfrak{P}$ is smooth at b.*

Let $f \in F \setminus \{0\}$. Then $Z_{[f]}$ is smooth at b if and only if $\mathrm{ord}_b(f) = 1$ if and only if $Z_{[\pi'(f)]} = \pi'(Z_{[f]})$. If this is the case, then $Z_{[f]}$ has exactly one intersection point with E, where $Z_{[\pi'(f)]}$ is smooth and intersects E transversally, and the restriction to $Z_{[\pi'(f)]}$ of π is an isomorphism from $Z_{[\pi'(f)]}$ onto $Z_{[f]}$.

We have $\mathrm{i}_b^{\mathfrak{P}} \geq \mu_b := \max_{f \in F\setminus\{0\}} \mathrm{ord}_b(f)$. If b is a simple base point of $\mathfrak{P}$, then $\pi'(\mathfrak{P})$ has no base points in E, and the mapping that assigns to each $[f] \in \mathrm{P}(F)$ the kernel $\mathrm{T}_b\, Z_{[f]}$ of the derivative at b of f is an isomorphism from $\mathrm{P}(F)$ onto E. On the other hand, if $\mathrm{i}_b^{\mathfrak{P}} > 1$, then $\pi'(\mathfrak{P})$ has exactly one base point $e \in E$, of multiplicity $\mathrm{i}_e^{\pi'(\mathfrak{P})} = \mathrm{i}_b^{\mathfrak{P}} - 1$, where e is equal to the common kernel of the derivatives at b of all $f \in F$.

If $\mu_b > 1$, then there is exactly one $[f_0] \in \mathrm{P}(F)$ such that $\mathrm{ord}_b(f_0) > 1$, when $\mathrm{ord}_b(f_0) = \mu_b$. We have $Z_{[\pi'(f_0)]} = \pi'(Z_{[f_0]}) + (\mu_b - 1)\, E$, and $\pi'(Z_{[f_0]}) \cdot E = \mu_b$. Furthermore, if $f \in F$ and $\mathrm{ord}_b(f) = 1$, then $\pi'(Z_{[f_0]})$ intersects $\pi'(Z_{[f]})$ near E only at the base point $e \in E$ of $\pi'(\mathfrak{P})$, and there with multiplicity $\mathrm{i}_b^{\mathfrak{P}} - \mu_b$. Finally, $\mu_b(F) - 1 \leq \mu_e(\pi'(F)) \leq m_b(F) - 1$.

Proof. We first prove the uniqueness of the base point of $\pi'(\mathfrak{P})$ in E. In view of Lemma 3.2.7 (iii), $e \in E$ is a base point of $\pi'(\mathfrak{P})$ if and only if for every $f \in F$ the restriction to e of the derivative at b of f is equal to zero. If this happens for two distinct one-dimensional linear subspaces e of $\mathrm{T}_b\, S$, then we have for every $f \in F$ that the derivative of f at b, which is a linear map from $\mathrm{T}_b\, S$ to L_b, is equal to zero. This would imply that $k > 1$, in contradiction to the assumption.

The equation $\pi'(Z_{[f_0]}) \cdot E = \mu_b$ follows from (3.2.7), and (3.2.8) implies that $\pi'(Z_{[f_0]}) \cdot_E \pi'(Z_{[f]}) = Z_{[f_0]} \cdot_b Z_{[f]} - \mu_b = m_b - \mu_b$. Finally, $\mu_e(\pi'(F)) \geq \mathrm{ord}_e(\pi'(f_0)) \geq \mu_b(F) - 1$, whereas on the other hand, $\mu_e(\pi'(F)) \leq m_e(\pi'(F)) = m_b(F) - 1$. The statement that the restriction of π to $Z_{\pi'(f)}$ is an isomorphism follows from Lemma 3.2.6. □

Corollary 3.2.9 *If the set of base points is finite and infinitely many members of $\mathfrak{P}$ are singular, then there is a base point b of $\mathfrak{P}$ in S such that every member of $\mathfrak{P}$ is singular at b. Furthermore, at most finitely many members of $\mathfrak{P}$ have singular points that are not base points of $\mathfrak{P}$.*

Proof. If the first conclusion does not hold, then $m = 1$ at every base point, and according to the last statement in Lemma 3.2.8 this remains true for all the proper transforms of the pencil that appear after blowing up base points. At every blowing up, except for finitely many, the members of the pencil are isomorphic to their proper transforms. Because in the end, when we have a pencil without base points, at most

finitely many members of the pencil are singular, it would follow that only finitely many members of $\mathfrak{P}$ are singular.

For the second conclusion we observe that the modification $\pi : \widehat{S} \to S$ is an isomorphism over the complement of the base points, which maps members of the pencil $\widehat{\mathfrak{P}}$ of curves in $\widehat{S}$, only finitely many of which have singular points, to members of the pencil $\mathfrak{P}$. □

With S replaced by a compact complex analytic manifold of any dimension, the second statement in Corollary 3.2.9 is known as *Bertini's theorem*; see Griffiths and Harris [74, p. 137].

3.3 Blowing Up $\mathbb{P}^1 \times \mathbb{P}^1$ at the Base Points

Throughout this section we will assume that the biquadratic polynomials p^0 and p^1 do not have a nonconstant common factor when there would not be a birational QRT transformation; see Section 3.1.2. This assumption is equivalent to the assumption that the set B of all base points of the pencil, the common zero-set of p^0 and p^1 in $\mathbb{P}^1 \times \mathbb{P}^1$, does not contain a curve. It follows that B is finite, and there are eight base points when counted with multiplicities; see Lemma 3.1.1.

Combining Lemma 3.1.5 with Subsection 3.2.1, we obtain the following rephrasing of Corollary 3.1.6.

Corollary 3.3.1 *Let $S = S^p$ denote the surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ defined by the equation* (2.5.3)*, where $\pi_2 : S \to \mathbb{P}^1 \times \mathbb{P}^1$ denotes the projection of S onto the second factor. Then S is a smooth submanifold of $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ if and only if the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by* (2.5.3) *has eight distinct base points, and then $\pi_2 : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ at these base points.*

If the surface S^p in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ defined by the equation (2.5.3) is not smooth, then, after successively blowing up base points and replacing the pencil by its proper transform, one always obtains, after at most eight steps, a smooth surface S on which the pencil of curves has no base points, and therefore defines a fibration of S. See Section 3.2.4. Here it is even allowed that all the members of our initial pencil $\mathfrak{P} = \mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are singular.

In the next subsections we will show that if at least one of the given biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ is smooth, then the appropriate line bundle on each surface is the anticanonical bundle, whose holomorphic sections are the holomorphic exterior two-vector fields, and we arrive at the surface S after blowing up exactly eight times. See Section 3.3.1 for the case that all members of the pencil $\mathfrak{P} = \mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are singular.

3.3.1 When All Biquadratic Curves are Singular

Assume that p^0 and p^1 are biquadratic polynomials without a nonconstant common factor, whereas on the other hand the pencil $\mathfrak{B}$ of biquadratic curves $z_0\, p^0 + z_1\, p^1 = 0$

has infinitely many singular members. We will readily see that this implies that each of the biquadratic curves is singular.

Let C be a member of the pencil of biquadratic curves that after all the blowing-up transformations in base points has a nonsingular proper transform $\widehat{C}$ in $\widehat{S}$, where we note that this condition does not hold for only finitely many members of $\mathfrak{B}$; see Section 3.2.4. First assume that C is irreducible. According to (6.2.9), (6.2.8), the genus of $\widehat{C}$ is equal to $\mathrm{g}(\widehat{C}) = \mathrm{vg}(C) - \sum_j m_j/(m_j - 1)/2$, where the virtual genus of C is defined as $\mathrm{vg}(C) = 1 + (K \cdot C + C \cdot C)/2$, and the m_j are the vanishing degrees of the local defining functions at the singular points of the proper images of C that appear in the iterated blowups. Here K is the canonical line bundle of $\mathbb{P}^1 \times \mathbb{P}^1$, and because Lemma 3.3.3 implies that the class of C is equal to $-K$, it follows that $\mathrm{vg}(C) = 1$. Because $\mathrm{g}(\widehat{C}) \geq 0$, C has at most one singular point, where $m = 2$, and after blowing up that point, C has already become nonsingular.

On the other hand, according to Lemma 3.2.9, there is a base point $b \in \mathbb{P}^1 \times \mathbb{P}^1$ of the pencil such that both p^0 and p^1 have a zero of order > 1 at b. It follows that b is the unique singular point of C where $m = 2$ and $\mathrm{g}(\widehat{C}) = 0$. The conclusion is that each nonsingular fiber of the fibration $\varphi : \widehat{S} \to \mathbb{P}^1$ is isomorphic to the complex projective line, that is, we have a fibration of $\widehat{S}$ into $\mathbb{P}^1$'s rather than a fibration into elliptic curves. On each nonsingular fiber the QRT map is an automorphism of the complex projective line, which is a projective linear transformation. By means of projective linear transformations in the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we can arrange that $b = ([1 : 0],\ [1 : 0])$ when the matrices A^j in (2.5.4) satisfy $A^j_{1\,2} = A^j_{2\,1} = A^j_{2\,2} = 0$ for $j = 0,\ 1$.

In the remaining case, when all the members C of $\mathfrak{B}$ are reducible, the only possibility of having a birational QRT transformation is that each C is the union of two $(1,\ 1)$-curves $C_\pm$, which are interchanged by the horizontal switch and by the vertical switch, and therefore each component $C_\pm$ is invariant under the QRT transformation. Because the restriction to $C_\pm$ of the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first (or second) factor is an isomorphism, we have $C_\pm \simeq \mathbb{P}^1$, and the QRT map acts on $C_\pm$ as a projective linear transformation. The generic case occurs when $\mathfrak{B}$ has two base points, not on the same horizontal or vertical axis, at each of which both p^0 and p^1 have a zero of order two. By means of projective linear transformations in the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we can arrange that the base points are $([1 : 0],\ [1 : 0])$ and $([0 : 1],\ [0 : 1])$, and we arrive at the normal form $p^j = A^j_{0\,2}\, {x_1}^2\, {y_0}^2 + A^j_{1\,1}\, x_0\, x_1\, y_0\, y_1 + A^j_{2\,0}\, {x_0}^2\, {y_1}^2$ for the biquadratic polynomials p^0 and p^1 that define the pencil $\mathfrak{B}$.

3.3.2 Holomorphic ExteriorTwo-Vector Fields

Let M be a smooth complex analytic manifold of complex dimension n. The space of all holomorphic sections of a holomorphic line bundle L over a complex analytic manifold M is denoted by $\mathrm{H}^0(M,\ \mathcal{O}(L))$. For example, the space of all holomorphic complex volume forms on M is denoted by $\mathrm{H}^0(M,\ \mathcal{O}(\mathrm{K}_M))$, where K_M is the canonical line bundle of M introduced in Section 2.1. If M is compact, then the

vector space $\mathrm{H}^0(M,\, \mathcal{O}(L))$ is finite-dimensional. This follows from the ellipticity, overdetermined if $n > 1$, of the Cauchy–Riemann operator. It can also be viewed as a special case of the finite-dimensionality of the cohomology of any coherent sheaf on M; see Griffiths and Harris [74, p. 700]. When $M = C$ is a compact Riemann surface, that is, $n = 1$ and C is compact and connected, then $\mathrm{H}^0(C,\, \mathcal{O}(\mathrm{K}_C))$ is the space of all holomorphic complex one-forms on M, and its dimension is equal to the genus $\mathrm{g}(C)$ of C; see Section 2.3.

Let W be a linear subspace of $\mathrm{H}^0(M,\, \mathcal{O}(L))$. For every $w \in W$, $w \neq 0$, the zero-set of w in M, or better the divisor $\mathrm{Div}(w)$, does not change if w is replaced by a multiple of w, and therefore depends only on the element $[w] := \mathbb{C}\, w \in \mathrm{P}(W)$ of the projective space $\mathrm{P}(W)$ of W. The family of divisors $\mathrm{Div}(w)$, $[w] \in \mathrm{P}(W)$, is called the *linear system defined by* W. It is parametrized by $\mathrm{P}(W)$, and $\dim \mathrm{P}(W) = \dim W - 1$ is called the *dimension* of the linear system defined by W. A linear system of dimension 1, 2, 3 is called a *pencil, net, web*, respectively. The common intersection of the linear system is called the *base locus* of the linear system. See Griffiths and Harris [74, p. 137].

If L is a holomorphic line bundle over M, then the dual spaces ${L_m}^*$, $m \in M$, form a holomorphic line bundle L^* over M, called the *dual* of the line bundle L.

Lemma 3.3.2 *Suppose that M is compact and connected. If both vector spaces $\mathrm{H}^0(M,\, \mathcal{O}(L))$ and $\mathrm{H}^0(M,\, \mathcal{O}(L^*))$ are nonzero, then both are one-dimensional and their nonzero elements have no zeros.*

Proof. Suppose that both $\lambda \in \mathrm{H}^0(M,\, \mathcal{O}(L))$ and $\lambda^* \in \mathrm{H}^0(M,\, \mathcal{O}(L^*))$ do not vanish identically in M. It follows from the maximum principle that the holomorphic function $c : m \mapsto \lambda^*_m(\lambda_m)$ on M is a constant. If $c = 0$, then M is equal to the union of the zero-sets Z_λ and Z_{λ^*} of λ and λ^*, respectively. Because λ is not identically zero, we have $Z_\lambda \neq M$; hence the complex analytic subset Z_λ is nowhere dense in M. But then the closed subset Z_{λ^*} of M is dense in M, hence equal to M, in contradiction to the assumption that λ^* does not vanish identically. Therefore $c \neq 0$, when both λ and λ^* have no zeros. Hence every other $\mu \in \mathrm{H}^0(M,\, \mathcal{O}(L))$ and $\mu^* \in \mathrm{H}^0(M,\, \mathcal{O}(L^*))$ is of the form $\mu = f\,\lambda$ and $\mu^* = f^*\,\lambda^*$, where f and f^* are holomorphic functions on M, and therefore are constants. □

We have met the first case when M is an elliptic curve and $L = \mathrm{K}_M$, where $\mathrm{H}^0(M,\, \mathcal{O}(\mathrm{K}^*_M))$ is the space of holomorphic vector fields on M; see Section 2.3.

If M and N are complex analytic manifolds of any dimensions, and Φ is a complex analytic mapping from M to N, then for each $m \in M$ we denote by $\mathrm{T}_m\Phi$ the complex-linear mapping from $\mathrm{T}_m M$ to $\mathrm{T}_{\Phi(m)}N$ whose matrix in local coordinates is equal to the Jacobi matrix of all first order partial derivatives at the point m. This linear mapping induces, for each positive integer n, a complex-linear mapping from $\Lambda^n\, \mathrm{T}_m M$ to $\Lambda^n\, \mathrm{T}_{\Phi(m)}N$, which we denote by $\Lambda^n\, \mathrm{T}_m\Phi$. See Section 2.1.7 for the defintion of exterior n-vectors.

Lemma 3.3.3 *Let $B := \mathrm{H}_{2,\,2}(\mathbb{C}^2 \times \mathbb{C}^2)$ denote the 9-dimensional vector space of all bihomogeneous polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ of bidegree (2, 2), and let π be the projection from $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ onto $\mathbb{P}^1 \times \mathbb{P}^1$. Then there exists, for every*

$p \in B$, a unique exterior two-vector field $w = \varpi(p)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that for every $u = ((x_0,\ x_1),\ (y_0,\ y_1)) \in (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$,

$$p(u)\ \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_0} \wedge \frac{\partial}{\partial y_1} = \left(x_0\ \frac{\partial}{\partial x_0} + x_1\ \frac{\partial}{\partial x_1}\right) \wedge \left(y_0\ \frac{\partial}{\partial y_0} + y_1\ \frac{\partial}{\partial y_1}\right) \wedge \widetilde{w}, \tag{3.3.1}$$

where $\widetilde{w}$ is any exterior two-vector such that $(\Lambda^2\, \mathrm{T}_u\pi)(\widetilde{w}) = w_{\pi(u)}$.

In the affine coordinates $x_0 = 1$, $x_1 = x$, and $y_0 = 1$, $y_1 = y$, we have

$$w = q(x,\ y)\ \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}, \tag{3.3.2}$$

where $q(x,\ y) = p((1,\ x),\ (1,\ y))$.

The mapping $p \mapsto \varpi(p)$ is a complex-linear isomorphism from B onto the space

$$\mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1,\ \mathcal{O}(\mathrm{K}^*_{\mathbb{P}^1 \times \mathbb{P}^1}))$$

of all holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$.

As a consequence, the space of all holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$ is 9-dimensional. There are no nonzero holomorphic complex area forms on $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The left hand side of (3.3.1) is bihomogeneous of bidegree (0, 0), and the same is true for the exterior product of the two Euler vector fields on the right-hand side of (3.3.1). Because these two Euler vector fields form a basis of the tangent space $\ker \mathrm{T}_u\pi$ of the fiber of π through the point u, the first statement follows from the fact that if F is an m-dimensional linear subspace of an n-dimensional complex vector space E, then the mapping

$$(f_1 \wedge \cdots \wedge f_m, (e_{m+1} + F) \wedge \cdots \wedge (e_n + F)) \mapsto f_1 \wedge \cdots \wedge f_m \wedge e_{m+1} \wedge \cdots \wedge e_n$$

induces an isomorphism from $\Lambda^m F \otimes \Lambda^{n-m}(E/F)$ onto $\Lambda^n E$.

Formula (3.3.2) follows from the observation that

$$\left(x_0\ \frac{\partial}{\partial x_0} + x_1\ \frac{\partial}{\partial x_1}\right) \wedge \left(y_0\ \frac{\partial}{\partial y_0} + y_1\ \frac{\partial}{\partial y_1}\right) \wedge \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_0} \wedge \frac{\partial}{\partial y_1}$$

when $x_0 = y_0 = 1$.

The mapping ϖ is obviously complex-linear and injective. In order to prove that it is surjective, we observe that in the affine local coordinates $(x,\ y)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, every holomorphic two-vector field w on $\mathbb{P}^1 \times \mathbb{P}^1$ is of the form (3.3.2), where q is a holomorphic function on $\mathbb{C} \times \mathbb{C}$. The substitution of variables $x = 1/\xi$ leads to $\partial/\partial x = -\xi^2\, \partial/\partial\xi$, and therefore

$$w = \xi^2\, q\left(\frac{1}{\xi},\ y\right)\ \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial y}, \tag{3.3.3}$$

where for each y, the function $g : \xi \mapsto \xi^2 q(\frac{1}{\xi},\ y)$, defined for $\xi \neq 0$, has to extend to a holomorphic function on $\mathbb{C}$. Writing $g(\xi) = g_0 + g_1\xi + g_2\xi^2 + \mathrm{O}(|\xi|^3)$ as $\xi \to 0$, it follows that $f(1/\xi) = g_0\xi^{-2} + g_1\xi^{-1} + g_2 + \mathrm{O}(|\xi|)$ as $\xi \to 0$, and therefore $h(x) := f(x) - (g_0x^2 + g_1x + g_2) = \mathrm{O}(1/|x|)$ as $|x| \to \infty$. In view of the maximum principle we conclude that $h(x) \equiv 0$, and the conclusion is that for every y, the function $x \mapsto q(x,\ y)$ is a polynomial function of degree at most two. With a similar argument we obtain that for each x, the function $y \mapsto q(x,\ y)$ is a polynomial function of degree at most two. That is, $q(x,\ y) = p((1,\ x),\ (1,\ y))$ for some $p \in B$.

The last conclusion follows from Lemma 3.3.2. □

Lemma 3.3.4 *Let S be a complex analytic surface and w a nonzero holomorphic exterior two-vector field on S, and $Z := \mathrm{Div}(w)$ the zero-set of w in S. For every $z \in Z$ where $\mathrm{T}_z w$ is nonzero, there is a unique $v = v_z \in \ker \mathrm{T}_z w$ such that $\mathrm{T}_z w(u) = u \wedge v$ for every $u \in \mathrm{T}_z S$. This defines a nowhere vanishing holomorphic tangent vector field v on the smooth part of the complex analytic curve Z in S. If $(x,\ y)$ are complex analytic local coordinates near z in S, where w has the form* (3.3.2) *for a locally defined holomorphic function $q(x,\ y)$, then v is equal to the restriction to $q(x,\ y) = 0$ of the Hamiltonian vector field*

$$v = \frac{\partial q(x,\ y)}{\partial y}\,\frac{\partial}{\partial x} - \frac{\partial q(x,\ y)}{\partial x}\,\frac{\partial}{\partial y} \tag{3.3.4}$$

defined by the function q. It follows that each smooth and compact connected component of Z, if it exists, is an elliptic curve.

Proof. The equation $\mathrm{T}_z w(u) = u \wedge v$ makes sense because at any zero z of a holomorphic section w of a vector bundle V the tangent map $\mathrm{T}_z w$ of w at z defines a linear map from $\mathrm{T}_z S$ to the fiber V_z over z, where $\mathrm{T}_{w(z)} V$ is projected onto V_z along the tangent space of the zero section. Write $E = \mathrm{T}_z S$ and $F = \ker \mathrm{T}_z w$. The mapping $\iota : v \mapsto (u \mapsto u \wedge v)$ is an injective linear mapping from the one-dimensional vector space F to the space L of linear mappings from E to $\Lambda^2 E$ that vanish on F. Because E/F and $\Lambda^2 E$ are one-dimensional, L is one-dimensional, and therefore ι is an isomorphism from F onto L. Because $\mathrm{T}_z w \in L$, the first statement follows. For (3.3.4) we observe that

$$\begin{aligned}\mathrm{T}_z w(\delta x,\ \delta y) &= \left(\frac{\partial q}{\partial x}\,\delta x + \frac{\partial q}{\partial y}\,\delta y\right)\,\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x} \\ &= \left(\delta x\,\frac{\partial}{\partial x} + \delta y\,\frac{\partial}{\partial y}\right) \wedge \left(\frac{\partial q}{\partial y}\,\frac{\partial}{\partial x} - \frac{\partial q}{\partial x}\,\frac{\partial}{\partial y}\right).\end{aligned}$$

Note that (3.3.4) implies that the vector field v is holomorphic on Z. □

Remark 3.3.5. If $s \in S$ and $w_s \neq 0$, then there is a unique $\omega_s \in \Lambda^2(\mathrm{T}_s S)^*$ such that $\omega_s(w_s) = 1$. This defines a meromorphic area form $\omega := 1/w$ on S with poles along the zero-set Z of w. The dual complex one-form $\nu = 1/v$ of v on Z^{reg} is the *Poincaré residue* of ω; see Griffiths and Harris [74, p. 147]. In this way Lemma 3.3.4

is a two-vector field version, or an anticanonical version, of the adjunction formula, as given in Section 6.2.1.

When $S = \mathbb{P}^1 \times \mathbb{P}^1$ and w is the exterior two-vector field of Lemma 3.3.3, then $\omega = 1/w$ is the unique meromorphic complex area form on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\pi^* \omega = (1/p)\ \mathrm{i}_{E_2}\ \mathrm{i}_{E_1}\ \Omega,$$

where π is the projection from $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ onto $\mathbb{P}^1 \times \mathbb{P}^1$, E_1 and E_2 are the Euler vector fields

$$E_1 := x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} \quad \text{and} \quad E_2 := y_0 \frac{\partial}{\partial y_0} + y_1 \frac{\partial}{\partial y_1},$$

respectively, and $\Omega := \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_0 \wedge \mathrm{d}y_1$ is the standard volume form on $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$.

Remark 3.3.6. The mapping $p \mapsto w = W(p)$ in Lemma 3.3.3 can be generalized to a map that assigns to every polyhomogeneous polynomial p on $\mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_k+1}$ of polydegree $(n_1 + 1, \ldots, n_k + 1)$ a holomorphic n-vector field w on $M = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where $n = n_1 + \cdots + n_k$ is the dimension of M. The case $k = 1, n_1 = 2$ is Lemma 4.1.1, and then the application of Lemma 3.3.4 to the resulting exterior two-vector fields w leads to Lemma 4.1.2.

If p is the product of p_i, $1 \le i \le n-1$, where each of the p_i is polyhomogeneous, and the hypersurfaces $p_i = 0$ in M are smooth and have transversal intersection along their common intersection curve C, then taking successive Poincaré residues of $1/w$ on $p_1 = 0$, $p_1 = p_2 = 0$, up to $p_1 = \cdots = p_{n-1} = 0$ leads to a holomorphic complex one-form without zeros on C, which implies that each connected component of C is an elliptic curve. For instance, each connected component of a transversal intersection of two quadrics in $\mathbb{P}^3$ is an elliptic curve. Similarly, each connected component of a transversal intersection of two trilinear surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is an elliptic curve. I expect that these cases are well known in the literature, but the only reference I have is Adler [1], where the invariant curves are intersections of pairs of trilinear surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. ⊘

3.3.3 Blowing Up for Anticanonical Pencils

If M and N are both n-dimensional complex analytic manifolds and Φ is a complex analytic mapping from M to N, then for each $m \in M$ the complex-linear mapping $\mathrm{T}_m\Phi : \mathrm{T}_m M \to \mathrm{T}_{\Phi(m)} N$ induces a complex-linear mapping from the complex one-dimensional vector space $\mathrm{K}^*_{M,m}$ to the complex one-dimensional vector space $\mathrm{K}^*_{M,\Phi(m)}$, which we denote by $\Lambda^n \mathrm{T}_m\Phi$. Its transpose $(\Lambda^n \mathrm{T}_m\Phi)^*$ is a complex-linear mapping from $\mathrm{K}_{N,\Phi(m)}$ to $\mathrm{K}_{M,m}$. Because of the one-dimensionality of the fibers, the mappings $\mathrm{T}_m\Phi$ and $(\mathrm{T}_m\Phi)^*$ are either linear isomorphisms or equal to zero, and the latter occurs if and only if $\mathrm{T}_m\Phi$ is not a linear isomorphism from $\mathrm{T}_m M$ to $\mathrm{T}_{\Phi(m)} N$. If ν is a volume form on N, then the *pullback of* ν *by* Φ is the volume

form $\mu = \Phi^* \nu$ on M defined by

$$\mu(m) = (\Lambda^n \, \mathrm{T}_m \Phi)^* (\nu(\Phi(m)), \quad m \in M.$$

If w is an n-vector field on M and Φ is bijective from M onto N, then we can similarly define the *pushforward of w by Φ* as the n-vector field $v = \Phi_* w$ on N given by

$$v(\Phi(m)) = (\Lambda^n \, \mathrm{T}_m \Phi)(w(m)), \quad m \in M. \tag{3.3.5}$$

In general, the formula (3.3.5) does not define a pushforward $v = \Phi_* w$ of an arbitrary n-vector field w on M if Φ is not bijective. However, in the case of blowing-up transformations of surfaces, where the projection contracts a complex projective line to a point and therefore is not bijective at all, we do have a well-defined pushforward. In the last statement of Proposition 3.3.7 we use the identification of the -1 curve E with the projective line $\mathrm{P}(\mathrm{T}_s S)$ of all one-dimensional linear subspaces of $\mathrm{T}_s S$.

Proposition 3.3.7 *Let S be a complex analytic surface, $b \in S$, and let $\pi : R \to S$ be the blowing up of S at the point b, with -1 curve $E = \pi^{-1}(\{b\})$, where the restriction of π to $R \setminus E$ is a complex analytic diffeomorphism from $R \setminus E$ onto $S \setminus \{b\}$. Then for every holomorphic two-vector field ϖ on R, the holomorphic two-vector field $\pi_*(\varpi|_{R \setminus E})$ on $S \setminus \{b\}$ has an extension to a holomorphic two-vector field on S, which we also denote by $\pi_* \varpi$. The pushforward by π defines a linear isomorphism π_* from $\mathrm{H}^0(R, \mathcal{O}(\mathrm{K}_R^*))$ onto the space of all $w \in \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S^*))$ that vanish at b. With the notation of Section 3.2.3, the anticanonical bundle K_R^* of R is isomorphic to $\pi_1^*(\mathrm{K}_S^*)$ and $\pi_* = \iota_1$. It follows that*

$$\dim \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S^*)) - 1 \leq \dim \mathrm{H}^0(R, \mathcal{O}(\mathrm{K}_R^*)) \leq \dim \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S^*)), \tag{3.3.6}$$

where the second inequality is an equality if and only if every holomorphic exterior two-vector field w on S vanishes at b.

If $w = \pi_* \varpi$, and w has a zero of order $m \geq 1$ at b, then ϖ has a zero of order $m - 1$ along E. We have $m = 1$ if and only if the derivative of w at b, which is a well-defined linear mapping from $\mathrm{T}_b S$ to the one-dimensional vector space $\mathrm{K}_{S,b}^*$, is nonzero. If $m = 1$, then $e := \ker(\mathrm{d}w)(b)$ is the unique zero of ϖ on E, and the derivative at e of ϖ is nonzero on $\mathrm{T}_e E$.

Proof. We need to investigate $w = \pi_* \varpi$ only near b. In local coordinates (x, y) that map b to $(0, 0)$, the projection π is given by (3.2.2) or (3.2.3). We concentrate on (3.2.2), since (3.2.3) is analogous.

The derivative of $\pi : (x, \eta) \mapsto (x, y) = (x, x\,\eta)$ sends $\partial/\partial x$ and $\partial/\partial \eta$ to $\partial/\partial x + \eta\, \partial/\partial y$ and $x\, \partial/\partial y$, respectively; hence it sends $\partial/\partial x \wedge \partial/\partial \eta$ to $x\, \partial/\partial x \wedge \partial/\partial y$. Therefore, if

$$w = f(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \quad \text{and} \quad \varpi = \phi(x, \eta) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \eta}, \tag{3.3.7}$$

the equation $w = \pi_* \varpi$ is equivalent to $f(x, y) = x\, \phi(x, \eta)$ when $y = x\, \eta$ and $(x, y) \neq (0, 0)$.

Because f is a holomorphic function on $U\setminus\{(0,\ 0)\}$, where U is an open neighborhood of $(0,\ 0)$ in $\mathbb{C}^2$, it follows from Hartogs's theorem, see Griffiths and Harris [74, p. 7], that f extends to a holomorphic function on U, which we also denote by f. Keeping η fixed and letting x go to zero in $f(x,\ \eta\, x) = x\, f(x,\ \eta)$, we obtain that $f(0,\ 0) = 0$.

If, conversely, f is a given holomorphic function on U such that $f(0,\ 0) = 0$, then

$$\phi(x,\ \eta) = \begin{cases} f(x,\ x\,\eta)/x & \text{for } x \neq 0, \\ \\ \partial_1 f(0,\ 0) + \partial_2 f(0,\ 0)\,\eta & \text{for } x = 0, \end{cases} \tag{3.3.8}$$

defines a holomorphic function $\phi(x,\ \eta)$, which has a zero of order $m-1$ along $x=0$ if f has a zero of order m at $(0,\ 0)$. If $\partial_2 f(0,\ 0) \neq 0$ then $\eta \mapsto \phi(0,\ \eta)$ has only one zero at $\eta = -\partial_1 f(0,\ 0)/\partial_2 f(0,\ 0)$, which is simple. If $\partial_2 f(0,\ 0) = 0$ but $\partial_1 f(0,\ 0) \neq 0$, then the zero of ϕ lies at infinity, and the last two sentences in the proposition follow from an analysis in the affine coordinates (3.2.3). □

Corollary 3.3.8 *Let W be a two-dimensional vector space of holomorphic exterior two-vector fields on S. Let b be a base point of the pencil $\mathfrak{P}$ of curves $Z_{[w]}$, $[w] \in \mathrm{P}(W)$, and let $\pi : R \to S$ be the blowing up of S at b, with the -1 curve $E = \pi^{-1}(\{b\})$. Assume that at least one of the members of $\mathfrak{P}$ is smooth at b. That is, $m = 1$ if $L = \mathrm{K}_S^*$ and $F = W$ in Section* 3.2.4.

Recalling that $\mathrm{K}_R^* \simeq \pi_1^*\,\mathrm{K}_S^*$, we have for each $w \in W$ that $\pi'(w)$ is the unique holomorphic two-vector field ϖ on R such that $\pi_*\varpi = w$. The mapping π' is a linear isomorphism from W onto a two-dimensional vector space $\pi'(W)$ of holomorphic two-vector fields on R, and the pencil of the curves $Z_{[\varpi]}$, $[\varpi] \in \mathrm{P}(\pi'(W))$, in R is equal to the proper transform $\pi'(\mathfrak{P})$ of the pencil $\mathfrak{P}$ of the curves $Z_{[w]}$, $[w] \in \mathrm{P}(W)$, in S.

All the conclusions of Lemma 3.2.8 *hold with L, F, $\pi_1^*(L)$, and π' replaced by K_S^*, W, K_R^*, and ${\pi_*}^{-1}$, respectively.*

Iterating the blowing-up transformations of Corollary 3.3.8 as long as the anticanonical pencils have a base point, we arrive at the following conclusions.

Corollary 3.3.9 *Let S be a nonsingular compact connected complex analytic surface. Let W be a two-dimensional vector space of holomorphic exterior two-vector fields on S such that for some $w \in W \setminus \{0\}$, $\mathrm{Div}(w)$ is a nonsingular and connected curve in S, hence an elliptic curve. Then the set B of base points of the pencil $\mathfrak{A}$ defined by W is finite. The number of base points, counted with multiplicities, is equal to the intersection number $m := \mathrm{K}_S^* \cdot \mathrm{K}_S^* = \mathrm{K}_S \cdot \mathrm{K}_S$ of any two members of $\mathfrak{A}$. If C is an irreducible curve in S, then $C \cdot C < 0$ if and only if C is an embedded complex projective line in S and either $C \cdot C = -1$ and $C \cdot A = 1$ for every member A of the pencil, or $C \cdot C = -2$, $\mathrm{K}_S \cdot C = 0$, and C is an irreducible component of a reducible member of $\mathfrak{A}$.*

If there are no base points, then W is equal to the space of all holomorphic two-vector fields on S. For every $s \in S$, $\kappa(s) := \{w \in W \mid w(s) = 0\}$ is a one-dimensional linear subspace of W, and $\kappa : S \to \mathrm{P}(W) \simeq \mathbb{P}^1$ is a relatively minimal

elliptic fibration in the sense of Definition 6.1.7. *A curve E in S is a holomorphic section of κ if and only if E is a -1 curve in S.*

Let $m > 0$. By induction over $j \geq 1$, let $s_{j-1} \in S_{j-1}$ be a base point of the pencil of curves in S_{j-1} defined by W_{j-1}, $\pi_j : S_j \to S_{j-1}$ the blowing up of S_{j-1} at s_{j-1}, and W_j the two-dimensional space of holomorphic exterior vector fields on S_j such that $(\pi_j)_*(W_j) = W_{j-1}$. Here $S_0 = S$, and we have the same properties with S, W, and m replaced by S_j, W_j, and $m - j$, respectively. The process stops at $j = m$, when the pencil defined by W_m has no base points and defines a relatively minimal elliptic fibration $\kappa : S_n \to \mathrm{P}(W_m)$. Furthermore, the -1 curve ${\pi_m}^{-1}(\{s_{m-1}\})$ which appears at the last blowup, is a holomorphic section of κ.

Proof. Let $A = \mathrm{Div}(w)$ for any nonzero holomorphic two-vector field w on S. If A is smooth and connected, then Lemma 3.3.4 implies that A is an elliptic curve. Furthermore, if the holomorphic two-vector field w' vanishes on A, then w'/w extends to a holomorphic function on S, hence a constant in view of the maximum principle, because S is compact and connected. If $w' \notin \mathbb{C}\, w$ is close to w, then $A' := \mathrm{Div}(w')$ is a smooth curve close to A and real analytic diffeomorphic to A, and therefore both A and A' are irreducible. Because $A' \neq A$, it follows that A' and A intersect at finitely many points, where the number of intersections, counted with multiplicities, is equal to $n := A \cdot A' \in \mathbb{Z}_{\geq 0}$, and $A \cdot A' = 0$ if and only if $A \cap A' = \emptyset$.

Lemma 2.1.2 implies that $[\mathrm{K}_S^*] = \delta(A)$ when (2.1.23) yields $A \cdot C = \mathrm{K}_S^* \cdot C = -\mathrm{K}_S \cdot C$ for any divisor C; hence $A \cdot C$ does not depend on the choice of w. If we apply this to $C = A'$, then we obtain that $0 \leq n = A \cdot A' = \mathrm{K}_S^* \cdot \mathrm{K}_S^* = \mathrm{K}_S \cdot \mathrm{K}_S$. If C is an irreducible curve and $C \cdot C < 0$, then $C \neq A$ because $A \cdot A = n \geq 0$, hence $-\mathrm{K}_S \cdot C = A \cdot C \geq 0$. Therefore (6.2.8) with A replaced by C implies that $\mathrm{vg}(C) - 1 < 0$; hence the nonnegative integer $\mathrm{vg}(C)$ is equal to zero, which implies that C is a smoothly embedded complex projective line. Moreover, $\mathrm{K}_S \cdot C + C \cdot C = -2$ with $\mathrm{K}_S \cdot C \leq 0$ and $C \cdot C < 0$ implies that either $\mathrm{K}_S \cdot C = -1$ and $C \cdot C = -1$, or $\mathrm{K}_S \cdot C = 0$ and $C \cdot C = -2$. In the second case, let $A' = \sum_i n_i \Theta_i$ be any member of $\mathfrak{A}$, where the Θ_i are the irreducible components of A' and $n_i \in \mathbb{Z}_{>0}$. If $C \neq \Theta_i$, hence $\Theta_i \cdot C \geq 0$ for every i, then $0 = A' \cdot C = \sum_i n_i\, \Theta_i \cdot C$ implies that $\Theta_i \cdot C = 0$ and hence $\Theta_i \cap C = \emptyset$ for every i, which in turn implies that $A' \cap C = \emptyset$. For every $s \in S$, let $\kappa(s) := \{w \in W \mid w(s) = 0\}$. Because $\dim W = 2$, we have $1 \leq \dim \kappa(s) \leq 2$, where $\dim \kappa(s) = 2$ if and only if s is a base point of the anticanonical pencil $\mathfrak{A}$. It follows that every $s \in S$ lies on a member of $\mathfrak{A}$; hence there exists $A' \in \mathfrak{A}$ such that $A' \cap C \neq \emptyset$ when C is an irreducible component of A'. Because $A' \cdot A' = n \geq 0$ and $C \cdot C = -2$, it follows that $C \neq A'$; hence A' is reducible.

If the pencil has no base points, then $\kappa : S \to \mathrm{P}(W)$ is a well-defined and holomorphic mapping, not constant because the w mentioned in the assumption is not identically equal to zero. Because the fiber of κ over $[w]$ is equal to $\mathrm{Div}(w)$ and there exist $w \in W \setminus \{0\}$ such that $A := \mathrm{Div}(w) = \kappa^{-1}(\mathbb{C}\, w)$ is an elliptic curve, κ is an elliptic fibration. The previous implies that if E is a -1 curve in S, then E is not contained in any fiber F of κ when $E \cdot F = 1$ implies that E intersects F in a unique point that is a smooth point of F, and where the intersection is transversal. That is,

E is a holomorphic section of κ, and we have proved in passing that the fibration $\kappa : S \to \mathrm{P}(W)$ is relatively minimal.

Now assume that the set B of all base points is not empty, that is, $n > 0$. If $\pi : R \to S$ denotes the blowing up of S at a base point b, then Corollary 3.3.8 implies that there is a unique two-dimensional vector space $\pi'(W)$ of holomorphic two-vector fields on R such that the pushforward π_* by π defines a linear isomorphism from $\pi'(W)$ onto W. Furthermore, Lemma 3.2.8 implies that C is a smooth member of the pencil defined by W if and only if its proper transform $\pi'(C)$ is a smooth member of the pencil defined by $\pi'(W)$ when $\pi|_{\pi'(C)}$ is a complex analytic diffeomorphism from $\pi'(C)$ onto C, and (3.2.8) implies that $\pi'(c) \cdot \pi'(C) = C \cdot C - 1 = n - 1$. □

3.3.4 Blowing Up for Pencils of Biquadratic Curves

In the remaining subsections of Section 3.3 we will assume that the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ has at least one smooth member, which according to Corollary 3.2.9 is equivalent to the condition that for every base point b of $\mathfrak{B}$ at least one member of $\mathfrak{B}$ is smooth at b. We refer to Section 3.3.1 for the case that all members of $\mathfrak{B}$ are singular.

We apply Corollary 3.3.9 with $S = S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $W = W_0$ equal to the two-dimensional vector space of holomorphic two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$ that corresponds as in Lemma 3.3.3 to the two-dimensional vector space of biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ that defines the pencil $\mathfrak{B}$. Note that Lemma 3.1.1 implies that $\mathfrak{B}$ has eight base points, counted with multiplicities.

Corollary 3.3.10 *Let $S = S_8$ be the surface obtained by successively blowing up at the base points of the anticanonical pencils, eight times, as in Corollary* 3.3.9*, starting with a pencil $\mathfrak{B}$ of biquadratic curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with at least one smooth member. Write $\pi = \pi_1 \circ \cdots \circ \pi_8 : S \to \mathbb{P}^1 \times \mathbb{P}^1$. Then the space W of all holomorphic two-vector fields on S is two-dimensional, and the mapping $\kappa : S \to \mathrm{P}(W) : s \mapsto \{w \in W \mid w(s) = 0\}$ is a relatively minimal elliptic fibration with a holomorphic section defining a rational elliptic surface in the sense of Definition* 9.1.4*. The pushforward π_* by π is a linear isomorphism from W onto the two-dimensional vector space of biquadratic polynomials that defines the pencil $\mathfrak{B}$. The member C of $\mathfrak{B}$ is smooth if and only if its proper transform $\pi'(C)$ is a smooth fiber of κ when $\pi|_{\pi'(C)}$ is a complex analytic diffeomorphism from $\pi'(C)$ onto C.*

If we identify $(z_0, z_1) \in \mathbb{C}^2$ with the element $w \in W$ such that $\pi_*(w)$ is the two-vector field on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the biquadratic polynomial $z_0\, p^0 + z_1\, p^1$ as in Lemma 3.3.3*, and we use the affine coordinate $z = -z_1/z_0$ on the $[z_0 : z_1]$ projective line, then $\kappa = (p^0/p^1) \circ \pi$ on the dense open subset where the right hand side is defined.*

Remark 3.3.11. At each blowing up the dimension of the space of holomorphic exterior two-vector fields either remains the same or goes down by 1; see (3.3.6). Because for $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and S_8 this dimension is equal to 9 and 2, respectively, the di-

mension goes down by 1 at every blowing up, except at one blowing up, where the dimension remains the same. This is related to the algebraic relation between the base points of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ as discussed in Remark 3.1.2.⊘

Remark 3.3.12. It follows from Section 3.3.3 that the blowing-up transformations can change the structure of the singular members of the anticanonical pencil, in particular each time a new irreducible component is added when the member has a singular point at the base point; see Corollary 3.3.8 = Lemma 3.2.8. Because there are many possible cases, starting from singular members of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, it is quite spectacular to see that every time after the completion of the eight blowing-up transformations, one ends up with one of the singular fibers in Kodaira's list as presented in Section 6.2.6. For instance, at the end, all the irreducible components of a reducible fiber always have self-intersection number equal to -2, which is not at all true for the irreducible components of the members of the anticanonical pencil at the intermediate stages. Section 12.1 is a catalogue of all the blowing-up transformations that can occur over base points of pencils of biquadratic curves. ⊘

Figure 3.3.1 shows the blowing-up diagram over the double base point in Figure 3.1.2. Here E_1' is the irreducible component that after the two blowing-up transformations has been added to the singular fiber. The -1 curve E_2 that appeared at the second blowing up is a holomorphic section of the elliptic fibration. The curve E_2 intersects each fiber, including each singular fiber, in exactly one point in the regular part of the fiber, where the intersection is transversal. Note that E_2 is contained in the total transform of the curve C_{sing}, which is singular at the base point. However, E_2 is not contained in the fiber $C''_{\text{sing}} \cup E_1'$ of the elliptic fibration that is projected by π onto C_{sing}.

3.3.5 Relation with the Surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$

Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the smooth surface that is obtained by successively blowing up, eight times, base points of the anticanonical pencils, starting with the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, as in Corollary 3.3.10, and let $\kappa : S \to \mathrm{P}(W) \simeq \mathbb{P}^1$ be the corresponding elliptic fibration of S.

Recall the surface S^p in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ defined by equation (2.5.3), as discussed in Section 3.1. Then the restriction π_1 to S^p of the projection $(z, (x, y)) \mapsto z$ exhibits $\pi_1 : S^p \to \mathbb{P}^1$ as an elliptic fibration, where the complex projective algebraic surface S^p may have singular points. The mapping $\psi : s \mapsto (\kappa(s), \pi(s))$ is a complex analytic map from S to $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ with $\psi(S) = S^p$ and $\pi_1 \circ \psi = \kappa$.

The mapping ψ precisely blows down the irreducible components of the reducible fibers of $\kappa : S \to \mathbb{P}^1$ that are equal to the proper transforms of the exceptional curves of the blowing up transformations π_i in $\pi = \pi_1 \circ \ldots \circ \pi_8 : S \to \P^1 \times \P^1$, where the exceptional curve of the last blowup over a given base point in $\P^1 \times \P^1$, being a section, is not contained in a fiber. Because the elliptic fibration $\kappa : S \to \mathbb{P}^1$ is relatively minimal, that is, no fiber of κ contains a -1 curve, no fiber of ψ contains a -1 curve.

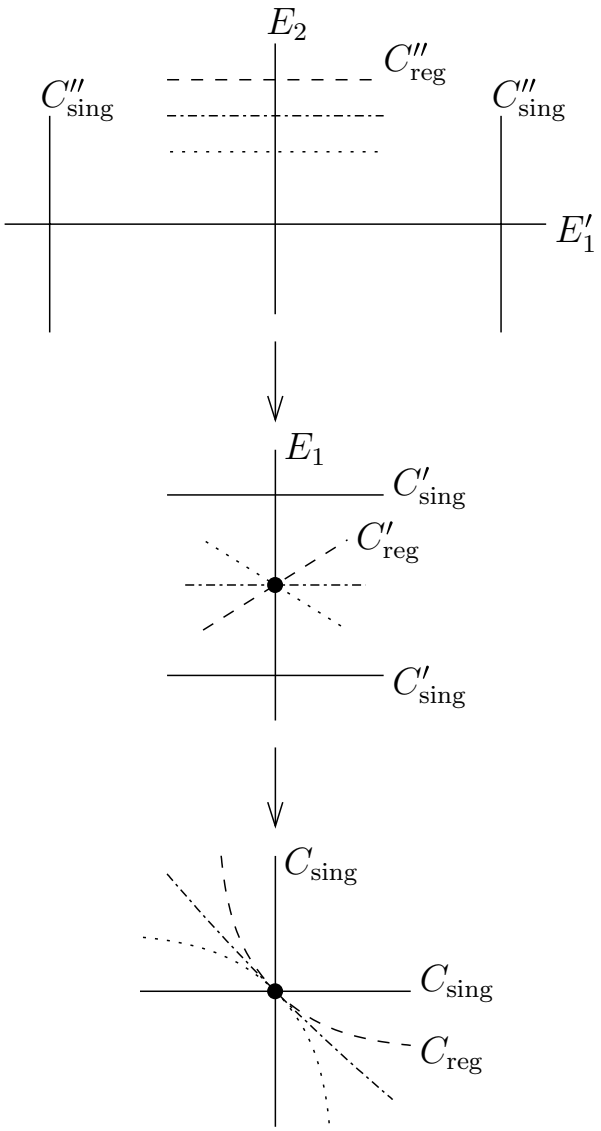


Fig. 3.3.1 Blowup diagram over the double base point in Figure 3.1.2.

Therefore the mapping $\psi : S \to S^p$ is a minimal resolution of singularities of S^p, where we refer to Section 6.2.13 for the basic facts about resolutions of singularities. Because minimal resolutions of singularities are unique up to isomorphisms, see Lemma 6.2.56, this description determines the elliptic surface S in terms of the complex analytic surface S^p in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$.

The mapping $\psi : S \to S^p$ is injective if and only if S^p is smooth, and then ψ is a complex analytic diffeomorphism from S onto S^p. Recall from Section 3.1.3 that S^p is smooth if and only if the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ has eight distinct base points.

3.3.6 The Weierstrass Model

Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the smooth surface that is obtained by successively blowing up, eight times, base points of the anticanonical pencils, starting with the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, as in Corollary 3.3.10, and let $\kappa : S \to \mathbb{P}^1$ be the corresponding elliptic fibration of S.

Let $b \in \mathbb{P}^1 \times \mathbb{P}^1$ be a base point of the pencil of biquadratic points and let E be the -1 curve that appears at the last blowing up over p. Then $E = \sigma(\mathbb{P}^1)$ for a unique holomorphic section $\sigma : \mathbb{P}^1 \to S$ of $\kappa : S \to \mathbb{P}^1$, and we have the Weierstrass model $S \xrightarrow{f} W \xrightarrow{p} \mathbb{P}^1$, with $p \circ f = \kappa$, as described in Theorem 6.3.6. Note that $p : W \to \mathbb{P}^1$ is an elliptic fibration, where it is allowed that the complex analytic surface W has singular points. Furthermore, $p : S \to W$ is a minimal resolution of singularities, where we refer to Section 6.2.13 for the basic facts about resolutions of singularities.

Because the degree of the Lie algebra bundle of $\kappa : S \to \mathbb{P}^1$ is equal to -1, see (ii) in Lemma 9.1.2, the respective sections g_2 and g_3 of O(4) and O(6) correspond to homogeneous polynomials of degree 4 and 6 in the variables $z = (z_0, z_1) \in \mathbb{C}^2$, which we also denote by g_2 and g_3. See Example 5.

It follows from Corollary 2.4.7 that $g_2(z) = D(\Delta_k(z_0\, p^0 + z_1\, p^1))$ and $g_3(z) = -E(\Delta_k(z_0\, p^0 + z_1\, p^1))$, and hence $\Delta(z) = g_2(z)^3 - 27\, g_3(z)^2 = -F(\Delta_k(z_0\, p^0 + z\, p^1))$, where $\Delta_2(p)(x)$ and $\Delta_1(p)(y)$ are the discriminants of $y \mapsto p(x, y)$ and $x \mapsto p(x, y)$, respectively, and $D(f)$, $E(f)$, and $F(f)$ are the Eisenstein invariants of any homogeneous polynomial f of degree four in two variables. This leads to an explicit computation of the homogeneous polynomials $g_2(z)$, $g_3(z)$, and $\Delta(z)$, of degree 4, 6, and 12, respectively, in the variables (z_0, z_1). Because the type of a singular fiber $S_{[z]}$ can be determined by the order of the zeros at $[z]$ of g_2, g_3, and Δ, see Table 6.3.2, this leads to the following corollary.

Corollary 3.3.13 *The fiber $S_{[z]}$ over $[z]$ is singular if and only if $\Delta(z) = 0$. The Kodaira type of the singular fiber $S_{[z]}$ can be determined from the order of the zeros at $[z]$ of the Weierstrass invariants g_2, g_3, and Δ, as in Table 6.3.2.*

Note that in general we do not have an explicit algebraic formula for the zeros of the homogeneous polynomial $z \mapsto \Delta(z)$ of degree 12. On the other hand, in almost all of the examples in Chapter 11 these zeros can be determined explicitly.

Corollary 3.3.14 *A triple X, Y, g_2 of homogeneous polynomials of respective degrees 2, 3, and 4 in two variables corresponds to a QRT map as in Proposition 2.5.6 if and only if g_2 and $g_3 := -Y^2 + 4\,X^3 - g_2\,X$ are homogeneous polynomials in*

two variables of the respective degrees 4 *and* 6 *such that* $\Delta := {g_2}^3 - 27\,{g_3}^3$ *is not identically equal to zero and for each* $[z] \in \mathbb{P}^1$ *such that* $\Delta(z) = 0$*,* g_2 *and* g_3 *do not have a zero at* $[z]$ *of order* ≥ 4 *and* ≥ 6*, respectively. The* $(g_2,\ g_3)$ *that can arise in this way are equal to the set of all* $(g_2,\ g_3)$ *such that* g_2 *and* g_3 *are homogeneous polynomials in two variables of the respective degrees four and six such that* ${g_2}^3 - 27\,{g_3}^2$ *is not identically zero, and at each* $[z] \in \mathbb{P}^1$ *such that* $\Delta(z) = 0$*,* g_2 *has a zero of order* < 4 *or* g_3 *has a zero of order* < 5*.*

Proof. For the first statement, see Example 5 and Section 6.3.3.

The last statement follows from the combination of Corollary 2.4.7, condition (v) in Lemma 9.1.2, Corollary 4.5.6, and the characterization of type II* in Table 6.3.2. □

Example 2. If $g_2(z) = 3\,{z_0}^4$ and $g_3(z) = {z_1}^6$, then $\Delta(z) = g_2(z)^3 - 27\,g_3(z)^2 = 27\,({z_0}^{12} - {z_1}^{12})$ is not identically equal to zero and has 12 simple zeros, lying on the regular 12-gon of all the 12th roots of unity. Also, g_3 is nonzero where g_2 is zero and vice versa, and it follows from Corollary 3.3.14 that there exists a pencil of biquadratic curves $z_0\,p^0 + z_1\,p^1 = 0$ with at least one smooth member such that $g_2(z) = 3\,{z_0}^4$ and $g_3(z) = {z_1}^6$. It follows from Corollary 3.3.14 that the QRT surface defined by this pencil of biquadratic curves is a rational elliptic surface with exactly 12 singular fibers, each of type I_1. I learned this example from Heckman and Looijenga [81, Example 2.3].

This proves the existence of a QRT surface with configuration of singular fibers equal to $12\,\mathrm{I}_1$, with almost no computations. An explicit example of a QRT surface with singular fibers over a regular 12-gon in $\mathbb{P}^1$ is given in Subsection 9.2.5, for instance when $b = 1/2 - a$ and $128\,a^6 - 192\,a^5 + 84\,a^4 - 4\,a^3 + 21\,a^2 - 12\,a + 2 = 0$. ⊘

Because for every reducible fiber $S_{[z]}$ of κ, the mapping $f : S \to W$ contracts the union of the irreducible components of $S_{[z]}$ that do not intersect E to a point, the number of these irreducible components is at least as large as the number of irreducible components that are contracted to a point by the mapping $\psi : S \to S^p$ in Section 3.3.5. I thought for some time that each irreducible component of a reducible fiber that is contracted to a point by $\psi : S \to S^p$ is also contracted to a point by $f : S \to W$. This would imply that S^p is "less singular than W" in the sense that there is a complex analytic mapping $\theta : S^p \to W$ such that $f = \theta \circ \psi$.

However, if the base point $b \in \mathbb{P}^1 \times \mathbb{P}^1$, the one for which the section E is equal to the last -1 curve that appears over b, is multiple, then the proper image F' of the previous -1 curve that appeared over b intersects E, and therefore F is not blown down to a point by $f : S \to W$. On the other hand, $\psi(F)$ is a point, and we conclude that there is no map $\theta : S^p \to W$ such that $f = \theta \circ \psi$. On the other hand, if $Z_{[z]}$ denotes the unique member of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $b \in Z_{[z]}$, then the proper image $Z'_{[z]}$ of $Z_{[z]}$ under the projection $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is an irreducible component of the fiber $S_{[z]}$ of $\kappa : S \to \mathbb{P}^1$ that is different from F', and therefore is mapped to a point by $f : S \to W$. Because $\psi(Z'_{[z]}) = \{[z]\} \times Z_{[z]}$ is not a point, it follows that there is no map $\mu : W \to S^p$ such that $\psi = \mu \circ f$.

3.4 The QRT Map on the QRT Surface

3.4.1 Bimeromorphic Transformations of Fibrations

The following lemma can be found in Iskovskikh and Shafarevich [94, §7.3, Theorem 3], where it is assumed that S and S' are projective algebraic surfaces, $C = C'$ is a projective algebraic curve, β is the identity, φ and φ' are algebraic morphisms, and α is a birational transformation from S to S' such that $\varphi' \circ \alpha = \varphi$. If the genus of the regular fibers is equal to zero when we have ruled surfaces as in Section 6.2.3, each zigzag is a counterexample to the conclusions of Lemma 3.4.1. Note that the equation $\varphi' \circ \alpha = \beta \circ \varphi$ means that for each $c \in C$, α maps the fiber S_c of φ over c to the fiber $S'_{c'}$ of φ' over the point $c' = \beta(c)$. If $s \in S$ and $\mathrm{T}_s\varphi \neq 0$, then the fiber of φ through s is smooth at s, with one-dimensional tangent space equal to $\ker \mathrm{T}_s\varphi$. Therefore the condition on the set S_0 in Lemma 3.4.1 is a quite weak way of requiring that at most points of S the tangent map of α maps tangent spaces of fibers of φ into tangent spaces of fibers of φ'. This condition is certainly satisfied if α, where defined, maps fibers of φ to fibers of φ'.

Lemma 3.4.1 *Let $\varphi : S \to C$ and $\varphi' : S' \to C'$ be relatively minimal fibrations with connected fibers as defined in Section* 6.1*, where the genus of the regular fibers is > 0. Let α be a bimeromorphic transformation from S to S'. Assume that there exists a subset S_0 of the domain of definition of α such that $(\mathrm{T}_s\alpha)(\ker \mathrm{T}_s\varphi) \subset \ker \mathrm{T}_{\alpha(s)}\varphi'$ for every $s \in S_0$, where S_0 is not contained in a closed complex analytic curve in S. Then α has a unique extension to a complex analytic diffeomorphism from S onto S', which we again denote by α, and there is a unique complex analytic diffeomorphism β from C onto C' such that $\varphi' \circ \alpha = \beta \circ \varphi$.*

If S, S' are projective algebraic surfaces and C, C' are projective algebraic curves, then α and β are isomorphisms of projective algebraic varieties, whereas φ and φ' are algebraic morphisms.

Proof. Let G denote the closure in $S \times S'$ of the graph of α, and π, π' the restriction to G of the respective projections $(s, s') \mapsto s$, $(s, s') \mapsto s'$. According to Definition 2.2.2, G is an irreducible complex analytic surface in $S \times S'$, where $\pi : G \to S$ and $\pi' : G \to S'$ are modifications of S and S' in the sense of Section 6.2.13. According to Lemma 6.2.53, there exists a minimal resolution of singularities $\rho : H \to G$ of G. Because $\pi \circ \rho : H \to S$ and $\pi' \circ \rho : H \to S'$ are modifications where H, S, and S' are nonsingular surfaces, it follows from Lemma 6.2.50 that $\pi \circ \rho$ and $\pi' \circ \rho$ are compositions of blowups, hence modifications in discrete subsets B and B' of S and S', and therefore π and π' are modifications of S and S' in discrete subsets D and D', respectively. Furthermore, α and α^{-1} have respective holomorphic extensions to the inverse of $\pi|_{\pi^{-1}(S\setminus D)}$ and $\pi'|_{(\pi')^{-1}(S'\setminus D')}$, followed by π' and π, denoted by α and α'. Note that α defines a diffeomorphism from $S \setminus (D \cup \pi((\pi')^{-1}(D')))$ onto $S' \setminus (D' \cup \pi'(\pi^{-1}(D)))$ with inverse equal to α', whereas $(\pi')^{-1}(D') \setminus D$ and $\pi^{-1}(D) \setminus D'$ are curves in $S \setminus D$ and $S' \setminus D'$ that are mapped by α and α' into the respective discrete subsets D' and D of S' and S.

We first find β. Let ψ denote the restriction to G of the mapping $(\varphi,\ \varphi') : S \times S' \to C \times C'$. The mapping $A : s \mapsto (s,\ \alpha(s))$ is a complex analytic diffeomorphism from $S \setminus D$ onto an open dense subset of G. If $s \in S_0$ then

$$\mathrm{T}_{A(s)}\psi \circ \mathrm{T}_s A = \mathrm{T}_s(\psi \circ A) = (\mathrm{T}_s\varphi,\ \mathrm{T}_s(\varphi' \circ \alpha)) = (\mathrm{T}_s\varphi,\ \mathrm{T}_{\alpha(s)}\varphi' \circ \mathrm{T}_s\alpha)$$

is equal to zero on the nonzero vector space $\ker \mathrm{T}_s\varphi$, that is, $\mathrm{T}_{A(s)}\psi$ is not injective. The set of all $g \in G$ at which the rank of ψ is < 1 is a closed complex analytic subset G_0 of G, where $S_0 \subset \pi(G_0)$. Because π is a proper holomorphic mapping, Remmert's proper mapping theorem yields that $\pi(G_0)$ is a closed complex analytic subset of S, and the assumption on S_0 implies that $\dim \pi(G_0) = 2$, hence $\pi(G_0) = S$, and therefore $A(S \setminus D) \subset G_0$. That is, for every $s \in S \setminus D$ the tangent map $\mathrm{T}_s(\psi \circ A)$ is not injective when there exists a nonzero $v \in \mathrm{T}_s S$ such that $\mathrm{T}_s\varphi(v) = 0$ and $\mathrm{T}_{\alpha(s)}\varphi' \circ \mathrm{T}_s\alpha(v) = 0$.

If S^{reg} denotes the set of all $s \in S$ such that $\mathrm{T}_s\varphi \neq 0$, then for each fiber F of φ, $F \cap S^{\mathrm{reg}}$ is the nonsingular part of F with tangent space at $s \in F \cap S^{\mathrm{reg}}$ equal to the one-dimensional space $\ker \mathrm{T}_s\varphi$. It follows from the above that the tangent map of the restriction to $(F \cap S^{\mathrm{reg}}) \setminus D$ of $\varphi' \circ \alpha$ is identically equal to zero; hence $\varphi' \circ \alpha$ is constant on every connected component of $(F \cap S^{\mathrm{reg}}) \setminus D$. Because D is a discrete subset of S and $\varphi : S \to C$ is proper, $\varphi(D)$ is a discrete subset of C. Because also the set C^{sing} of singular values of φ is a discrete subset of C, $\varphi(D) \cup C^{\mathrm{sing}}$ is a discrete subset of C; hence $C \setminus (\varphi(D) \cup C^{\mathrm{sing}})$ is a dense open subset of C. For every $c \in C_1 := C \setminus (\varphi(D) \cup C^{\mathrm{sing}})$ the fiber S_c of φ over c, which was assumed to be connected, is contained in $S^{\mathrm{reg}} \setminus D$, and therefore $\varphi' \circ \alpha$ is constant on S_c, which means that there is a unique $c' = \beta(c) \in C'$ such that $\varphi' \circ \alpha = \beta \circ \varphi$ on S_c. This defines a mapping $\beta : C_1 \to C'$, with graph equal to $\psi(A(\varphi^{-1}(C_1)))$. Because $\varphi^{-1}(C_1)$ is dense in $S \setminus D$ and $A(S \setminus D)$ is dense in G, the graph of β is dense in $\psi(G)$.

Because $\psi := (\varphi,\ \varphi') \circ (\pi,\ \pi')$, the properness of the holomorphic mappings π, π', φ, and φ' implies that ψ is a proper holomorphic mapping from G to $C \times C'$. Therefore Remmert's proper mapping theorem implies that $\psi(G)$ is a complex analytic subset of $C \times C'$. Because the mappings $\pi : G \to X$ and $\varphi : S \to C$ are surjective, $\varphi \circ \pi : G \to C$ is surjective, where $\varphi \circ \pi$ is equal to ψ followed by the projection $C \times C' \to C : (c,\ c') \mapsto c$. Therefore the restriction to $\psi(G)$ of the projection $(c,\ c') \mapsto c$ is surjective, and we have verified the conditions (i), (iii), and (ii) in Remark 2.2.3 with X, Y, f, and G replaced by C, C', β, and $\psi(G)$, respectively. In other words, β is a meromorphic map from C to C'. Because the set of indeterminacy of a meromorphic map has codimension ≥ 2 and $\dim C = 1$, the set of indeterminacy of β is empty, which means that β has a unique extension to a holomorphic mapping from C to C', which is also denoted by β. We have $\varphi' \circ \pi' = \varphi' \circ \alpha \circ \pi = \beta \circ \varphi \circ \pi$ on the dense open subset $\pi^{-1}(S \setminus D)$ of G, and therefore by continuity $\varphi' \circ \pi' = \beta \circ \varphi \circ \pi$ on G. The same argument with π and π' interchanged leads to a holomorphic mapping $\beta' : C' \to C$ such that $\varphi \circ \pi = \beta' \circ \varphi' \circ \pi'$. It follows that $\beta \circ \beta' \circ \varphi' \circ \pi' = \beta \circ \varphi \circ \pi = \varphi' \circ \pi'$, and because $\varphi' \circ \pi'$ is surjective as the composition of the surjective maps π' and φ', it

follows that $\beta \circ \beta'$ is the identity on C'. Similarly $\beta' \circ \beta$ is the identity on C, and the conclusion is that β is a complex analytic diffeomorphism from C onto C'.

We next prove that D and D' are empty, which implies that α is a diffeomorphism. Recall that $D \subset B$ and $D' \subset B'$, where B and B' are the discrete subsets of S and S' such that $\varpi := \pi \circ \rho : H \to S$ and $\varpi' := \pi' \circ \rho : H \to S'$ are compositions of finitely many blowups over each of the points of B and B', respectively. Let $s' \in B'$ and let E be the -1 curve in H that appears at the last blowup over s'. Because the resolution of singularities $\rho : H \to G$ is minimal, $\rho(E)$ is not a point; hence it is an irreducible complex analytic curve in G. Because $\varpi'(E) = \{s'\}$, it follows that $\rho(E) = \varpi(E) \times \{s'\}$, where $\varpi(E)$ is an irreducible complex analytic curve in S. Because $\beta \circ \varphi \circ \pi \circ \rho = \varphi' \circ \pi' \circ \rho$ maps E to the point $c' = \varphi'(s')$, it follows that $\varpi(E)$ is contained in the fiber S_c of φ over $c := \beta^{-1}(c')$. If $\varpi(E) \cap B = \emptyset$, then the fact that ϖ is a complex analytic diffeomorphism from $H \setminus \varpi^{-1}(B)$ onto $S \setminus B$ implies that $\varpi(E)$ is a -1 curve in S, contained in the fiber S_c, in contradiction to the assumption that the fibration $\varphi : S \to C$ is relatively minimal. Therefore the finite set $\varpi(E) \cap B$ is not empty.

Assume that S_c is irreducible. Then $\varpi(E) = S_c$ and hence $\varpi(E) \cdot \varpi(E) = S_c \cdot S_c = 0$; see Lemma 6.1.2. At every blowup the self-intersection number of the proper image of an irreducible curve C decreases by k^2, if k is the order of C at the blowup point b, see (3.2.8). On the other hand, the proper image E of $\varpi(E)$ after the last blowup in ϖ has self-intersection number equal to -1. Therefore $\varpi(E) \cap B = \{b\}$, b is a smooth point of $\varpi(E)$, and over an open neighborhood of $\varpi(E)$ the mapping $\varpi = \pi \circ \rho$ is the single blowup at b. In view of Lemma 3.2.6 we conclude that $\varpi|_E$ is a complex analytic diffeomorphism from $E \simeq \mathbb{P}^1$ onto $\varpi(E)$. That is, $S_c = \varpi(E)$ is a smoothly embedded $\mathbb{P}^1$ in S, in contradiction to the assumption that the smooth fibers of φ have strictly positive genus.

It follows that S_c is reducible, with $\varpi(E)$ as one of its irreducible components. Then (6.2.13) implies that $\varpi(E) \cdot \varpi(E) \leq -1$. Because at each blowup the self-intersection number of the proper image strictly decreases, and at the final stage $E \cdot E = -1$, we again arrive at a contradiction. The conclusion is that $B' = \emptyset$, hence $D' \subset B'$ is empty, and interchanging the roles of α and α' we also obtain that $D = \emptyset$. Therefore α is a complex analytic diffeomorphism from S onto S' with G as its graph, hence $\pi' = \alpha \circ \pi$. Now $\varphi' \circ \alpha \circ \pi = \varphi' \circ \pi' = \beta \circ \varphi \circ \pi$ in combination with the surjectivity of π yields that $\varphi' \circ \alpha = \beta \circ \varphi$.

The last statement in Lemma 3.4.1 follows from Theorem 2.2.4. □

If $\varphi : S \to C$ and $\varphi' : S' \to C'$ are relatively minimal elliptic fibrations, then the genus of each smooth fiber is equal to 1, and it follows from Lemma 3.4.1 that every bimeromorphic transformation from S to S' that, where defined, maps the fibers of φ to the fibers of φ' actually is a complex analytic diffeomorphism from S onto S', where $\varphi' \circ \alpha = \beta \circ \varphi$ for a unique complex analytic diffeomorphism β from C onto C'.

If the elliptic surface S is compact and $\varphi : S \to C$ has at least one holomorphic section, then it follows from Corollary 6.2.28 that S is complex analytic diffeomorphic to a complex projective algebraic surface. The same conclusions hold for the diffeomorphic fibration $\varphi' : S' \to C'$. Proposition 2.1.6 implies that C and C' are

complex projective algebraic curves. With respect to the complex projective algebraic structures on our surfaces and curves, compatible with the complex analytic structures, Lemma 3.4.1 implies that every birational transformation α from S to S' that, where defined, maps the fibers of φ to the fibers of φ' is an isomorphism, where $\varphi' \circ \alpha = \beta \circ \varphi$ for a unique isomorphism β from C onto C'. Furthermore, φ and φ' are algebraic morphisms of complex projective algebraic varieties. Note that Theorem 2.2.4 implies that the transformation α is birational if and only if it is bimeromorphic.

3.4.2 Application to the QRT Map

Recall the QRT map τ, which is a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by (1.1.4), (1.1.5), (1.1.6) where $\tau = \iota_2 \circ \iota_1$ and ι_1 and ι_2 are the involutory birational transformations of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by (1.1.4) and (1.1.5), respectively. We also recall that these rational maps are not defined at the points where the numerator and denominator in their defining formulas vanish simultaneously.

Corollary 3.4.2 *With the assumption and notation of Corollary* 3.3.10*, write* $\pi = \pi_1 \circ \pi_2 \ldots \circ \pi_8 : S \to \mathbb{P}^1 \times \mathbb{P}^1$*. Then there are unique automorphisms* $\iota_1^S : S \to S$ *and* $\iota_2^S : S \to S$ *such that for* $i = 1, 2$, $\pi \circ \iota_i^S(s) = \iota_i \circ \pi(s)$ *for all* $s \in S$ *such that* ι_i *is well defined at the point* $\pi(s) \in \mathbb{P}^1 \times \mathbb{P}^1$*. As a consequence,* $\tau^S := \iota_2^S \circ \iota_1^S$ *is the unique automorphism of* S *such that* $\pi \circ \tau^S = \tau \circ \pi$ *wherever the right-hand side is defined.*

Proof. Because the involutions ι_1 and ι_2 are locally defined by rational formulas and equal to their own inverses, they define birational maps ι_1^S and ι_2^S from S to S, which preserve the fibers of $\kappa : S \to \mathrm{P}(W)$. It follows from Corollary 3.3.10 that $\kappa : S \to P$ is a rational elliptic surface. This implies that the generic fiber of κ is an elliptic curve, and therefore has genus > 0, and no fiber of κ contains a -1 curve. We therefore can apply Lemma 3.4.1 with $S' = S$, $C' = C = \mathrm{P}(W)$, $\varphi' = \varphi = \kappa$, $\alpha = \iota_i^S$, and β equal to the identity in $\mathrm{P}(W)$. The conclusion is that ι_1^S and ι_2^S are automorphisms of S. Because $\tau^S = \iota_2^S \circ \iota_1^S$, the same conclusion holds for τ^S. □

We also have the following uniqueness statement.

Corollary 3.4.3 *Let* $\varphi' : S' \to C'$ *be a relatively minimal algebraic fibration of a complex projective algebraic surface* S' *over a complex projective algebraic curve* C' *as in Lemma* 3.4.1*, and let* ϖ *be a birational transformation from* S' *to* $\mathbb{P}^1 \times \mathbb{P}^1$ *that, where defined, maps the* φ'*-fibers to curves of the biquadratic pencil. Then there are unique isomorphisms* $\iota : S \to S'$ *and* $\beta : \mathbb{P}^1 \to C'$ *such that* $\varphi' \circ \iota = \beta \circ \kappa$*. The mapping* $\tau^{S'} := \iota \circ \tau^S \circ \iota^{-1}$ *is an automorphism of* S' *that preserves the fibers of* φ' *and extends the birational transformation* $\varpi^{-1} \circ \tau \circ \varpi$ *of* S'*.*

Proof. This follows from Lemma 3.4.1 with α replaced by $\iota := \varpi^{-1} \circ \pi : S \to S'$. □

In the following corollary, recall that each regular fiber of an elliptic fibration is an elliptic curve, with the group of translations as the idenity component of its automorphism group.

Corollary 3.4.4 *We use the notation of Corollary* 3.4.2. *If $w \in W$, that is, w is a holomorphic exterior two-vector field on S, then $(\iota_1^S)_* w = -w$, $(\iota_2^S)_* w = -w$, and $(\tau^S)_* w = w$. It follows that $\tau^S \in \mathrm{Aut}(S)^+_\kappa$, that is, τ^S is an automorphism of S that acts as a translation on each regular fiber of $\kappa : S \to P$.*

Proof. We may assume that $w \neq 0$. Let z be an affine coordinate defined on an affine coordinate neighborhood V of the complex projective line $P = \mathrm{P}(W)$, and let v be the holomorphic vector field on $U = \kappa^{-1}(V)$ defined by (6.2.21) with $\varphi = \kappa$. Note that V is the complement of a point in P, and therefore U is the complement of a fiber of κ in S. The Hamiltonian vector field defined by the function $z \circ \kappa$ is tangent to the fibers in U of κ and has no zeros on any smooth fiber in U that is not equal to the fiber $\kappa^{-1}(\{[w]\})$, the zero-set of w.

The involutions ι_1 and ι_2 in $\mathbb{P}^1 \times \mathbb{P}^1$, the horizontal and vertical switches on each member of the pencil of biquadratic curves, act as inversions on the smooth members of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$; see the text preceding Proposition 2.5.2. According to Corollary 3.3.10, the restriction of π to each smooth fiber of κ is an isomorphism onto such a smooth biquadratic curve. In combination with $\pi \circ \iota_i^S = \iota_i \circ \pi$, see Corollary 3.4.2, this implies that ι_i^S defines an inversion on each smooth fiber of κ. In turn, this implies that $(\iota_i^S)^* v = -v$. Because $\kappa \circ \iota_i^S = \kappa$, we have

$$(\iota_i^S)^*(\kappa^*(\mathrm{d}z)) = (\kappa \circ \iota_i^S)^*(\mathrm{d}z) = \kappa^*(\mathrm{d}z),$$

which in combination with $(\iota_i^S)^* v = -v$ and (6.2.21) yields that $(\iota_i^S)_* w = -w$ on $U \setminus \kappa^{-1}(\{[w]\})$. Because $U \setminus \kappa^{-1}(\{[w]\})$ is equal to the complement of at most two fibers of κ in S, it is dense in S and we obtain that $(\iota_i^S)_* w = -w$ on S by continuity. This implies in turn that

$$(\tau^S)_* w = (\iota_2^S \circ \iota_1^S)_* w = (\iota_2^S)_*((\iota_1^S)_* w) = (\iota_2^S)_*(-w) = -(\iota_2^S)_* w = -(-w) = w.$$

□

Remark 3.4.5. If $w \in W$, $w \neq 0$, is a nonzero holomorphic exterior two-vector field, with zero-set equal to the fiber $\mathrm{Z}_{[w]} = \kappa^{-1}(\{[w]\})$ of κ over $[w] \in \mathrm{P}(W)$, then the equation $w\,\omega = 1$ defines a holomorphic complex two-form = complex area form = complex symplectic form ω on $S \setminus \mathrm{Z}_{[w]}$, which, however, has a pole along $Z_{[w]}$. Because w is invariant under τ^S, the area form $\omega = 1/w$ is invariant under τ^S as well. These ω are the invariant area forms found by Roberts and Quispel [175, p. 168]. See also Remark 6.2.22 for an interpretation in terms of integrable systems with two degrees of freedom. ⊘

3.4.3 Singularity Confinement

Let S_0 be a complex analytic surface. When a meromorphic transformation $\alpha_0 : S_0 \to S_0$ is iterated, it may happen that we reach a point s_0 for which the next value

is undetermined, because both the numerator and the denominator of one of the coordinates of $\alpha_0(s)$ are equal to zero at the point where $s = s_0$. This leads to a blowing up of the point s_0 by the map α_0. Grammaticos, Ramani and Papageorgiou [67] say that the transformation T_0 satisfies *singularity confinement* if "this blowing up is canceled out after a finite number of steps, and the memory of the initial condition is not lost."

One way to obtain a well-defined map is to have a smooth complex analytic manifold S, a bimeromorphic map $\pi : S \to S_0$, and an everywhere defined complex analytic mapping α from S to S such that $\alpha_0 \circ \pi(s) = \pi \circ \alpha(s)$ for all $s \in S$ such that α_0 is well defined at the point $\pi(s) \in S_0$. In other words, there is a modification S of S_0 on which the mapping is a globally defined complex analytic mapping. This is a strong form of singularity confinement, which was introduced in Veselov [199, p. 34], and which may be called *geometric singularity confinement*. The complex analytic mapping $\alpha : S \to S$ such that $\alpha_0 \circ \pi = \pi \circ \alpha$ wherever defined is called a *regular lift* of α_0, and the dynamics of α_0 can be studied by studying the iterates of the regular lift α of α_0.

It follows from Corollary 3.4.2 that the QRT map satisfies geometric singularity confinement, with $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$, α_0 equal to the QRT map τ defined by (1.1.4), (1.1.5), (1.1.6), $S = S_8$ as in Corollary 3.3.10, and π and $\alpha = \tau^S$ as in Corollary 3.4.2.

Hietarinta and Viallet [83] showed that the birational transformation $\alpha_0 : (x,\ y) \mapsto (y,\ -x + y + a/y^2)$ on the one hand satisfies singularity confinement, but on the other hand exhibits chaotic behavior. They link the chaotic behavior to the fact that the degrees of the maps α_0^n (after cancellations of common factors) have exponential growth $(3 + \sqrt{5})/2)^n$ as $n \to \infty$. Takenawa [192], [193] showed that the mapping α_0 satisfies geometric singularity confinement by constructing a surface S obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by 14 successive blowing-up transformations such that α_0 lifts to an automorphism α, that is, a complex analytic diffeomorphism, of S.

The group $\mathrm{H}_2(X,\ \mathbb{Z})$ of homology classes of real two-dimensional cycles in X is a lattice of rank 16, which will be denoted by L. For the action α^L of α on L, the maximum of the absolute values of the eigenvalues is equal to $(3 + \sqrt{5})/2 > 1$. This implies that the action of α^n on L has exponential growth as $n \to \infty$, which corresponds to the exponential growth of the degree of α^n as $n \to \infty$ found by Hietarinta and Viallet [83].

If $\alpha \in \mathrm{Aut}(S)$ is an automorphism of a rational elliptic surface S, then its action A on L belongs to $\mathrm{Aut}(L)_f$, see Lemma 9.2.11, and therefore some finite power A^m of A acts as the identity on Q, which implies that $A^m = 1 + N$ for a transformation N of L such that $N^3 = 0$. See Lemma 7.3.2. That is, the action on $L = \mathrm{H}_2(S,\ \mathbb{Z})$ of the iterates of an automorphism of a rational surface, in particular of any QRT map, has at most quadratic growth, in strong contrast with the exponential growth of the Hietarinta–Viallet map.

(Geometric) singularity confinement is a rare phenomenon and not well understood in general. It is believed that the chaotic behavior of the iterates of α_0 is related to the exponential behaviour of the iterates of α^L. Veselov [199, p. 34] mentioned the definition of Arnol'd [4] of the *complexity* of the discrete dynamical system defined by α in terms of the intersection numbers $\alpha^n(C) \cdot C'$, where C and C' are curves in S

and $n \in \mathbb{Z}_{>0}$. The latter numbers for $n = 1$, and $\mathrm{H}(C'])$, $\mathrm{H}(C')$ varying over a basis of $L = \mathrm{H}_2(M, \mathbb{Z})$, determine the action α^L of α on L; hence they determine $(\alpha^L)^n$, and therefore the intersection numbers $\alpha^n(C) \cdot C'$, for all $n \in \mathbb{Z}$.

3.4.4 The Universal Biquadratic Curve

Let $\mathcal{B}$ denote the nine-dimensional space of all biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$, as in Remark 2.5.7, and $\mathrm{P}(\mathcal{B})$ the eight-dimensional projective space of all its one-dimensional linear subspaces. Then

$$U := \{([p], ([x], [y])) \in \mathrm{P}(\mathcal{B}) \times (\mathbb{P}^1 \times \mathbb{P}^1) \mid p(x, y) = 0\}$$

is a nine-dimensional complex projective algebraic variety. The restriction π_2 to U of the projection $([p], ([x], [y])) \mapsto ([x], [y])$ exhibits U as a locally trivial $\mathbb{P}^7$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. In particular the complex projective variety U is nonsingular.

The projection $\pi_1 : U \to \mathrm{P}(\mathcal{B}) : ([p], ([x], [y])) \mapsto [p]$ is a surjective morphism = complex analytic map, of which the fiber $U_{[p]}$ over $[p]$ is canonically isomorphic to the biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p(x, y) = 0$. In this way U can be viewed as the "universal biquadratic curve." Because the fiber $U_{[p]}$ of π_1 over $[p]$ is an elliptic curve if and only if the curve $p = 0$ is smooth if and only if $[p]$ is a regular value of the projection π_1, U can be viewed as an elliptic fibration over $\mathrm{P}(\mathcal{B})$. The curve $p = 0$ is smooth if and only if $\Delta(p) := -F(\Delta_k(p)) \neq 0$; see Proposition 2.4.3. In the sequel the zero-set of Δ in $\mathrm{P}(\mathcal{B})$ will be denoted by Z_Δ. It is an algebriac hypersurface in the projective space $\mathrm{P}(\mathcal{B})$ of degree 12.

The horizontal switch is well-defined on each biquadratic curve $p = 0$ which does not contain a horizontal axis. If the biquadratic curve contains a horizontal axis, then one can use a smooth perturbation of the biquadratic curve to obtain a horizontal switch on the horizontal axis. However, the resulting involution on the horizontal axis depends on the choice of the perturbation. For example, the generic biquadratic curve $p = 0$ which contains the horizontal axis at level $y_1 = 0$ and a $(2, 1)$-curve through the points $([1 : 0], [1 : 0])$ and $([0 : 1], [1 : 0])$ is characterized by $p_{0\,1} = p_{0\,2} = p_{1\,2} = p_{2\,1} = p_{2\,2} = 0$ and $p_{1\,1} = 1$. If we approximate this curve by $\epsilon\, q = 0$, then the limit for $\epsilon \to 0$ of the horizontal switch at the level $y_1 = 0$ is equal to $x \mapsto q_{2\,2}/(q_{0\,2}\, x)$, which is well-defined if $q_{2\,2}$ and $q_{0\,2}$ are not equal to zero. The limit horizontal switch is in bijective correspondence with the ratio $q_{2\,2}/q_{0\,2}$. Therefore, in contrast with the situation in Corollary 3.4.2, the horizontal switch in U, which is defined on the complement of the union of the horizontal axes contained in biquadratic curves, does not extend to a globally defined continuous map from U to U. It remains to be investigated how far this can be remedied by passing to a suitable modification of U.

The surface S^p in Section 3.1 is equal to $\pi_1^{-1}(\mathrm{P}(\mathcal{A}))$, where $\mathcal{A}$ is the two-dimensional linear subspace of $\mathcal{B}$ which defines the pencil of biquadratic curves, and $\mathrm{P}(\mathcal{A})$ is viewed as a complex projective line in the eight-dimensional complex projective space $\mathrm{P}(\mathcal{B})$. The next paragraphs contain a generalization which can be

viewed as a description of the *curve-dependent McMillan maps of Iatrou and Roberts* [93] in terms of the universal biquadratic curve $\pi_1 : U \to \mathrm{P}(\mathcal{B})$.

Let C be a complex analytic curve, a connected complex analytic manifold of complex dimension one, and $\gamma : C \to \mathrm{P}(\mathcal{B})$ a complex analytic map, for instance obtained from a nonzero complex analytic map from C to $\mathcal{B}$. Then the pullback γ^*U by γ of the fibration $\pi_1 : U \to \mathrm{P}(\mathcal{B})$ is defined as follows. The set $\gamma^*U := \{(c,\, u) \in C \times U \mid \gamma(c) = \pi_1(u)\}$ is a complex analytic subset of $C \times U$. Let $p_1 : \gamma^*U \to C$ and $p_2 : \gamma^*U \to U$ denote the restrictions to γ^*M of the respective projections $C \times U \to C : (c,\, u) \mapsto c$ and $C \times U \to M : (c,\, u) \mapsto u$. Both p_1 and p_2, as restrictions to the complex analytic set γ^*U of complex analytic maps, are complex analytic maps. For each $c \in C$, the restriction of p_2 to the fiber $(\gamma^*U)_c$ of p_1 over c is a complex analytic diffeomorphism from $(\gamma^*U)_c$ onto the fiber $U_{\gamma(c)}$ of π over $\gamma(c)$. It follows that each fiber of p_1 is nonempty, and therefore the mapping $p_1 : \gamma^*M \to C$ is surjective.

In the sequel we will assume that not all biquadratic curves $M_{\gamma(c)}$, $c \in C$ are singular, which is equivalent to the condition that the image $\gamma(C)$ of C in $\mathrm{P}(\mathcal{B})$ is not entirely contained in Z_Δ. It follows that $C^{\mathrm{sing}} := \{c \in C \mid \gamma(c) \in Z_\Delta\}$ is a discrete subset of C. As C and the fibers of π are connected and complex one-dimensional, γ^*M is connected and complex two-dimensional. That is, γ^*M is a complex analytic surface, which can have singularities. A straightforward computation shows that $(c,\, u) \in \gamma^*U$ is a singular point of γ^*U if and only if u is a singular point of the biquadratic curve $U_{\gamma(c)}$ and $\mathrm{T}_c\gamma(\mathrm{T}_cC)$ is contained in the codimension one linear subspace $\mathrm{T}_u\pi_1(\mathrm{T}_uU)$ of $\mathrm{T}_{\gamma(c)}(\mathrm{P}(\mathcal{B}))$. Note that the first condition implies that $c \in C^{\mathrm{sing}}$.

Let Σ denote the set of all singular points of γ^*U. If Σ is not a discrete subset of γ^*u, then its one-dimensional part is a union of irreducible curves, each of which is an irreducible component of multiplicity > 1 of a biquadratic curve. If this happens then γ^*M is not normal. Normalizing γ^*M we obtain a surface with only isolated singularities, which can be resolved by a finite sequence of blowups. This leads to the so-called minimal resolution of singularities $\rho : S \to \gamma^*M$, where the minimality means that the blowups stop when the surface is smooth, and no additional blowups have been performed. Here ρ is a surjective holomorphic map from S onto γ^*M, and a complex analytic diffeomorphism from $\rho^{-1}((\gamma^*M) \setminus \Sigma)$ onto $(\gamma^*M) \setminus \Sigma$. The set $(\gamma^*M) \setminus \Sigma$ is open and dense in γ^*M, and $\rho^{-1}((\gamma^*M) \setminus \Sigma)$ is an open and dense subset of S.

The surjective holomorphic map $\varphi := p_1 \circ \rho : S \to C$ is an elliptic fibration of S, meaning that the smooth fibers of φ, the fibers over the points $c \in C^{\mathrm{reg}} := C \setminus C^{\mathrm{sing}}$, are elliptic curves. On each biquadratic curve $U_{[p]}$ which does not contain a horizontal or vertical axis, we have the horizontal and vertical switch $\iota_1^{[p]}$ and $\iota_2^{[p]}$, respectively. The mapping $p_2 \circ \rho : S \to U$ intertwines these switches with respective transformations ι_1^S and ι_2^S of S, defined on dense open subsets of S. Because ι_i^S, as far as defined, leaves the fibers of φ invariant, it follows from Lemma 3.4.1 that ι_i^S extends to globally defined complex analytic mappings from S to S, which will again be denoted by ι_i^S. Because the original domain of definition of ι_i^S is dense in S, any continuous extensions of it is uniquely determined. Moreover, because the original

mapping satisfies $\iota_i^S(\iota_i^S(s)) = s$ for all s in a dense subset of S, it follows from the continuity of ι_i^S that $\iota_i^S \circ \iota_i^S$ is equal to the identity on S. That is, ι_i^S is a two-sided inverse of ι_i^S, and therefore ι_i^S is a globally defined complex analytic diffeomorphism of S. It follows that $\tau^S := \iota_2^S \circ \iota_1^S$ is a globally defined complex analytic diffeomorphism of S, leaving each fiber of φ invariant and acting on each smooth fiber, which is an elliptic curve, as a translation. Note that τ^S is the common extension to S of the QRT maps on the biquadratic curves, as far the latter were defined. The statement that τ^S is a complex analytic diffeomorphism of the smooth surface S is a strong geometric form of singularity confinement. The point of the construction is that we not only have an elliptic fibration $\varphi : S \to C$ in the sense of Kodaira [109, II], but that it comes equipped with the involutions ι_1^S and ι_2^S of S preserving the fibers, and the ensuing element $\tau^S := \iota_2^S \circ \iota_1^S$ of the Mordell–Weil group $\mathrm{Aut}(S)_\varphi^+$ of the elliptic fibration $\varphi : S \to C$, the group of all automorphisms of S which act as a translation on each smooth fiber. of φ.

The rational elliptic surfaces occur precisely when $C \simeq \mathbb{P}^1$ and $\gamma : C \to \mathrm{P}(\mathcal{B})$ is an embedding of C as a projective line in the eight-dimensional projective space $\mathrm{P}(\mathcal{B})$. However, the construction of Iatrou and Roberts yields a much larger class of elliptic fibrations. At one trivial extreme, if the mapping $\gamma : C \to \mathrm{P}(\mathcal{B})$ is constant, with image point $[p] \in \mathrm{P}(\mathcal{B}) \setminus Z_\Delta$, then $S = C \times U_{[p]}$ with φ equal to the projection onto the first component. On the other hand there exist maps γ with arbitrarily high order of contact with the hypersurface Z_Δ, which means that the construction of Iatrou and Roberts can lead to elliptic fibrations with singular fibers of arbitrary high multiplicity = Euler number of the singular fiber. Note that the elliptic fibrations $\varphi : S \to C$ with trivial Mordell–Weil group $\mathrm{Aut}(S)_\varphi^+$ are not Iatrou–Roberts. The rational elliptic surfaces in Lemma 4.5.1 represent the two isomorphism classes of rational elliptic surfaces which are not Iatrou–Roberts. We leave the discussion of the universal biquadratic curve at this point as most QRT mappings in the examples in the literature, see Chapter 11, are defined by pencils of biquadratic curves.

Chapter 4
Cubic Curves in the Projective Plane

4.1 From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and Back

Let $b = (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, and let $\pi_b : U \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point b, with the exceptional fiber $E_b := {\pi_b}^{-1}(\{b\})$, which is a -1 curve in U. Let $L_1 := \mathbb{P}^1 \times \{y\}$ and $L_2 = \{x\} \times \mathbb{P}^1$ denote the horizontal and the vertical axes through b. Because the parallel axes are homologous and don't intersect, we have $L_1 \cdot L_1 = 0$ and $L_2 \cdot L_2 = 0$. Let $\pi_b^*(L_j)$ and $\pi_b'(L_j)$ denote the total and proper transforms of L_j in U, as defined in Section 3.2.4. Because L_1 and L_2 are smooth at b, it follows that $\pi_b^*(L_j) = \pi_b'(L_j) + E_b$, and therefore (3.2.8) yields that $\pi_b'(L_j) \cdot \pi_b'(L_j) = -1$, $j = 1, 2$. Furthermore, the restriction to $\pi_b'(L_j)$ of π_b defines a complex analytic diffeomorphism from $\pi_b'(L_j)$ onto L_j, which in turn is complex analytic diffeomorphic to $\mathbb{P}^1$, and therefore $\pi_b'(L_1)$ and $\pi_b'(L_2)$ are -1 curves in U. Because L_1 intersects L_2 only at b and transversally, it follows that the -1 curves $\pi_b'(L_1)$ and $\pi_b'(L_2)$ in U are *disjoint*. The -1 curve E_b intersects the -1 curves $\pi_b'(L_1)$ and $\pi_b'(L_2)$ in exactly one point b_1' and b_2', respectively, and the intersection is transversal.

Applying the Castelnuovo–Enriques criterion, Theorem 3.2.4, twice, we obtain a smooth surface T with two distinct points $b_1, b_2 \in T$, and a double blowing-up map $\psi_{b_1,b_2} : U \to T$ such that $\pi_b'(L_1) = {\psi_{b_1,b_2}}^{-1}(\{b_1\})$ and $\pi_b'(L_2) = {\psi_{b_1,b-2}}^{-1}(\{b_2\})$. According to Proposition 3.3.7, the pushforward $(\pi_b)_*$ and $(\psi_{b_1,b_2})_*$ of holomorphic exterior two-vector fields defines a linear isomorphism from the space of all holomorphic exterior two-vector fields on U onto the space of all holomorphic exterior two-vector fields w and $\widetilde{w}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and T such that $w(b) = 0$ and $\widetilde{w}(b_1) = 0$, $\widetilde{w}(b_2) = 0$, respectively. It follows that pushforward by the birational transformation

$$\rho = \rho_{b;\,b_1,b_2} := \psi_{b_1,b_2} \circ {\pi_b}^{-1}$$

defines a linear isomorphism from the space of all holomorphic two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$ that vanish at b onto the spaces of all holomorphic two-vector fields on T that vanish at b_1 and b_2.

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_4,
© Springer Science+Business Media, LLC 2010

In order to understand the surface T and the holomorphic exterior two-vector fields on T, we make the constructions more explicit. By means of suitable projective linear transformations in each of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we can arrange that $x = \infty = [0 : 1]$ and $y = \infty = [0 : 1]$, that is, $b = (\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$. Consider the birational transformation $\rho = \rho_\infty$ from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ defined by

$$\rho_\infty([x_0 : x_1], [y_0 : y_1]) = [x_0\, y_0 : x_1\, y_0 : y_1\, x_0], \tag{4.1.1}$$

of which the inverse $\sigma = \rho^{-1}$ is given by

$$\sigma([u_0 : u_1 : u_2]) = ([u_0 : u_1], [u_0 : u_2]). \tag{4.1.2}$$

The only point b where ρ is not determined is $x_0 = y_0 = 0$, that is, $b = (\infty, \infty) = b$. The only points where σ is not determined are $b_1 = [0 : 0 : 1]$, the image under ρ of the horizontal axis through b, and $b_2 = [0 : 1 : 0]$, the image under ρ of the vertical axis through b. More precisely, $\rho = \psi_{b_1, b_2} \circ {\pi_b}^{-1}$, where $\pi_b : S_b \to \mathbb{P}^1 \to \mathbb{P}^1$ and $\psi_{b_1, b_2} : S_b \to T$ are blowing up maps as above, but now with $T = \mathbb{P}^2$. Because the surface T is unique up to isomorphisms = complex analytic diffeomorphisms, the conclusion is that T is isomorphic to $\mathbb{P}^2$ and the isomorphism can be arranged such that $\rho := \psi_{b_1, b_2} \circ {\pi_b}^{-1}$ is given by (4.1.1). Note that σ maps the "line at infinity" in $\mathbb{P}^2$, the set of all $[u_0 : u_1 : u_2]$ such that $u_0 = 0$, onto the point $b = (\infty, \infty)$. In other words, the -1 curve $E_b = \pi^{-1}(\{b\})$ is equal to the proper transform $(\psi_{b_1, b_2})'(L)$ in U of the projective line $L = b_1\, b_2$ in $\mathbb{P}^2$ that passes through b_1 and b_2.

The biquadratic curve defined by the equation

$$\sum_{i,\, j=0}^{2} {x_0}^i\, {x_1}^{2-i}\, A_{ij}\, {y_0}^j\, {y_1}^{2-j} = 0 \tag{4.1.3}$$

passes through $b = (\infty, \infty) = ([0 : 1], [0 : 1])$ if and only if $A_{00} = 0$. If in (4.1.3) we substitute $x_0 = y_0 = u_0$, $x_1 = u_1$, $y_1 = u_2$, corresponding to (4.1.1), (4.1.2), and divide out the common factor u_0 that appears due to $A_{00} = 0$, then we arrive at the equation

$$\sum_{0 \le i,\, j \le 2,\, i+j>0} A_{ij}\, {u_0}^{i+j-1}\, {u_1}^{2-i}\, {u_2}^{2-j} = 0, \tag{4.1.4}$$

which defines a cubic curve in $\mathbb{P}^2$, which passes through the points $b_1 = [0 : 0 : 1]$ and $b_2 = [0 : 1 : 0]$. Conversely, every cubic curve in $\mathbb{P}^2$ that contains b_1 and b_2 is of the form (4.1.4), because the homogeneous polynomials of degree 3 in (u_0, u_1, u_3) that appear on the left-hand side of (4.1.4) are precisely those for which the coefficients of ${u_2}^3$ and ${u_1}^3$ are equal to zero. Note that the division by u_0 corresponds to the fact that the line $u_0 = 0$ is blown down by ρ to the point b.

Any homogeneous polynomial q of degree 3 in three variables x_0, x_1, x_2, also called a *ternary cubic form*, is of the form

$$q(x_0,\, x_1,\, x_2) = a\, {x_1}^3 + b\, {x_1}^2\, x_2 + c\, {x_1}^2\, x_0 + d\, x_1\, {x_2}^2 + e\, x_1\, x_2\, x_0 + f\, x_1\, {x_0}^2 + g\, {x_2}^3 + h\, {x_2}^2\, x_0 + i\, x_2\, {x_0}^2 + j\, {x_0}^3, \quad (4.1.5)$$

in which $a,\, \ldots,\, j$ are the coefficients. That is, the ternary cubic forms form a 10-dimensional vector space. If $q \neq 0$, then $\{[x] \in \mathbb{P}^2 \mid q(x) = 0\}$ is called the *cubic curve in* $\mathbb{P}^2$ defined by q.

We have the following analogue of Lemma 3.3.3, which identifies the holomorphic exterior two-vector fields on $\mathbb{P}^2$ with the ternary cubic forms.

Lemma 4.1.1 *Let* $\mathrm{H}_3(\mathbb{C}^3)$ *denote the* 10*-dimensional vector space of all homogeneous polynomials on* $\mathbb{C}^3$ *of degree* 3*, and let* π *be the projection from* $\mathbb{C}^3 \setminus \{0\}$ *onto* $\mathbb{P}^2$*. Then there exists, for every* $q \in \mathrm{H}_3(\mathbb{C}^3)$*, a unique exterior two-vector field* $w = \varpi(q)$ *on* $\mathbb{P}^1 \times \mathbb{P}^1$ *such that for every* $u = (x_0,\, x_1,\, x_2) \in \mathbb{C}^3 \setminus \{0\}$*, we have*

$$q(u)\, \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} = \left(x_0\, \frac{\partial}{\partial x_0} + x_1\, \frac{\partial}{\partial x_1} + x_2\, \frac{\partial}{\partial x_2} \right) \wedge \widetilde{w}, \quad (4.1.6)$$

where $\widetilde{w}$ *is any exterior two-vector such that* $(\Lambda^2\, \mathrm{T}_u \pi)(\widetilde{w}) = w_{\pi(u)}$.

In the affine coordinates $x_0 = 1$*,* $x_1 = x$*, and* $x_2 = y$*, we have*

$$w = r(x,\, y)\, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \quad (4.1.7)$$

where $r(x,\, y) = q(1,\, x,\, y)$.

The mapping $q \mapsto W(q)$ *is a complex-linear isomorphism from* $\mathcal{P}^3(\mathbb{C}^3)$ *onto the space*

$$\mathrm{H}^0(\mathbb{P}^2,\, \mathcal{O}(\mathrm{K}^*_{\mathbb{P}^2}))$$

of all holomorphic exterior two-vector fields on $\mathbb{P}^2$.

As a consequence, the space of all holomorphic exterior two-vector fields on $\mathbb{P}^2$ *is* 10*-dimensional. There are no nonzero holomorphic area forms on* $\mathbb{P}^2$.

Proof. In affine local coordinates $(x,\, y)$ on $\mathbb{P}^2$, every holomorphic exterior two-vector field w on $\mathbb{P}^2$ is of the form (3.3.2), where r is a holomorphic function on $\mathbb{C} \times \mathbb{C}$. We have $(x,\, y) = [1 : x : y] = [1/x : 1 : y/x] = [1/y : x/y : 1]$. The substitution of variables $1/x = \xi$, $y/x = \eta$ leads to $\partial/\partial x = -\xi^2\, \partial/\partial\xi - \xi\, \eta\, \partial/\partial\eta$, $\partial/\partial y = \xi\, \partial/\partial\eta$, and therefore

$$w = -\xi^3\, r\left(\frac{1}{\xi},\, \frac{\eta}{\xi}\right)\, \frac{\partial}{\partial\xi} \wedge \frac{\partial}{\partial\eta}. \quad (4.1.8)$$

Therefore the function $(\xi,\, \eta) \mapsto \xi^3\, r(1/\xi,\, \eta/\xi)$, defined for $\xi \neq 0$, has to extend to a holomorphic function on $\mathbb{C} \times \mathbb{C}$. As in the proof of Lemma 3.3.3, this implies that for each fixed η, the function $x \mapsto r(x,\, \eta\, x)$ is a polynomial of degree at most 3. Similarly, for each fixed ξ the function $y \mapsto r(\xi\, y,\, y)$ is a polynomial of degree at most 3, and we obtain that for each $(x,\, y)$ the function $h_{x,\, y} : u \mapsto r(u\, x,\, u\, y)$ is a polynomial of degree at most 3. Now $r(x,\, y) = h_{x,\, y}(1) = \sum_{j=0}^{3} h^{(j)}_{x,\, y}(0)/j!$ shows

that r is a polynomial of degree ≤ 3 in $(x,\ y)$, and therefore $r(x,\ y) = q(1,\ x,\ y)$ for a unique $q \in \mathrm{H}_3(\mathbb{C}^3)$. This proves that the mapping W is surjective.

The last conclusion follows from Lemma 3.3.2. □

Application of Lemma 3.3.4 to the w in Lemma 4.1.1 leads to the following analogue of Lemma 2.4.5, which, among other things, implies that every smooth cubic curve in $\mathbb{P}^2$ is an elliptic curve.

Lemma 4.1.2 *Suppose that q is a cubic polynomial on $\mathbb{C}^3$, let C^{reg} be the smooth part of the curve C in $\mathbb{P}^2$ defined by the zero-set of q, and write $r(x,\ y) = q(1,\ x,\ y)$. Then the Hamiltonian vector field*

$$-\frac{\partial r(x,\ y)}{\partial y}\,\frac{\partial}{\partial x} + \frac{\partial r(x,\ y)}{\partial x}\,\frac{\partial}{\partial y}$$

on the smooth part of the curve $r(x,\ y) = 0$ in the affine $(x,\ y)$-plane in $\mathbb{P}^2$ extends to the holomorphic tangent vector field v on C^{reg} without zeros given in Lemma 3.3.4 *with $w = W(q)$ as in Lemma* 4.1.1.

Let $b \in \mathbb{P}^1 \times \mathbb{P}^1$ and let b_1, b_2 be two distinct points in $\mathbb{P}^2$. Let ρ be a birational transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ that blows up b to the projective line L passing through b_1 and b_2 and blows down the horizontal and the vertical axes L_1 and L_2 through b to b_1 and b_2, respectively. That is, if ρ_∞ is the birational transformation of $\mathbb{P}^2$ defined by (4.1.1), then $\rho = \beta \circ \rho_\infty \circ \alpha$, where α is an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ that maps b to $([0 :\ 1],\ [0 :\ 1])$ and β is an automorphism of $\mathbb{P}^2$ that maps b_1 to $[0 :\ 0 :\ 1]$ and b_2 to $[0 :\ 1 :\ 0]$. The pushforward by ρ defines a linear isomorphism from the eight-dimensional vector space of all holomorphic exterior two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$ that vanish at b onto the eight-dimensional vector space of all holomorphic exterior two-vector fields on $\mathbb{P}^2$ that vanish at b_1 and at b_2. The birational transformation ρ maps the biquadratic curves C in $\mathbb{P}^1 \times \mathbb{P}^1$ that pass through b onto the cubic curves C' in $\mathbb{P}^2$ that pass through b_1 and b_2. A repeated application of Corollary 3.3.8 yields that C' is smooth if and only if C is smooth, and if this is the case, then ρ defines an isomorphism from the elliptic curve C onto the elliptic curve C'.

Assume that q^0 and q^1 are linearly independent ternary cubic forms. The *pencil of cubic curves in $\mathbb{P}^2$ defined by q^0 and q^1* is the family of curves

$$C_{[z]} = \{[x] \in \mathbb{P}^2 \mid z_0\, q^0(x) + z_1\, q^1(x) = 0\} \tag{4.1.9}$$

in $\mathbb{P}^2$, parametrized by $[z] = [z_0 :\ z_1] \in \mathbb{P}^1$. The theorem of Bézout implies that $C \cdot C' = 3 \cdot 3 = 9$ for any pair of cubic curves $C,\ C'$ in $\mathbb{P}^2$. Therefore the pencil has nine base points when counted with multiplicities. The previous discussions lead to the following conclusions.

Proposition 4.1.3 *The birational transformation ρ induces a bijection from the collection of all pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with a smooth member and having b as a base point onto the collection of all pencils of cubic curves in $\mathbb{P}^2$ with a smooth member and having b_1 and b_2 as base points. If $\mathfrak{B}$ is a pencil of biquadratic*

curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with at least one smooth member and having b as a base point, and S is the elliptic surface obtained by successively blowing up at the base points of the anticanonical pencils, eight times, starting with $\mathfrak{B}$ as in Corollary 3.3.10*, then S is isomorphic to the elliptic surface obtained by successively blowing up at the base points of the anticanonical pencils, nine times, starting with the pencil $\rho(\mathfrak{B})$ of cubic curves in $\mathbb{P}^2$, with at least one smooth member and having b_1 and b_2 as base points.*

We conclude this section with an observation that according to Griffiths and Harris [74, p. 673] goes back to Euler (1748). See also Griffiths and Harris [74, p. 704]. Proposition 4.1.4, which is analogous to Proposition 3.1.3, shows that the nine base points of a pencil of cubic curves cannot be chosen arbitrarily in $\mathbb{P}^2$.

Proposition 4.1.4 *Let C be a smooth cubic curve in $\mathbb{P}^2$, and let D be a cubic curve in $\mathbb{P}^2$ not equal to C. If E is a cubic curve that passes through eight of the nine intersection points of D with C, where multiplicities are allowed, then E passes through the ninth intersection point of D with C. Furthermore, in this case E belongs to the pencil of cubic curves that contains C and D.*

Proof. Let r and s be cubic polynomials on $\mathbb{C}^3$ such that D and E are given by the zero-sets of r and s, respectively. Then s/r is a meromorphic function, homogeneous of degree zero, and therefore can be viewed as a meromorphic function on $\mathbb{P}^2$. The restriction φ of s/r to C is a meromorphic function on C with divisor equal to $\sum_i P_i - \sum_i Q_i = \sum_i (P_i - Q_i)$, where the P_i and the Q_i for $1 \leq i \leq 9$ denote the intersection points of E and D with C, where multiplicities are allowed. Because C is an elliptic curve, it carries a nonzero holomorphic complex one-form ω, unique up to a constant factor, such that for each $O \in C$, the mapping that assigns to each $P \in C$ the integral of ω over a curve from O to P defines an isomorphism from C onto $\mathbb{C}/P$, where P is the period lattice. See Section 2.3.1. Because $\sum_i (P_i - Q_i)$ is equal to the divisor of a rational function on C, it follows from Abel's theorem, as formulated in Griffiths and Harris [74, p. 235], that the sum over all i of the integrals of ω over curves in C running from Q_i to P_i belongs to the period lattice P. In other words, the sum over all i of the P_i is equal to the sum over all i of the Q_i, viewed as an identity in the additive group $\mathbb{C}/P \simeq C$. If $P_i = Q_i$ for all $1 \leq i \leq 8$, then this implies that $P_9 = Q_9$.

Now assume that $P_i = Q_i$ for all $1 \leq i \leq 9$. Then φ is a holomorphic function on C, and because C is compact and connected, it follows from the maximum principle that φ is equal to a constant $c \in \mathbb{C}$. The equation $s/r = c$ on C is equivalent to $s - r\,c = 0$ on the zero-set of q. Because q only has simple zeros in $\mathbb{C}^3 \setminus \{0\}$, it follows that the rational function $(s - c\,r)/q$ is holomorphic on $\mathbb{C}^3 \setminus \{0\}$, which, moreover, is homogeneous of degree zero, and therefore defines a holomorphic function ψ on $\mathbb{P}^2$. Because $\mathbb{P}^2$ is compact and connected, it follows that ψ is equal to a constant $d \in \mathbb{C}$, and the conclusion is that $s = c\,r + d\,q$. In other words, E belongs to the pencil of cubic curves that contains C and D. □

4.2 Manin Transformations

Let C be a smooth cubic curve in $\mathbb{P}^2$, that is an elliptic curve, and $b \in C$. Let $x \in C$, $x \neq b$. Because C is a cubic curve, it intersects the projective line $x\,a$ through x and b in three points (counted with multiplicities), and because x and b both belong to $C \cap (x\,b)$, the third point of intersection $\iota_{C,b}(x) \in C \cap (x\,b)$ is uniquely determined. If $x = b$, then we replace $x\,b$ by the projective line in $\mathbb{P}^2$ that is tangent to C at b, and we again obtain a unique third point of intersection $\iota_{C,b}(x) \in C \cap (x\,b)$. We have $\iota_{C,b}(b) = b$ if and only if b is a *flex point of* C, meaning that the projective line L tangent to C at b has a contact of order 3 with C, or equivalently $L \cap C = \{b\}$. This defines an involutory automorphism $\iota_{C,b} : C \to C$.

If b_1 and b_2 are two distinct points on C, then we have the nontrivial automorphism

$$\tau_{C,b_1,b_2} := \iota_{C,b_1} \circ \iota_{C,b_2} \tag{4.2.1}$$

of C.

Figures 4.2.1 and 4.2.2 Figure 4.2.1 shows (4.2.1), where C is the Hesse curve (11.1.1) for $z_0 = 1$, $z_1 = 5$, and b_1 and b_2 are randomly chosen points on C.

In Figure 4.2.2 the curve C is the Weierstrass curve $x_2{}^2 - 4\,x_1{}^3 + 2\,x_1 - 2 = 0$, and b_1 is the point on it at infinity. In Proposition 2.5.6, the point b_2 to which the point at infinity is mapped has the affine coordinates $(X,\ Y)$. In Figure 4.2.2 it appears that $Y = 0$, a coincidence related to the fact that "arbitrary choices" of humans often are not so arbitrary. We have $Y = 0$ if and only if the Manin transformation has order two.

Lemma 4.2.1 *For any $b \in C$, the automorphism $\iota_{C,b}$ is an inversion on the elliptic curve C. For any $b_1,\ b_2 \in C$, τ_{C,b_1,b_2} is the unique translation on C that maps b_1 to b_2.*

Proof. The fixed points of $\iota_{C,b}$ are the points $x \in C$ such that the projective line $x\,b$ through b and x is tangent to C at the point x. There are four such points, and therefore $\iota_{C,b}$ is a nontrivial involution on C with fixed points that is an inversion on C. It follows that τ_{C,b_1,b_2}, as the composition of two inversions, is a translation on the elliptic curve C.

The projective line $b_1\,b_2$ through b_1 and b_2 intersects C in b_1, b_2, and a unique third point $c \in C \cap (b_1\,b_2)$. We have $\iota_{C,b_2}(b_1) = c$ and $\iota_{C,b_1}(c) = b_2$, hence $\tau_{C,b_1,b_2}(b_1) = b_2$. □

Remark 4.2.2. Lemma 4.2.1 can be found in Poincaré [164, pp. 181, 182], together with the observation that if C is defined over $\mathbb{Q}$ and b, b_1, b_2 are rational points on C, then $\iota_{C,b}$ and τ_{C,b_1,b_2} map rational points on C to rational points on C. Poincaré did not give any reference, but A. Weil [210, p. 108] observed that the geometric construction of $\iota_{C,b}$ and τ_{C,b_1,b_2}, together with its application to the generation of rational points on a cubic plane curve, had been found in its full generality in the late 1670s by Newton [151, pp. 112, 114 (Latin) = pp. 113, 115 (English)]. These observations of Newton apparently became known only much later. ⊘

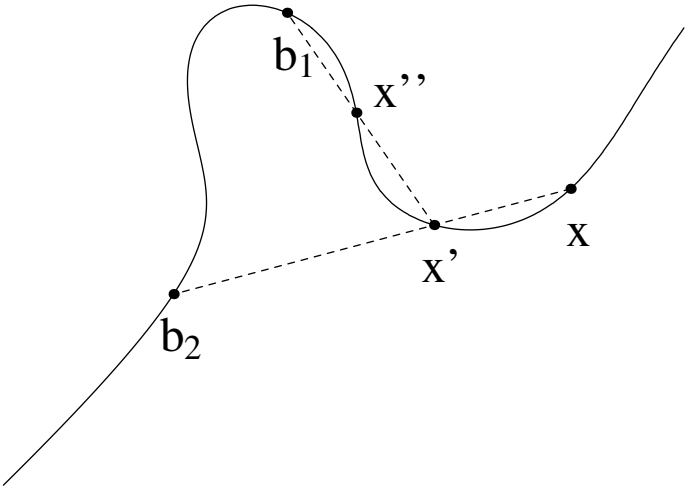


Fig. 4.2.1 The transformation $\tau_{C,\,b_1,\,b_2} : x \mapsto x''$ on a Hesse curve.

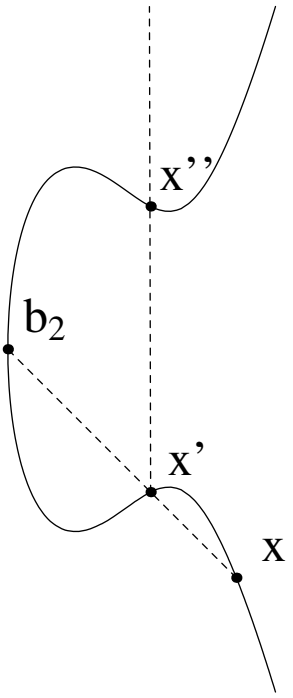


Fig. 4.2.2 $\tau_{C,\,\infty,\,b_2} : x \mapsto x''$ on the Weierstrass curve with $g_2 = 2$, $g_3 = -2$.

Now suppose that $\mathfrak{C}$ is a pencil of cubic curves in $\mathbb{P}^2$ with at least one smooth member. Let $b \in \mathbb{P}^2$ be a base point of $\mathfrak{C}$. If $x \in \mathbb{P}^2$ is not a base point of $\mathfrak{C}$, there is a unique member C of $\mathfrak{C}$ such that $x \in C$, and if C is smooth, we can define $\iota_{\mathfrak{C},b}(x) := \iota_{C,b}(x) \in C$. This defines a birational transformation $\iota = \iota_{\mathfrak{C},b}$ from $\mathbb{P}^2$ to $\mathbb{P}^2$, which is an involution in the sense that $\iota \circ \iota$, wherever defined, is equal to the identity in $\mathbb{P}^2$. If b_1 and b_2 are two distinct base points of $\mathfrak{C}$, then the nontrivial birational transformation

$$\tau_{\mathfrak{C},b_1,b_2} := \iota_{\mathfrak{C},b_1} \circ \iota_{\mathfrak{C},b_2} : \mathbb{P}^2 \to \mathbb{P}^2 \tag{4.2.2}$$

will be called the *Manin transformation defined by* $\mathfrak{C}$*,* b_1*, and* b_2.

Remark 4.2.3. In projective coordinates, the definition of the Manin transformation leads to an explicit formula for the birational transformation, in terms of the coefficients of two of the cubic forms that define the pencil $\mathfrak{C}$ and the projective coordinates of the base points b_1 and b_2. We like to emphasize that here the information of the two base points b_1 and b_2 is essential.

Given the pencil $\mathfrak{C}$, in general one has no explicit formula for its base points, because the determination of the base points involves the solution of a polynomial equation in one variable of degree 9. In contrast with this, the explicit formula for the QRT map in Lemma 1.1.1 is given in terms of only the coefficients of two of the biquadratic forms that define the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, and no information on the base points of $\mathfrak{B}$ is needed.

However, in many special cases one has explicit formulas for pairs of distinct base points b_1, b_2 of the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$, and then one also has an explicit rational formula for the Manin transformation $\tau_{\mathfrak{C},b_1,b_2}$. ⊘

Let S be the rational elliptic surface obtained by successively blowing up at the base points of the anticanonical pencil, nine times, starting with the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$, and let $\pi : S \to \mathbb{P}^2$ be the corresponding projection. For equivalent characterizations of rational elliptic surfaces, see Theorem 9.1.3. The same proof as for Corollary 3.4.2 yields that for each base point b of $\mathfrak{C}$, there is a unique complex analytic diffeomorphism $\iota^S_{\mathfrak{C},b} : S \to S$ such that $\pi(\iota^S_{\mathfrak{C},b}(s)) = \iota_{\mathfrak{C},b}(\pi(s))$ for all $s \in S$ such that $\iota_{\mathfrak{C},b}$ is well defined at the point $\pi(s)$. As a consequence, if b_1 and b_2 are two distinct base points of the pencil $\mathfrak{C}$, then

$$\tau^S_{\mathfrak{C},b_1,b_2} := \iota^S_{\mathfrak{C},b_1} \circ \iota^S_{\mathfrak{C},b_2} \tag{4.2.3}$$

is the unique automorphism of S such that $\pi \circ \tau^S_{\mathfrak{C},b_1,b_2} = \tau_{\mathfrak{C},b_1,b_2} \circ \pi$ wherever the right-hand side is defined. We will call $\tau^S_{\mathfrak{C},b_1,b_2}$ the *Manin automorphism of* S *defined by the base points* b_1*,* b_2 *of the pencil* $\mathfrak{C}$. We have $\tau^S_{\mathfrak{C},b_1,b_2} \in \mathrm{Aut}(S)^+_\kappa$, that is, $\tau^S_{\mathfrak{C},b_1,b_2}$ preserves the fibers of the elliptic fibration $\kappa : S \to P$ and acts as a translation on each smooth fiber.

Lemma 4.2.4 *If* E_1 *and* E_2 *are* -1 *curves in* S*, then there is a unique* $\alpha = \alpha_{E_1,E_2} \in \mathrm{Aut}(S)^+_\kappa$ *such that* $\alpha(E_1) = E_2$.

If E_1 and E_2 are the last -1 curves in S that appeared in the blowing up over b_1 and b_2, respectively, then E_1 and E_2 are disjoint and $\tau^S_{\mathfrak{C},b_1,b_2} = \alpha_{E_1,E_2}$.

Proof. The first statement follows from Lemma 9.2.1 and Lemma 7.1.1.

Now let E_1 and E_2 be the last -1 curves in S that appeared in the blowing up over b_1 and b_2, respectively. Then $\pi(E_1) = \{b_1\}$ and $\pi(E_2) = \{b_2\}$. Because $b_1 \neq b_2$, hence $\pi(E_1 \cap E_2) \subset \pi(E_1) \cap \pi(E_2) = \emptyset$, we have that $E_1 \cap E_2 = \emptyset$.

The smooth fibers of the elliptic fibration of S are the proper transforms $\widetilde{C} = \pi'(C)$ of the smooth members C of the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$, where $\pi|_{\widetilde{C}}$ is an isomorphism from $\widetilde{C}$ onto C. If $s \in \widetilde{C} \cap E_1$ then it follows from Lemma 4.2.1 that

$$\pi \circ \tau^S_{\mathfrak{C},b_1,b_2}(s) = \tau_{\mathfrak{C},b_1,b_2} \circ \pi(s) = \tau_{\mathfrak{C},b_1,b_2} \circ \pi(s) = \tau_{\mathfrak{C},b_1,b_2}(b_1) = b_2,$$

which implies that $\tau^S_{\mathfrak{C},b_1,b_2}$ maps the set E_1^s of all intersection points with E_1 of the smooth fibers of κ into $\pi^{-1}(\{b_2\})$. Because every -1 curve in S is a holomorphic section of κ, see Lemma 9.2.1, every element of E_1 belongs to a unique fiber of κ. Because there are only finitely many nonsmooth fibers, E_1^s is dense in E_1, and because $\tau^S_{\mathfrak{C},b_1,b_2}$ is continuous and $\pi^{-1}(\{b_2\})$ is a closed subset of S, it follows that $\tau^S_{\mathfrak{C},b_1,b_2}(E_1) \subset \pi^{-1}(\{b_2\})$. Because $\tau^S_{\mathfrak{C},b_1,b_2}(E_1)$ is a -1 curve in S and E_2 is the unique -1 curve in $\pi^{-1}(\{b_2\})$, it follows that $\tau^S_{\mathfrak{C},b_1,b_2}(E_1) = E_2$. □

Remark 4.2.5. The birational transformations $\iota_{\mathfrak{C},b}$ of $\mathbb{P}^2$ and automorphisms $\iota^S_{\mathfrak{C},b}$ of S, where b runs over the base points of the pencil $\mathfrak{C}$ of cubic curves, correspond to the transformations T_i and R_i in Manin [129, p. 95], respectively. The automorphisms $\tau^S_{\mathfrak{C},b_1,b_2} = \alpha_{E_1,E_2}$ correspond to the automorphisms $R_j \circ R_1$ in Manin [129, p. 96].

Actually, in Manin [129, §4] it is assumed that the pencil $\mathfrak{C}$ of cubic curves has nine distinct base points b_i, $1 \leq i \leq 9$, and that $\mathfrak{C}$ has no reducible members. Under these assumptions Manin [129, Theorem 6] obtained that the subgroup of $\mathrm{Aut}(S)^+_\kappa$ generated by the automorphisms $\tau_{\mathfrak{C},b_i,b_j}$, $1 \leq i,\ j \leq 9$, has index 3 in $\mathrm{Aut}(S)^+_\kappa$. See Theorem 4.3.4 below.

The following lemma is a converse to Lemma 4.2.4.

Lemma 4.2.6 *Let S be a rational elliptic surface, and suppose that E_1 and E_2 are two disjoint -1 curves in S. Then there exists a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with at least one smooth member, and two distinct base points b_1 and b_2, such that $\tau^S_{\mathfrak{C},b_1,b_2}(E_1) = E_2$.*

Proof. According to Remark 9.1.5 we may start the blowing down from S to $\mathbb{P}^2$, in the proof of (c) ⇒ (a) of Theorem 9.1.3, by first blowing down the -1 curve E_1 in $S = S_0$, and then the image $\pi_1(E_2)$ of E_2 in S_1, which is a -1 curve in S_1 because π_1 is an isomorphism from $S \setminus E_1$ onto $S_1 \setminus \pi_1(E_1)$ and $E_2 \subset S \setminus E_1$ because $E_1 \cap E_2 = \emptyset$. It follows that in the blowing-up procedure at base points of the anticanonical pencils from $\mathbb{P}^2$ to S, E_1 and E_2 belong to the set of -1 curves in S that are obtained at the last blowing up over base points of the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$. Because for each base point of $\mathfrak{P}$ there is only one such last -1 curve,

see Lemma 3.2.7 (iii) with $k = 1$, there are two distinct base points b_1 and b_2 of $\mathfrak{C}$ such that the projection $\pi : S \to \mathbb{P}^2$ maps E_1 to b_1 and E_2 to b_2. It follows from Lemma 4.2.4 that $\tau^S_{\mathfrak{C}, b_1, b_2}(E_1) = \alpha_{E_1, E_2}(E_1) = E_2$. □

4.3 Manin QRT Automorphisms

The following lemma, in which the QRT automorphisms are identified with Manin automorphisms, was given by Veselov [199, p. 35] and by Tsuda [196, Figures 1 and 2], without reference to Manin [129].

Lemma 4.3.1 *Let $\mathfrak{B}$ be a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with at least one smooth member, and b a base point of $\mathfrak{B}$. Let $\mathfrak{C} = \rho(\mathfrak{B})$ be the corresponding pencil of cubic curves in $\mathbb{P}^2$ according to Proposition* 4.1.3*, with b_1 and b_2 as base points. Then ρ conjugates the QRT transformation $\tau = \tau_{\mathfrak{B}}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\mathfrak{B}$ with the Manin transformation $\tau_{\mathfrak{C}, b_1, b_2}$, in the sense that*

$$\rho \circ \tau_{\mathfrak{B}} = \tau_{\mathfrak{C}, b_1, b_2} \circ \rho. \tag{4.3.1}$$

The rational elliptic surface S that is obtained by successively blowing up at the base points (eight times) of the anticanonical pencils, starting with the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, is equal to the surface that is obtained by successively blowing up at the base points (nine times) of the anticanonical pencils, starting with the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$.

Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi' : S \to \mathbb{P}^2$ denote the corresponding projections. Let $\tau^S_{\mathfrak{B}}$ and $\tau^S_{\mathfrak{C}, b_1, b_2}$ be the automorphism of S such that $\pi \circ \tau^S_{\mathfrak{B}} = \tau_{\mathfrak{B}} \circ \pi$ and $\pi' \circ \tau^S_{\mathfrak{C}, b_1, b_2} = \tau_{\mathfrak{C}, b_1, b_2} \circ \pi'$, respectively. Then $\tau^S_{\mathfrak{B}} = \tau^S_{\mathfrak{C}, b_1, b_2}$.

Proof. Let $\pi_b : U \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the base point $b = (x, y)$ of the pencil $\mathfrak{B}$ as in Section 4.1. Then $\psi_{b_1, b_2} : U \to \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ at the base points b_1 and b_2 of the pencil $\mathfrak{C}$, and the birational transformation $\rho := \psi_{b_1, b_2} \circ {\pi_b}^{-1}$ from $\mathbb{P}^1 \times \mathbb{P}^1$ maps the members of the pencil $\mathfrak{B}$ to the members of the pencil $\mathfrak{C}$; see Proposition 4.1.3. Furthermore, ρ maps the horizontal axes $\mathbb{P}^1 \times \{y'\}$, $y' \in \mathbb{P}^1$, in $\mathbb{P}^1 \times \mathbb{P}^1$ to the projective lines through the point b_2, because the vertical axis $\{x\} \times \mathbb{P}^1$ through b, which is a holomorphic section of the fibration of $\mathbb{P}^1 \to \mathbb{P}^1$ by the horizontal axes, is blown down by ρ to the point b_2. Therefore ρ conjugates the horizontal switch ι_1, which on each member of $\mathfrak{B}$ interchanges the two intersection points on every horizontal axis, with the mapping $\iota_{\mathfrak{C}, b_2}$, which on each member of $\mathfrak{C}$ interchanges the two intersection points with the projective line through b_2 other than b_2. Similarly ρ maps the vertical axes in $\mathbb{P}^1 \times \mathbb{P}^1$ to the projective lines through the point b_1, and ρ conjugates the vertical switch ι_2 with $\iota_{\mathfrak{C}, b_1}$. It follows that ρ conjugates the QRT mapping $\tau_{\mathfrak{B}} = \iota_2 \circ \iota_1$ with $\tau_{\mathfrak{C}, b_1, b_2} = \iota_{\mathfrak{C}, b_1} \circ \iota_{\mathfrak{C}, b_2}$, that is, we have (4.3.1).

The second statement follows from the observation that the anticanonical pencil $\mathfrak{A}$ in U obtained from $\mathfrak{B}$ by blowing up the base point $b \in \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathfrak{B}$ is equal to the

aniticanonical pencil in U obtained from $\mathfrak{C}$ by blowing up the base points $b_1,\ b_2 \in \mathbb{P}^2$ of $\mathfrak{C}$. Therefore the surface $\theta : S \to U$ obtained by successively blowing up at base points of the anticanonical pencils until there are no longer any base points, starting with the anticanonical pencil $\mathfrak{A}$ of curves in U, is equal both to the surface obtained by successively blowing up at base points starting with the anticanonical pencil $\mathfrak{B}$ of curves in $\mathbb{P}^1 \times \mathbb{P}^1$, where we begin with b, and to the surface obtained by successively blowing up at base points starting with the anticanonical pencil $\mathfrak{C}$ of curves in $\mathbb{P}^2$, where we begin with b_1 and b_2.

In the category of birational transformations, equation (4.3.1), where $\rho = \psi_{b_1,\,b_2} \circ {\pi_b}^{-1}$, is equivalent to

$$\pi_b{}^{-1} \circ \tau_{\mathfrak{B}} \circ \pi_b = \psi_{b_1,\,b_2}{}^{-1} \circ \tau_{\mathfrak{C},\,b_1,\,b_2} \circ \psi_{b_1,\,b_2}.$$

Therefore we have the identity

$$\begin{aligned}\tau^S_{\mathfrak{B}} &= \pi^{-1} \circ \tau_{\mathfrak{B}} \circ \pi = \theta^{-1} \circ \pi_b{}^{-1} \circ \tau_{\mathfrak{B}} \circ \pi_b \circ \theta \\ &= \theta^{-1} \circ \psi_{b_1,\,b_2}{}^{-1} \circ \tau_{\mathfrak{C},\,b_1,\,b_2} \circ \psi_{b_1,\,b_2} \circ \theta = \pi'^{-1} \circ \tau_{\mathfrak{C},\,b_1,\,b_2} \circ \pi' = \tau^S_{\mathfrak{C},\,b_1,\,b_2}\end{aligned}$$

of birational transformations of S. Therefore the automorphisms $\tau^S_{\mathfrak{B}}$ and $\tau^S_{\mathfrak{C},\,b_1,\,b_2}$ of S agree on the dense subset of S of all points where all the intermediate birational transformations in the above sequence of identities are defined. Because of the continuity of both automorphisms, it follows that these automorphisms are the same. □

For various characterizations of rational elliptic surfaces, see Theorem 9.1.3. On each elliptic surface $\varphi : S \to C$, each smooth fiber of φ is an elliptic curve; see Section 6.1. The group of automorphisms of S that act as translations on each smooth fiber of φ is denoted by $\mathrm{Aut}(S)^+_\varphi$. It acts freely and transitively of the set Σ_φ of all sections of φ; see Lemma 7.1.1. In other words, after a choice of a holomorphic section $E_0 \in \Sigma_\varphi$, the mapping $\alpha \mapsto \alpha(E_0)$ is bijective from $\mathrm{Aut}(S)^+_\varphi$ onto Σ_φ. By means of this bijection the group structure of $\mathrm{Aut}(S)^+_\varphi$ can be carried over to Σ_φ, and with this group structure Σ_φ is called the Mordell–Weil group of S. A curve in a rational elliptic surface is a -1 curve if and only if it is a holomorphic section of the elliptic fibration; see Lemma 9.2.1. With this background information, the following theorem can be viewed as a characterization of the QRT automorphisms and the Manin automorphisms in terms of the Mordell–Weil group of the rational elliptic surface.

Theorem 4.3.2 *Let $\kappa : S \to P \simeq \mathbb{P}^1$ be a rational elliptic surface, and let $\alpha \in \mathrm{Aut}(S)^+_\kappa$. Then the following four conditions for α are equivalent:*

(i) *There exists a pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that S is equal to a successive blowing up at base points of the anticanonical pencils starting with $\mathfrak{B}$, and α is equal to the QRT automorphism $\tau^S_{\mathfrak{B}}$ of S defined by $\mathfrak{B}$.*

(ii) *There exists a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with two distinct base points b_1 and b_2 in $\mathbb{P}^2$ such that S is equal to a successive blowing up at base points of the*

anticanonical pencils starting with $\mathfrak{C}$*, and* α *is equal to the Manin automorphism* $\tau^S_{\mathfrak{C},b_1,b_2}$ *of* S *defined by* $\mathfrak{C}$*,* b_1 *and* b_2*.*

(iii) *For some (every)* -1 *curve* E *in* S*, we have* $\alpha(E) \cap E = \emptyset$*. Note that Lemma* 9.2.1 *implies that* E *is a* -1 *curve in* S *if and only if* E *is a holomorphic section of* κ*.*

(iv) *The automorphism* α *has no fixed points in the set* S^{reg} *of all regular points of* κ *in* S*.*

Proof. The equivalence between (i) and (ii) follows from Lemma 4.3.1. The equivalence between (ii) and (iii) follows from Lemma 4.2.4 and Lemma 4.2.6 with $E_1 = E$ and $E_2 = \alpha(E)$. Finally, the equivalence between (iii) and (iv) follows from the discussion in Definition 7.4.5, in combination with the fact that -1 curves and sections are the same in rational elliptic surfaces. □

I propose to call any $\alpha \in \mathrm{Aut}(S)^+_\kappa$ that satisfies any of the equivalent conditions (i)–(iv) in Theorem 4.3.2 a *Manin QRT automorphism* of the rational elliptic surface $\kappa : S \to P$.

Theorem 4.3.3 *Let* $\kappa : S \to P \simeq \mathbb{P}^1$ *be a rational elliptic surface. Then the Manin QRT automorphisms of* S *generate the Mordell–Weil group* $\mathrm{Aut}(S)^+_\kappa$ *of* S*. There are at most* 240 *Manin QRT automorphisms of* S*, with equality if and only if all the singular fibers are irreducible, that is, of type* I_1 *or* II*. If* α *is a Manin QRT automorphism of* S*, then the number of* k*-periodic fibers, counted with multiplicities, of* α *is equal to*

$$\nu(\alpha^k) = -1 + \frac{1}{2} \sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha^k) + \left[1 - \frac{1}{2} \sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha) \right] k^2. \qquad (4.3.2)$$

Furthermore, every nontrivial element α *of* $\mathrm{Aut}(S)^+_\kappa$ *of finite order is a Manin QRT automorphism of* S*, and conversely a Manin QRT automorphism of* S *is of finite order if and only if the sum of the contributions* $\mathrm{contr}_r(\alpha)$ *of the reducible fibers* S_r *to* α *is equal to* 2*.*

Proof. Let $e_0 \in \Lambda = \mathrm{NS}(S)$ be the homology class of a holomorphic section E_0 of κ. According to Theorem 7.2.7, the mapping $\alpha \mapsto \alpha^\Lambda(e_0) - e_0 + Q^{\mathrm{irr}}$ is an isomorphism from $\mathrm{Aut}(S)^+_\kappa$ onto Q/Q^{irr}, and therefore $\alpha \mapsto q(\alpha) := \alpha^\Lambda(e_0) - e_0 + \mathbb{Z}\, f$ is an injective mapping q from $\mathrm{Aut}(S)^+_\kappa$ to $Q := f^\perp/\mathbb{Z}, f$. If $\alpha \neq 1$, then $\alpha(E_0) \neq E_0$, and because E_0 and $\alpha(E_0)$ are irreducible curves, it follows that $\alpha(E_0) \cdot E_0 \geq 0$ with equality if and only if $\alpha(E_0) \cap E_0 = \emptyset$. Therefore $q(\alpha) \cdot q(\alpha) = \alpha(E_0) \cdot \alpha(E_0) - 2\alpha(E_0) \cdot E_0 + E_0 \cdot E_0 = 2\, E_0 \cdot E_0 - 2\alpha(E_0) \cdot E_0 = -2\,(1 + \alpha(E_0) \cdot E_0) \leq -2$, with equality if and only if α is a Manin QRT automorphism of S. Because Q with respect to minus the intersection form is a root lattice of type E_8, see Lemma 9.2.3, the elements in it of norm squared equal to 2 are the roots, of which there are 240, see Bourbaki [23, p. 213 and 268], it follows that there are at most 240 Manin QRT automorphisms of S. Furthermore, if κ has no reducible fibers, then $Q^{\mathrm{irr}} = \{0\}$, the mapping q is an isomorphism from $\mathrm{Aut}(S)^+_\kappa$ onto Q, and S has 240 Manin QRT automorphisms. On the other hand, if Θ is an irreducible component of

a reducible fiber S_r of κ, then Lemma 6.2.10 implies that $\Theta \cdot \Theta = -2$. Furthermore, $(\alpha^{\Lambda}(e_0) - e_0) \cdot [\Theta]$ is equal to 0, -1, or 1, because E_0 and $\alpha(E_0)$ intersect S_r in exactly one of its irreducible components, and it follows that there is no $\alpha \in \mathrm{Aut}(S)^+_\kappa$ such that $q(\alpha) = [\Theta] + \mathbb{Z} f$. This shows that the number of Manin QRT automorphisms of S is strictly smaller than 240 if κ has reducible fibers.

Theorem 4.3.2 (iii) says that $\alpha \in \mathrm{Aut}(S)^+_\kappa$ is a Manin QRT automorphism of S if and only if $\nu(\alpha) = 0$, and Lemma 9.1.2 (iii) implies that $\chi(S, \mathcal{O}) = 1$. Therefore (4.3.2) follows from (7.5.2), and the equivalence (i) $\Leftrightarrow$ (iii) in Corollary 7.5.5 says that $\alpha \in \mathrm{Aut}(S)^+_\kappa$ has finite order if and only if α is a Manin QRT automorphism and the sum of the contributions to α is equal to 2.

The Manin QRT automorphisms of S are the elements of the Mordell–Weil group of S that map some (any) holomorphic section to a disjoint one. Theorem 2.5 of Oguiso and Shioda [155] says that for any rational elliptic surface S, these elements generate the Mordell–Weil group of S. Their proof runs as follows. If B is a subgroup of a commutative group A, and a_i and b_j are elements of A and B such that the $a_i + B$ generate A/B and the b_j generate B, respectively, then the a_i and b_j generate A. Because the nontrivial elements of the torsion subgroup T of $A := \mathrm{Aut}(S)^+_\kappa$ are Manin QRT automorphisms of S, it suffices to prove that there are Manin QRT automorphisms α_i of S such that the $\alpha_i + T$ generate A/T.

Let π^0 denote the orthogonal projection from Q to the orthogonal complement of Q^{irr} in $Q \otimes \mathbb{Q}$. Then Lemma 7.5.1, Theorem 7.6.6, and Lemma 9.2.3 imply that $\mu^0 : \alpha \mapsto \pi^0(\alpha^{\Lambda}(e) - e + \mathbb{Z} f)$ is an isomorphism from A/T onto the dual lattice $(Q^0)^*$ of Q^0, where Q^0, the narrow Mordell–Weil lattice, is the orthogonal complement of Q^{irr} in Q. In view of (7.5.1) with $\chi(S, \mathcal{O}) = 1$ it therefore suffices to prove that $(Q^0)^*$ is generated by elements $c = \mu^0(\alpha) \neq 0$ such that $1 > \nu(\alpha) = -1 - (c \cdot c)/2 + m/2$, or equivalently $-c \cdot c + m < 4$, where m denotes the maximum over all elements β of the Mordell–Weil group of the sum of contributions to β of the reducible fibers. The contributions are given in Lemma 7.5.3 in terms of the intersection diagram of the reducible fiber, and the numbers m can be read off from the list of Oguiso and Shioda [155, pp. 84–86]. For any positive definite lattice Λ, let $M(\Lambda)$ denote the minimum over all $\mathbb{Z}$-bases c_i of Λ of $\max_i \; c_i \cdot c_i$. It suffices to prove that $M((Q^0)^*) + m < 4$, where $(Q^0)^*$ is the Mordell–Weil lattice with respect to minus the intersection form.

In each case of the list, the narrow Mordell–Weil lattice Q^0, provided with minus the intersection form, is an orthogonal direct sum of root lattices of type A_n, D_n, E_n, or $\langle k \rangle$. In 8 of the 74 cases Q^0 has a different structure, but then the intersection matrix with respect to a suitable $\mathbb{Z}$-basis is explicitly given. Here $\langle k \rangle = \mathbb{Z}$ with $1 \cdot 1 = k$, when $\langle k \rangle^* = \langle 1/k \rangle$, and $M = 1/k$. Note that only even k occur because Q is an even lattice, and $\mathrm{A}_1 \simeq \langle 2 \rangle$.

The dual of the root lattice is the weight lattice, see Bourbaki [23, p. 167], and the fundamental weights $\varpi = \varpi_i$ form a $\mathbb{Z}$-basis of the weights. However, their norms squared $\varpi \cdot \varpi$ become too large. For the type A_l with the ϖ_i as in Bourbaki [23, p. 250], the weights ϖ_1 and $\varpi_i - \varpi_{i-1}$, $2 \leq i \leq l$, form another $\mathbb{Z}$-basis of the weight lattice A_l^*, all of whose elements have norm squared equal to $l/(l+1)$, and $M(\mathrm{A}_l^*) = l/(l+1)$. For the type D_l, $l \geq 4$, with the ϖ_i as in Bourbaki [23, p.

256, 257], the weights ϖ_1, $\varpi_i - \varpi_{i-1}$ for $2 \le i \le l-2$, $\varpi_l - \varpi_{l-1}$, and ϖ_{l-1} form a $\mathbb{Z}$-basis of the weight lattice D_n^*, all of whose elements have norm squared equal to 1, except for the last one, where $\varpi_{l-1} \cdot \varpi_{l-1} = l/4$, and $M(\mathrm{D}_l^*) = l/4$. For the type E_6, with the ϖ_i as in Bourbaki [23, p. 261], the weights ϖ_6, $\varpi_5 - \varpi_6$, $\varpi_4 - \varpi_5$, $\varpi_3 - \varpi_4$, $\varpi_2 - \varpi_3$, and ϖ_1 form a $\mathbb{Z}$-basis of the weight lattice E_6^*, all of whose elements have norm squared equal to $4/3$, and $M(\mathrm{E}_6^*) = 4/3$. For the type E_7, with the ϖ_i as in Bourbaki [23, p. 265], the weights ϖ_7, $\varpi_6 - \varpi_7$, $\varpi_5 - \varpi_6$, $\varpi_4 - \varpi_5$, $\varpi_1 - \varpi_7$, $\varpi_2 - \varpi_1$, and $\varpi_3 - \varpi_2$ form a $\mathbb{Z}$-basis of the weight lattice E_7^*, all of whose elements have norm squared equal to $3/2$, and $M(\mathrm{E}_7^*) = 3/2$. Finally, $\mathrm{E}_8^* = \mathrm{E}_8$, and $M(\mathrm{E}_8^*) = M(\mathrm{E}_8) = 2$. With this information, the number $M((Q^0)^*)$ can be determined for each case, and Oguiso and Shioda verified that always $M((Q^0)^*) + m < 4$, thus completing the proof of the statement that the Manin QRT automorphisms generate the Mordell–Weil group. □

The following theorem is due to Manin [129, §4, Theorem 6]. See Shioda [184, Theorem 10.11] for a different proof. Theorem 4.3.4 says that *for the given pencil* $\mathfrak{C}$ *of planar cubic curves*, the group generated by the Manin automorphisms defined by the pencil $\mathfrak{C}$ is almost equal to $\mathrm{Aut}(S)_\kappa^+$, but not quite.

Theorem 4.3.4 *Let* $\mathfrak{C}$ *be a pencil of cubic curves in* $\mathbb{P}^2$ *such that* (i) $\mathfrak{C}$ *has nine distinct base points* b_i, $1 \le i \le 9$, *and* (ii) *every member of* $\mathfrak{C}$ *is irreducible.*

Let $\kappa : S \to P \simeq \mathbb{P}^1$ *be the rational elliptic surface obtained by blowing up* $\mathbb{P}^2$ *in the points* b_i, $1 \le i \le 9$. *Then Manin's automorphisms* $\tau^S_{\mathfrak{C}, b_i, b_j}$ *for* $1 \le i,\ j \le 9$, $i \ne j$, *generate a subgroup of index* 3 *in* $\mathrm{Aut}(S)_\kappa^+$.

Proof. Because of (i) no base point is a singular point of any member of $\mathfrak{C}$, and therefore each fiber of κ is isomorphic to a member of $\mathfrak{C}$, and therefore irreducible in view of (ii). It follows that $L^{\mathrm{irr}} = \mathbb{Z} f$, and therefore Theorem 7.2.7 implies that the mapping $\alpha \mapsto \mathrm{H}(\alpha(E)) - \mathrm{H}(E) + \mathbb{Z} f$ is an isomorphism from $\mathrm{Aut}(S)_\kappa^+$ onto $f^\perp/\mathbb{Z} f = Q$, which does not depend on the section E. Here $\mathrm{H}(D) \in \mathrm{H}_2(S, \mathbb{Z})$ denotes the homology class of any divisor D in S. In the notation of the proof of Lemma 9.2.3, Manin's automorphisms $\tau^S_{\mathfrak{C}, b_i, b_j}$ maps e_i to e_j and therefore corresponds to the element $e_j - e_i + \mathbb{Z} f$ of Q. This subgroup is already generated by the elements $\alpha_k + \mathbb{Z} f = e_k - e_{k+1}$, $1 \le k \le 8$, because if $j < i$ then

$$e_j - e_i = \sum_{k=j}^{i-1} (e_k - e_{k+1}).$$

In the proof of Lemma 9.2.3 we have seen that the $\alpha_k + \mathbb{Z} f$ with $1 \le k \le 7$, together with $\alpha_0 + \mathbb{Z} f$, form a $\mathbb{Z}$-basis of Q. Furthermore, α_8 is equal to $f - 3\alpha_0$ modulo an integral linear combination of the α_i with $1 \le i \le 7$. This completes the proof of the theorem. □

For the conditions (i) and (ii) in Theorem 4.3.4, see Proposition 9.2.22.

Question Let $\mathfrak{C}$ be a given pencil of cubic curves in $\mathbb{P}^2$ with at least one smooth member. Do Manin's automorphisms $\tau^S_{\mathfrak{C}, b, b'}$, for all pairs of base points b, b' of $\mathfrak{C}$

always generate a subgroup of finite index in the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$, that is, even if conditions (i) and (ii) in Theorem 4.3.4 are not both satisfied?

4.4 Aronhold's Invariants

Aronhold [6] found the two polynomial projective invariants S and T for polynomials q of degree three in three variables. Such q have been given explicitly in (4.1.5). The invariants S and T are homogeneous polynomials in the coefficients of q of degree 4 and 6, respectively, whereas the equation $R := S^3 - T^2 = 0$ is the discriminant of q, characterizing when the cubic curve $q = 0$ in $\mathbb{P}^2$ is singular. The only difference with Aronhold's notation is a change of sign in R. The explicit recipe for S and T, called the *Aronhold invariants of ternary cubics*, is given in terms of the symbolic manipulations descibed in the remainder of this subsection. Sylvester [191] proved that Aronhold's invariants S and T are basic in the sense that every polynomial projective invariant for ternary cubics is a polynomial in S and T.

For any $n \times n$ matrix a, let $\mathrm{C}(a)$ denote *Cramer's comatrix* of a, for which $\mathrm{C}(a) \circ a$ is equal to the determinant of a times the identity matrix. The mapping $a \mapsto \mathrm{C}(a)$, from the space M_n of all $n \times n$ matrices to itself, is a homogeneous polynomial mapping of degree $n - 1$. For $n = 2$ we have a unique symmetric bilinear mapping $(a, b) \mapsto (a\, b)$ from $\mathrm{M}_3 \times \mathrm{M}_3$ to M_3 such that $(a\, a)/2 = \mathrm{C}(a)$ for all $a \in \mathrm{M}_3$. Explicitly,

$$\begin{aligned}(ab)_{i,\,j} :=& a_{i+1,\,j+1}\, b_{i+2,\,j+2} + b_{i+1,\,j+1}\, a_{i+2,\,j+2} - a_{i+1,\,j+2}\, b_{i+2,\,j+1} \\ &- b_{i+1,\,j+2}\, a_{i+2,\,j+1},\end{aligned} \tag{4.4.1}$$

where the indices are counted modulo 3. Note that if a is symmetric, then $\mathrm{C}(a)$ is symmetric, and therefore $(a\, b)$ is symmetric if a and b are symmetric.

If $q(x_0,\, x_1,\, x_2)$ is a homogeneous polynomial of degree 3 in three variables, then we define the coefficients $a_{i,\,j,\,k} = a_{i,\,j,\,k}(q)$ by

$$a_{i,\,j,\,k} = 6^{-1}\partial^3 q/\partial x_i\, \partial x_j\, \partial x_k, \quad 0 \le i,\ j,\ k \le 2, \tag{4.4.2}$$

and for each i, a_i denotes the symmetric 3×3 matrix defined by

$$(a_i)_{j,\,k} = a_{i,\,j,\,k}, \quad 0 \le j,\ k \le 2. \tag{4.4.3}$$

Then

$$S = S(q) := \sum_{i,\,j=0}^{2} (a_k\, a_k)_{i,\,j}\, (a_i\, a_j)_{k,\,k} = 2 \sum_{i,\,j=0}^{2} (a_k\, a_l)_{i,\,j}\, (a_i\, a_j)_{k,\,l}, \tag{4.4.4}$$

where the right hand side is independent of the choice of $k \in \{0,\, 1,\, 2\}$ or $k,\, l \in \{0,\, 1,\, 2\}$ such that $k \ne l$. The expression $S(q)$ is a homogeneous polynomial of

degree 4 in the coefficients of q, and is a projective invariant of q in the sense that if L is a linear substitution of variables, then $S(L^*(q)) = (\det L)^4\, S(q)$.

For the second projective invariant T one also uses the *Hesse determinant* $r = \mathrm{H}(q)$ of q, which is defined as the determinant of the Hesse matrix $\partial^2 q(x)/\partial x_i\, \partial x_j$ of q. See Hesse [82]. Because the coefficients of the Hesse matrix depend linearly on x, r is again a homogeneous polynomial of degree 3 in three variables. The coefficients $b_{i,\,j,\,k}$, defined as

$$b_{i,\,j,\,k} = 6^{-2} a_{i,\,j,\,k}(\mathrm{H}(q)), \quad 0 \le i,\ j,\ k \le 2, \tag{4.4.5}$$

are homogeneous polynomials of degree 3 in the coefficients of q. Then

$$2T = 2T(q) := \sum_{i,\,j,\,l=0}^{2} (a_k\, b_l)_{i,\,j}\, (a_i\, a_j)_{k,\,l}, \tag{4.4.6}$$

which does not depend on k. The expression $T(q)$ is a homogeneous polynomial of degree 6 in the coefficients of q, and is a projective invariant in the sense that $T(L^*(q)) = (\det L)^6\, T(q)$ for any linear substitution of variables L. Finally, if

$$R = R(q) := S(q)^3 - T(q)^2, \tag{4.4.7}$$

then $R(q) = 0$ is the discriminant of q, which is equivalent to the condition that the cubic curve in $\mathbb{P}^2$ defined by the equation $q = 0$ has singular points.

We have seen in Section 2.3.2 that every elliptic curve is complex analytic diffeomorphic to a Weierstrass curve. For nonsingular cubic curves in $\mathbb{P}^2$ we have the following, classically known, stronger statement.

Lemma 4.4.1 *Every nonsingular cubic curve C in $\mathbb{P}^2$ can be brought into the Weierstrass normal form* (2.3.6) *by means of a projective linear transformation.*

Proof. Let q be the homogeneous polynomial of degree 3 on $\mathbb{C}^3$ such that C corresponds to the zero-set of q. If r denotes the Hesse determinant of q, the determinant of the matrix of the second-order partial derivatives of q, then Hesse [82, p. 104] observed that the flex points of C are the points of C where $r = 0$. Because r is also a homogeneous polynomial of degree 3, it follows from the theorem of Bézout that the curves defined by $q = 0$ and $r = 0$ intersect at $3 \cdot 3 = 9$ points. In particular, C has at least one flex point o.

By means of a projective linear transformation we can arrange that $o = [0 : 0 : 1]$ and that $x_0 = 0$ corresponds to the tangent line of C at o. This means that in (4.1.5) we have $g = 0$ because $[0 : 0 : 1] \in C$, $d = 0$, and $h \neq 0$ because C is smooth at o and is tangent to $x_0 = 0$ at o, and finally $b = 0$ because o is a flex point of C. We also have $a \neq 0$, because otherwise $x_0 = 0$ would be a component of C, in contradiction to the smoothness, hence irreducibility of C.

Replacing x_2 by $x_2 - x_0\, i/2h$, we obtain in addition that $i = 0$, where some of the other coefficients are changed, but we will use the same letters for the new coefficients. Subsequently replacing x_2 by $x_2 - x_1\, e/2h$, we obtain in addition that

$e = 0$. Then replacing x_1 by $x_1 - x_0\, c/3a$, we obtain in addition that $c = 0$. With a rescaling of the variables we can finally arrange that q is equal to the left-hand side of (2.3.6). □

Remark 4.4.2. Hesse [82] proposed the normal form ${x_0}^3 + {x_1}^3 + {x_2}^3 + 6c\, x_0\, x_1\, x_2 = 0$ for a cubic curve in $\mathbb{P}^2$, which is older than Weierstrass's normal form (2.3.6). See Section 11.1. ⊘

The following proposition can be viewed as an analogue for cubic curves in $\mathbb{P}^2$ of Lemma 2.3.10 and Corollary 2.4.7 for biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 4.4.3 *Let q be a homogeneous polynomial of degree three in three variables, with Aronhold invariants S and T, and let C be the cubic curve in $\mathbb{P}^2$ that corresponds to the zero-set of q in $\mathbb{C}^3$. Then C is smooth, that is, q has only simple zeros in $\mathbb{C}^3 \setminus \{0\}$, if and only if $R := S^3 - T^2 \neq 0$. If C is smooth, then it is an elliptic curve. If P is the period lattice of the Hamiltonian vector field in Lemma 4.1.2, then $g_2(P) = 27\, S$ and $g_3(P) = -27\, T$, which are homogeneous polynomials of degree 4 and 6 in the coefficients of q, respectively. It follows that $\Delta = {g_2}^3 - 27\, {g_3}^2 = 3^9\, R$ and the modulus of C is equal to*

$$J = {g_2}^3/\Delta = S^3/R = S^3/(S^3 - T^2). \tag{4.4.8}$$

Proof. If q is equal to one-half of the left-hand side of (2.3.6), then the application of the recipe for the computation of the Aronhold invariants yields that $S(q) = g_2/27$ and $T(q) = -g_3/27$. It follows from the paragraph in Section 2.3.2 starting with "Now assume conversely ..." that $g_2 = g_2(P)$ and $g_3 = g_3(P)$ if $P = P(q)$ is the period lattice of the Hamiltonian vector field (2.3.7) on the curve (2.3.6), which is equal to the Hamiltonian vector field in Lemma 4.1.2 if q is equal to the left-hand side of (2.3.6).

Let q be any homogeneous polynomial of degree 3 in three variables, $P = P(q)$ the period lattice the Hamiltonian vector field H_q in Lemma 4.1.2, and $L \in \mathrm{GL}(3, \mathbb{C})$. It follows from Lemmas 4.1.1 and 4.1.2 that the projective linear transformation of $\mathbb{P}^2$ induced by L intertwines $\mathrm{H}_{q \circ L}$ on the curve in $\mathbb{P}^2$ defined by $q \circ L = 0$ with $(\det L)$ times H_q on the curve in $\mathbb{P}^2$ defined by $q = 0$. Because period lattices of intertwined vector fields are the same, it follows that $P(q \circ L) = (\det L)^{-1}\, P(q)$, and therefore (2.3.4) implies that $g_2(P(q \circ L)) = (\det L)^4\, g_2(P(q))$ and $g_3(P(q \circ L)) = (\det L)^6\, g_2(P(q))$. Because $S(q \circ L) = (\det L)^4\, S(q)$ and $T(q \circ L) = (\det L)^6\, T(q)$, and $g_2 \circ P - 27\, S = 0$ and $g_3 \circ P + 27\, T = 0$ on q as in the left hand side of (2.3.6), it follows from Lemma 4.4.1 that these identities hold on the set of all cubic polynomials q for which the curve in $\mathbb{P}^2$ defined by $q = 0$ is smooth. □

Remark 4.4.4. The mapping that assigns to each cubic polynomial q in three variables the period lattice $P(q)$ of the Hamiltonian vector field in Lemma 4.1.2 is highly transcendental. Also the mapping that assigns to any lattice P the Weierstrass invariants $g_2(P)$ and $g_3(P)$ in (2.3.4) is transcendental. It is quite remarkable that

the compositions $q \mapsto g_2(P(q))$ and $q \mapsto g_3(P(q))$ are equal to the *polynomial* mappings $27\,S$ and $-27\,T$, respectively. This remark is analogous to Remark 2.4.8. ⊘

The following corollaries are analogous to Corollary 3.3.13 and Corollary 3.3.14.

Corollary 4.4.5 *With the notation* $q_z = z_0\,q^0 + z_1\,q^1$, *write* $g_2(z) = g_2(P(q_z))$, $g_3(z) = g_3(P(q_z))$, *and* $\Delta(z) = g_2(z)^3 - 27\,g_3(z)^2$. *These are explicitly computable homogeneous polynomials of degree* 4, 6, *and* 12 *in* $z = (z_0,\ z_1)$, *respectively. The fiber* $S_{[z]}$ *over* $[z]$ *is singular if and only if* $\Delta(z) = 0$. *The Kodaira type of the singular fiber* $S_{[z]}$ *can be determined from the order of the zeros at* $[z]$ *of* g_2, g_3, *and* Δ *as in Table* 6.3.2.

Note that in general we do not have an explicit algebraic formula for the zeros of the polynomial $\Delta(z)$ of degree 12, but it happens often that in examples these zeros can be computed explicitly.

Corollary 4.4.6 *The set of all pairs* $(g_2,\ g_3)$ *that can occur for pencils* $z_0\,q^0 + z_1\,q^1 = 0$ *of cubic curves in* $\mathbb{P}^2$ *with at least one smooth member is equal to the set of all* $(g_2,\ g_3)$ *such that* g_2 *and* g_3 *are homogeneous polynomials of degree* 4 *and* 6 *in two variables, respectively, such that* ${g_2}^3 - 27\,{g_3}^2$ *is not identically zero, and for each* $[z] \in \mathbb{P}^1$, g_2 *has a zero of order* < 4 *or* g_3 *has a zero of order* < 6 *at* $[z]$.

Proof. This follows from the combination of Proposition 4.4.3 with condition (v) in Lemma 9.1.2. □

In the following corollary we describe the relation between the Eisenstein invariants of the partial discriminants of biquadratic polynomials and the Aronhold invariants of the cubic polynomials.

Corollary 4.4.7 *Let* A *and* B *be surjective linear mappings from* $\mathbb{C}^3$ *to* $\mathbb{C}^2$ *such that* $\ker A \neq \ker B$. *Let* ν *be a nonzero linear form on* $\mathbb{C}^3$ *that vanishes on* $\ker A$ *and on* $\ker B$. *Let* $\mathcal{B}$ *denote the vector space of all biquadratic polynomials on* $\mathbb{C}^2 \times \mathbb{C}^2$ *that vanish on* $A(\ker B) \times B(\ker A)$, *and let* $\mathcal{Q}$ *denote the vector space of all cubic polynomials on* $\mathbb{C}^3$ *that vanish on* $\ker A$ *and on* $\ker B$. *Then the mapping* $p \mapsto q(p) := (p \circ (A,\ B))/\nu$ *defines a linear isomorphism from* $\mathcal{B}$ *onto* $\mathcal{Q}$. *Furthermore, there is a constant* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that*

$$D(\Delta_k(p)) = 27\,\lambda^4\,S(q(p)) \quad \text{and} \quad E(\Delta_k(p)) = 27\,\lambda^6\,T(q(p))$$

for every $p \in \mathcal{B}$. *We have* $\lambda = 1$ *if* $A(u_0,\ u_1,\ u_2) = (u_0,\ u_1)$, $B(u_0,\ u_1,\ u_2) = (u_0,\ u_2)$, *and* $\nu(u) = u_0$, *as in (4.1.2).*

Proof. In view of the invariance properties of the Eisenstein invariants and the Aronhold invariants, it suffices to prove the identities in the situation of (4.1.2), where they follow from Corollary 2.4.7 and Proposition 4.4.3. I must admit that I couldn't resist the temptation of also checking the identities by means of a formula manipulation program. □

4.4.1 A Mathematica Notebook

For general cubic polynomials q in three variables, the explicit formulas for the Aronhold invariant S, T, and R of q, as homogeneous polynomials of degree 4, 6, and 12 in the ten coefficients of q, are quite unwieldy. The formulas for S and T contain 25 and 103 terms, respectively, whereas the formula for R is several pages long.

However, Aronhold's recipes can easily be turned into a formula manipulation computer program, which efficiently computes the Aronhold invariants S and T of any ternary cubic q. From these, the discriminant $R = S^3 - T^2$ and the modulus $J = S^3/R$ of the elliptic curve defined by the equation $q = 0$ are easily obtained.

Below follows a Mathematica notebook to this effect. Enter the coefficients c_0–c_9 of the ternary cubic in the first block. Then do SHIFT-ENTER at the end of each block.

```
Array[x, 3];
pol[x_] :=c0*x[1]^3+c1*x[1]^2*x[2]+c2*x[1]^2*x[3]+c3*x[1]*x[2]^2
+c4*x[1]*x[2]*x[3]+c5*x[1]*x[3]^2+c6*x[2]^3+c7*x[2]^2*x[3]
+c8*x[2]*x[3]^2 + c9*x[3]^3;
c0= ;c1= ;c2 = ;c3 = ;c4 = ;c5 = ;c6 = ;c7 = ;c8 = ;c9 = ;
Array[p, {3, 3, 3}];
Do[p[i,j,k]=D[pol[x],x[i],x[j],x[k]]/6,{i,1,3},{j,1,3},{k,1,3}];
Array[hesse,{3, 3}];
Do[hesse[i,j]=D[pol[x],x[i],x[j]],{i,1,3},{j,1,3}];
hesse=Table[hesse[i,j],{i,1,3},{j,1,3}];
deth[x_]:=Det[hesse]/36;
Array[h,{3,3,3}];
Do[h[i,j,k]=D[deth[x],x[i],x[j],x[k]]/6,{i,1,3},{j,1,3},{k,1,3}];

bracket[a_, b_]:=Table[a[Mod[i,3]+1,Mod[j,3]+1]
*b[Mod[Mod[i,3]+1,3]+1,Mod[Mod[j,3]+1,3]+1]
+a[Mod[Mod[i,3]+1,3]+1,Mod[Mod[j,3]+1,3]+1]
*b[Mod[i,3]+1,Mod[j,3]+1]-a[Mod[i,3]+1,Mod[Mod[j,3]+1,3]+1]
*b[Mod[Mod[i,3]+1,3]+1,Mod[j,3]+1-a[Mod[Mod[i,3]+1,3]+1,Mod[j,3]+1]
*b[Mod[i,3]+1,Mod[Mod[j,3]+1,3]+1], {i,1,3}, {j,1,3}];

Do[Array[p[i],{3,3}];
Do[p[i][j,k]=p[i,j,k], {j,1,3}, {k,1,3}], {i,1,3}];
aronholdfirst=Sum[bracket[p[1],p[1]][[i, j]]
*bracket[p[i],p[j]][[1, 1]], {i, 1, 3}, {j, 1, 3}]

Do[Array[p[i],{3,3}];
Do[p[i][j,k]=p[i,j,k], {j,1,3}, {k,1,3}], {i,1,3}];
Do[Array[h[i],{3,3}];
Do[h[i][j,k]=h[i,j,k], {j,1,3}, {k,1,3}], {i,1,3}];
aronholdsecond=Sum[bracket[p[1],h[l]][[i,j]]
*bracket[p[i],p[j]][[1,l]]/2, {i,1,3}, {j,1,3}, {l,1,3}]
```

4.5 Pencils of Cubic Curves with Only One Base Point

If $\mathfrak{C}$ is a pencil of cubic curves in $\mathbb{P}^2$ with only one base point, then there is no pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathfrak{C} = \rho(\mathfrak{B})$ as in Proposition 4.1.3, and there are no Manin transformations (4.2.2) defined by $\mathfrak{C}$ related to a QRT transformation as in (4.3.1). Frans Oort and Frits Beukers immediately pointed out to me the following example of a pencil of cubic curves in $\mathbb{P}^2$ with only one base point. For the Kodaira types of the singular fibers mentioned in this section, see Section 6.2.6.

Figures 4.5.1 and 2.3.2 Figure 4.5.1 shows the base point $[0 : 0 : 1]$ of order 9 of the pencil (4.5.1), in the affine coordinates $x_2 = 1$, $x_1 = x$, $x_0 = y$. The left- and right-hand pictures in Figure 2.3.2 show the affine real curves of the pencils (4.5.1) and (4.5.2), respectively. In both pictures in Figure 2.3.2 the base point of order 9 at infinity betrays its presence, even if the affine coordinates in the picture do not become very large.

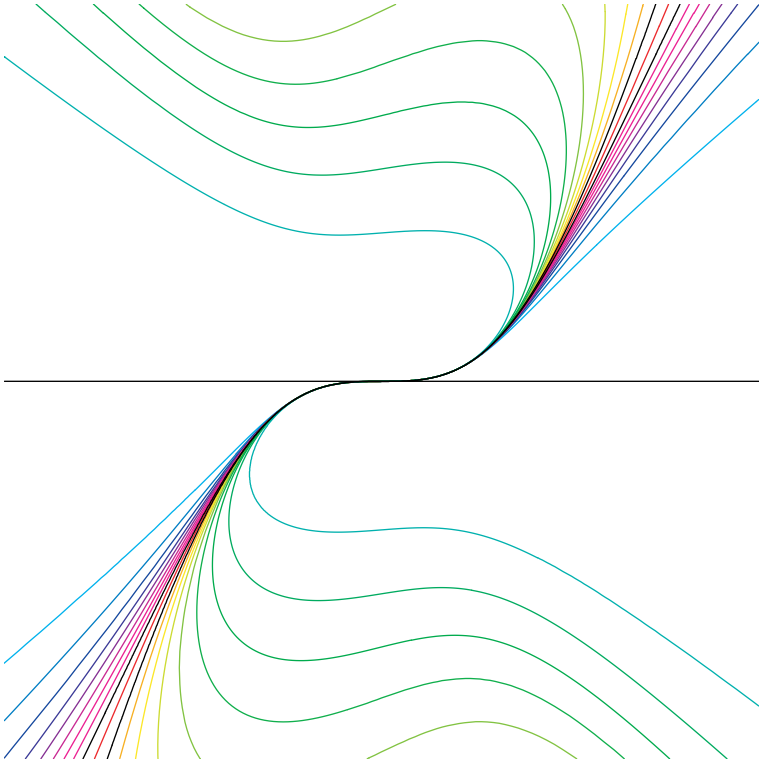


Fig. 4.5.1 The base point of order 9 of the pencil (4.5.1).

Lemma 4.5.1 *Let $\mathfrak{C}$ be a pencil of cubic curves in $\mathbb{P}^1$ with a base point b such that a smooth member C of $\mathfrak{C}$ has a flex at b. Let L be the projective line through b and*

tangent to C, and let $D = 3L$ be the singular cubic curve that is equal to L with multiplicity 3. Then b is the only base point of $\mathfrak{C}$ if and only if D is a member of $\mathfrak{C}$.

If this is the case, then in suitable projective coordinates $[x_0 : x_1 : x_2]$ the pencil $z_0\, q^0 + z_1\, q^1 = 0$ can be brought into one of the following two forms:

$$q^0(x_0,\, x_1,\, x_2) = (x_0\, {x_2}^2 - 4\, {x_1}^3 + {x_0}^2\, x_1)/2, \quad q^1(x_0,\, x_1,\, x_2) = {x_0}^3/2, \tag{4.5.1}$$

$$\text{or} \quad q^0(x_0,\, x_1,\, x_2) = (x_0\, {x_2}^2 - 4\, {x_1}^3)/2, \quad q^1(x_0,\, x_1,\, x_2) = {x_0}^3/2. \tag{4.5.2}$$

Let S be the elliptic surface that is obtained by successively blowing up, nine times, at the successive base points of the pencils of holomorphic exterior two-vector fields. Then the configuration of the singular fibers is $\mathrm{II}^*\ 2\,\mathrm{I}_1$ *when* (4.5.1) *and* $\mathrm{II}^*\ \mathrm{II}$ *when* (4.5.2). *In both cases the group* $\mathrm{Aut}(S)^+_\kappa$, *or equivalently the Mordell–Weil group of S, is trivial.*

Proof. The intersection number at b of C with L is equal to 3, and therefore the intersection number at b of C with D is equal to 9. Because the total intersection number of two cubic curves, counted with multiplicities, is equal to 9, it follows that the pencil $\mathfrak{C}'$ of cubic curves containing C and D has b as its only base point. Therefore, if $D \in \mathfrak{C}$, then $\mathfrak{C} = \mathfrak{C}'$ has b as its only base point.

Now assume that b is the only base point of $\mathfrak{C}$. Let $C' \in \mathfrak{C}$ and let $q' = 0$, $q = 0$, and $l = 0$ be the defining equations of C', C, and L, where q', q, and l are homogeneous polynomials of degree 3, 3, and 1, respectively. Because C' has an order of contact ≥ 9 with C at b, in local coordinates $q'|_C$ has a zero of order ≥ 9 at b, whereas we have seen in the previous paragraph that also $l^3|_C$ has a zero of order 9 at b. It follows that the rational function $f := (q'/l^3)|_C$ on C is regular at b, and because C and D intersect only at b, it is also regular at all other points of C. Because C is a compact connected complex analytic manifold, it follows from the maximum principle that the complex analytic function f on C is a constant. That is, $q' - f\, l^3 = 0$ when $q = 0$. Because $\mathrm{d}q = 0$ when $q = 0$, this implies that $g := (q' - f\, l^3)/q$ defines a complex analytic function on $\mathbb{P}^2$, which is a constant as another consequence of the maximum principle. Therefore $q' = f\, l^3 + g\, q$ for some constants f and g. In other words, the pencil $\mathfrak{C}$ is defined by the two-dimensional vector space of cubic polynomials that is spanned by l^3 and g, which implies that $D \in \mathfrak{C}$.

By means of a suitable projective linear transformation we can arrange that $b = [0 : 0 : 1]$ and L corresponds to $x_0 = 0$. It follows from the proof of Lemma 4.4.1 that by a linear transformation that preserves b and L we can arrange that the curve C is given by an equation of the form

$$q^0 := a\, {x_1}^3 + f\, {x_0}^2\, x_1 + h\, x_0\, {x_2}^2 + j\, {x_0}^3 = 0,$$

where $a \neq 0$ and $h \neq 0$. By adding to q^0 a suitable multiple of $q^1 := {x_0}^3$, we can arrange that $j = 0$. If $f \neq 0$, then we can arrange by means of a scaling of x_0, x_1, and x_2 that $a = f = h = 1$, and we have arrived at (4.5.1). If $f = 0$, then a scaling leads to (4.5.2).

The member D' of the anticanonical pencil in S that by the projection $\pi : S \to \mathbb{P}^2$ is mapped to $D = 3L$ has at least nine irreducible components, namely the strict transform of D and the strict transforms of the -1 curves that appear in the eight blowing-up transformations preceding the last blowing up, where the last blowing up creates a -1 curve in S that is a section of the elliptic fibration $\kappa : S \to P$, and therefore does not belong to the fiber D'. It follows from Kodaira's classification of singular fibers, see Section 6.2, that this leaves as possibilities II^*, I_4^*, and I_9, with intersection diagrams $\mathrm{E}_8^{(1)}$, $\mathrm{D}_8^{(1)}$, and $\mathrm{A}_8^{(1)}$, respectively. Because the strict transform $\pi'(D)$ of D is one of the irreducible components of D' and has multiplicity 3, we conclude that D' is of type II^*, with intersection diagrams $\mathrm{E}_8^{(1)}$. It follows from the list of Persson [156, pp. 7–14] that the only possible configurations of singular fibers are II^* II and II^* $2\,\mathrm{I}_1$.

In the case (4.5.1), the Aronhold invariants yield the Weierstrass invariants $g_2 = -2^2\, {z_0}^4$ and $g_3 = 2^2\, {z_0}^5\, z_1$, with discriminant $\Delta = -2^4\, {z_0}^{10}\, (2^2\, {z_0}^2 + 3^3\, {z_1}^2)$. The zero $z_0 = 0$ of multiplicity 10 of the discriminant corresponds to the singular fiber of type II^*, whose Euler number is equal to 10; see Section 6.2.10. Because the discriminant has two more zeros that are simple, the configuration of the singular fibers is II^* $2\,\mathrm{I}_1$.

In the case (4.5.2) the Aronhold invariants yield the Weierstrass invariants $g_2 = 0$ and $g_3 = 2^2\, {z_0}^5\, z_1$, with discriminant $\Delta = -2^4\, 3^3\, {z_0}^{10}\, {z_1}^2$. Again $z_0 = 0$ corresponds to a fiber of type II^*, whereas the double zero $z_1 = 0$ corresponds to the curve ${x_1}^3 + x_2\, {x_0}^2 = 0$ in $\mathbb{P}^2$, which is smooth except at the point $[0 : 0 : 1] \neq b = [1 : 0 : 0]$, where it has a cusp. The member C of the anticanonical pencil in S that corresponds to this curve C_0 in $\mathbb{P}^2$ is equal to the strict transform of C_0, and therefore C also has only one singular point, which is a cusp. That is, C is of type II and we have the configuration of the singular fibers II^* II. It follows from Lemma 9.2.6 that if there is a singular fiber of type II^* with intersection diagram $\mathrm{E}_8^{(1)}$, then the Mordell–Weil group of S is trivial. □

The pencil (4.5.2) also appears in (9.2.3), where the identity component of the automorphism group of the corresponding rational elliptic surface is isomorphic to $\mathbb{C}^\times$.

Remark 4.5.2. The cubic curve $q^0 + g_3\, q^1 = 0$ in (4.5.1) and (4.5.2) is equal to the Weierstrass curve (2.3.6) with $g_2 = 1$ and $g_2 = 0$, respectively. This explains why the Weierstrass invariants g_2, g_3, and Δ of the cubic polynomial $z_0\, q^0 + z_1\, q^1$, computed from the Aronhold invariants as in Proposition 4.4.3, are equal to $g_2 = {z_0}^4$, $g_3 = {z_0}^5\, z_1$, and $\Delta = {z_0}^{10}\, ({z_0}^2 - 27\, {z_1}^2)$ if q^0 and q^1 are as in (4.5.1), and $g_2 = 0$, $g_3 = {z_0}^5\, z_1$, and $\Delta = -27\, {z_0}^{10}\, {z_1}^2$ if q^0 and q^1 are as in (4.5.2). In view of Table 6.3.2, this leads to an alternative determination of the configuration of the singular fibers as II^* $2\,\mathrm{I}_1$ and II^* II for (4.5.1) and (4.5.2) when the modulus function $J := {g_2}^3/\Delta = {z_0}^2/({z_0}^2 - 27\, {z_1}^2)$ and $J = 0$ have degree $j = 2$ and $j = 0$, respectively, in accordance with (6.2.48).

In the Weierstrass model, see Section 6.3, every elliptic surface is obtained from a one-parameter family of Weierstrass curves, where the Weierstrass invariants g_2 and g_3 are suitable functions of the parameter. It would be confusing to call the pencils in

(4.5.1) and (4.5.2) the Weierstrass pencils, because the corresponding elliptic surfaces S are extremely special, as shown by their characterization in Corollary 4.5.6. In particular, these are the only rational elliptic surfaces that are not QRT surfaces.

In view of Example 5, this means that there are no homogeneous polynomials $X(z)$ and $Y(z)$ in $z = (z_0,\ z_1)$ of degree 2 and 3, respectively, such that $Y(z)^2 - 4\,X(z)^3 + {z_0}^4\,X(z) + {z_0}^5\,z_1 = 0$ or $Y(z)^2 - 4\,X(z)^3 + {z_0}^5\,z_1 = 0$, corresponding to (4.5.1) or (4.5.2), respectively. Moreover, modulo linear substitutions of variables in $(z_0,\ z_1)$ these are the only homogeneous polynomials $g_2(z)$ and $g_3(z)$ of degree 4 and 6, respectively, such that $g_2(z)^3 - 27\,g_3(z)^2$ is not identically equal to zero, there are no nonzero $z \in \mathbb{C}^2$ where g_2 and g_3 have a zero of order ≥ 4 and ≥ 6, respectively, and the equation $Y(z)^2 - 4\,X(z)^3 + g_2(z)\,X(z) + g_3(z) = 0$ does not have solutions $X(z)$ and $Y(z)$ that are rational functions of $z = (z_0,\ z_1)$, homogeneous of degree 2 and 3, respectively. I do not see a proof of these statements directly in terms of homogeneous rational functions of two variables. ⊘

In the following lemma we describe the only case of a pencil of cubic curves in $\mathbb{P}^2$ with a smooth member and only one base point, other than the case in Lemma 4.5.1. For a characterization of the corresponding rational elliptic surface, see Proposition 12.1.6.

Lemma 4.5.3 *A pencil of cubic curves in $\mathbb{P}^2$ with at least one smooth member has only one base point b such that b is not a flex point for the members that are smooth at b if and only if in suitable projective coordinates it is given by $z_0\,q^0 + z_1\,q^1 = 0$, where*

$$q^0 = x_1\,{x_2}^2 + {x_1}^2\,x_0 - x_2\,{x_0}^2, \tag{4.5.3}$$

$$q^1 = {x_1}^3 + {x_2}^3 - x_1\,x_2\,x_0. \tag{4.5.4}$$

In this case the member that is not smooth at the only base point $b = [1 : 0 : 0]$ is an irreducible cubic curve with an ordinary double point at b, and every member that is smooth at b has a contact of order 8 at b with one of the local components of the member with the double point at b.

If S is the elliptic surface that is obtained by successively blowing up, nine times, at the successive base points of the pencils of holomorphic exterior two-vector fields, then the configuration of the singular fibers is $\mathrm{I}_9\ 3\,\mathrm{I}_1$, and the Mordell–Weil group of S is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

S is also isomorphic to the elliptic surface that is obtained by successively blowing up at the base points of the pencil of cubic curves $z_0\,\widetilde{q}^0 + z_1\,\widetilde{q}^1 = 0$, where

$$\widetilde{q}^0 = x_0\,{x_1}^2 + x_1\,{x_2}^2 + x_2\,{x_0}^2, \tag{4.5.5}$$

$$\widetilde{q}^1 = x_0\,x_1\,x_2. \tag{4.5.6}$$

This pencil has the three points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ as its base points, and the transformation of S induced by the symmetry $[x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0]$ of the pencil generates the Mordell–Weil group of S.

Proof. We can arrange that $b = [1 : 0 : 0]$ and let $q^0 = 0$ be a member that is smooth at b but does not have b as a flex point. We write the cubic forms $q = q^0$

and $q = q^1$ as (4.1.5), with the coefficients $a, \ldots, j$ replaced by $a_0, \ldots, j_0$ and $a_1, \ldots, j_1$, respectively. We assume that $q^1 = 0$ has a singular point at $[1 : 0 : 0]$.

The condition that $q^0(1, 0, 0) = 0$ means that $j^0 = 0$, and the condition that $[1 : 0 : 0]$ is a smooth point of $q^0 = 0$ means that f_0 and i_0 are not both equal to 0. By means of a linear change of coordinates we can arrange that the curve $q^0 = 0$ is tangent at $[1 : 0 : 0]$ to the projective line $x_2 = 0$ in $\mathbb{P}^2$, which means that $f_0 = 0$, and by scaling x_2 we can arrange that $i_0 = -1$. In this situation the solution curve $q^0([1 : x : y]) = 0$ near $(x, y) = (0, 0)$ can be parametrized by a power series $y = y(x) = \sum_{l \geq 2} y_l x^l$, and we find that $y_2 = c_0$. Therefore the condition that the curve q^0 does not have a flex at $[1 : 0 : 0]$ is equivalent to the condition that $c_0 \neq 0$. By scaling x_1 we can arrange that $c_0 = 1$. The linear substitutions of variables that preserve the point $x_1 = x_2 = 0$ and the line $x_2 = 0$ in $\mathbb{P}^2$, and preserve $i_0 = -1$, $c_0 = 1$, are the substitutions $x_2 \mapsto a_{22}\, x_2$, $x_1 \mapsto a_{11}\, x_1 + a_{12}\, x_2$, and $x_0 \mapsto a_{00}\, x_0 + a_{01}\, x_1 + a_{02}\, x_2$, with $a_{22}\, {a_{00}}^2 = {a_{11}}^2\, a_{00} = 1$. Then a_0 changes into $a_0\, a_{11} + a_{10}$, which can be made equal to zero. Retaining $a_0 = 0$ amounts to keeping $a_{01} = 0$ in future substitutions. Then b_0 changes into $b_0\, a_{11}^2\, a_{22} + {a_{11}}^2\, a_{02}$, which can be made equal to zero. Retaining $b_0 = 0$ amounts to keeping also $a_{02} = 0$ in future substitutions. Then e_0 changes into $2 a_{11}\, a_{12}\, a_{00} + e_0\, a_{11}\, a_{22}\, a_{00}$, which can be made equal to zero. Retaining this amounts to keeping also $a_{12} = 0$, which means that we still can perform only a scaling of each of the variables, and therefore we cannot make more coefficients of q^0 equal to zero by means of linear substitutions of variables. At this stage we have arrived at

$$q^0 = x^2 + d_0\, x\, y^2 + g_0\, y^3 + h_0\, y^2 - y$$

in the affine coordinates $[1 : x : y]$. Iterating the identity $y(x) = x^2 + d^0\, x\, y(x)^2 + g^0\, y(x)^3 + h^0\, y(x)^2$ a few times, we arrive at

$$y(x) = x^2 + h_0\, x^4 + d_0\, x^5 + (g_0 + 2\, {h_0}^2)\, x^6 + 4\, d_0\, h_0\, x^7 + \mathrm{O}(x^8).$$

Because we assumed that $q^1 = 0$ has a singular point at $[1 : 0 : 0]$, we have

$$q^1(1, x, y) = a_1\, x^3 + b_1\, x^2\, y + c_1\, x^2 + d_1\, x\, y^2 + e_1\, x\, y + g_1\, y^3 + h_1\, y^2.$$

The curve $q^1 = 0$ has a contact of order 9 with the curve q^0 at the point $[1 : 0 : 0]$ if and only if the coefficients of x^k for $0 \leq k \leq 8$ in the powers series expansion of $q^1(1, x, y(x))$ are equal to zero, where those for $k \leq 1$ are automatically satisfied. This leads to the seven linear equations $c_1 = 0, a_1 + e_1 = 0, b_1 + h_1 = 0, d_1 + h_0\, e_1 = 0, d_0\, e_1 + g_1 + h_0\, b_1 + 2\, h_0\, h_1 = 0, d_0\, b_1 + 2\, h_0\, d_1 + (g_0 + 2\, {h_0}^2)\, e_1 + 2\, d_0\, h_1 = 0$, and $2\, d_0\, d_1 + 4\, d_0\, h_0\, e_1 + 3\, h_0\, g_1 + (g_0 + 2\, {h_0}^2)\, b_1 + (2\, g_0 + 5\, {h_0}^2)\, h_1 = 0$ for the seven coefficients of q^1. Using the first five equations in order to express a_1, b_1, c_1, d_1, and g_1 in terms of e_1 and h_1, we are left with the two linear equations $g_0\, e_1 + d_0\, h_1 = 0$ and $-d_0\, h_0\, e_1 + g_0\, h_1 = 0$ for the two unknowns e_1 and h_1. If the determinant ${g_0}^2 + {d_0}^2\, h_0$ is not equal to zero, then $e_1 = h_1 = 0$, which in turn implies that all the coefficients of q^1 are equal to zero, which is not what we want. Therefore ${g_0}^2 + {d_0}^2\, h_0 = 0$. If $d_0 = 0$, then $g_0 = 0$, when $q^0 = x_0\, ({x_1}^2 + h_0\, {x_2}^2 - x_0\, x_2)$ is

reducible, in contradiction to the assumption that the curve $q^0 = 0$ is nonsingular. Therefore $d_0 \neq 0$, and it follows that $e_1 \neq 0$ because $e_1 = 0$ and $g_0\, e_1 + d_0\, h_1 = 0$ would imply that also $h_1 = 0$ and hence $q^1 = 0$. Scaling q^1, we can take $e_1 = 1$, and using $h_0 = -{g_0}^2/{d_0}^2$, we arrive at $h_1 = -g_0/d_0$, $c_1 = 0$, $a_1 = -1$, $b_1 = g_0/d_0$, $d_1 = {g_0}^2/{d_0}^2$, and $g_1 = -d_0 - {g_0}^3/{d_0}^3$. Writing $d^0 = d$ and $g^0/d^0 = g$, this leads to

$$q^0 = {x_1}^2\, x_0 + d\, x_1\, {x_2}^2 + d\, g\, {x_2}^3 - g^2\, {x_2}^2\, x_0 - x_2\, {x_0}^2,$$
$$q^1 = -{x_1}^3 + g\, {x_1}^2\, x_2 + g^2\, x_1\, {x_2}^2 + x_1\, x_2\, x_0 - (d + g^3)\, {x_2}^3 - g\, {x_2}^2\, x_0,$$

where $d \neq 0$. The substitution of variables $x_0 = \delta\, y_0 + 2g\, y_1$, $x_1 = y_1 + (g/\delta)\, y_2$, $x_2 = (1/\delta)\, y_2$, with $\delta^3 = d$, now leads to

$$q^0 = -{y_1}^3 + y_1\, y_2\, y_0 - {y_2}^3,$$
$$q^1 = -2g\, q^1 + \delta\, (y_1\, {y_2}^2 + {y_1}^2\, y_0 - y_2\, {y_0}^2).$$

This completes the proof that the pencil has the required properties if and only if in suitable projective coordinates it is the pencil $z_0\, q^0 + z_1\, q^1 = 0$ with q^0, q^1 as in (4.5.3), (4.5.4).

The curve $q^1 = 0$ is irreducible and has only one singular point, namely at $[1 : 0 : 0]$, and this is an ordinary double point, whereas the curve $q^0 = 0$ is smooth at $[1 : 0 : 0]$ and has a contact of order 8 at $[1 : 0 : 0]$ with one of the local components of $q^1 = 0$. After the first blowing up, the total transform of $q^1 = 0$ is the union of two projective lines, intersecting each other transversally, and after all the nine blowing-up transformations the total transform of $q^1 = 0$ is a cycle of nine projective lines, which form a singular fiber of the fibration $\kappa : S \to \mathbb{P}^1$, and one exceptional curve attached to one of the projective lines in the cycle. This exceptional curve is a holomorphic section of the fibration $\kappa : S \to \mathbb{P}^1$ and is not contained in the singular fiber. This shows that the singular fiber that contains the proper transform of $q^1 = 0$ is of Kodaira type I_9, with the intersection diagram $\mathrm{A}_8^{(1)}$. According to Persson [156, pp. 7–14] the only possible configuration of singular fibers is $\mathrm{I}_9\ 3\,\mathrm{I}_1$. In Oguiso and Shioda [155, pp. 84–86], this corresponds to the case No. 63, where the Mordell–Weil group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

The singular fiber F of type I_9 is a cycle of nine rational curves C_i, $i \in \mathbb{Z}/9\mathbb{Z}$, and the fact that the Mordell–Weil group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ implies that there are three -1 curves E_j, $j \in \mathbb{Z}/3\mathbb{Z}$, in S, where E_j intersects F exactly once in its smooth part, at C_{3j}. We have $C_i \cdot C_i = -2$ for every $i \in \mathbb{Z}/9\mathbb{Z}$. The sections E_j are disjoint from each other, because after identifying one of them as the zero element of the Mordell–Weil group, every other section defines a nontrivial element of the Mordell–Weil group of finite order, and it follows from Corollary 7.5.5 that the two sections are disjoint. Blowing down the sections E_j, F is mapped to a cycle F' of nine rational curves C'_i, where $C'_{3j} \cdot C'_{3j} = -1$, and $C'_i \cdot C'_i = -2$ if $i \notin 3\mathbb{Z}/9\mathbb{Z}$. Blowing down the C'_{3j}, we are left with a hexagon of rational -1 curves. Blowing down three nonadjacent ones, we have performed nine blowdowns and therefore arrive at a surface that is isomorphic to $\mathbb{P}^2$. Under this blowdown F

is mapped to a triangle of rational curves with self-intersection numbers 1, which therefore is a triangle of projective lines in $\mathbb{P}^2$. By choosing suitable projective coordinates we can arrange that this triangle is given by the equation $x_0\,x_1\,x_2 = 0$. Under this blowdown the elliptic fibration is mapped to a pencil of cubic curves with base points equal to the vertices $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $x_0\,x_1\,x_2 = 0$ as one of its members. Furthermore, after renumbering the coordinates if necessary, we have that a smooth member $q^0 = 0$ is tangent to $x_i = 0$ at the point $x_i = x_{i+1} = 0$. Substracting a suitable multiple of $q^1 = x_0\,x_1\,x_2$, it follows that $q^0 = c\,{x_0}^2\,x_2 + d\,x_0\,{x_1}^2 + i\,x_1\,{x_2}^2$ for nonzero coefficients c, d, i, and by a suitable rescaling of the variables we can arrange that $q^0 = {x_0}^2\,x_2 + x_0\,{x_1}^2 + x_1\,{x_2}^2$. This proves the last statement in Lemma 4.5.3. □

Remark 4.5.4. In both representations in Lemma 4.5.3, the Weierstrass invariants g_2, g_3, Δ, and the modulus J, computed from the Aronhold invariants as in Proposition 4.4.3, are given by

$$\begin{aligned} 2^2\,3\,g_2 &= z_1\,(24\,{z_0}^3 + {z_1}^3),\\ -2^3\,3^3\,g_3 &= 216\,{z_0}^6 + 36\,{z_0}^3\,{z_1}^3 + {z_1}^6,\\ -\Delta &= {z_0}^9\,(27\,{z_0}^3 + {z_1}^3),\\ -2^6\,3^3\,J &= {z_1}^3\,(24\,{z_0}^3 + {z_1}^3)^3/{z_0}^9\,(27\,{z_0}^3 + {z_1}^3). \end{aligned}$$

The solution $z_0 = 0$ of the discriminant equation $\Delta = 0$ of order 9 corresponds to the singular fiber of type I_9. The other three solutions are $z_1 = -3\,u\,z_0$, where u runs over the *third roots of unity*, the three solutions of $u^3 = 1$. These correspond to the three singular fibers of type I_1.

The modulus function J has degree 12, which in view of (6.2.48) agrees with the configuration of the singular fibers $\mathrm{I}_9\ 3\,\mathrm{I}_1$.

Remark 4.5.5. Wall [206, §3] described the singular fibers that appear after successively blowing up $\mathbb{P}^2$ nine times in the base points of a pencil of cubic curves in $\mathbb{P}^2$ with smooth members. (A member can contain more than one nonsimple base point.) The list contains 31 cases, of which the cases that lead to a singular fiber of Kodaira type I_n are lumped together in case (2), and it is not specified how the various I_n arise. In particular, it is not discussed that $n \le 9$ and that $n = 9$ occurs. ⊘

Corollary 4.5.6 *Let $\kappa : S \to P \simeq \mathbb{P}^1$ be a rational elliptic surface. Then the following statements are equivalent:*

(i) *The group $\mathrm{Aut}(S)^+_\kappa$ of all automorphisms (= complex analytic diffeomorphisms) of S that act as a translation on each smooth fiber of κ (which is an elliptic curve) is trivial.*
(ii) *The fibration $\kappa : S \to P$ has only one section.*
(iii) *There is only one -1 curve in S.*
(iv) *S is not a QRT surface. That is, S is not isomorphic to the successive blowing up, eight times, at base points of the anticanonical pencils defined by a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with a smooth member.*

(v) *S is isomorphic to one of the two rational elliptic surfaces described in Lemma* 4.5.1.
(vi) *S contains a fiber of type* II^**, with intersection diagram* $\mathrm{E}_8^{(1)}$.

Proof. The equivalence between (i), (ii), and (iii) follows from Lemma 7.1.1 and Lemma 9.2.1.

If (iii) holds then it follows from the implication (i) $\Rightarrow$ (iii) in Theorem 4.3.2 that (iv) holds.

Now suppose that (iv) holds. It follows from (a) in Theorem 9.1.3 that there is a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ such that S is isomorphic to the successive blowing up, nine times, at the base points of the anticanonical pencils defined by the pencil $\mathfrak{C}$. It follows from Proposition 4.1.3 that $\mathfrak{C}$ has only one base point, and therefore in suitable projective coordinates $\mathfrak{C}$ is equal to the pencil described in Lemma 4.5.1 or Lemma 4.5.3. Because in the case of Lemma 4.5.3, S can also be obtained from a cubic pencil with three base points, in contradiction to (iv), we conclude (v).

Since after the choice of a holomorphic section as the neutral element, the set of all sections of κ is identified with the Mordell–Weil group of S, the implication (v) $\Rightarrow$ (ii) follows from Lemma 4.5.1.

Finally (v) $\Rightarrow$ (vi) follows from Lemma 4.5.1, whereas (vi) $\Rightarrow$ (i) follows from Lemma 9.2.6. $\square$

Chapter 5
The Action of the QRT Map on Homology

5.1 The Action of the QRT Map on Homology Classes

Let S be the surface that is obtained from successively blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ eight times as in Corollary 3.3.10, where the fibration $\kappa : S \to P$ of S over the projective line $P = \mathrm{P}(W)$ exhibits S as a rational elliptic surface as in Definition 9.1.4. It follows from Theorem 9.2.4 that then the Néron–Severi group $\mathrm{NS}(S)$, the Picard group $\mathrm{Pic}(S)$, the cohomology group $\mathrm{H}^2(S, \mathbb{Z})$, and the homology group $\mathrm{H}_2(S, \mathbb{Z})$ are all canonically isomorphic to each other. Provided with the intersection form, $L := \mathrm{H}_2(S, \mathbb{Z})$ is an integral lattice of rank 10 and modulus equal to 1.

We begin with the introduction of a $\mathbb{Z}$-basis in the group $\mathrm{H}_2(S, \mathbb{Z})$ of homology classes of real two-dimensional cycles in S, which will be convenient for the description of the action of the QRT map on $\mathrm{H}_2(S, \mathbb{Z})$.

Lemma 5.1.1 *Let e_i, $1 \leq i \leq 8$, be the homology classes of the total transforms in S of the -1 curves that appear at each of the blowing-up transformations. Let l_1 and l_2 denote the classes of the total transforms of any horizontal and vertical axes $\mathbb{P}^1 \times \{y\}$ and $\{x\} \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively. Then*

$$L := \mathrm{H}_2(S, \mathbb{Z}) = \mathbb{Z}\, l_1 \oplus \mathbb{Z}\, l_2 \oplus \bigoplus_{i=1}^{8} \mathbb{Z}\, e_i, \tag{5.1.1}$$

where $l_1 \cdot l_1 = 0$, $l_1 \cdot l_2 = 1$, $l_2 \cdot l_2 = 0$, $l_i \cdot e_j = 0$, $e_i \cdot e_i = -1$, and $e_i \cdot e_j = 0$ when $i \neq j$.

Let $f \in L := \mathrm{H}_2(S, \mathbb{Z})$ denote the homology class of any fiber of κ, where we note that all fibers of a fibration are homologous. Then $f \cdot f = 0$ and

$$f = 2l_1 + 2l_2 - \sum_{i=1}^{8} e_i. \tag{5.1.2}$$

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_5,
© Springer Science+Business Media, LLC 2010

Proof. Let $H = \mathbb{P}^1 \times \{y\}$ and $V = \{x\} \times \mathbb{P}^1$ be any horizontal and vertical axes in $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $H \cdot V = 1$. Furthermore, if H' is any other horizontal axis, then H' is homologous to H, and therefore $H \cdot H = H \cdot H' = 0$. Similarly $V \cdot V = 0$. Taking H and V disjoint from the base points, we obtain that $l_1 \cdot l_2 = 1$, $l_1 \cdot l_1 = 0$, and $l_2 \cdot l_2 = 0$.

Because the Riemann sphere $\mathbb{P}^1$ is simply connected, there is no torsion in the homology, and it follows from the Künneth formula for homology, see Spanier [190, p. 235], that the mapping $(h, v) \mapsto h\,\mathrm{H}(H) + v\,\mathrm{H}(V)$ is an isomorphism from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathrm{H}_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$, a sort of dual statement to (2.4.3). The formula (5.1.1) now follows from (3.2.4). It follows from (3.2.6) and the definition of the e_i that $e_i \cdot e_i = -1$ when $1 \le i \le 8$, whereas all the other intersection numbers are equal to zero because the curves can be chosen to be disjoint.

We have $f \cdot f = 0$ because distinct fibers are disjoint. The intersection number of each smooth member of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with a horizontal axis and with a vertical axis is equal to 2, the degree of the biquadratic form with respect to the first and second variable, respectively. Using (3.2.6) at every blowing up, we conclude that $f \cdot l_1 = f \cdot l_2 = 2$. On the other hand, it follows from Corollary 3.3.8 = Lemma 3.2.8 that the intersection number of any smooth member of the pencil with the -1 curve that appears is equal to 1, and again using (3.2.6) we obtain that $f \cdot e_i = 1$ for $1 \le i \le 8$. This proves (5.1.2). □

Let ι_1, ι_2, and $\tau = \iota_2 \circ \iota_1$ denote the action on $L = \mathrm{H}_2(S, \mathbb{Z})$ of the diffeomorphisms ι_1^S, ι_2^S, and τ^S of S, respectively. The following lemma shows that the facts that ι_1 is an automorphism of the lattice L preserving the intersection numbers such that $\iota_1(f) = f$, $\iota_1(l_1) = l_1$, and $\iota_1 \circ \iota_1 = 1$, already almost determines ι_1. However, further investigations of the various cases are needed in order to determine the subsets I_1, I_2 and the permutations p_1 and p_2 of the index set I.

Lemma 5.1.2 *There exist $d_1 \in \mathbb{Z}_{\ge 0}$, $I_1 \subset I := \{1, \dots, 8\}$, and a permutation p_1 of I such that $\#(I_1) = 2d_1$, $p_1 \circ p_1 = 1$, $p_1(I_1) = I_1$, and*

$$\iota_1(l_1) = l_1, \tag{5.1.3}$$

$$\iota_1(l_2) = d_1\, l_1 + l_2 - \sum_{i \in I_1} e_i, \tag{5.1.4}$$

$$\iota_1(e_i) = \begin{cases} l_1 - e_{p_1(i)} & \text{when } i \in I_1, \\ e_{p_1(i)} & \text{when } i \notin I_1. \end{cases} \tag{5.1.5}$$

The same statement holds with ι_1, d_1, I_1, p_1, l_1, and l_2 replaced by ι_2, d_2, I_2, p_2, l_2, and l_1, respectively.

Proof. Equation (5.1.3) is obvious. Because $\iota_1(a \cdot b) = a \cdot b$ and $\iota_1 \circ \iota_1 = 1$, we have $\iota_1(a) \cdot = \iota_1(a) \cdot \iota_1(\iota_1(b)) = a \cdot \iota_1(b)$. Therefore $\iota_1(l_2) \cdot l_1 = l_2 \cdot \iota_1(l_1) = l_2 \cdot l_1 = 1$ implies that

$$\iota_1(l_2) = d_1\, l_1 + l_2 + \sum_{i \in I} c_i\, e_i,$$

for uniquely determined $d_1 \in \mathbb{Z}$, $c_i \in \mathbb{Z}$. For each $i \in I$ we have

$$\iota_1(e_i) = \alpha_i\, l_1 + \beta_i\, l_2 + \sum_{j=1}^{8} \gamma_{ji}\, e_j$$

for uniquely determined $\alpha_i,\ \beta_i,\ \gamma_{j,\,i} \in \mathbb{Z}$. We have

$$\begin{aligned} \alpha_i &= \iota_1(e_i) \cdot l_2 = \iota_1(\iota_1(e_i)) \cdot \iota_1(l_2) = e_i \cdot \iota_2(l_2) = -c_i, \\ \beta_i &= \iota_1(e_i) \cdot l_1 = \iota_1(\iota_1(e_i)) \cdot \iota_1(l_1) = e_i \cdot l_1 = 0, \\ -\gamma_{ji} &= \iota_1(e_i) \cdot e_j = e_i \cdot \iota_1(e_j) = -\gamma_{ij}, \end{aligned}$$

where the last equation shows that the matrix γ is symmetric. Furthermore, applying the formulas for ι_1 twice, we have

$$e_i = \iota_1(\iota_1(e_i)) = -(c_i + \sum_{j\in I} \gamma_{ji}\, c_j)\, l_1 + \sum_{k\in I} \sum_{j\in I} \gamma_{ji}\, \gamma_{kj}\, e_k,$$

for every $i \in I$, which implies that $1 = \gamma\,\gamma = \gamma\,{}^{\mathrm{t}}\gamma$, that is, γ is an orthogonal matrix with integral coefficients. It follows that each row and each column of γ has only one nonzero entry, which is equal to ± 1. In other words, there is a unique permutation p_1 of I such that $\gamma_{ji} \in \{\pm 1\}$ if $j = p_1(i)$ and $\gamma_{ji} = 0$ if $j \neq p_1(i)$, where $\gamma^2 = 1$ implies that $p_1 \circ p_1 = 1$.

Finally, we use that

$$2\, l_1 + 2\, l_2 - \sum_{i\in I} e_i = f = \iota_1(f) = \left(2 + 2\, d_1 + \sum_{i\in I} c_i\right) l_1 + 2\, l_2 + \sum_{i\in I} (2\, c_1 - \gamma_{i,\, p_1(i)})\, e_i,$$

which implies that $2\, d_1 + \sum_{i\in I}\, c_i = 0$ and $2\, c_i - \gamma_{i,\, p_1(i)} = -1$ for every $i \in I$. It follows that $c_i = 0$ if $\gamma_{i,\, p_1(i)} = 1$ and $c_i = -1$ if $\gamma_{i,\, p_1(i)} = -1$. The conclusions of the lemma follow if I_1 is the set of all $i \in I$ such that $\gamma_{i,\, p_1(i)} = -1$. □

The number $d_1 = \iota_1(l_2) \cdot l_2$ in Lemma 5.1.2 is equal to the degree of the mapping

$$\alpha : \mathbb{P}^1 \to \mathbb{P}^1 : y \mapsto \frac{f_0(y) - f_1(y)\, x}{f_1(y) - f_2(y)\, x},$$

for a generic choice of the point $x \in \mathbb{P}^1$. See (1.1.4).

In terms of the mapping $\iota_1^S : S \to S$, the mapping α can be described as follows. Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowdown map in Corollary 3.4.2, and let B be the set of base points in $\mathbb{P}^1 \times \mathbb{P}^1$, where $\#(B) \leq 8$. Let $x \in \mathbb{P}^1$ be such that $(\{x\} \times \mathbb{P}^1) \cap B = \emptyset$. Note that π is a complex analytic diffeomorphism from $S \backslash \pi^{-1}(B)$ onto $(\mathbb{P}^1 \times \mathbb{P}^1) \backslash B$. Then α is equal to $\mathbb{P}^1 \ni y \mapsto \pi^{-1}(x,\, y)$, followed by ι_1^S, followed by π, and concluded by the projection $(x',\, y) \mapsto x'$ from $\mathbb{P}^1 \times \mathbb{P}^1$ onto $\mathbb{P}^1$.

Definition 5.1.3. Because the $f_i(y)$ are homogeneous polynomials in $Y = ({y_1}^2,\ y_0\, y_1,\ {y_0}^2)$ of degree 2, it follows that for a generic pencil of biquadratic curves we have $d_1 = 4$. A lower degree d_1 occurs if and only if the polynomials $f_1(y)$, $f_2(y)$, and $f_3(y)$ have a common factor of degree $4 - d_1$. We will call $d_1 = \iota_1(l_2) \cdot l_2$ the

degree of the horizontal switch ι_1. Similarly, $d_2 = \iota_2(l_1) \cdot l_1$ is called the *degree of the vertical switch* ι_2.

Examples show that any of the numbers 0, 1, 2, 3, 4 can occur as the degree of the horizontal switch for a pencil of biquadratic curves with a smooth member. ⊘

Remark 5.1.4. If $i \in I_1$, then $\iota_1(e_i) \cdot l_2 = (l_1 - e_{p_1(i)}) \cdot l_2 = 1$. This implies that if $\pi : S \to S_0$ denotes the projection from S onto $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then the image under $\pi \circ \iota_1^S$ of the curve E_i in S, of which e_i was the homology class, intersects each vertical axis $\{x\} \times \mathbb{P}^1$. Because (1.1.4) implies that $\pi \circ \iota_1^S(E_i)$ is contained in the horizontal axis $L_{i,1} = \mathbb{P}^1 \times \{y\}$ through the base point of E_i, we conclude that $\pi \circ \iota_1^S(E_i) = L_{i,1}$. If $\pi'(L_{i,1})$ denotes the proper transform in S of this horizontal axis, then it follows that $\pi'(L_{i,1}) \subset \iota_1^S(E_i)$. ⊘

Remark 5.1.5. We have $d_1 = 0$ if and only if the polynomials $f_0(y)$, $f_1(y)$, $f_2(y)$ in (1.1.3) are proportional to each other if and only if the reflection (1.1.4) does not depend on y.

Let $d_1 = 0$. With a suitable choice of affine coordinates in the first $\mathbb{P}^1$-axis, we can arrange that (1.1.4) is equal to the reflection $x \mapsto -x$, which means that $f_0(y) \equiv 0$ and $f_2(y) \equiv 0$. This means that $(A^0\,Y) \times (A^1\,Y) \in [0 : 1 : 0]$, which in turn is equivalent to $(A^0\,Y)_1 \equiv 0$ and $(A^1\,Y)_1 \equiv 0$, or $A^0_{10} = A^0_{11} = A^0_{12} = 0$ and $A^1_{10} = A^1_{11} = A^1_{12} = 0$. In other words, after a suitable linear substitution of variables in the (x_0, x_1)-coordinates, the middle rows of both A^0 and A^1 are equal to zero. The QRT mapping has order two, see case (i) in Lemma 5.2.4. Similarly $d_2 = 0$ implies that $\tau^2 = 1$. ⊘

Lemma 5.1.6 *The following statements are equivalent.*

(i) $d_1 < 4$.

(ii) *A horizontal axis* $\mathbb{P}^1 \times \{[y_0 : y_1]\}$ *is a component of a member* $C_{[z_0:z_1]}$ *of the pencil* (2.5.3) *of biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$.

(iii) *There exist two different base points* (x, y), (x', y') *of the pencil of biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$ *such that* $y = y'$, *or there exists a multiple base point* (x, y) *such that in the notation of Corollary* 3.3.8 = *Lemma* 3.2.8, *with* $S = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $b = (x, y)$, *we have that* $e = \mathrm{T}_x\mathbb{P}^1 \times \{0\}$.

Proof. We use the notation $p_z(x, y) = X\,(z_0\,A^0 + z_1\,A^1)\,Y$ for the biquadratic polynomial p_z in (2.5.3), where $X = ({x_0}^2, x_0\,x_1, {x_1}^2)$ and $Y = {}^{\mathrm{t}}({y_0}^2, y_0\,y_1, {y_1}^2)$.

We have $d_1 < 4$ if and only if $f_1(y)$, $f_2(y)$, and $f_3(y)$ have a common factor if and only if there exists $(y_0, y_1) \in \mathbb{C}^2$, $(y_0, y_1) \neq (0, 0)$, such that $(A^0\,Y) \times (A^1\,Y) = f(y) = 0$; see (1.1.3). Now $(A^0\,Y) \times (A^1\,Y) = 0$ if and only if $A^0\,Y$ and $A^1\,Y$ are linearly dependent if and only if there exist $(z_0, z_1) \in \mathbb{C} \setminus \{(0, 0)\}$ such that $z_0\,A^0\,Y + z_1\,A^1\,Y = 0$, where the latter equation means that $p_z(x, y) = X\,(z_0\,A^0 + z_1\,A^1)\,Y = 0$ for every $x \in \mathbb{P}^1$. This proves (i) ⇔ (ii), because the vectors $X = ({x_1}^2, x_0\,x_1, {x_0}^2)$ span $\mathbb{C}^3$.

Suppose that (ii) holds, that is, $\mathbb{P}^1 \times \{y\}$ is contained in a member C of the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Because each other member D of $\mathfrak{B}$, which is a

$(2,\ 2)$-curve, has intersection number equal to 2 with the $(0,\ 1)$-curve $\mathbb{P}^1 \times \{y\}$, and because the intersection points are intersection points of C with D, and therefore base points of the pencil $\mathfrak{B}$, (iii) follows.

Conversely, let $([x_0 : x_1],\ [y_0 : y_1])$ and $([x_0' : x_1'],\ [y_0 : y_1])$ be two different base points. Note that $(x,\ y)$ is a base point if and only if $X\ (z_0\ A^0 + z_1\ A^1)\ Y = 0$ for every z_0, z_1 if and only if $X\ A^0\ Y = 0$ and $X\ A^1\ Y = 0$. Then $X\ A^0\ Y = 0$, $X\ A^1\ Y = 0$, and $X'\ A^0\ Y = 0$, $X'\ A^1\ Y = 0$, where X and X' are linearly independent. It follows that $A^0\ Y$ and $A^1\ Y$ belong to the common orthogonal complement of X and X', which is one-dimensional, and therefore $A^0\ Y$ and $A^1\ Y$ are linearly dependent. In the case of a multiple base point as in (iii), we replace X' by a nonzero tangent vector to the quadric P at X. This completes the proof of (iii) $\Rightarrow$ (ii). □

We now investigate the action of ι_1 on e_i which are the total transforms of -1 curves that appear at blowing up over base points b in $\mathbb{P}^1 \times \mathbb{P}^1$ such that the horizontal axis through b is not contained in a member of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. We begin with a simple observation.

Lemma 5.1.7 *The total transform in S of any horizontal axis is invariant under ι_1^S, and the total transform in S of every vertical axis is invariant under ι_2^S.*

Proof. Let $p_2 : (x,\ y) \mapsto y$ denote the projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the second component. Let C be a smooth member of the pencil of biquadratic curves. Because for each $y \in \mathbb{P}^1$ the restriction to C of ι_1 is the map that switches the two intersection points of C with the horizontal axis $\mathbb{P}^1 \times \{y\}$, we have $p_2 \circ \iota_1 = p_2$ on C. Furthermore, we have that $\pi \circ \iota_1^S = \iota_1 \circ \pi$ on $\pi^{-1}(C)$; see Corollary 3.4.2. It follows that $p_2 \circ \pi \circ \iota_1^S = p_2 \circ \pi$ on $\pi^{-1}(C)$. Because the union of the $\pi^{-1}(C)$'s is dense in S and the mapping $p_2 \circ \pi : S \to \mathbb{P}^1$ is complex analytic, hence continuous, we conclude that $p_2 \circ \pi \circ \iota_1^S = p_2 \circ \pi$ on S. It follows that for each $y \in \mathbb{P}^1$ and each $s \in \pi^{-1}(\mathbb{P}^1 \times \{y\})$, we have $p_2 \circ \pi(s) = y$, hence $(p_2 \circ \pi)(\iota_1^S(s)) = p_2 \circ \pi(s) = y$, and therefore $\iota_1^S(s) \in \pi^{-1}(\mathbb{P}^1 \times \{y\})$. The proof of the statement for ι_2 is similar. □

Lemma 5.1.8 *Let $(x,\ y) \in \mathbb{P}^1 \times \mathbb{P}^1$ be a base point of multiplicity k of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be as in Corollary* 3.3.10*, and let e_{l+j}, $1 \le j \le k$, be the homology class of the total transform in S of the exceptional curve that appears at the jth blowing up over the point $(x,\ y)$.*

If the horizontal axis $\mathbb{P}^1 \times \{y\}$ is not contained in a member of the pencil, then $l + j \in I_1$ and $\iota_1(e_{l+j}) = l_1 - e_{l+k-j+1}$, hence $p_1(l + j) = l + k - j + 1$, for every $1 \le j \le k$.

If the vertical axis $\{x\} \times \mathbb{P}^1$ is not contained in a member of the pencil, then $l + j \in I_2$ and $\iota_2(e_{l+j}) = l_2 - e_{l+k-j+1}$, hence $p_2(l + j) = l + k - j + 1$, for every $1 \le j \le k$.

If both the horizontal and the vertical axes through $(x,\ y)$ are not contained in a member of the pencil, then $p_1(l + j) = p_2(l + j)$ for every $1 \le j \le k$.

Proof. We refer to Sections 3.2.2 and 3.3.3 for the basic facts about blowing up that will be used in the proof.

Let L_1^* and L_1 denote the total and proper transform in S of the horizontal axis $\mathbb{P}^1 \times \{y\}$ through the base point $(x,\ y)$, respectively. It follows from Lemma 5.1.7 that $\iota_1^S(L_1^*) = L_1^*$.

Let E_{l+j} denote the proper transforms in S of the -1 curve that appears at the jth blowing up over the point $(x,\ y)$. Each of these proper transforms is a smooth curve in S, and rational and therefore isomorphic to a complex projective line. It follows from the proof of (iii) $\Rightarrow$ (ii) in Lemma 5.1.6 that if $\mathbb{P}^1 \times \{y\}$ contains a base point other than $(x,\ y)$ or the smooth members are tangent to $\mathbb{P}^1 \times \{y\}$ at $(x,\ y)$, then $\mathbb{P}^1 \times \{y\}$ is contained in a member of the pencil, in contradiction with the assumption. Therefore there are no other base points on $\mathbb{P}^1 \times \{y\}$, and if $k > 1$, the base point which appears on the -1 curve after the first blowing up does not lie on the proper transform of $\mathbb{P}^1 \times \{y\}$. Therefore

$$L_1^* = L_1 \cup \bigcup_{j=1}^{i} E_{l+j}.$$

Furthermore, the assumption that $\mathbb{P}^1 \times \{y\}$ is not contained in a member of the biquadratic pencil implies that L_1 intersects E_{l+1} and transversally, but does not intersect E_{l+j} with $j > 1$; see Lemma 5.1.6. It follows from Corollary 3.3.8 = Lemma 3.2.8 that if $1 \le j,\ j' \le i$ and $j \ne j'$, then $E_{l+j} \cdot E_{l+j'} = 1$ when $j' = j \pm 1$, and $E_{l+j} \cdot E_{l+j'} = 0$ otherwise. In other words, the irreducible components L_1, E_{l+j}, $1 \le j \le k$, of L_1^* form a chain of rational curves, beginning at L_1 and ending at E_{l+k}. Because e_{l+j} is equal to the homology class of the total transform E_{l+j}^* of the -1 curve that appeared at the jth blowing up, where E_{l+j}^* is equal to the union of the E_{l+h} with $j \le h \le k$, we have

$$l_1 = [L_1^*] = [L_1] + \sum_{j=1}^{k} [E_{l+j}], \quad \text{and} \quad e_{l+j} = [E_{l+j}^*] = \sum_{h=j}^{k} [E_{l+h}], \quad 1 \le j \le k. \tag{5.1.6}$$

The rational curve E_{l+k} is the -1 curve that appeared at the last blowing up, and therefore equal to a curve E_i as in Remark 5.1.4. It follows that $L_1 \subset \iota_1^S(E_{l+k})$, where we have equality because E_{l+k} is irreducible; hence $\iota_1^S(E_{l+k})$ is irreducible. In view of the chain structure of L_1^*, we conclude that $\iota_1^S([E_{l+k}]) = [L_1]$, $\iota_1^S([E_{l+j}]) = [E_{l+k-j}]$ when $1 \le j \le k-1$, and $\iota_1^S([L_1]) = [E_{l+k}]$. Combining this with (5.1.6), we obtain that $\iota_1(e_{l+j}) = l_1 - e_{l+k-j+1}$ for every $1 \le j \le k$.

It follows from Lemma 5.1.2 that $l + j \in I_1$ for every $1 \le j \le k$. This completes the proof of the first statement. The proof of the second statement is entirely analogous. The last statement follows immediately from the first and the second one. □

In the following corollary, we use the Eichler–Siegel transformations E_q introduced in (7.3.1).

Corollary 5.1.9 *Let the horizontal switch ι_1 and the vertical switch ι_2 both have the maximal degree* 4*, which in view of Lemma* 5.1.6 *is equivalent to the condition that*

no member of the pencil of biquadratic curves contains a horizontal or a vertical axis. Let $\tau^S : S \to S$ be the diffeomorphism of S defined in Corollary 3.4.2 *and let $\tau = (\tau^S)^L$ denote its action on $L = \mathrm{H}_2(S, \mathbb{Z})$. Then $\tau(l_1) = l_1 + 4l_2 - \sum_{i=1}^8 e_i$, $\tau(l_2) = 4l_1 + 9l_2 - 3\sum_{i=1}^8 e_i$, and $\tau(e_i) = l_1 + 3l_2 - \sum_{j\neq i} e_j$ for $1 \le i \le 8$.*

The action of the QRT map on the homology group $\mathrm{H}_2(S, \mathbb{Z})$ is equal to the Eichler–Siegel transformation E_q, with $q = l_2 - l_1 + \mathbb{Z} f$. It follows that for every $k \in \mathbb{Z}$, $((\tau^S)^k)^L = \tau^k = \mathrm{E}_{kq}$. The number of k-periodic fibers of τ^S, counted with multiplicities, is equal to $k^2 - 1$.

Proof. Because no member of the pencil of biquadratic curves contains a horizontal or a vertical axis, Lemma 5.1.8 implies that $I_1 = I_2 = \{1, \ldots, 8\}$ and $p_1 = p_2$. The formulas for the action $\tau = \iota_2 \circ \iota_1$ of the QRT map on $\mathrm{H}_2(S, \mathbb{Z})$ now follow from Lemma 5.1.2.

Let $q = l_2 - l_1 + \mathbb{Z} f$. A direct computation shows that $v \cdot v = -2$, $\mathrm{E}_q(l_1) = l_1 + 4l_2 - \sum e_i = \tau(l_1)$, $\mathrm{E}_q(l_2) = 4l_1 + 9l_2 - 3\sum e_i = \tau(l_2)$, and $\mathrm{E}_q(e_i) = e_i + l_1 + 3l_2 - \sum e_i = \tau(e_i)$, $1 \le i \le 8$, and therefore $\mathrm{E}_q = \tau$. The conclusion about the number of k-periodic fibers of τ^S follows from Corollary 7.4.7 and the observation that $\nu(\tau) = 0$ because

$$\begin{aligned}\tau(e_i) \cdot e_i &= \iota_2(\iota_1(e_i)) \cdot e_i = \iota_1(e_i) \cdot \iota_2(e_i)\\ &= (l_1 - e_{p_1(i)}) \cdot (l_2 - e_{p_2(i)}) = 1 + e_{p_1(i)} \cdot e_{p_2(i)} = 0,\end{aligned}$$

where we have used that $p_1 = p_2$. □

The conclusions of Corollary 5.1.9 hold in particular if $\kappa : S \to P$ has no reducible fibers.

Corollary 5.1.10 *If $\kappa : S \to P$ has no reducible fibers, then $d_1 = d_2 = 4$ and τ^S is a primitive element of $\mathrm{Aut}(S)^+_\kappa$. That is, if $l \in \mathbb{Z}$ and $\tau^S = \alpha^l$ for an automorphism α of S that acts as translations on the elliptic fibers, then $l = \pm 1$.*

Proof. That $d_1 = d_2 = 4$ if κ has no reducible fibers follows from Lemma 5.1.6 (ii).

Let $\tau^S = \alpha^l$ for an $\alpha \in \mathrm{Aut}(S)^+_\kappa$. Because $\kappa : S \to P$ has no reducible fibers, it follows from Corollary 9.2.23 that $\alpha^L = T_{v+\mathbb{Z} f}$ for some $v \in f^\perp$. Therefore

$$T_{l_2 - l_1 + \mathbb{Z} f} = \tau = (\alpha^l)^L = (\alpha^L)^l = T_{l\, v + \mathbb{Z} f},$$

which implies that $l_2 - l_1 + \mathbb{Z} f = l\, v + \mathbb{Z} f$, and therefore $-2 = (l_2 - l_1) \cdot (l_2 - l - 1) = l^2\, (v \cdot v)$. Because $v \cdot v \in \mathbb{Z}$, it follows that $l = \pm 1$. □

It follows from Lemma 7.3.2 that the action of $\alpha \in \mathrm{Aut}(S)^+_\kappa$ on $\Lambda = \mathrm{NS}(S) \simeq \mathrm{H}_2(S, \mathbb{Z})$ is an Eichler–Siegel transformation if and only if α leaves each irreducible component of each reducible fiber of κ invariant. Because there are only finitely many such irreducible components, we have that some finite iterate $(\tau^S)^m$ of τ^S acts as an Eichler–Siegel transformation ε on $\mathrm{H}_2(S, \mathbb{Z})$. If ε is nontrivial, then ε has infinite order, hence τ^S has infinite order, and the number of k-periodic fibers, counted with multiplicities, has quadratic growth as a function of k as $k \to \infty$; see (7.5.5). On

the other hand, if the Eichler–Siegel transformation ε is trivial, then it follows from the last statement in Theorem 7.2.7 with $\Lambda = \mathrm{H}_2(S, \mathbb{Z})$ that $(\tau^S)^m$ is equal to the identity on S, that is, the QRT mapping has some finite order m. In Section 5.2 we will take a closer look at QRT mappings of finite order, which form the extreme opposite to the case of Corollary 5.1.9.

For a general formula for the number of k-periodic fibers, counted with multiplicities, in terms of the way the QRT automorphism permutes the irreducible components of the reducible fibers, we can apply (4.3.2) with $\alpha = \tau^S$.

It follows from the above considerations that in order to understand the action of τ^S on $\mathrm{H}_2(S, \mathbb{Z})$ in the case that some member of the pencil of biquadratic curves contains a horizontal or a vertical axis, we need to know how τ^S permutes the irreducible components of the reducible fibers of κ. In the appendix in Section 12.1, we have listed the possible singular fibers of κ and described how τ^S permutes their irreducible components. In some cases where it is not immediately obvious how τ^S permutes the components of the reducible fibers, we use Lemmas 12.1.2 and 12.1.3, which are analogues of Lemma 5.1.8 in the case that a horizontal or vertical axis through the base point is contained in a member of the pencil of biquadratic curves. The list in Section 12.1 implies the following conclusion.

Proposition 5.1.11 *Let S_r be a reducible fiber of κ. Then the QRT automorphism permutes the irreducible components of S_r in a nontrivial way if and only if the member $\pi(S_r)$ of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ contains a horizontal or a vertical axis.*

Corollary 5.1.12 *The following conditions for the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are equivalent:*

(i) *The horizontal and vertical switches ι_1 and ι_2 both have the maximal degree* 4.
(ii) *No member of the pencil of biquadratic curves contains a horizontal or a vertical axis.*
(iii) *The QRT automorphism τ^S leaves each irreducible component of each fiber of κ invariant.*
(iv) *The action on $\mathrm{H}_2(S, \mathbb{Z})$ of the QRT automorphism τ^S is an Eichler–Siegel transformation.*

Proof. The equivalence between (i) and (ii) follows from Lemma 5.1.6. The equivalence between (ii) and (iii) follows from Proposition 5.1.11, where we observe that τ^S always leaves the fibers of κ invariant, in particular the irreducible fibers. The equivalence between (iii) and (iv) follows from Lemma 7.3.2. □

5.2 QRT Transformations of Finite Order

Inspired by the paper of Tsuda [196], we discuss in this section when the QRT mapping is *periodic* or synonymously *of finite order*, meaning that there is a positive integer m such that the mth iterate of the QRT mapping is equal to the identity. In

this case the *order* or *period* of τ is defined as the smallest positive integer m such that $\tau^m = 1$, where for any $m \in \mathbb{Z}$ we have $\tau^m = 1$ if and only if m is equal to an integral multiple of the order of τ. If τ is not periodic, then one says that the order of τ is infinite.

The QRT mapping τ is of order m if and only if the induced automorphism $\tau^S \in \mathrm{Aut}(S)^+_\kappa$ of the QRT surface S is of order m. Conversely, it follows from Theorem 4.3.3 that for every rational elliptic surface $\kappa : S \to P$, every nontrivial element $\alpha \in \mathrm{Aut}(S)^+_\kappa$ of finite order is a Manin QRT automorphism of S. That is, there exists a pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that S is isomorphic to the successive blowing up, eight times, at base points of the anticanonical pencils starting with $\mathfrak{B}$, and α is equal to the QRT automorphism τ^S of S induced by the QRT mapping τ of the pencil $\mathfrak{B}$.

The elements of finite order of the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ together form the torsion subgroup T of the Mordell–Weil group. It follows from Lemma 9.2.6 that the only nontrivial torsion subgroups that can appear are isomorphic to $\mathbb{Z}/m\,\mathbb{Z}$ for $2 \le m \le 6$, $(\mathbb{Z}/2\,\mathbb{Z})^2$, $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/4\,\mathbb{Z})$, or $(\mathbb{Z}/3\,\mathbb{Z})^2$. Therefore the period of a periodic QRT mapping can only be 2, 3, 4, 5, or 6. Conversely, the fact that every nontrivial element of finite order of the Mordell–Weil group can be realized as a QRT automorphism implies that for each of these numbers there exist QRT mappings of this order.

In the following criteria for periodicity, the contributions to the map of the reducible fibers are described in Lemma 7.5.3, and Manin's function $\mu(z)$ is defined by (2.5.16) in terms of the Weierstrass data $g_2(z)$, $g_3(z)$, $X(z)$, $Y(z)$ of Corollaries 2.5.10 and 2.5.13. Criterion (ii) will be used in finding pencils of biquadratic curves for which the QRT map has a prescribed finite order, whereas criterion (iii) is an efficient test for finiteness of the order of the QRT map for a given pencil of biquadratic curves.

Lemma 5.2.1 *The following conditions are equivalent:*

(i) *The QRT mapping has finite order.*
(ii) *The sum of over all reducible fibers S_r of the contributions* $\mathrm{contr}_r(\tau)$ *is equal to* 2.
(iii) *Manin's function $\mu(z)$ is identically equal to zero.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 4.3.3, and (i) $\Leftrightarrow$ (iii) follows from the last conclusion in the first paragraph of Proposition 2.5.20. □

It follows from Corollary 7.3.3 that the representation of the torsion subgroup of the Mordell–Weil group as permutations of the set R of all irreducible components of the reducible fibers of φ is faithful. It follows that if α has finite order, then its order is equal to the smallest positive integer m such that α^m leaves each irreducible component of each reducible fiber invariant.

Let $g_2(z)$, $g_3(z)$, $X(z)$, and $Y(z)$ be the Weierstrass data of the QRT map as defined in Corollaries 2.5.10 and 2.5.13. Using the group law on the Weierstrass curve given by (7.8.2), (7.8.3), one can obtain the rational functions $X_m(z)$, $Y_m(z)$

that are the affine coordinates of the image point of the point $[0 : 0 : 1]$ at infinity on the Weierstrass curve under the mth power τ^m of the QRT map. We have that $\tau^m = 1$ if and only if on the Weierstrass curves τ^m maps the point at infinity to itself, that is, the denominators of $X_m(z)$ and $Y_m(z)$ are identically equal to zero. These denominators are contants times $\delta_m(z)^2$ and $\delta_m(z)^3$, respectively, where $\delta_m(z)$ is a polynomial. For $2 \le m \le 6$, using a formula manipulation program, I obtained the following formulas for $\delta_m(z)$, where I have suppressed the dependence on z in the notation.

Table 5.2.2

$$\delta_2 = Y,$$

$$\delta_3 = (g_2 - 12\,X^2)^2 - 48\,X\,Y^2,$$

$$\delta_4 = \delta_2\,[-(g_2 - 12\,X^2)^3 + 48\,X\,(g_2 - 12\,X^2)\,Y^2 + 32\,Y^4],$$

$$\begin{aligned}\delta_5 = {} & (g_2 - 12\,X^2)^6 - 144\,X\,(g_2 - 12\,X^2)^4\,Y^2 \\ & + 128\,(g_2 - 12\,X^2)^2\,(g_2 + 42\,X^2)\,Y^4 - 6144\,X\,(g_2 + 6\,X^2)\,Y^6 - 4096\,Y^8,\end{aligned}$$

$$\begin{aligned}\delta_6 = \delta_2\,\delta_3\,[& 3\,(g_2 - 12\,X^2)^6 - 240\,X\,(g_2 - 12\,X^2)^4\,Y^2 \\ & - 384\,(g_2 - 12\,X^2)^2\,(g_2 - 18\,X^2)\,Y^4 \\ & + 18432\,X\,(g_2 - 6\,X^2)\,Y^6 + 8192\,Y^8].\end{aligned}$$

For each $2 \le m \le 6$, the equation $\delta_m = 0$ leads to an explicit determination of the Weierstrass data of the QRT maps of order m. It turns out that the dimensions of the moduli spaces of isomorphism classes of QRT maps of orders 2, 3, and 4 are equal to 4, 2, and 1, respectively, whereas the QRT maps of orders 5 and 6 are unique up to isomorphisms. See Sections 5.2.1–5.2.5. For the dimension 8 of the moduli space of all QRT maps; see Section 6.3.3

Remark 5.2.3. The degree of δ_k is equal to $k^2 - 1$, which is precisely the number of k-periodic fibers of the QRT map, counted with multiplicities, if no member of the pencil of biquadratic curves contains a horizontal or a vertical axis. See the last statement in Corollary 5.1.9. Since the fiber over z is k-periodic if and only if $(X_k(z),\ Y_k(z))$ is the point at infinity on the Weierstrass curve, this means that the condition that no member of the pencil of biquadratic curves contains a horizontal or a vertical axis implies that the numerators in the rational expressions for $X_k(z)$, $Y_k(z)$ have no zeros at the zeros of the denominators in such a way that at these z, at least one of $X_k(z)$ and $Y_k(z)$ has finite values. ⊘

The Weierstrass data $X(z)$, $Y(z)$, $g_2(z)$, and $g_3(z)$ of the QRT map in Corollaries 2.5.13 and 2.5.10 are explicit homogeneous polynomials of the respective degrees 2, 3, 4, and 6 in the coefficients of the matrices A^0 and A^1 that define the pencil of biquadratic curves (2.5.3), (2.5.4). In Section 5.2.1 we will find an explicit description of the families of pairs of matrices A^0, A^1 whose QRT map has order two. For each QRT map of the respective orders five and six, unique up to isomorphism, we will find in Sections 5.2.4 and 5.2.5 a pair A^0, A^1 representing it. The QRT roots of the symmetric families of order two have order four and represent each isomorphism

class. In Section 5.2.3 we also find QRT maps of order four defined by pairs A^0, A^1. For the order three we find a family of pairs A^0, A^1 in Section 5.2.2 in which each isomorphism class of a generic order-three QRT is represented.

The strategy for finding the pairs A^0, A^1 is to use criterion (ii) in Lemma 5.2.1 in order to determine the way the QRT transformation τ^S permutes the irreducible components of the reducible fibers such that τ has the given period m. Then we use Section 12.1 in order to find pencils of biquadratic curves with the desired properties. For each $2 \le m \le 6$, Tsuda [196, Example 3.5] gave examples of pairs of matrices A^0, A^1 such that the corresponding QRT maps have order m. Tsuda mentioned that it can easily be checked that the QRT mappings in his examples have the desired period, but did not tell how he found the matrices.

5.2.1 Order 2

According to Table 5.2.2, we have $\delta_2 = 0$ if and only if $Y(z_0, z_1) \equiv 0$ in the Weierstrass model, when the equation $-4\,X^3 + g_2\,X + g_3 = 0$ determines g_3 in terms of X and g_2. Conversely, if for any homogeneous polynomials X and g_2 in (z_0, z_1) of the respective degrees 2 and 4 we define $g_3 = 4\,X^3 - g_2\,X$, then the mapping that on the Weierstrass curve translates the point at infinity to $(X, 0)$ is an element of order two in the Mordell–Weil group of the rational elliptic surface with the Weierstrass data g_2 and g_3. We have $g_2(z_0, z_1) = g_3(z_0, z_1) = 0$ if and only if $X(z_0, z_1) = g_2(z_0, z_1) = 0$, and the condition that there is no $(z_0, z_1) \ne (0, 0)$ at which g_2 and g_3 have zeros of respective orders ≥ 4 and ≥ 6, see (v) in Lemma 9.1.2, is equivalent to the condition that there is no (z_0, z_1) at which X and g_2 have a common zero of respective orders ≥ 2 and ≥ 4. The condition that the discriminant $\Delta = {g_2}^3 - 27\,{g_3}^2 = (g_2 - 12\,X^2)^2\,(g_2 - 3\,X^2)$ is not identically equal to zero is equivalent to the condition that the polynomial g_2 is not equal to $12\,X^2$ or to $3\,X^2$. Working modulo linear substitutions of variables in (z_0, z_1), we obtain a moduli space of QRT maps of order two of dimension $(2 + 1) + (4 + 1) - 4 = 4$, as compared to the dimension $(2 + 1) + (3 + 1) + (4 + 1) - 4 = 8$ of the moduli space of all QRT automorphisms, determined by the Weierstrass data X, Y, and g_2. The list of Persson [156] contains 41 configurations of singular fibers where the Mordell–Weil group has elements of order two. I don't know how hard it is to find these from the above description in terms of X and g_2.

According to Table 6.3.2, we have a singular fiber of type other than I_b if and only if g_2 and g_3 have a common zero if and only if X and g_2 have a common zero if and only if the two factors $g_2 - 12\,X^2$ and $g_2 - 3\,X^2$ in Δ have a common zero. It follows that the configuration of the singular fibers is $4\ \mathrm{I}_2\ \ 4\ \mathrm{I}_1$ if and only if the resultant of X and g_2 is nonzero, the discriminant of $g_2 - 12\,X^2$ is nonzero, and the discriminant of $g_2 - 3\,X^2$ is nonzero. QRT maps with such X and g_2, where $g_3 = 4\,X^3 - g_2\,X$ and $Y = 0$, will be called the *generic QRT maps of order two*. According to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z}^4 \times (\mathbb{Z}/2\,\mathbb{Z})$ or $\mathbb{Z}^4$. Because the QRT map is an element of the Mordell–Weil group of order two,

we conclude that the Mordell–Weil group is isomorphic to $\mathbb{Z}^4 \times (\mathbb{Z}/2\,\mathbb{Z})$, where the generic QRT map generates the torsion subgroup of the Mordell–Weil group.

The QRT map is of order two if and only if $Y = 0$ if and only if $\det(z_0\,A^0 + z_1\,A^1) = 0$ for all $(z_0,\ z_1) \in \mathbb{C}^2$; see Corollary 2.5.9.

Lemma 5.2.4 *We have* $\det(z_0\,A^0 + z_1\,A^1) \equiv 0$ *if and only if we are in one of the following cases:*

(i) *There is a nonzero* $e \in \mathbb{C}^3$ *such that* $e\,A^0 = 0$ *and* $e\,A^1 = 0$.
(ii) *There is a nonzero* $f \in \mathbb{C}^3$ *such that* $A^0\,f = 0$ *and* $A^1\,f = 0$.
(iii) *There are two-dimensional linear subspaces* E *and* F *of* $\mathbb{C}^3$ *such that for every* $e \in E$ *and* $f \in F$, *we have* $e\,A^0\,f = 0$ *and* $e\,A^1\,f = 0$.

Proof. Assume that we are not in case (i) or case (ii). Then we have linearly independent vectors f_0 and f_1 such that $A^0\,f_0 = 0$ and $A^1\,f_1 = 0$. Assume that $A^0\,f_1$ and $A^1\,f_0$ are linearly independent, and choose $e \in \mathbb{C}^3$ such that $e \neq 0$ and $e\,A^0\,f_1 = e\,A^1\,f_0 = 0$. If $z_0 \neq 0$ and $z_1 \neq 0$, then $z_0\,A^0 + z_1\,A^1$ maps $\mathbb{C}\,f_0 + \mathbb{C}\,f_1$ onto the two-dimensional vector space $\mathbb{C}\,A^0\,f_1 + \mathbb{C}\,A^1\,f_0$, and because the dimension of the range of $z_0\,A^0 + z_1\,A^1$ is ≤ 2, it follows that $e\,(z_0\,A^0 + z_1\,A^1)\,f = 0$ for all $f \in \mathbb{C}^3$, that is, we are in case (i), a contradiction. The conclusion is that $A^0\,f_1$ and $A^1\,f_0$ span a one-dimensional vector space. Let E be the two-dimensional vector space of all $e \in \mathbb{C}^3$ such that $e\,A^0\,f_1 = e\,A^1\,f_0 = 0$. If F denotes the vector space spanned by f_0 and f^1, we have $e\,A^0\,f = e\,A^1\,f = 0$ if $e \in E$ and $f \in F$, that is, we are in case (iii). □

The biquadratic polynomial defined by the 3×3 matrix A is of the form $p(x,\ y) = e\,A\,f$, where $x = (x_0,\ x_1)$, $y = (y_0,\ y_1)$, e is the row vector with entries $e_0 = {x_0}^2$, $e_1 = x_0\,x_1$, and $e_2 = {x_1}^2$, and f is the column vector with entries $f_0 = {y_0}^2$, $f_1 = y_0\,y_1$, and $f_2 = {y_1}^2$. Note that e and f lie on the quadratic cones Q defined by $e_0\,e_2 - {e_1}^2 = 0$ and $f_0\,f_2 - {f_1}^2 = 0$, respectively. The action on $p(x,\ y)$ of a linear transformation in x or y corresponds to the action on e or f of a linear transformation in $\mathbb{C}^3$ that leaves Q invariant, that is, an element of the projective orthogonal group $\mathrm{PO}(Q)$ of Q. This leads to a surjective homomorphism of $\mathrm{GL}(2, \mathbb{C})$ onto the identity component $\mathrm{PO}(Q)^\circ$ of $\mathrm{PO}(Q)$. Because $\mathrm{PO}(Q)^\circ$ acts locally transitively on the connected open subset $\mathbb{C}^3 \setminus Q$ of $\mathbb{C}^3$, $\mathrm{PO}(Q)^\circ$ acts transitively on $\mathbb{C}^3 \setminus Q$, and similarly we obtain that $\mathrm{PO}(Q)^\circ$ acts transitively on $Q \setminus \{0\}$, and even on the set of all pairs of distinct one-dimensional linear subspaces of $\mathbb{C}^3$ that are contained in Q.

Suppose that we are in case (i) with $e \in Q$. Then, using an element of $\mathrm{PO}(Q)^\circ$, which corresponds to a linear substitution in x, we can arrange that $e = (1,\ 0,\ 0)$ that means that the first rows of A^0 and A^1 are equal to zero. However, this implies that for each z_0 and z_1 the biquadratic polynomial $p_z(x,\ y) = z_0\,p^0(x,\ y) + z_1\,p^1(x,\ y)$ has x_2 as a factor, and therefore the biquadratic curve defined by $p_z(x,\ y) = 0$ is reducible, hence singular. Since this is ruled out by the assumption that the pencil of biquadratic curves has at least one smooth member, we have $e \in \mathbb{C}^3 \setminus Q$. But then we can arrange that $e = (0,\ 1,\ 0)$, which means that the middle rows of A^0 and A^1 are equal to zero. In other words, we are in the case of Remark 5.1.5, when the degree d_1 of the horizontal switch ι_1 is equal to zero.

In the, admittedly somewhat arbirary, example

$$A^0 = \begin{pmatrix} 0\,1\,1 \\ 0\,0\,0 \\ 1\,0\,0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 1\;\;0\;\;0 \\ 0\;\;0\;\;0 \\ 0\,{-1}\,1 \end{pmatrix},$$

(2.5.1) and Corollary 2.4.7 yield $X = (2\,{z_0}^2 + z) - z_1 + 2\,{z_1}^2)/3$ and $g_2 = 4\,({z_0}^4 - 2\,{z_0}^3\,z_1 + 15\,{z_0}^2\,{z_1}^2 - 2\,z_0\,{z_1}^3 + {z_1}^4)/3$, and we have a generic QRT map of order two. Furthermore, the derivative at this special $(A^0,\ A^1)$ of the coefficients of X and g_2 with respect to the coefficients of arbitrary pairs of matrices with zero middle rows is surjective. In view the implicit function theorem it follows that the set R(i) of $(X,\ g_2)$ for such pairs of matrices has a nonempty interior.

A subset of an algebraic variety X is called *constructible* if it is the union of finitely many sets $A \setminus B$, where A and B are algebraic subsets of X. *Chevalley's theorem on constructible sets* [36, Section III, Theorem 3] states that any set that is algebraically related to a constructible set is constructible. That is, if X and Y are algebraic varieties, C is a constructible subset of X, and R is an algebraic subset of $X \times Y$, then the set of all $y \in Y$ such that $(x,\ y) \in R$ for some $x \in C$ is a constructible subset of Y. See for instance Łojasiewicz [125, Chapter VII, §8] for a detailed exposition. It follows that the set R(i) is constructible and therefore contains the complement of an algebraic subset of positive codimension in $\mathrm{H}_2(\mathbb{C}^2) \times \mathrm{H}_4(\mathbb{C}^2)$, if $\mathrm{H}_k(\mathbb{C}^2)$ denotes the space of all homogeneous polynomials of degree k in two variables.

Case (ii) is case (i) with the roles of $(x_0,\ x_1)$ and $(y_0,\ y_1)$ interchanged. If the matrices A^0 and A^1 in case (i) are chosen to be symmetric, then we are also in case (ii). Furthermore, the QRT root, see Section 10.1, has order 4. In Section 5.2.3 we verify that, up to isomorphism, each rational elliptic surface with an element of order four in the Mordell–Weil group can be realized in this way.

Suppose that we are in case (iii). Because Q is nonsingular, $E \cap Q$ is the union of two distinct one-dimensional linear subspaces of $\mathbb{C}^3$. By means of an element of $\mathrm{PO}(Q)$ coming from a linear substitution of variables in x, we can arrange that these are $\mathbb{C}\,(1,\ 0,\ 0)$ and $\mathbb{C}\,(0,\ 0,\ 1)$. Similarly we can arrange that $F \cap Q$ is the union of $\mathbb{C}\,(1,\ 0,\ 0)$ and $\mathbb{C}\,(0,\ 0,\ 1)$. The condition that $e\,A^0\,f = e\,A^1\,f = 0$ if $e \in E$ and $f \in F$ now means that $A^0_{00} = A^0_{02} = A^0_{20} = A^0_{22} = 0$ and $A^1_{00} = A^1_{02} = A^1_{20} = A^2_{22} = 0$, that is, the four corners of both matrices A^0 and A^1 are equal to zero. This is the family of QRT mappings of order two in Example 3.5(a) of Tsuda [196]. This condition is equivalent to the statement that each of the four points $([1 : 0],\ [1 : 0])$, $([1 : 0],\ [0 : 1])$, $([0 : 1],\ [1 : 0])$, and $([0 : 1],\ [0 : 1])$ is a base point.

In the example

$$A^0 = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,1 \\ 0\,0\,0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0\,0\;\;0 \\ 1\,1\,{-1} \\ 0\,1\;\;0 \end{pmatrix},$$

(2.5.1) and Corollary 2.4.7 yield $X = (-4\,{z_0}^2 - 4\,z_0\,z_1 + 5\,{z_1}^2)/12$ and $g_2 = (16\,{z_0}^4 - 16\,{z_0}^3\,z_1 - 24\,{z_0}^2\,{z_1}^2 + 8\,z_0\,{z_1}^3 + 25\,{z_1}^4)/12$, and we have a generic

QRT map of order two. Furthermore, the derivative at this special $(A^0,\ A^1)$ of the coefficients of X and g_2 with respect to the coefficients of arbitrary pairs of matrices with zero corners is surjective, which implies that the set R(iii) of $(X,\ g_2)$ for such pairs of matrices has a nonempty interior. As for case (i) it follows that R(iii) contains the complement of an algebraic subset of positive codimension in $\mathrm{H}_2(\mathbb{C}^2) \times \mathrm{H}_4(\mathbb{C}^2)$. If the matrices A^0 and A^1 in case (iii) are chosen to be symmetric, then the QRT root, see Section 10.1, has order 4. According to Section 5.2.3, all isomorphism classes of rational elliptic surfaces with an element of order four in the Mordell–Weil group are realized in this manner.

I found it a bit surprising that each of the sets R(i), R(ii), and R(iii) is so large. I do not know whether any of the cases (i), (ii), (iii) yields all QRT maps of order two. Note that every QRT map of order two is realized by one in case (i) or in case (iii).

5.2.2 Order 3

The derivative $\mathrm{d}y/\mathrm{d}x$ at $x = X$ and $y = Y$ on the Weierstrass curve $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$ is equal to $D := (12\,X^2 - g_2)/2\,Y$. With this substitution, the equation $\delta_3 = 0$, see Table 5.2.2, is equivalent to $D^2 - 12\,X = 0$. Since any pole in D appears with multiplicity two in D^2 and is not compensated, the rational function D has no poles, hence is a polynomial. Because $g_2 - 12\,X^2$ and Y are homogeneous of the respective degrees 4 and 3, D is a homogeneous polynomial of degree one, a linear form in $(z_0,\ z_1)$.

Conversely, for any homogeneous polynomials D and Y of the respective degrees one and three, the polynomials $X = D^2/12$, $g_2 = D^4/12 - 2\,D\,Y$, and $g_3 = -D^6/216 + D^3\,Y/6 - Y^2$ define a Weierstrass model in which $(X,\ Y)$ represents an element of the Mordell–Weil group of order 3. The point $(z_0,\ z_1) \neq (0,\ 0)$ is a common zero of g_2 and g_3 if and only if it is a common zero of D and Y when $\mathrm{ord}_{(z_0,\,z_1)}\,g_2 \geq 4$ if and only if $D \equiv 0$ or $\mathrm{ord}_{(z_0,\,z_1)}\,Y \geq 3$. If $D \equiv 0$ then $\mathrm{ord}_{(z_0,\,z_1)}\,g_3 \geq 6$ if and only if $\mathrm{ord}_{(z_0,\,z_1)}\,Y \geq 3$, whereas $\mathrm{ord}_{(z_0,\,z_1)}\,g_3 \geq 6$ also holds if $D(z_0,\ z_1) = 0$ and $\mathrm{ord}_{(z_0,\,z_1)}\,Y \geq 3$. That is, $\mathrm{ord}_{(z_0,\,z_1)}\,g_2 < 4$ or $\mathrm{ord}_{(z_0,\,z_1)}\,g_3 < 6$ if and only if $D(z_0,\ z_1) \neq 0$ or $\mathrm{ord}_{(z_0,\,z_1)}\,Y < 3$. Furthermore, the discriminant $\Delta = Y^3\,(D^3 - 27\,Y)$ is not identically zero if and only if Y is not identically equal to zero or to $D^3/27$. The dimension of the moduli space of QRT maps of order three is equal to $(1+1)+(3+1)-4 = 2$. In the list of Persson [156, pp. 7–14] there are nine configurations of singular fibers with elements of order three in the Mordell–Weil group. It should not be too hard to find these from the above description in terms of D and Y.

In view of Table 6.3.2 we have the configuration of singular fibers $3\ \mathrm{I}_3\ \ 3\ \mathrm{I}_1$ if and only if the resultant of D and Y is nonzero, the discriminant of Y is nonzero, and the discriminant of $D^3 - 27\,Y$ is nonzero. The QRT maps with such D and Y will be called the *generic QRT maps of order three*. According Lemma 9.2.6 the Mordell–Weil group is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/3\,\mathbb{Z})$, where the generic QRT map of order three generates the torsion subgroup of the Mordell–Weil group.

In order to find a pencil of biquadratic curves of which the QRT map has order three, we observe that (A) in Lemma 7.5.3 implies that the singular fibers of type I_1 have contribution 0, and those of type I_3 have contribution $2/3$ or 0, depending on whether or not the components are permuted. In view of (ii) in Lemma 5.2.1, each singular fiber of type I_3 must have contribution $2/3$, and therefore its irreducible components must be permuted. It follows from the classification in Section 12.1 that the only cases of a singular fiber of type I_3 with contribution $2/3$ to the QRT automorphism are (4a2), its counterpart (5a2) with the roles of the horizontal and vertical axes interchanged, and (6a1). We will work out the case that each of the three members belongs to the case (6a1). That is, for $1 \le k \le 3$ we have a member $L_1^{(k)} \cup L_2^{(k)} \cup D^{(k)}$, where $L_1^{(k)}$, $L_2^{(k)}$, and $D^{(k)}$ are a horizontal axis, a vertical axis, and an irreducible $(1,\,1)$ curve, respectively. The curves $L_1^{(k)}$, $L_2^{(k)}$, and $D^{(k)}$ form a triangle such that none of the intersection points of sides is a base point. Because these three members don't have a component in common, since otherwise such a common component would be a component of each member of the pencil, the three horizontal axes and the three vertical axes are distinct from each other. By means of a uniquely determined projective linear transformation in each of the factors $\mathbb{P}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$, we can arrange that $L_1^{(1)} = \{y_0 = 0\}$, $L_1^{(2)} = \{y_1 = 0\}$, $L_1^{(3)} = \{y_0 = y_1\}$, and $L_2^{(1)} = \{x_0 = 0\}$, $L_2^{(2)} = \{x_1 = 0\}$, $L_2^{(3)} = \{x_0 = x_1\}$. On the other hand, we can arrange by means of a projective linear transformation in the $[z_0 : z_1]$ projective line that the member $L_1^{(k)} \cup L_2^{(k)} \cup D^{(k)}$ with $k = 1,\, 2,\, 3$ corresponds to $[z_0 : z_1]$ equal to $[1 : 0]$, $[0 : 1]$, $[1 : 1]$, respectively. This amounts to $A^0_{02} = A^0_{12} = A^0_{20} = A^0_{21} = A^0_{22} = 0$, $A^1_{00} = A^1_{01} = A^1_{02} = A^1_{10} = A^1_{20} = 0$, and $A^0_{01} = A^0_{10} = -A^0_{00}$, $A^1_{12} = A^1_{21} = -A^1_{22}$, $A^0_{11} + A^1_{11} = A^0_{00} + A^1_{22}$. That is,

$$A^0 = \begin{pmatrix} a & -a & 0 \\ -a & p & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & -b \\ 0 & -b & b \end{pmatrix} \quad \text{where} \quad p + q = a + b.$$

It follows from (2.5.1) and Corollary 2.4.7 that $X = (p\,z_0 + q\,z_1)^2/12$, $Y = a\,(p + q - a)\,(a - p)\,z_0\,z_1\,(z_0 - z_1)$, and $12\,g_2 = (p\,z_0 + q\,z_1)((p\,z_0 + q\,z_1)^3 + 24\,Y)$, and therefore $D := (12\,X^2 - g_2)/2\,Y = -(p\,z_0 + q\,z_1)$. This defines a generic QRT mapping of order three if and only if $p \ne 0$, $q \ne 0$, $p + q \ne 0$, $a \ne 0$, $b = p + q - a \ne 0$, and $a - p = q - b \ne 0$. In this case the degrees d_1 and d_2 of ι_1 and ι_2, see Definition 5.1.3, are both equal to 1. For the generic QRT mapping of order three we can arrange by means of a linear substitution of variables in $(z_0,\, z_1)$ that Y is a multiple of $z_0\,z_1\,(z_0 - z_1)$, where for our family we have the same Y and arbitrary D. It follows that every generic QRT map of order three can be realized by one in our family, which is a symmetric QRT mapping as discussed in Chapter 10. The family in Example 3.5(b) of Tsuda [196] resembles ours, but is not symmetric.

5.2.3 Order 4

According to Table 5.2.2 with $D := (12\,X^2 - g_2)/2\,Y$, the equation $\delta_4/\delta_2 = 0$ is equivalent to $D^3 - 12\,D\,X + 4\,Y = 0$. Since any pole in D appears with multiplicity three in D^3 which is not compensated, D has no poles, hence is a polynomial in $(z_0,\ z_1)$, homogeneous of degree one.

Conversely, for any homogeneous polynomials D and X of the respective degrees one and two, the polynomials $Y = -D^3/4 + 3\,D\,X/2$, $g_2 = (D^4 - 12\,D^2\,X + 24\,X^2)/2$, and $g_3 = (D^2 - 8\,X)\,(-D^4 + 8\,D^2\,X + 16\,X^2)/16$ define a Weierstrass model in which $(X,\ Y)$ represents an element of the Mordell–Weil group of order 4. The point $(z_0,\ z_1) \neq (0,\ 0)$ is a common zero of g_2 and g_3 if and only if it is a common zero of D and X when $\mathrm{ord}_{(z_0,\,z_1)}\,g_2 \geq 4$ and $\mathrm{ord}_{(z_0,\,z_1)}\,g_3 \geq 6$ if and only if $\mathrm{ord}_{(z_0,\,z_1)}\,X \geq 2$. That is, we have to require that if D and X have a common zero, then the two zeros of X are simple. Furthermore, the discriminant $\Delta = (D^2 - 12\,X)^4\,D^2\,(5\,D^2 - 48\,X)/256$ is not identically zero if and only if D is not identically equal to zero and X is not identically equal to $D^2/12$ or to $5\,D^2/48$. The dimension of the moduli space of QRT maps of order four is equal to $(1 + 1) + (2 + 1) - 4 = 1$.

In view of Table 6.3.2 we have the configuration of singular fibers $2\ \mathrm{I}_4\ \ \mathrm{I}_2\ \ 2\ \mathrm{I}_1$ if and only if the resultant of D and X, the discriminant of $5\,D^2 - 48\,X$, and the discriminant of $D^2 - 12\,X$ all are nonzero. The QRT maps with such D and X will be called the *generic QRT maps of order four*. According to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/4\,\mathbb{Z})$.

If the resultant of D and X is nonzero, but the discriminant of $5\,D^2 - 48\,X$ is equal to zero when the discriminant of $D^2 - 12\,X$ is nonzero, then the configuration of the singular fibers is equal to $2\ \mathrm{I}_4\ \ 2\,\mathrm{I}_2$, and Lemma 9.2.6 implies that the Mordell–Weil group is isomorphic to $(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$. If the resultant of D and X is nonzero, but the discriminant of $D^2 - 12\,X$ is equal to zero, when the discriminant of $5\,D^2 - 48\,X$ is nonzero, then the configuration of the singular fibers is equal to $\mathrm{I}_8\ \ \mathrm{I}_2\ \ 2\ \mathrm{I}_1$, and Lemma 9.2.6 implies that the Mordell–Weil group is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$. Finally, if the zero of D is a simple zero of X, then the configuration of the singular fibers is equal to $\mathrm{I}_1^*\ \ \mathrm{I}_4\ \ \mathrm{I}_1$, and Lemma 9.2.6 implies that the Mordell–Weil group is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$. This case also occurs in Section 11.5.3 for $a = -\omega - 2 = 0$ and in Section 11.6.2 for $\lambda = 0$, where the map is $(x,\ y) \mapsto (y,\ -x)$. It is easily verified that for each of the nongeneric configurations of singular fibers for elements of the Mordell–Weil group of order four, the corresponding rational elliptic surface is uniquely determined up to isomorphism. See the proof of Proposition 9.2.19 for $\mathrm{I}_8\ \mathrm{I}\ 2\ \mathrm{I}_1$, the proof for the other two configurations of the singular fibers $2\ \mathrm{I}_4\ \ 2\,\mathrm{I}_2$ and $\mathrm{I}_1^*\ \ \mathrm{I}_4\ \ \mathrm{I}_1$ is analogous. The list of Persson [156, pp. 7–14] shows the same four configurations of singular fibers with an element of order four in the Mordell–Weil group.

As observed in Section 5.2.1, cases (i) and (ii), the QRT root for the pencil of symmetric biquadratic curves defined by the symmetric matrices

$$A^0 = \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ b & 0 & c \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} p & 0 & q \\ 0 & 0 & 0 \\ q & 0 & r \end{pmatrix}$$

has order four. The formulas (10.1.12), (10.1.13) and Corollary 2.4.7 yield $3\,X_\rho = (5\,b^2 - a\,c)\,{z_0}^2 + (10\,b\,q - (a\,r + c\,p))\,z_0\,z_1 + (5\,q^2 - p\,r)\,{z_1}^2$ and $D = (12\,{X_\rho}^2 - g_2)/2\,Y_\rho = -4\,(b\,z_0 + q\,z_1)$, where X_ρ and Y_ρ play the role of X and Y in the previous paragraphs. Up to isomorphism every rational elliptic surface with an element of order four in its Mordell–Weil group occurs in this family. The configuration of the singular fibers is readily determined from the resultant $16\,(a\,q - b\,p)\,(b\,r - c\,q)/3$ of D and X_ρ, the discriminant $256\,(a\,r - c\,p)^2$ of $5\,D^2 - 48\,X_\rho$, and the discriminant $16\,((a\,r - c\,p)^2 + 4\,(a\,q - b\,p)\,(b\,r - c\,q))$ of $D^2 - 12\,X_\rho$.

Doing the same with the symmetric case (iii) in Section 5.2.1,

$$A^0 = \begin{pmatrix} 0 & d & 0 \\ d & e & f \\ 0 & f & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & u & 0 \\ u & v & w \\ 0 & w & 0 \end{pmatrix},$$

we arrive at $D = (e\,z_0 + v\,z_1)$ and $12\,X_\rho = (e^2 + 4\,d\,f)\,{z_0}^2 + (4\,(f\,u + d\,w) + 2\,e\,v)\,z_0\,z_1 + (v^2 + 4\,u\,w)\,{z_1}^2$, with similar conclusions.

In order to find a pencil of biquadratic curves of which the QRT map has order four, we observe that (A) in Lemma 7.5.3 implies that the singular fibers of type I_4 have contribution 0, 3/4, or 1, depending on whether τ^S cyclically permutes the irreducible components 0, ± 1, or 2 steps, where the order of the permutation is 1, 4, or 2, respectively. The singular fiber of type I_2 has contribution 0 or 1/2, depending on whether τ^S leaves the irreducible components invariant or switches them. The singular fibers of type I_1, which are irreducible, have contribution 0. If the permutation of the irreducible fibers of each singular fiber has order ≤ 2, then $(\tau^S)^2$ leaves all irreducible components invariant, and it follows from Corollary 7.3.3 that $(\tau^S)^2$ is equal to the identity, that is, the QRT mapping would be of order 2 instead of order 4. Therefore at least one of the singular fibers of type I_4 must have contribution 3/4. According to (ii) in Lemma 5.2.1, the other singular fiber of type I_4 must have contribution 3/4 as well, and the fiber of type I_2 must have contribution 1/2.

It follows from the classification in Section 12.1 that the cases of a singular fiber of type I_4 with contribution 3/4 to the QRT automorphism are (4a3), (4a4) with $\mathrm{m}_b + 1 = 4$, their counterparts (5a3), (5a4) with the roles of the horizontal and vertical axes interchanged, and (6a1) with $b = 4$. We will concentrate on the case that both singular fibers of type I_4 come from members of the pencil of biquadratic curves as in 6a1). That is, for $k = 1,\ 2$ we have a member of the form $C^{(k)} = L_1^{(k)} \cup L_2^{(k)} \cup D^{(k)}$, where $L_1^{(k)}$ is a horizontal axis, $L_2^{(k)}$ is a vertical axis, and $D^{(k)}$ is an irreducible $(1,\ 1)$ curve that intersects $L_1^{(k)}$ and $L_2^{(k)}$ at points $a_1^{(k)}$ and $a_2^{(k)}$ not equal to the intersection point of $L_1^{(k)}$ and $L_2^{(k)}$. Because the two singular members do not have an irreducible component in common, since this would lead to an irreducible component contained in every member, contradicting the assumption that the pencil has at least one smooth member, we have that $L_1^{(1)} \neq L_1^{(2)}$ and $L_2^{(1)} \neq L_2^{(2)}$. That is, the horizontal and

vertical axes form a rectangle. In order to have $b = 4$ for each of the members in (6a1), we need that $a_1^{(1)}$ or $a_2^{(1)}$ is a base point and $a_1^{(2)}$ or $a_2^{(2)}$ is a base point, in each case of multiplicity 2.

Of the considerable number of possibilities that we still have at this stage, we will concentrate on the case that $a_1^{(1)} \in L_2^{(2)}$ and $a_1^{(2)} \in L_2^{(1)}$. By means of projective linear transformations in each of the factors $\mathbb{P}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ we can arrange that $L_1^{(1)} = \mathbb{P}^1 \times \{[0 : 1]\}$, $L_1^{(2)} = \mathbb{P}^1 \times \{[1 : 0]\}$, $L_2^{(1)} = \{[1 : 0]\} \times \mathbb{P}^1$, and $L_2^{(2)} = \{[0 : 1]\} \times \mathbb{P}^1$, where we have the base points $a_1^{(1)} = ([0 : 1], [0 : 1]) \in D^{(1)}$ and $a_1^{(2)} = ([1 : 0], [1 : 0]) \in D^{(2)}$. We also can arrange that $C^{(1)}$ and $C^{(2)}$ correspond to $[z_0 : z_1] = [1 : 0]$ and $[z_0 : z_1] = [0 : 1]$, respectively. In terms of the matrices A_{ij}^0, A_{ij}^1 in Chapter 1, the condition that $L_1^{(1)} \cup L_2^{(1)} \subset C^{(1)}$ means that $A_{00}^0 = A_{10}^0 = A_{20}^0 = A_{21}^0 = A_{22}^0 = 0$, whereas $L_1^{(2)} \cup L_2^{(2)} \subset C^{(2)}$ means that $A_{00}^1 = A_{01}^1 = A_{02}^1 = A_{12}^1 = A_{22}^1 = 0$. Furthermore, $a_1^{(1)} \in D^{(1)}$ and $a_2^{(2)} \in D^{(2)}$ mean that $A_{01}^0 = 0$ and $A_{21}^1 = 0$, respectively.

Next we use the freedom in scaling each of the variables z_0, z_1 in order to arrange that the member for $[z_0 : z_1] = [1 : 1]$ leads to the singular fiber of type I_2 with contribution $1/2$, that is, a member as in (4a1) of (5a1) in Section 12.1. In the case (4a1), the horizontal axis L_1 in the member that is different from $L_1^{(1)}$ and $L_1^{(2)}$ contains the base points in $L_1 \cap L_2^{(1)}$ and $L_1 \cap L_2^{(2)}$, and because a horizontal axis contains two base points, it would follow that $L_1 \cap L_2^{(k)} = L_1 \cap D^{(k)}$, hence the intersection point of $L_2^{(k)}$ and $D^{(k)}$ is a base point, in contradiction to the above description of $C^{(k)}$. Therefore the member is equal to the union of a vertical axis and an irreducible $(1, 2)$ curve. Using the freedom of scaling in each of the variables x_0, x_1, we can arrange that the vertical axis is given by the equation $[x_0 : x_1] = [1 : 1]$. In terms of the matrices A_{ij}^0, A_{ij}^1, which means that the sum of each of the three columns of $A^0 + A^1$ is equal to zero. Using the freedom of scaling each of the variables y_0, y_1, we arrive at

$$A^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ -t & -1 & 0 \\ t & 0 & 0 \end{pmatrix},$$

in which t is a nonzero complex number. Note that the respective degrees d_1 and d_2 of ι_1 and ι_2, see Definition 5.1.3, are equal to 2 and 1. For this family the Weierstrass data of Corollaries 2.5.13 and 2.5.10 yield $D = z_0 - z_1$, $12\,X = {z_0}^2 - 2\,(1 - 2\,t)\,z_0\,z_1 + {z_1}^2$. The linear form D is a factor of D if and only if $t = 0$ if and only if $X = D^2/12$. Therefore we have to require $t \neq 0$ when the resultant of D and X and the discriminant of $D^2 - 12\,X = -4\,t\,z_0\,z_1$ are not both equal to zero. Because the discriminant of $5\,D^2 - 48\,X$ is equal to $64\,t\,(1 + 4\,t)$, we conclude that we have a generic order four QRT map if $t \neq 0$ and $t \neq -1/4$, with the configuration of singular fibers $2\ \mathrm{I}_4\ \mathrm{I}_2\ 2\ \mathrm{I}_1$, whereas for $t = -1/4$ the configuration of the singular fibers is $2\ \mathrm{I}_4\ 2\ \mathrm{I}_2$.

Since the construction aimed at having two singular fibers of type I_4 and at least one of type I_2, it is no surprise that the configurations of singular fibers $\mathrm{I}_8\ \mathrm{I}_2\ 2\,\mathrm{I}_1$ and $\mathrm{I}_1^*\ \mathrm{I}_4\ \mathrm{I}_1$ do not appear in our family.

For the 3-parameter family

$$A^0 = \begin{pmatrix} 0 & p & 0 \\ 0 & -1 & q \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & r \\ 1 & 0 & 0 \end{pmatrix}$$

in Example 3.5(c) of Tsuda [196], we have $D = -z_0$ and $12\,X = {z_0}^2 + 4\,p\,q\,z_0\,z_1 + 4\,p\,r\,{z_1}^2$, and we have to require that $p \neq 0$ and $(q,\ r) \neq (0,\ 0)$. Since the respective discriminants of $5\,D^2 - 48\,X^2$ and $D^2 - 12\,X^2$ are equal to $64\,p\,(4\,p\,q^2 + r)$ and $16\,p^2\,q^2$, the generic configuration of singular fibers $2\,\mathrm{I}_4\ \mathrm{I}_2\ 2\,\mathrm{I}_1$ occurs if and only if $p \neq 0$, $q \neq 0$, $r \neq 0$, and $4\,p\,q^2 + r \neq 0$. If $p \neq 0$, $r \neq 0$, and $4\,p\,q^2 + r = 0$, the configuration of singular fibers is $2\,\mathrm{I}_4\ 2\,\mathrm{I}_2$. If $p \neq 0$, $r \neq 0$, and $q = 0$, the configuration of singular fibers is $\mathrm{I}_8\ \mathrm{I}_2\ 2\,\mathrm{I}_1$. Since Tsuda required that $p\,q \neq 0$, he excluded this case. Finally, if $p \neq 0$, $q \neq 0$, and $r = 0$, the configuration of singular fibers is $\mathrm{I}_1^*\ \mathrm{I}_4\ \mathrm{I}_1$. Therefore Tsuda's family catches all possible configurations of singular fibers for order-four QRT mappings.

Remark 5.2.5. One of the configurations of the singular fibers for which the Mordell–Weil group has elements of order 4 is $\mathrm{I}_8\ \mathrm{I}_2\ 2\,\mathrm{I}_1$. Because the rank of the Mordell–Weil group is equal to $8 - 7 - 1 = 0$, the narrow Mordell–Weil lattice Q^0 is trivial, and (7.6.10) with $\varphi = \kappa$ yields $8 \cdot 2 = \#((\mathrm{Aut}(S)_\kappa^+)_{\mathrm{tor}})^2$. It follows that $\#((\mathrm{Aut}(S)_\kappa^+)_{\mathrm{tor}}) = 4$, and because $\mathrm{Aut}(S)_\kappa^+$ contains an element of order four, the conclusion is that $\mathrm{Aut}(S)_\kappa^+ = (\mathrm{Aut}(S)_\kappa^+)_{\mathrm{tor}} \simeq \mathbb{Z}/4\,\mathbb{Z}$. This is the structure of the Mordell–Weil group as given in Persson [156, p. 7], whereas the $E(K) = \mathrm{Aut}(S)_\kappa^+ \simeq (\mathbb{Z}/2\,\mathbb{Z})^2$ in No. 70 in Oguiso and Shioda [155, p. 86] probably is a typo. ⊘

5.2.4 Order 5

According to Table 5.2.2 with $D := (12\,X^2 - g_2)/2\,Y$, the equation $\delta_5 = 0$ is equivalent to $(D^2 - 12\,X)^3 - 16\,D\,(D^2 - 12\,X)\,Y - 64\,Y^2 = 0$. Since any pole in D appears with multiplicity six in the term D^6, which is not compensated, D has no poles, hence is a polynomial in $(z_0,\ z_1)$, homogeneous of degree one. Note that $D^2 - 12\,X \equiv 0$ would imply that $Y \equiv 0$, which is excluded because the QRT map τ has order five, not order two. The substitution $E := Y/(D^2 - 12\,X)$ leads to the equation $D^2 - 12\,X - 16\,D\,E - 64\,E^2 = 0$. Since any pole in E appears with multiplicity two in the term $-64\,E^2$ which is not compensated, E has no poles, hence is a polynomial in $(z_0,\ z_1)$, homogeneous of degree one.

Conversely, for any homogeneous polynomials D and E of degree one, the polynomials $X = (D^2 - 16\,D\,E - 64\,E^2)/12$, $Y = 16\,(D + 4\,E)\,E^2$, $g_2 = (D^4 - 32\,D^3\,E - 256\,D^2\,E^2 + 512\,D\,E^3 + 4096\,E^4)/12$, and $g_3 = -(D^2 + 8\,D\,E + 32\,E^2)\,(D^4 - 56\,D^3\,E + 416\,D^2\,E^2 + 7424\,D\,E^3 + 19456\,E^4)/216$ define a Weierstrass model

in which $(X,\ Y)$ represents an element of the Mordell–Weil group of order 5. Since the discriminant is equal to $\Delta = 1024\,(D+4\,E)^5\,E^5\,(D^2-36\,D\,E-176\,E^2)$, we have to require that the two linear forms D and E be linearly independent. Since any pair of linearly independent linear forms in $(z_0,\ z_1)$ is mapped to any other such pair by a uniquely determined linear substitution of variables, the QRT map of order five is unique up to isomorphism.

Table 6.3.2 and Lemma 9.2.6 yield that the configuration of singular fibers is equal to $2\ \mathrm{I}_5\ \ 2\ \mathrm{I}_1$ and the Mordell–Weil group is isomorphic to $\mathbb{Z}/5\,\mathbb{Z}$. This is the only case in Persson's list [156, pp. 7–14] with an element of order five in the Mordell–Weil group, but the statement that the latter is unique up to isomorphism is stronger. It follows from (ii) in Lemma 5.2.1 that the QRT map shifts the irreducible components over ±1 unit in one of the I_5 fibers and over ±2 units in the other.

The Lyness map for $a = 1$, see (11.4.4), is a QRT root of order five. Its square, the QRT map of the given pencil of biquadratic curves, also has order five. For the Lyness pencil with $a = 1$ the formulas (10.1.12), (10.1.13) and Corollary 2.4.7 yield $D = z_0 + z_1$, $E = -z_0/4$, $X = (z_0{}^2 + 6\,z_0\,z_1 + z_1{}^2)/12$, $Y = z_0{}^2\,z_1$, $g_2 = (z_0{}^4 - 12\,z_0{}^3\,z_1 + 14\,z_0{}^2\,z_1{}^2 + 12\,z_0\,z_1{}^3 + z_1{}^4)/12$, $g_3 = -(z_0{}^2 + z_1{}^2)\,(z_0{}^4 - 18\,z_0{}^3\,z_1 + 74\,z_0{}^2\,z_1{}^2 + 18\,z_0\,z_1{}^3 + z_1{}^4)/216$, and $\Delta = z_0{}^5\,z_1{}^5\,(z_0{}^2 - 11\,z_0\,z_1 - z_1{}^2)$.

5.2.5 Order 6

According to Table 5.2.2 with $D := (12\,X^2 - g_2)/2\,Y$, the equation $\delta_6/\delta_2\,\delta_3 = 0$ is equivalent to $3\,(D^2 - 12\,X)^2\,(D^2 + 4\,X) + 48\,D\,(D^2 - 12\,X)\,Y + 128\,Y^2 = 0$. Since any pole in D appears with multiplicity six in the term $3\,D^6$ which is not compensated, D has no poles, hence is a polynomial in $(z_0,\ z_1)$, homogeneous of degree one. Note that $D^2 - 12\,X \equiv 0$ would imply that $Y \equiv 0$, which is excluded because the QRT map τ has order five, not order two. The substitution $E := Y/(D^2 - 12\,X)$ leads to the equation $3\,(D^2 + 4\,X) + 48\,D\,E + 128\,E^2 = 0$. Since any pole in E appears with multiplicity two in the term $128\,E^2$ which is not compensated, E has no poles, hence is a polynomial in $(z_0,\ z_1)$, homogeneous of degree one.

Conversely, for any homogeneous polynomials D and E of degree one, the polynomials $X = -(3\,D^2 + 48\,D\,E + 128\,E^2)/12$, $Y = 4\,(D + 8\,E)\,(D + 4\,E)\,E$, $g_2 = (3\,D + 16\,E)\,(3\,D^3 + 48\,D^2\,E + 384\,D\,E^2 + 1024\,E^3)/12$, and $g_3 = (3\,D^2 - 64\,E^2)\,(9\,D^4 + 288\,D^3\,E + 3072\,D^2\,E^2 + 13824\,D\,E^3 + 22528\,E^4)/216$ define a Weierstrass model in which $(X,\ Y)$ represents an element of the Mordell–Weil group of order 6. Since the discriminant is equal to $\Delta = 16\,(D + 4\,E)^6\,(D + 8\,E)^3\,E^2\,(9\,D + 40\,E)$, we have to require that the two linear forms D and E be linearly independent. Since any pair of linearly independent linear forms in $(z_0,\ z_1)$ is mapped to any other such pair by a uniquely determined linear substitution of variables, the QRT map of order six is unique up to isomorphism.

Table 6.3.2 and Lemma 9.2.6 yield that the configuration of singular fibers is equal to $\mathrm{I}_6\ \ \mathrm{I}_3\ \ \mathrm{I}_2\ \ \mathrm{I}_1$ and the Mordell–Weil group is isomorphic to $\mathbb{Z}/6\,\mathbb{Z}$. This is the only case in Persson's list [156, pp. 7–14] with an element of order six in the

Mordell–Weil group, but the statement that the latter is unique up to isomorphism is stronger. It follows from (ii) in Lemma 5.2.1 that the QRT map shifts the irreducible components over ± 1 unit in each of the fibers of type I_6, I_3, and I_2.

The Lyness map for $a = 0$, see (11.4.4), is a QRT root of order six. For the Lyness pencil with $a = 0$ the formulas (10.1.12), (10.1.13) and Corollary 2.4.7 yield $D = -2\,z_0 + z_1$, $E = (z_0 - z_1)/4$, $X_\rho = (4\,{z_0}^2 - 8\,z_0\,z_1 + {z_1}^2)/12$, $Y_\rho = z_0\,z_1\,(z_0 - z_1)$, $g_2 = (2\,z_0 + z_1)\,(8\,{z_0}^3 - 12\,{z_0}^2\,z_1 + 6\,z_0\,{z_1}^2 + {z_1}^3)/12$, $g_3 = -(8\,{z_0}^2 - 4\,z_0\,z_1 - {z_1}^2)\,(8\,{z_0}^4 - 8\,{z_0}^3\,z_1 - 8\,z_0\,{z_1}^3 - {z_1}^4)/216$, and $\Delta = {z_0}^6\,{z_1}^3\,(z_0 - z_1)^2\,(8\,z_0 + z_1)$.

The QRT map of the Lyness pencil of biquadratic curves for $a = 0$ is equal to the square of the Lyness map, and therefore has order three. In order to find a pencil of biquadratic curves with an order-six QRT map, we observe that in Section 12.1 the only cases of a singular fiber of type I_6 with contribution $5/6$ to the QRT automorphism are (4a4) with $\mathrm{m}_b + 1 = 6$, its counterpart (5a4) with the roles of the horizontal and vertical axes interchanged, and 6a1) with $b = 6$. As we have seen in Section 5.2.2, the I_3 fiber with contribution $2/3$ comes from a member from (4a2), (5a2) or (6a1) in Section 12.1. The I_2 with contribution $1/2$ comes from a member from (4a1) or (5a1) in Section 12.1. We will restrict ourselves to the case that the member C corresponding to the fiber of type I_6 is as in (6a1) with $b = 6$. It is the union of a horizontal axis L_1, a vertical axis L_2, and an irreducible $(1,\ 1)$ curve D not passing through $L_1 \cap L_2$, where the point $a_1 \in L_1 \cap D$ is a base point of multiplicity 4 and the points in $L_1 \cap L_2$ and $L_2 \cap D$ are not base points.

If the member C' corresponding to the I_3 fiber were as in (6a1) with $b = 3$, then it would be equal to the union of a horizontal axis $L'_1 \neq L_1$, a vertical axis $L'_2 \neq L_2$, and an irreducible $(1,\ 1)$ curve $D' \neq D$, but this case cannot occur, since C' has to intersect C at a_1 with multiplicity 4, whereas $D' \cdot L_1 = 1$ and $D' \cdot D = 2$ leads to a multiplicity ≤ 3. Therefore C' is as in (4a2) or (5a2). If C' is as in (5a2), then it is the union of a vertical axis $L'_2 \neq L_2$ and an irreducible $(1,\ 2)$ curve C'_1, where one of the two intersection points a' of L'_2 and C'_1 is a base point of order 2 and C'_1 and D intersect at a_1 with multiplicity 3. The latter implies that C'_1 and D have no other intersection points; hence the base point $a' \in L'_2 \cap C'_1$ has to lie on L_1, which contradicts the fact that C'_1 and L_1 have only one intersection point, which is $a_1 \neq a'$. We conclude that C' is as in (4a2), the union of a horizontal axis $L'_1 \neq L_1$ and an irreducible $(2,\ 1)$ curve C'_2, where one of the two intersection points b' of L'_1 and C'_2 is a base point of order 2 and C'_2 and D intersect at a_1 with multiplicity 3, and $b' \in L'_1 \cap L_2$. Because a_1 and b' are base points of multiplicity 4 and 2, respectively, there are only two more base points, namely the second intersection point c of C'_2 with L_1, and the intersection point d of D with L'_1.

If the member C'' corresponding to the I_2 fiber were as in (4a1), then the intersection point with L_2 of the horizontal axis $L''_1 \subset C''$ would be a base point, and because $b' \in L'_1 \cap L_2$ is the only base point on L_2, we would have $L''_1 = L'_1$; hence this horizontal axis would be contained in every member, in contradiction to the existence of at least one smooth, hence irreducible, member. Therefore C'' is the union of a vertical axis $L''_2 \neq L_2$ and an irreducible $(1,\ 2)$ curve C''_1. Because $C''_1 \cdot L_2 = 2$, $L''_2 \cap L_2 = \emptyset$, and C'' and C intersect at b' with multiplicity 2, we have that C''_1 and L_2 intersect at b' with multiplicity 2. If $a_1 \in L''_2$, then, because neither of the intersection

points of C_1'' with L_2'' is a base point, $a_1 \notin C_1''$, and because $L_2'' \cdot L_1 = L_2'' \cdot D = 1$, we would conclude that L_2'' intersects C at a_1 with multiplicity 2, in contradiction to the fact that a_1 is base point of multiplicity 4. Therefore $a_1 \notin L_2''$; hence $a_1 \in C_1''$. Because $C_1'' \cdot L_1 = 1$, we have that C_1'' and D have a contact of order 3 at a_1, and because $C_1'' \cdot D = 3$ it follows that $C_1'' \cap D = \{a_1\}$. Because $C_1'' \cdot C = 6$, we have $C_1'' \cap C = \{b',\ a_1\}$, and the remaining two base points c and d therefore must lie on L_2''. That is, $c \in D \cap L_1' \cap L_2''$ and $d \in L_1 \cap L_2''$.

By means of suitable projective linear transformations of $[x_0 : x_1]$, $[y_0 : y_1]$, $[z_0 : z_1]$, we can arrange that $L_1 = \mathbb{P}^1 \times \{[0 : 1]\}$, $L_2 = \{[1 : 0]\} \times \mathbb{P}^1$, $L_1' = \mathbb{P}^1 \times \{[1 : 0]\}$, $L_2'' = \{[0 : 1]\} \times \mathbb{P}^1$, and $a_1 = ([1 : 1], [0 : 1])$. Note that this implies that $b' = ([1 : 0], [1 : 0])$, $c = ([0 : 1], [1 : 0])$, and $d = ([0 : 1], [0 : 1])$. We also still can arrange that the intersection point e of D with L_2 is equal to $([1 : 0], [1 : 1])$. Furthermore, we can arrange that C, C', and C'' correspond to $[z_0 : z_1] = [0 : 1]$, $[z_0 : z_1] = [1 : 0]$, and $[z_0 : z_1] = [1 : 1]$, respectively.

In terms of the matrices A_{ij}^0, A_{ij}^1 in Chapter 1, the condition that $L_1 \cup L_2 \subset C$ means that $A_{00}^1 = A_{10}^1 = A_{20}^1 = A_{21}^1 = A_{22}^1 = 0$, whereas $L_1' \subset C'$ means that we have $A_{01}^0 = A_{02}^0 = A_{12}^0 = A_{22}^0 = 0$. Then $c \in D$ implies that $A_{02}^1 = 0$, and $a_1 \in D$ and $e \in D$ imply that $A_{01}^1 + A_{11}^1 = 0$ and $A_{11}^1 + A_{12}^1 = 0$, respectively. Similarly $b' \in C_2'$ and $d \in C_2'$ imply that $A_{21}^0 = 0$ and $A_{00}^0 = 0$, respectively. Then $a_1 \in C_2'$ implies that $A_{10}^0 + A_{20}^0 = 0$. Next the condition that D and C_2' have contact of order 3 at a_1 means that $A_{01}^0 + A_{11}^0 = A_{10}^0$ and $A_{01}^0 + A_{01}^0 = 0$. The condition that $L_2'' \subset C''$ means that $A_{01}^1 + A_{10}^0 = 0$. After a scaling of all the matrix coefficients with a common factor, we arrive at

$$A^0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Corollaries 2.5.13 and 2.5.10 yield that the Weierstrass data of the QRT map are equal to the previously mentioned Weierstrass data of the Lyness map for $a = 0$. The degrees d_1 and d_2 of ι_1 and ι_2, see Definition 5.1.3, are both equal to 3.

Chapter 6
Elliptic Surfaces

In this chapter we put the discussion into the perspective of the theory of elliptic surfaces. In order to make the exposition more self-contained, we give full proofs of all the basic facts about elliptic surfaces. Because many of these basic facts are highly nontrivial, this chapter has grown into a small book within the book.

6.1 Fibrations

Let S and C be a connected complex analytic surface and curve, respectively. A nonconstant proper complex analytic mapping $\varphi : S \to C$ is called a *fibration* of the surface S over the curve C. Here we recall that the properness of φ means that $\varphi^{-1}(K)$ is a compact subset of S for every compact subset K of C. In this subsection we will show that the seemingly weak conditions for φ have quite strong consequences.

For any $s \in S$ the *tangent mapping* $\mathrm{T}_s\varphi$ of φ at s is the complex linear mapping from the complex two-dimensional tangent space $\mathrm{T}_s S$ of S at s to the complex one-dimensional tangent space $\mathrm{T}_{\varphi(s)}C$ of C at $\varphi(s)$, which in local coordinates corresponds to the total derivative, the Jacobi matrix, or the gradient of φ at the point s. A point $s \in S$ is called a *regular point* for φ if the tangent map $\mathrm{T}_s\varphi$ of φ at s surjective, which in view of the one-dimensionality of $\mathrm{T}_{\varphi(s)}C$ is equivalent to the condition that $\mathrm{T}_s\varphi \neq 0$. We denote the set of all regular points for φ in S by S^{reg}. Its complement is the set $S^{\mathrm{sing}} := S \setminus S^{\mathrm{reg}} = \{s \in S \mid \mathrm{T}_s\varphi = 0\}$ of all singular points for φ in S. We write $C^{\mathrm{sing}} := \varphi(S^{\mathrm{sing}})$ and $C^{\mathrm{reg}} := C \setminus C^{\mathrm{sing}}$ for the set of all *singular values* and *regular values* of φ in C, respectively. The notation is very much relative to the fibration φ, since both S and C are nonsingular manifolds.

The fibers $S_c := \varphi^{-1}(\{c\})$ such that $c \in C^{\mathrm{reg}}$ and $c \in C^{\mathrm{sing}}$ are called the *regular* and *singular fibers* of the fibration $\varphi : S \to C$, respectively. Let z be a complex analytic local coordinate on an open neighborhood of c in C such that $z(c) = 0$. If $c \in C^{\mathrm{sing}}$, then it can happen that the function $z \circ \varphi$ vanishes to order > 1 along some irreducible components of S_c. *In this case the fiber S_c is defined as the divisor* $\mathrm{Div}(z \circ \varphi)$.

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_6,
© Springer Science+Business Media, LLC 2010

Lemma 6.1.1 *The mapping $\varphi : S \to C$ is surjective. The set C^{sing} of singular values of φ is a locally finite subset of C. For each regular value $c \in C^{\mathrm{reg}}$ the fiber $S_c := \varphi^{-1}(\{c\})$ in S of φ over c is a nonsingular compact complex analytic curve in S, and the restriction of φ to $\varphi^{-1}(C^{\mathrm{reg}})$ is a real analytic locally trivial fiber bundle over C^{reg}. The surface S is compact if and only if the curve C is compact. If this is the case, then φ has only finitely many singular values.*

Proof. The set S^{sing} of singular points in S is a closed analytic subset of S. As discussed in Section 2.1.1, it has a locally finite decomposition into irreducible components A_i, of which the regular part A_i° is a connected nonsingular complex analytic submanifold of S, and the complement $A_i^* = A_i \setminus A_i^\circ$ in A_i is a closed complex analytic subset of S. Continuing this decomposition with the A_i^* we obtain that S^{sing} is equal to the union of a locally finite collection of connected nonsingular complex analytic submanifolds Σ_ν of S. For every $s \in \Sigma_\nu \subset S^{\mathrm{sing}}$ we have $\mathrm{T}_s\varphi = 0$, hence $\mathrm{T}_s(\varphi|_{\Sigma_\nu}) = \mathrm{T}_s\varphi|_{\mathrm{T}_s\Sigma_\nu} = 0$. Therefore $\mathrm{T}(\varphi|_{\Sigma_\nu}) = 0$, and hence φ is locally constant on Σ_ν, and therefore φ is constant on Σ_ν because Σ_ν is connected. For any compact subset K of C, $\varphi^{-1}(K)$ is a compact subset of S, and because S^{sing} is a closed subset of S it follows that $\varphi^{-1}(K) \cap S^{\mathrm{sing}}$ is a compact subset of S^{sing}, and therefore meets only finitely many Σ_ν's. It follows that $K \cap C^{\mathrm{sing}} = K \cap \varphi(S^{\mathrm{sing}}) = \varphi(\varphi^{-1}(K) \cap S^{\mathrm{sing}})$ is finite. Because C is locally compact it follows that C^{sing} is a locally finite subset of C.

If $\varphi(S) \subset C^{\mathrm{sing}}$, then S is equal to the union of the disjoint open subsets $\varphi(U)$, where the U's are disjoint open neighborhoods in C of the singular values of φ. Because S is connected, this would imply that there is a unique $c \in C$ such that $\varphi(S) = \{c\}$, in contradiction to the assumption that φ is not constant. It follows that $\varphi^{-1}(C^{\mathrm{reg}})$ is a nonempty open subset of S. The restriction to $\varphi^{-1}(C^{\mathrm{reg}})$ of φ is a *submersion* in the sense that $\mathrm{T}_s\varphi$ is surjective for each $s \in \varphi^{-1}(C^{\mathrm{reg}})$. It follows from the open mapping theorem that $\varphi(\varphi^{-1}(C^{\mathrm{reg}}))$ is a nonempty open subset of C^{reg}. Because the restriction to $\varphi^{-1}(C^{\mathrm{reg}})$ of φ is a proper mapping from $\varphi^{-1}(C^{\mathrm{reg}})$ to C^{reg}, $\varphi(\varphi^{-1}(C^{\mathrm{reg}}))$ is a closed subset of C^{reg}, and because C^{reg} is connected as the complement of the locally finite subset C^{sing} of the connected curve C, it follows that $\varphi(\varphi^{-1}(C^{\mathrm{reg}})) = C^{\mathrm{reg}}$. The properness of φ implies that $\varphi(S)$ is a closed subset of C, and because $\varphi(S)$ contains the dense subset C^{reg}, it follows that $\varphi(S) = C$.

Every proper real analytic submersion to a connected manifold is a real analytic locally trivial fiber bundle, a classical result due to Ehresmann [56, Proposition 1] in the C^∞ category and for maps defined on compact manifolds. In the real analytic category and for noncompact manifolds one may use the embedding theorem of Grauert [68] in order to find a real analytic Riemannian structure on the manifold, and then use the lifts of analytic curves in the base space that are orthogonal to the fibers in order to construct real analytic local trivializations. Because the restriction to $\varphi^{-1}(C^{\mathrm{reg}})$ of φ is a real analytic proper submersion from $\varphi^{-1}(C^{\mathrm{reg}})$ to C^{reg}, it exhibits $\varphi^{-1}(C^{\mathrm{reg}})$ as a real analytic locally trivial fiber bundle over C^{reg}.

The continuity and properness of φ respectively imply that $C = \varphi(S)$ is compact if S is compact and $S = \varphi^{-1}(C)$ is compact if C is compact. If C is compact then the locally finite subset C^{sing} of C is finite. □

The fibration $\varphi : S \to C$ need not be a topological locally trivial fiber bundle, because there can exist singular fibers that are not homeomorphic to the regular fibers, as happens in almost all examples in this book. Also, in general, the real analytic local trival fiber bundle $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ *is not a complex analytic locally trivial fiber bundle*, because it can happen that two regular fibers are not complex analytic diffeomorphic to each other.

Lemma 6.1.2 *All fibers of φ, viewed as divisors, are homologous to each other. We have $S_c \cdot S_{c'} = 0$ for all $c,\ c' \in C$, including the case that $c = c'$.*

Proof. The relation $c \sim c'$ when the real two-dimensional oriented cycles S_c and $S_{c'}$ are homomologous is an equivalence relation in C. We will prove that the equivalence classes are open, and because C is connected it then follows that $c \sim c'$ for all $c,\ c' \in C$.

Let $c \in C$. There exists an open neighborhood V of c in C such that $V \cap C^{\mathrm{sing}} \subset \{c\}$, and there exists a complex analytic coordinate $z : V \to \mathbb{C}$ such that $z(c) = 0$ and $z(V)$ is an open disk in $\mathbb{C}$ with center at the origin. Let $c' \in V$, $c' \neq c$, and let l denote the line segment in $\mathbb{C}$ from $z(c)$ to $z(c')$. Then $l \subset z(V)$ and $B := z \circ \varphi^{-1}(l)$ is a compact real three-dimensional semianalytic subset of S, with the preimages S_z and $S_{z'}$ of the endpoints of l as complex analytic subsets. According to the Łojasiewicz triangulation theorem, see Section 2.1.6, there are respective triangulations K_c, $K_{c'}$, and K of S_c, $S_{c'}$, and B such that K_c and $K_{c'}$ are subtriangulations of K. Let $l' = l \setminus \{z(c),\ z(c')\}$. Because $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a real analytic locally trivial fibration and $z^{-1}(l') \subset C^{\mathrm{reg}}$, the set $B := z \circ \varphi^{-1}(l')$ is a smooth real three-dimensional submanifold of S, and it follows that the boundary of the real three-dimensional chain B in S is equal to the union of S_c and $S_{c'}$. Because $c' \in C^{\mathrm{reg}}$, near $S_{c'}$ the set B is a half-space with $S_{c'}$ as its smooth boundary. Let Θ be an irreducible component of S_c, of multiplicity $\mu = \mu_\Theta$, and let $s \in S_c$ be a point in the complement Θ' of the other irreducible components in the regular part Θ° of Θ. In an open neighborhood S_0 of s we have $z \circ \varphi = g^\mu$, where $g : S_0 \to \mathbb{C}$ is a complex analytic function with simple zeros along $S_c \cap S_0$. Let $a \in \mathbb{C}$ be such that $a^\mu = z(c')$, where $a \neq 0$ because $z(c') \neq 0$. For $0 < t \ll 1$, the equation $t\, z(c') = z(\varphi(s)) = g(s)^\mu$ is equivalent to the μ equations $g(s) = a\, u\, t^{1/\mu}$, where $u \in \mathbb{C}$ runs over the μth roots of unity. This shows that Θ' is the common boundary of μ sheets of B. Because the complement of the Θ' in S_c is finite, it follows that $\partial B = S_{c'} - \sum_\Theta \mu_\Theta\, \Theta = S_{c'} - S_c$, where the irreducible components of the divisor S_c have been counted with their multiplicities. Therefore the divisor S_c is homologous to $S_{c'}$. Because this holds for any $c \in C$ and any c' near c, it follows that the equivalence classes are open subsets of C.

If $c' \neq c$ then $S_c \cap S_{c'} = \emptyset$; hence $S_c \cdot S_{c'} = 0$. Because $S_{c'}$ is homologous to S_c and the intersection number is a homology invariant, we also have $S_c \cdot S_c = S_c \cdot S_{c'} = 0$. □

The following lemma shows how to pass from a fibration with disconnected fibers to a fibration with connected fibers. And conversely, for each $c \in C$ the number of connected components of the fiber of φ over c is equal to the number of elements

in $\pi^{-1}(\{c\})$. Lemma 6.1.3 is a special case of the *Stein factorization theorem*, see Grauert and Remmert [72, pp. 213, 214].

Lemma 6.1.3 *There exist a fibration $\psi : S \to \widetilde{C}$ over a connected complex analytic curve $\widetilde{C}$ and a branched covering map $\pi : \widetilde{C} \to C$ of finite degree such that $\varphi = \pi \circ \psi$, and the fibers of ψ are the connected components of the fibers of φ. All connected components of all regular fibers of φ are compact Riemann surfaces of the same genus, and homologous to each other in S. If f and g in $\mathrm{H}_2(S, \mathbb{Z})$ denote the common homology class of a fiber of φ and ψ, respectively, then $f = (\deg \pi)\, g$. Some regular fiber of φ is connected if and only if every fiber of φ is connected if and only if $\deg \pi = 1$ if and only if π is a complex analytic diffeomorphism from $\widetilde{C}$ onto C.*

Proof. For each $c \in C$, let $\widetilde{C}_c$ denote the set of all connected components of the fiber $S_c = \varphi^{-1}(\{c\})$ of φ over c. Because S_c is a compact complex analytic subset of S, the set $\widetilde{C}_c$ is finite. We have $\varphi = \pi \circ \psi$ if $\psi : S \to \widetilde{C}$ maps $s \in S$ to the connected component of $S_{\varphi(s)}$ and $\pi : \widetilde{C} \to C$ maps each connected component of S_c to c. By construction, every fiber of ψ is connected.

Because $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a locally trivial fibration, the set $\pi^{-1}(C^{\mathrm{reg}})$ has a unique structure of a topological space such that $\pi : \pi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a covering map, which in turn implies that $\pi^{-1}(C^{\mathrm{reg}})$ has a unique complex analytic structure such that $\pi : \pi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a holomorphic covering map. The equation $\varphi = \pi \circ \psi$ then implies that $\psi : \varphi^{-1}(C^{\mathrm{reg}}) \to \pi^{-1}(C^{\mathrm{reg}})$ is a complex analytic fibration, each fiber of which is regular and connected, a compact Riemann surface. Because $\varphi^{-1}(C^{\mathrm{sing}})$ is a complex analytic subset of S, not equal to S, and S is connected, every irreducible component of $\varphi^{-1}(C^{\mathrm{sing}})$ has complex dimension $< \dim_{\mathbb{C}} S$, and therefore the complement $\varphi^{-1}(C^{\mathrm{reg}})$ of $\varphi^{-1}(C^{\mathrm{sing}})$ in S is connected. It follows that $\pi^{-1}(C^{\mathrm{reg}}) = \psi(\varphi^{-1}(C^{\mathrm{reg}}))$ is connected as well. Because $\psi : \varphi^{-1}(C^{\mathrm{reg}}) \to \pi^{-1}(C^{\mathrm{reg}})$ is a real analytic locally trivial fiber bundle over a connected manifold, all fibers are homeomorphic and therefore have the same genus. This implies in turn that all connected components of all regular fibers of φ are Riemann surfaces of the same genus. If A and B are connected components of a regular fiber of φ, then, because both are contained in the connected manifold $\varphi^{-1}(C^{\mathrm{reg}})$, there exists a smooth path $\gamma : [0,\ 1] \to \varphi^{-1}(C^{\mathrm{reg}})$ such that $\gamma(0) \in A$ and $\gamma(1) \in B$. Then $\delta := \varphi \circ \gamma : [0,\ 1] \to C^{\mathrm{reg}}$ is smooth, the pullback $\delta^* S$ under δ of the bundle $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a smooth bundle over $[0,\ 1]$ with real two-dimensional fibers, and the connected component Σ of $\delta^* S$ that contains γ is a real three-dimensional chain in S with boundary $\partial\Sigma = B - A$; hence B is homologous to A in S.

Choose a complex analytic coordinate function $z : V \to \mathbb{C}$ on an open neighborhood V of c in C such that $z(c) = 0$. By shrinking V if necessary, we can arrange that $V \cap C^{\mathrm{sing}} = \{c\}$ and that z is a complex analytic diffeomorphism from V onto $D_r := \{u \in \mathbb{C} \mid |u| < r\}$, where $r \in \mathbb{R}_{>0}$. Let U be a connected component of $\varphi^{-1}(V)$. Let $\pi : \widetilde{V} \to V$ and $\psi : U \to \widetilde{V}$ be as above, with the fibration $\varphi : S \to C$ replaced by the fibration $\varphi : U \to V$. The lift of a loop γ_0 once around c in $V^{\mathrm{reg}} = V \setminus \{c\}$ defines a deck transformation δ in $\pi^{-1}(V^{\mathrm{reg}})$. Because $\pi^{-1}(V^{\mathrm{reg}})$

is connected, any two points in a fiber of π over $c' \in V \setminus \{c\}$ can be connected by a path in $\pi^{-1}(V^{\mathrm{reg}})$, which is projected by π to a loop γ in $V \setminus \{c\}$ based at c'. Because the fundamental group of $V \setminus \{c\} \simeq D_r \setminus \{0\}$ is isomorphic to $\mathbb{Z}$, it follows that γ is homotopic to a multiple of γ_0. It follows that the iterates of δ act transitively on the fibers. More precisely, if $m \in \mathbb{Z}_{>0}$ denotes the degree of the covering $\pi : \pi^{-1}(V^{\mathrm{reg}}) \to V^{\mathrm{reg}}$, then $k \mapsto \delta^k$ is a surjective homomorphism from $\mathbb{Z}$ onto the group Δ of all deck transformations, with kernel equal to $m\,\mathbb{Z}$, $\Delta \simeq \mathbb{Z}/m\,\mathbb{Z}$, and $\pi : \pi^{-1}(V^{\mathrm{reg}}) \to V^{\mathrm{reg}}$ is a principal Δ-bundle. Write $\rho = r^{1/m}$. It follows that, with the notation $\mathrm{pow}(m) : \zeta \mapsto \zeta^m$, there is a complex analytic diffeomorphism $\chi : \pi^{-1}(V^{\mathrm{reg}}) \to D_\rho \setminus \{0\}$, unique up to multiplication by an mth root of unity, such that $z \circ \pi = \mathrm{pow}(m) \circ \chi$ on $\pi^{-1}(V^{\mathrm{reg}})$.

Let A_1 and A_2 be disjoint open subsets of U such that $S_c \cap U \subset A_1 \cup A_2$. The second paragraph in the proof of Lemma 3.2.1 yields $0 < r' \leq r$ such that $(z \circ \varphi)^{-1}(D_{r'}) \cap U \subset A_1 \cup A_2$. Because $(z \circ \varphi)^{-1}(D_{r'} \setminus \{0\}) \cap U$ is connected, $(z \circ \varphi)^{-1}(D_{r'} \setminus \{0\}) \cap A_k = \emptyset$ for $k = 1$ or $k = 2$. If $s \in S_c \cap A_k$ then, because $S_c \cap A_k$ is a complex one-dimensional complex analytic subset of A_k, there exists $s' \in A_k \setminus S_c$ arbitrarily close to s when $s' \in (z \circ \varphi)^{-1}(D_{r'} \setminus \{0\})$, in contradiction to $(z \circ \varphi)^{-1}(D_{r'} \setminus \{0\}) \cap A_k = \emptyset$. Therefore $S_c \cap A_k = \emptyset$ for $k = 1$ or $k = 2$, and we have proved that $S_c \cap U$ is connected.

Because $S_c \cap U$ is a connected component of S_c, we have $\widetilde{c} := S_c \cap U \in \widetilde{C}$, where $\pi(\widetilde{c}) = c$ and $V^{\mathrm{reg}} = V \setminus \{c\}$. With $\chi(\widetilde{c}) = 0$ we obtain an extension of the complex analytic diffeomorphism $\chi : \pi^{-1}(V^{\mathrm{reg}}) \to D_\rho \setminus \{0\}$ to a bijective mapping χ from $\widetilde{V} := \pi^{-1}(V^{\mathrm{reg}}) \cup \{\widetilde{c}\}$ onto D_ρ. It follows that χ is a complex analytic coordinate on $\widetilde{V}$ for a unique complex structure on $\widetilde{V}$ that extends the complex structure on $\widetilde{V} \setminus \{\widetilde{c}\} = \pi^{-1}(V^{\mathrm{reg}})$. With respect to this complex structure, the mapping $\pi = z^{-1} \circ \mathrm{pow}(m) \circ \chi : \widetilde{V} \to V$ is holomorphic, with $\widetilde{c}$ as a ramification point of order m. The mapping $\psi : U \to \widetilde{V}$ is holomorphic on $U \setminus S_c$ and converges to $\widetilde{c}$ when approaching S_c. Hence Riemann's theorem on removable singularties implies that the proper mapping $\psi : U \to \widetilde{V}$ is complex analytic. Gluing the constructions for all connected components U of $\varphi^{-1}(V)$ and all $c \in C^{\mathrm{reg}}$ together with the construction in $\varphi^{-1}(C^{\mathrm{reg}})$, the desired properties of $\widetilde{C}$, π, and ψ follow. □

Definition 6.1.4. A *holomorphic section* of a fibration $\varphi : S \to C$ is a holomorphic mapping $\sigma : C \to S$ such that $\varphi \circ \sigma = 1_C$, the identity in C. It follows that σ is an embedding from C onto the smooth curve $E := \sigma(C)$ in S, where E intersects each fiber S_c, $c \in C$, exactly once in a smooth point of S_c, and transversally. In particular $E \subset S^{\mathrm{reg}}$. Assume conversely that E is a smooth connected curve in S such that $E \cdot F = 1$ for some fiber F of φ; hence $E \cdot F = 1$ for every fiber F because Lemma 6.1.2 implies that all fibers are homologous to each other and the intersection number is a homology invariant. Then the restriction to E of φ is a surjective holomorphic mapping from E to C of degree one, hence a complex analytic diffeomorphism, and $\sigma = (\varphi|_E)^{-1}$, viewed as a mapping from C to S, is a holomorphic section of φ. For this reason sections are identified with their image curves in S. ⊘

Lemma 6.1.5 *Let $\varphi : S \to C$ be a fibration, and F any fiber of φ. Then we have the implications* (i) ⇔ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) *between the following statements:*

(i) *There is a holomorphic section for* φ.
(ii) *There is an effective divisor* E *in* S *such that* $E \cdot F = 1$.
(iii) *There is a real two-dimensional cycle* E *in* S *such that* $E \cdot F = 1$.
(iv) *There exists no irreducible compact curve* C *in* S *and* $d \in \mathbb{Z}_{>1}$ *such that* $\mathrm{H}(F) = d\ \mathrm{H}(C)$.
(v) *All fibers of* φ *are connected.*

Proof. The implication (i) $\Rightarrow$ (ii) has been discussed in Definition 6.1.4, and (ii) $\Rightarrow$ (iii) is obvious. If (iii) holds and C is a compact real two-dimensional cycle such that $\mathrm{H}(F) = d\ \mathrm{H}(C)$, then because the intersection number is a homology-invariant integer, we have $1 = E \cdot F = d\, E \cdot C$, hence $d = E \cdot C = \pm 1$, which proves (iv). The implication (iv) $\Rightarrow$ (v) follows from Lemma 6.1.3.

Assume (ii), which implies (v). Lemma 6.1.2 and the homology invariance of the intersection number implies that $E \cdot F = 1$ for every fiber F of φ. There exist a locally finite system of distinct irreducible curves E_i in S and $n_i \in \mathbb{Z}_{>0}$ such that $E = \sum_i n_i E_i$. If F is a smooth fiber of φ, then F is irreducible because connected, and therefore $E_i \cdot F \geq 0$ for every i. Since F is compact, the set I of all i such that $E_i \cap F \neq \emptyset$ is finite, and $1 = E \cdot F = \sum_{i \in I} n_i E_i \cdot F$ implies that there is an $i \in I$ such that $n_i = E_i \cdot F = 1$ and $E_j \cdot F = 0$ for every $j \neq i$. Therefore we have (ii) for an irreducible curve E. Let $\rho : \widetilde{E} \to E \subset S$ be a minimal resolution of singularities as discussed after Lemma 6.2.2. Then $\varphi \circ \rho : \widetilde{E} \to C$ is a branched covering. Because $E \cdot F = 1$ for every fiber F of φ, the degree $\varphi|_E$ is equal to one, and because the degree of ρ is also equal to one, the degree of $\varphi \circ \rho$ is equal to one, and therefore $\varphi \circ \rho$ is a complex analytic diffeomorphism. It follows that $\sigma : \rho \circ (\varphi \circ \rho)^{-1} : C \to S$ is a complex analytic mapping such that $\varphi \circ \sigma$ is equal to the identity in C, that is, σ is a holomorphic section of φ. □

Let L be a holomorphic line bundle over a compact and connected complex analytic surface S. Then the set $\Gamma(L) := \mathrm{H}^0(S,\ \mathcal{O}(L))$ of all holomorphic sections of L is a finite-dimensional vector space; see Section 6.2.8. Assume that $\dim \Gamma(L) \geq 1$. Let F be the maximal effective divisor D in S such that $D \leq \mathrm{Div}(\lambda)$ for every $\lambda \in \Gamma(L)$. Such F exist and is unique because for each nonzero $\lambda \in \Gamma(L)$ there are finitely many distinct irreducible curves C_j and $m_j \in \mathbb{Z}_{>0}$ such that $\mathrm{Div}(\lambda) = \sum_j m_j C_j$. According to Lemma 2.1.2 there exists a holomorphic line bundle L_{f} and a nonzero holomorphic section ϕ of L_{f} such that $[L_{\mathrm{f}}] = \delta(F)$ and $\mathrm{Div}(\phi) = F$. Note that ϕ is a nonzero constant if and only if $F = 0$ or equivalently L_{f} is trivial. Inspired by Griffiths and Harris [74, p. 137], we call the holomorphic line bundles L_{f} and $L_{\mathrm{m}} := L\, {L_{\mathrm{f}}}^{-1}$ the *fixed and moving part of* L, respectively.

For every nonzero $\lambda \in \Gamma(L)$ we have $\mathrm{Div}(\phi) = F \leq \mathrm{Div}(\lambda)$; hence the meromorphic section λ/ϕ of L_{m} is holomorphic. It follows that multiplication by ϕ is a bijective linear mapping from $\Gamma(L_{\mathrm{m}})$ onto $\Gamma(L)$, and $\dim \Gamma(L_{\mathrm{m}}) = \dim \Gamma(L)$. It follows from the definition of the fixed part that a holomorphic line bundle L_{m} has no fixed part if and only if for every irreducible curve C in S the set $\Gamma(L_{\mathrm{m}})_C := \{\lambda \in \Gamma(L_{\mathrm{m}}) \mid C \leq \mathrm{Div}(\lambda)\}$ is a linear subspace of $\Gamma(L_{\mathrm{m}})$ of positive codimension. If $F \neq 0$ then there exist finitely many distinct irreducible curves F_i and $l_i \in \mathbb{Z}_{>0}$ such that $F = \sum_i l_i F_i$, and there exists $\lambda \in \Gamma(L_{\mathrm{m}}) \setminus \bigcup_i \Gamma(L_{\mathrm{m}})_{F_i}$,

which means that the intersection Σ of the supports of F and $\mathrm{Div}(\lambda)$ does not contain a curve, and therefore is a finite subset of S. It follows that $L_{\mathrm{f}} \cdot L_{\mathrm{m}} = F \cdot \mathrm{Div}(\lambda) \geq 0$. If $\psi \in \Gamma(L_{\mathrm{f}})$, $\psi \neq 0$, then $\psi\,\lambda \in \Gamma(L)$, hence $\psi\,\lambda = \phi\,\mu$ for a unique $\mu \in \Gamma(L_{\mathrm{m}})$, and therefore $\mathrm{Div}(\psi) + \mathrm{Div}(\lambda) = \mathrm{Div}(\phi) + \mathrm{Div}(\mu)F + \mathrm{Div}(\mu)$. It follows that $\mathrm{Div}(\phi) = F \leq \mathrm{Div}(\psi)$ in $S \setminus \Sigma$; hence ψ/ϕ extends to a holomorphic function on $S \setminus \Sigma$, which in view of Hartogs's theorem extends to a holomorphic function on S, hence a constant as a consequence of the maximum principle. We conclude that ψ is a constant multiple of ϕ, and we have proved that $\dim \Gamma(L_{\mathrm{f}}) = 1$, which is also true if $F = 0$, when L_{f} is trivial. It follows that L has no moving part if and only if $\dim \Gamma(L) = 1$.

We say that *the linear system of L contains a fibration of S* if there exists a two-dimensional linear subspace V of $\Gamma(L)$ such that the pencil defined by V has no base points. That is, for each $s \in S$, $\kappa_V(s) := \{\lambda \in V \mid \lambda(s) = 0\}$ is a one-dimensional linear subspace of $\Gamma(L)$. This defines a holomorphic mapping $\kappa_V : S \to \mathrm{P}(V) \simeq \mathbb{P}^1$. If $\lambda \in V$ has no zeros then L is trivial; hence $\dim \Gamma(L) = 1$, in contradiction to the assumption. Therefore every nonzero $\lambda \in V$ has a zero $s \in S$ when $\lambda \in \kappa_V(s)$, and hence $\mathbb{C}\,\lambda = \kappa_V(s)$. This proves that the mapping $\kappa_V : S \to \mathrm{P}(V)$ is surjective, hence a fibration of S over a complex projective line. This is the fibration alluded to in the definition. The following lemma is useful in the search for fibrations contained in linear systems.

Lemma 6.1.6 *Let L be a holomorphic line bundle over a compact connected surface S. If the linear system of L contains a fibration κ_V, then $L \cdot L = 0$, for every irreducible curve C in S we have $L \cdot C \geq 0$ with equality if and only if C is contained in a fiber of κ_V, and L has no fixed part. If the fibers of κ_V are connected, then* $\dim \Gamma(L) = 2$, *that is, $V = \Gamma(L)$.*

Assume conversely that $L \cdot L = 0$, $\dim(\Gamma(L)) \geq 2$, and $L \cdot C \geq 0$ for every irreducible curve C in S. Then the linear system of the moving part L_{m} of L contains a fibration, and the fixed part L_{f} of L satisfies $L \cdot L_{\mathrm{f}} = L_{\mathrm{f}} \cdot L_{\mathrm{f}} = 0$.

Proof. Assume that the linear system of L contains a a fibration $\kappa_V : S \to \mathrm{P}(V)$ and C is an irreducible curve in S. If $s \in C$, then $\kappa_V(s) = \mathbb{C}\,\lambda$ for a nonzero $\lambda \in V \subset \Gamma(L)$, and Lemma 2.1.2 implies that $[L] = \delta(\mathrm{Div}(\lambda))$. There are finitely many distinct irreducible curves F_i in S and $n_i \in \mathbb{Z}_{>0}$ such that $\mathrm{Div}(\lambda) = \sum_i n_i F_i$, and $L \cdot C = \mathrm{Div}(\lambda) \cdot C = \sum_{i \in I} n_i F_i \cdot C$, where I denotes the nonempty set of all i such that $F_i \cap C \neq \emptyset$. If $C \neq F_i$ hence $F_i \cdot C > 0$ for every $i \in I$, then $L \cdot C > 0$. If $C = F_i$, then C is disjoint from the support of $\mathrm{Div}(\mu)$ for every nonzero $\mu \in V$ such that $\mathbb{C}\,\lambda \neq \mathbb{C}\,\mu$; hence $L \cdot C = \mathrm{Div}(\mu) \cdot C = 0$. If D is an effective divisor such that $D \leq \mathrm{Div}(\lambda)$ for every nonzero $\lambda \in \gamma(L)$, then this holds in particular for all nonzero $\lambda \in V$, and because the supports of $\mathrm{Div}(\lambda)$ and $\mathrm{Div}(\mu)$ are disjoint if $\mathbb{C}\,\lambda \neq \mathbb{C}\,\mu$, it follows that $D = 0$. This proves that L has no fixed part. Let $\lambda \in V$, $\lambda \neq 0$, be such that $\mathrm{Div}(\lambda)$ is smooth and connected. Then for every $\nu \in \Gamma(L)$ sufficiently close to λ we have that $\mathrm{Div}(\nu)$ is smooth and connected, hence irreducible, when $L \cdot \mathrm{Div}(\nu) = L \cdot L = 0$ implies that $\mathrm{Div}(\nu)$ is contained in a fiber of κ_V, that is, $\mathrm{Div}(\nu) \leq \mathrm{Div}(\mu)$ for a nonzero $\mu \in V$. It follows that μ/ν extends to a nonzero holomorphic function on S, which in view of the maximum principle

and the assumption that S is compact and connected implies that μ/ν is a nonzero constant; hence $\nu \in \mathbb{C}\,\mu \subset V$. It follows that there exists an open neighborhood U of λ in $\Gamma(L)$ such that $U \subset V$; hence $\Gamma(L) = V$.

Now assume that $L \cdot L = 0$, $\dim(\Gamma(L)) \geq 2$, and $L \cdot C \geq 0$ for every irreducible curve C in S, hence $L \cdot D \geq 0$ for every effective divisor D in S. We have $[L_{\mathrm{f}}] = \delta(F)$ for an effective divisor F such that $\mathrm{Div}(\lambda) - F$ is an effective divisor for every nonzero $\lambda \in \Gamma(L)$. Therefore $0 = L \cdot L = L \cdot \mathrm{Div}(\lambda) = L \cdot (\mathrm{Div}(\lambda) - F) + L \cdot F$, in combination with $L \cdot (\mathrm{Div}(\lambda) - F) \geq 0$ and $L \cdot F \geq 0$ implies that $L \cdot L_{\mathrm{m}} = L \cdot (\mathrm{Div}(\lambda) - F) = 0$ and $L \cdot L_{\mathrm{f}} = L \cdot F = 0$. For every irreducible curve C in S the set $\Gamma(L)_C := \{\lambda \in \Gamma(L) \mid C \leq \mathrm{Div}(\lambda)\}$ is a linear subspace of $\Gamma(L_{\mathrm{m}})$ of positive codimension. If λ is a nonzero element of $\Gamma(L_{\mathrm{m}})$, and C_j are the finitely many irreducible components of $\mathrm{Div}(\lambda)$, then there exists $\mu \in \Gamma(L_{\mathrm{m}}) \setminus \bigcup_j \Gamma(L_{\mathrm{m}})_{C_j}$, no irreducible component of $\mathrm{Div}(\mu)$ is equal to any irreducible component of $\mathrm{Div}(\lambda)$, and hence $L_{\mathrm{m}} \cdot L_{\mathrm{m}} = \mathrm{Div}(\lambda) \cdot \mathrm{Div}(\mu) \geq 0$. Since we have already observed in the text preceding Lemma 6.1.6 that $L_{\mathrm{f}} \cdot L_{\mathrm{m}} \geq 0$, it follows from $0 = L \cdot L_{\mathrm{m}} = L_{\mathrm{f}} \cdot L_{\mathrm{m}} + L_{\mathrm{m}} \cdot L_{\mathrm{m}}$ that $L_{\mathrm{f}} \cdot L_{\mathrm{m}} = L_{\mathrm{m}} \cdot L_{\mathrm{m}} = 0$. It follows in turn that $L_{\mathrm{f}} \cdot L_{\mathrm{f}} = L \cdot L_{\mathrm{f}} - L_{\mathrm{m}} \cdot L_{\mathrm{f}} = 0$. Because $\mathrm{Div}(\lambda) \cdot \mathrm{Div}(\mu) = L_{\mathrm{m}} \cdot L_{\mathrm{m}} = 0$, each irreducible component of $\mathrm{Div}(\lambda)$ is disjoint from each irreducible component of $\mathrm{Div}(\mu)$; hence the pencil defined by $V := \mathbb{C}\,\lambda + \mathbb{C}\,\mu$ has no base points, and therefore defines a fibration $\kappa_V : S \to \mathrm{P}(V)$. □

Definition 6.1.7. A *ruled surface* and an *elliptic surface* are fibrations $\varphi : S \to C$ with connected fibers such that some, hence each regular fiber of φ is a compact Riemann surface of genus zero and one, respectively. Note that Lemma 3.2.1 implies that if some regular fiber is connected then all fibers are connected and all regular fibers have the same genus. In the case of a ruled and an elliptic surface, the regular fibers are complex projective lines and elliptic curves, respectively, which explains the names. The fibration $\varphi : S \to C$, which is part of the definition, is called a *ruling* and an *elliptic fibration* of S, respectively. Kodaira [109, II, p. 563] used the phrase "analytic fibre space of elliptic curves" instead of "elliptic surface" or "elliptic fibration."

By successively blowing down exceptional curves of the first kind contained in fibers of a fibration, one arrives at a fibration with no exceptional curves of the first kind contained in any fiber. See Theorem 3.2.4. Such fibrations are called *relatively minimal*; see Barth, Hulek, Peters and van de Ven [11, Chapter V, Section 7]. ⊘

All Riemann surfaces of genus zero are complex analytic diffeomorphic to the complex projective line $\mathbb{P}^1$, and Proposition 6.2.5 implies that every relatively minimal ruled surface $\varphi : S \to C$ is a locally trivial complex analytic $\mathbb{P}^1$-bundle over C, without singular fibers.

If $\varphi : S \to C$ is an elliptic fibration, then for each $c \in C^{\mathrm{reg}}$ the fiber S_c over c, which is an elliptic curve, has a modulus $J(c) := \mathrm{J}(S_c)$; see (2.3.10). This defines a function $J : C^{\mathrm{reg}} \to \mathbb{C}$, called the *modulus function* of the elliptic fibration $\varphi : S \to C$. It follows from Corollary 6.2.42 that J extends to a meromorphic function on C, where Table 6.2.40 yields more qualitative properties of J. Because the elliptic curves S_c and $S_{c'}$ are complex analytic diffeomorphic to each other if and

only if $J(c) = J(c')$, it follows that $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is not a locally trivial complex analytic fiber bundle if its modulus function is not constant. Conversely, Lemma 6.4.10 implies that if the modulus function is constant, then $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ is a locally trivial complex analytic fiber bundle. For the case that for each $a \in \mathbb{C}$ there is exactly one fiber with modulus equal to a, see Remark 6.3.13.

6.2 The Singular Fibers

6.2.1 The Adjunction Formulas

Let A be a complex $(n-1)$-dimensional complex analytic submanifold of an n-dimensional complex analytic manifold B. For every $a \in A$, the tangent space $\mathrm{T}_a A$ of A at a is an $(n-1)$-dimensional linear subspace of the n-dimensional tangent space $\mathrm{T}_a B$ of B at a. For every linear form λ on $\mathrm{T}_a B$ that vanishes on $\mathrm{T}_a A$ and every complex $(n-1)$-form ω' on $\mathrm{T}_a A$ there is a unique complex n-form ω on $\mathrm{T}_a B$ such that

$$\omega(v_1, \ldots, v_{n-1}, w) = \omega'(v_2, \ldots, v_n)\, \lambda(w) \tag{6.2.1}$$

for every $v_1, \ldots, v_{n-1} \in \mathrm{T}_a A$ and $w \in \mathrm{T}_a B$. The space of all complex linear forms on $\mathrm{T}_a B$ that vanish on $\mathrm{T}_a A$ is identified with the space $(\mathrm{T}_a B/\mathrm{T}_a A)^*$ of all complex linear forms on $\mathrm{T}_a B/\mathrm{T}_a A$, and because $\mathrm{T}_a B/\mathrm{T}_a A$ is one-dimensional, $(\mathrm{T}_a B/\mathrm{T}_a A)^*$ is one-dimensional as well. Also the spaces $\Lambda^{n-1}(\mathrm{T}_a A)^*$ and $\Lambda^n(\mathrm{T}_a B)^*$ of complex $(n-1)$-forms and complex n-forms on $\mathrm{T}_a A$ and $\mathrm{T}_a B$, respectively, are complex one-dimensional, and the formula (6.2.1) defines an isomorphism

$$\alpha_a : \Lambda^{n-1}(\mathrm{T}_a A)^* \otimes (\mathrm{T}_a B/\mathrm{T}_a A)^* \xrightarrow{\sim} \Lambda^n(\mathrm{T}_a B)^*. \tag{6.2.2}$$

The complex one-dimensional spaces $\Lambda^{n-1}(\mathrm{T}_a A)^*$, $a \in A$, and $\Lambda^n(\mathrm{T}_b B)^*$, $b \in B$, form the canonical bundles K_A and K_B of A and B, respectively; see (2.1.27). The $\mathrm{T}_a B/\mathrm{T}_a A$ and the $(\mathrm{T}_a B/\mathrm{T}_a A)^*$, $a \in A$, form a holomorphic complex line bundle over A, called the *normal bundle* $\mathrm{N}_B(A)$ and the *conormal bundle* $\mathrm{N}_B(A)^*$ of A in B, respectively. Therefore, if $\iota_A : A \to B$ denotes the identity on A, viewed as a holomorphic mapping from A to B, the isomorphism (6.2.2) induces an isomorphism

$$\alpha : \mathrm{K}_A \otimes \mathrm{N}_B(A)^* \xrightarrow{\sim} \iota_A^* \mathrm{K}_B, \tag{6.2.3}$$

which is called the *adjunction formula* for smooth hypersurfaces. See Griffiths and Harris [74, pp. 146, 147] for equivalent formulations of the adjunction formula. See also Remark 3.3.5.

Remark 6.2.1. Let f be a holomorphic complex-valued function on the complex analytic manifold B, such that $\mathrm{d}f$ has no zeros on the zero-set A of f. If ω is a holomorphic volume form on B, a holomorphic section of K_B, then (6.2.1) with $\lambda = \mathrm{d}f$ defines a holomorphic volume form ω' on A, a holomorphic section of A,

which is called the *relative quotient* $\omega/\,\mathrm{d}f$ of ω by $\mathrm{d}f$. As a formula,

$$\omega_a(v_1,\,\ldots,\,v_{n-1},\,w) = (\omega/\,\mathrm{d}f)_a(v_1,\,\ldots,\,v_{n-1})\,(\mathrm{d}f)(w) \tag{6.2.4}$$

for every $a \in A$, $v_1,\,\ldots,\,v_{n-1} \in \mathrm{T}_a A$, and $w \in \mathrm{T}_a B$. The relative quotient $\omega/\,\mathrm{d}f$ has no zeros on A if and only if ω has no zeros on A.

In the real setting, the definition of the relative quotient was introduced by Jacobi [96, 14. Vorlesung, p. 115] in his theory of "multipliers" = volume forms that are invariant under the flow of a dynamical system. His point was that if the volume form ω and the function f both are invariant under the flow, then the codimension-one level sets of f and the volume forms $\omega/\,\mathrm{d}f$ on the level sets of f are also invariant under the flow ⊘

Lemma 6.2.2 *Let A be a smooth compact connected curve in the complex surface S. Then we have the numerical adjunction formula*

$$\mathrm{K}_S \cdot A + A \cdot A = 2\,\mathrm{g}(A) - 2, \tag{6.2.5}$$

where $\mathrm{g}(A)$ *is the genus of* A.

Proof. If in (6.2.3) for $B = S$ we take the degree of the left- and right-hand sides, we obtain $-\deg(\mathrm{N}_S(A)) + \deg(\mathrm{K}_A) = \deg(\iota_A{}^*\,\mathrm{K}_S)$. The formula (6.2.5) follows in view of (2.1.25), (2.1.15), and (2.1.22) with $C = A$ and $L = \mathrm{K}_S$. □

In the remainder of this subsection we discuss the generalization of (6.2.3) and (6.2.5) to curves A with singularities.

Assume that A is an irreducible curve germ at a point a in a smooth complex analytic surface S. The *theorem of Puiseux* [166], see for instance Łojasiewicz [125, Chapter II, §6], says that there exist a local holomorphic coordinate system $(x,\,y)$ in an open neighborhood of a in S with $x(a) = y(a) = 0$, $m \in \mathbb{Z}_{>0}$, and a holomorphic function Y on an open disk D in $\mathbb{C}$ with $Y(0) = 0$, such that the mapping $u : t \mapsto (t^m,\,Y(t))$ maps D homeomorphically onto an open neighborhood of a in A. Shrinking D if necessary, it can be arranged that a is the only singular point of $u(D)$, and $u : D \setminus \{0\} \to u(D) \setminus \{a\}$ is a complex analytic diffeomorphism. Such a map germ $u : (\mathbb{C},\,0) \to (A,\,a)$ is called a *uniformization* of A at a, and it follows from the theorem on removable singularities that any other uniformization is of the form $u \circ v$, where $v : (\mathbb{C},\,0) \to (\mathbb{C},0)$ is a germ of a complex analytic diffeomorphism preserving the origin. It follows that the lines $\mathbb{C}\,u(t) \in \mathbb{P}^1$ have a unique limit l when $t \neq 0$, $t \to 0$. By means of a linear substitution of variables in the $(x,\,y)$-plane we can arrange that $l = [1 : 0]$, that is, $Y(t)$ vanishes to order $> m$ as $t \to 0$. In this situation, $(A, a) = \mathrm{Div}(f)$, where f is the germ of a function of the form $f(x,\,y) = y^m + \sum_{i=1}^{m} c_i(x)\,y^{m-i}$. Here each $c_i(x)$ is a symmetric homogeneous polynomial of degree i in the zeros y of $f(x,\,y)$ and therefore is the germ of a holomorphic function of x, where $t \mapsto c_i(t^m)$ vanishes to order $> i\,m$ as $t \to 0$, and hence $c_i(x)$ vanishes to order $> i$ as $x \to 0$. It follows that f vanishes to order m at the origin. The pullback $u^*((\mathrm{d}x \wedge \mathrm{d}y)/\,\mathrm{d}f) = u^*(\,\mathrm{d}x/\partial_2 f) = p\,t^{m-1}\,\mathrm{d}t/u^*(\partial_2 f)$ by u of the relative quotient $(\mathrm{d}x \wedge \mathrm{d}y)/\,\mathrm{d}f$, see Remark 6.2.1, is a meromorphic

one-form on D, with a pole of order $c \in \mathbb{Z}_{\geq 0}$, where $c = 0$ if and only if $m = 1$ if and only if A is smooth at a. Indeed, $(\partial_2 f)(x,\ y) := \partial f(x,\ y)/\partial y$ is the sum of $m\, y^{m-1}$ and the terms $c_i(x)\,(m-i)\, y^{m-i-1}$ for $1 \leq i \leq m-1$. Substitution of $x = t^m$ and $y = Y(t)$ yields that each term is a holomorphic function of t that vanishes to order $> m-1$ if $m > 1$, whence the claim. If the germ $(A,\ a)$ has other irreducible components, then it is equal to $\mathrm{Div}(f\, g)$, where g is the germ at a of a holomorphic function such that $g(a) = 0$. Because $\mathrm{d}(f\, g) = g\ \mathrm{d}f + f\ \mathrm{d}g = g\ \mathrm{d}f$ along $f = 0$, we have $u^*((\mathrm{d}x \wedge \mathrm{d}y)/\mathrm{d}(f\, g)) = (u^*((\mathrm{d}x \wedge \mathrm{d}y)/\mathrm{d}f))/u^*(g)$, which is a meromorphic one-form with a pole at the origin, of order equal to c plus the strictly positive vanishing order of $u^*(g)$ at 0.

Assume that A is a complex analytic curve in the complex analytic complex two-dimensional manifold S when its set Σ of singular points is discrete. The uniformizations at the irreducible components of the local germs of A piece together to a minimal resolution of singularities of A, a smooth compact irreducible complex analytic curve $\widetilde{A}$ together with a proper complex analytic mapping $\rho : \widetilde{A} \to A$ such that the restriction of ρ to $\rho^{-1}(A \setminus \Sigma)$ is a complex analytic diffeomorphism from $\rho^{-1}(A \setminus \Sigma)$ onto $A \setminus \Sigma$. Furthermore, for each $a \in A$ the fiber $\rho^{-1}(\{a\})$ is finite, where $\#(\rho^{-1}(\{a\}))$ equals the number of irreducible components of the germ of A at a. If $\rho' : \widetilde{A}' \to A$ is another such mapping, then there is a unique complex analytic diffeomorphism ψ from $\widetilde{A}$ onto $\widetilde{A}'$ such that $\rho = \rho' \circ \psi$. If $\widetilde{a} \in \widetilde{A}$, $a = \rho(\widetilde{a})$, $(A,\ a) = \mathrm{Div}(f)$ for a holomorphic function f on an open neighborhood of a in S, and ω is a nonvanishing holomorphic two-form on an open neighborhood of a in S, then the previous paragraph yields that $\rho^*(\omega/\mathrm{d}f)$ is a meromorphic one-form on an open neighborhood of $\widetilde{a}$ in $\widetilde{A}$. If $c = c_{\widetilde{a}} \in \mathbb{Z}_{\geq 0}$ denotes the order of its pole at $\widetilde{a}$, then $c > 0$ if and only if $a \in \Sigma$. This order does not depend on the choice of ρ, ω, and f.

The effective divisor $\mathfrak{c}(A) := \sum_{\widetilde{a}} c_{\widetilde{a}}\, \widetilde{a}$ on $\widetilde{A}$ is called the *conductor* of A. Let $\alpha \mapsto U_\alpha$ be a sufficiently fine open covering of S, f_α holomorphic functions on U_α such that $\mathrm{Div}(f_\alpha) = A \cap U_\alpha$, and ω_α nonvanishing holomorphic two-forms on U_α. Then the nonvanishing holomorphic functions $f_{\alpha\beta} = f_\alpha/f_\beta$ on $U_\alpha \cap U_\beta$ are the transition functions that define the holomorphic line bundle $\delta(A)$, more precisely the holomorphic line bundle on S whose isomorphism class is equal to $\delta(A)$; see Section 2.1.3. Let s be a holomorphic section of the holomorphic line bundle $\delta(\mathfrak{c})$ on $\widetilde{A}$ such that $\mathrm{Div}(s) = \mathfrak{c}$, see Lemma 2.1.2, when $s_\alpha = \pi_2 \circ \tau_\alpha \circ s$ is a holomorphic function on $\rho^{-1}(U_\alpha)$ with divisor equal to $\mathfrak{c} \cap \rho^{-1}(U_\alpha)$, if τ_α is a local trivialization of $\delta(\mathfrak{c})$ over $\rho^{-1}(U_\alpha)$. Then $\nu_\alpha := \rho^*(\omega_\alpha/\mathrm{d}f_\alpha)\, s_\alpha$ is a nonvanishing holomorphic section of $\mathrm{K}_{\widetilde{A}}$ over $\rho^{-1}(U_\alpha)$. For any holomorphic line bundle L, any nonvanishing local holomorphic section λ_α defines a local trivialization $\tau_\alpha : l \mapsto l/\lambda_\alpha$, and L is defined by the transition functions $(\pi_2 \circ \tau_\alpha)/(\pi_2 \circ \tau_\beta) = \lambda_\beta/\lambda_\alpha$. Because $\mathrm{d}f_\alpha = f_{\alpha\beta}\ \mathrm{d}f_\beta + f_\beta\ \mathrm{d}f_{\alpha\beta} = f_{\alpha\beta}\ \mathrm{d}f_\beta$ on $A \cap U_\alpha \cap U_\beta$, we have $\nu_\beta/\nu_\alpha = (\omega_\beta/\omega_\alpha)\,(f_\alpha/f_\beta)\,(s_\beta/s_\alpha)$. This proves the generalization

$$\mathrm{K}_{\widetilde{A}} = \rho^*(\mathrm{K}_S \otimes \delta(A)) \otimes \delta(\mathfrak{c}(A))^{-1} \tag{6.2.6}$$

of (6.2.3) as an identity between isomorphism classes of holomorphic line bundles over $\widetilde{A}$. If A is compact and irreducible, then $\widetilde{A}$ is a compact Riemann surface, and (6.2.6) implies

$$2\ \mathrm{g}(\widetilde{A}) - 2 = \deg(\mathrm{K}_{\widetilde{A}}) = \mathrm{K}_S \cdot A + A \cdot A - \deg(\mathfrak{c}(A)), \tag{6.2.7}$$

a generalization of the numerical adjunction formula (6.2.7). The first identity in (6.2.7) follows from (2.1.15). For the second identity in (6.2.7) we have used (2.1.22), (2.1.21), and the observation that in (2.1.21) the zero section of L can be replaced by any continuous section λ when for $L = \rho^*(\mathrm{K}_S \otimes \delta(A))$ it can be arranged that λ is nonzero at the finite set $\rho^{-1}(\Sigma)$, and $\deg(L) = (\mathrm{K}_S \otimes \delta(A)) \cdot A = \mathrm{K}_S \cdot A + A \cdot A$ follows because ρ maps $\widetilde{A} \setminus \rho^{-1}(\Sigma)$ diffeomorphically onto $A \setminus \Sigma$. The term $\deg(\mathfrak{c}(A))$ in (6.2.7) is equal to the sum of the numbers $c_{\widetilde{a}}$ in the previous paragraph. Therefore $\deg(\mathfrak{c}(A)) \in \mathbb{Z}_{\geq 0}$ and $\deg(\mathfrak{c}(A)) = 0$ if and only if A is smooth.

The *virtual genus* $\mathrm{vg}(A)$ of a compact complex analytic curve A in a nonsingular complex analytic surface S is defined by the equation

$$\mathrm{K}_S \cdot A + A \cdot A = 2\ \mathrm{vg}(A) - 2. \tag{6.2.8}$$

If A is irreducible, then (6.2.7) is equivalent to

$$\mathrm{vg}(A) = \mathrm{g}(\widetilde{A}) + \deg(\mathfrak{c}(A))/2. \tag{6.2.9}$$

Here the *genus* $\mathrm{g}(\widetilde{A})$ *of the desingularization* $\widetilde{A}$ of A is a nonnegative integer, and $\mathrm{vg}(A) = \mathrm{g}(\widetilde{A})$ if and only if $A \simeq \widetilde{A}$ is smooth. Note that the definition (6.2.8) implies that the virtual genus is a homology invariant, see Section 2.1.6, which implies that the virtual genus is invariant under deformations of the curve and the surface in which the curve is embedded. The formulas (6.2.8), (6.2.9) are called the *numerical adjunction formulas for irreducible singular curves*.

In the remainder of this subsection we discuss the resolution of singularities of the curve A in the surface S by means of blowing up the singular points of A, repeating the process if necessary with the proper transforms of the curve. I learned this from Griffiths and Harris [74, pp. 498–507]. Let $a \in \Sigma$ and let $\pi : R \to S$ be the blowup of S at a, with the -1 curve $E = \pi^{-1}(\{a\})$, and $B = \pi'(A)$ the proper image of A. If A_i is an irreducible component of the germ (A, a), then its proper transform $B_i = \pi'(A_i)$ has a unique point of intersection b_i with E. In the above local coordinates (x, y) we have $b_i = [1 : 0] \in \mathbb{P}^1 \simeq E$ and $\pi : (x, \eta) \mapsto (x, y)$ with $y = x\,\eta$. If $(A, a) = \mathrm{Div}(f)$, then $(B, b) = \mathrm{Div}(g)$ with $\pi^*(f) = x^m\, g$, if m denotes the vanishing degree of f at a. Furthermore, if ω is a nonvanishing holomorphic two-form on an open neighborhood of a in A, then $\pi^*(\omega) = x\,\nu$, where ν is a nonvanishing holomorphic two-form on a neighborhood of B in R. If $u_i(t)$ is a uniformization of A_i at a with x-coordinate t^{m_i}, then $u_i = \pi \circ v_i$ for a uniformization of $B_i = \pi'(A_i)$ at b_i, and ${u_i}^*(\omega/\mathrm{d}f) = (t^{m_i})^{m-1}\, {v_i}^*(\nu/\mathrm{d}g$, and therefore $c_b(B_i) = c_a(A_i) - m_i\,(m-1)$. Since m is equal to the sum of the m_i, it follows that $\deg(\mathfrak{c}(B)) = \deg(\mathfrak{c}(A)) - m\,(m-1)$.

Because the degrees of the conductors, which are nonnegative integers, cannot keep decreasing, the process of repeating the blowups at singular points of the proper

transforms of the curves stops after finitely many steps when the proper image $\widetilde{A}$ of A is smooth and the restriction to $\widetilde{A}$ of the iterated blow-down map $\pi : \widetilde{S} \to S$ is a minimal resolution of singularities of A, isomorphic to the previous $\rho : \widetilde{A} \to A$. Since $\deg(\mathfrak{c}(\widetilde{A})) = 0$, it follows that

$$\deg(\mathfrak{c}) = \sum_j m_j\,(m_j - 1), \tag{6.2.10}$$

where the m_j's are the vanishing degrees of the local defining functions at the singular points of all the proper images of A that appear in the iterated blowing up procedure.

The blowing up can also be used to give an alternative proof of (6.2.7). Proposition 3.3.7 and (3.2.9) yield $\mathrm{K}_R \simeq (\pi^*\mathrm{K}_S) \otimes \delta(E)$. In the additive notation we therefore have $\mathrm{K}_S \cdot A = \pi^*(\mathrm{K}_S) \cdot \pi^*(A) = (\mathrm{K}_R - E) \cdot (\pi'(A) + m\,E) = \mathrm{K}_R \cdot B + d\ \mathrm{K}_R \cdot E - E \cdot B - m\,E \cdot E = \mathrm{K}_R \cdot B - m$. Here we have used (3.2.6), (3.2.5), and $\pi'_j(A) = B$ in the first three identities. The fourth identity follows from $E \cdot E = -1$, $\mathrm{K}_R \cdot E = 2\ \mathrm{g}(E) - 2 - E \cdot E = -1$, see (6.2.5) with S and A replaced by R and E, and $E \cdot B = m$, see (3.2.7). Similarly we have $A \cdot A = \pi^*(A) \cdot \pi^*(A) = (B + m\,E) \cdot (B + m\,E) = B \cdot B + 2\,m\,B \cdot E + m^2\,E \cdot E = B \cdot B + m^2$. Combining the two results we arrive at $\mathrm{K}_R \cdot B + B \cdot B = \mathrm{K}_S \cdot A + A \cdot A - m\,(m-1)$. Therefore the number $\mathrm{K}_S \cdot A + A \cdot A$ decreases at each blowup by the same even integer $m\,m(-1)$ as the degree $\deg(\mathfrak{c}(A))$ of the conductor of A. In the end, when we arrive at the smooth curve $\widetilde{A}$, we have $\mathrm{K}_{\widetilde{S}} \cdot \widetilde{A} + \widetilde{A} \cdot \widetilde{A} = 2\,\mathrm{g}(\widetilde{A}) - 2$, see (6.2.5), and therefore (6.2.7) follows with $\deg(\mathfrak{c}(A))$ replaced by $\sum_j m_j\,(m_1 - 1)$.

Remark 6.2.3. The definition of the conductor and the formula (6.2.7) are due to Kodaira [108, pp. 839, 852] when S is compact Kähler, where the assumption that S is Kähler has been removed in [109, I, Section 2]. Note that we did not even assume that S is compact. The virtual genus of A was originally defined by Zariski [212, p. 30] in a different manner. The adjunction formula (6.2.6) is due to Kodaira [109, I, (2.2)]. The identity (6.2.7) with $\deg(\mathfrak{c})$ replaced by $\sum_j m_j\,(m_j - 1)$ was been obtained for arbitrary compact complex analytic surfaces S by Hirzebruch [86, p. 118, 119], where his H, H^*, $e(H)$, and c_1 are equal to A, $\widetilde{A}$, the topological Euler number $\chi_{\mathrm{top}}(\widetilde{A}) = 2 - \mathrm{g}(\widetilde{A})$ of $\widetilde{A}$, and the homology class of K_S^*, respectively. If we triangulate $\widetilde{A}$ so that all points of $\rho^{-1}(\Sigma)$ are vertices, then the image of the triangulation under ρ is a triangulation of A, and it follows that $\chi_{\mathrm{top}}(\widetilde{A}) = \chi_{\mathrm{top}}(A) + \#(\rho^{-1}(\Sigma)) - \#(\Sigma)$. Hirzebruch [86, p. 118] defined $e(H)$ as the topological Euler number of $\widetilde{A}$, not of A.

According to the very thorough overview of Berzolari [15] of the classical theory of singularities of algebraic curves in $\mathbb{P}^2$, the idea of resolving the singularities by blowing up the singular points goes back to Kronecker [114] and, independently, Noether [154]. The formula of Noether [154, (2) on p. 337] coincides with (6.2.9) if $\deg(\mathfrak{c})$ is replaced by the sum of the numbers $m_j\,(m_j - 1)$, and the number p of Noether is replaced by the genus of $\widetilde{A}$, where the number $(m-1)\,(m-2)/2$ of Noether is equal to the virtual genus of every curve of degree m in $\mathbb{P}^2$. More information on singularities of curves in smooth surfaces can be found in Casas–Alvero [28]. ⊘

6.2.2 The Adjunction Formula for Fibers of a Fibration

Let $\varphi : S \to C$ be a fibration with connected fibers, and let g be the common genus of the regular fibers of φ. For any $c \in C$, let $S_c = \varphi^{-1}(\{c\})$ denote the fiber of φ over c, viewed as a divisor as in Section 6.1. If S_c^{irr} denotes the finite set of all irreducible components of the divisor S_c and $\mu_\Theta \in \mathbb{Z}_{>0}$ the multiplicity with which $\Theta \in S_c^{\mathrm{irr}}$ occurs in S_c, then

$$S_c = \sum_{\Theta \in S_c^{\mathrm{irr}}} \mu_\Theta \, \Theta. \tag{6.2.11}$$

Let $\Theta \in S_c^{\mathrm{irr}}$. If $c' \neq c$, then $\Theta \cap S_{c'} = \emptyset$; hence $\Theta \cdot S_{c'} = 0$. Because all fibers are homologous to each other, even if they are singular, see Lemma 6.1.2, and the intersection number is a homology invariant, it follows that $\Theta \cdot S_c = \Theta \cdot S_{c'} = 0$. In view of (6.2.11) this leads to the equation

$$\sum_{\Theta' \in S_c^{\mathrm{irr}}} \mu_{\Theta'} \, \Theta \cdot \Theta' = 0, \quad \Theta \in S_c^{\mathrm{irr}}, \quad c \in C. \tag{6.2.12}$$

This is Kodaira [109, I, (4.15)]. Note that $\Theta \cdot S_c = 0$ for all $\Theta \in S_c^{\mathrm{irr}}$ implies that $S_c \cdot S_c = 0$.

If S_c is reducible, then the fact that S_c is connected implies that for each $\Theta \in S_c^{\mathrm{irr}}$ there is at least one $\Theta' \in S_c^{\mathrm{irr}}$ such that $\Theta \neq \Theta'$ and $\Theta \cap \Theta' \neq \emptyset$; hence $\Theta \cdot \Theta' > 0$, whereas $\Theta \cdot \Theta' \geq 0$ for all $\Theta' \neq \Theta$. Because $\mu_{\Theta'} > 0$ for all Θ', it follows from (6.2.12) that

$$\Theta \cdot \Theta < 0 \text{ for every } \Theta \in S_c^{\mathrm{irr}} \quad \text{if} \quad S_c \text{ is reducible.} \tag{6.2.13}$$

If S_c is irreducible then $\Theta \cdot \Theta = 0$, because $0 = S_c \cdot \Theta = \mu_\Theta \, \Theta \cdot \Theta$ and $\mu_\Theta > 0$.

If $c \in C^{\mathrm{reg}}$, then (6.2.5) for $A = S_c$, in combination with $S_c \cdot S_c = 0$, implies that $\mathrm{K}_S \cdot S_c = 2\,g - 2$. Again using that all fibers are homologous to each other and the intersection number is a homology invariant, we obtain that $\mathrm{K}_S \cdot S_c = 2\,g - 2$ holds for every $c \in C$. In view of (6.2.11) and $\mathrm{K}_S \cdot \Theta = 2\,\mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta$, see (6.2.8) with $A = \Theta$, it follows that

$$\sum_{\Theta \in S_c^{\mathrm{irr}}} \mu_\Theta \, (2\,\mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta) = 2\,g - 2, \quad c \in C. \tag{6.2.14}$$

For $g = 1$ this is Kodaira [109, I, (4.16)]. Recall from (6.2.9) that $\mathrm{vg}(\Theta) = \mathrm{g}(\widetilde{\Theta}) + c_\Theta$, where $\widetilde{\Theta}$ denotes the desingularization of Θ, $c_\Theta \in \mathbb{Z}_{\geq 0}$, and $c_\Theta = 0$ if and only if Θ is smooth.

6.2.3 Ruled Surfaces

I learned most of the results in this subsection from Iskovskikh and Shafarevich [94, §13.1].

Lemma 6.2.4 *Let M be a projective algebraic manifold. For every holomorphic vector bundle L over M, the vector space of all meromorphic sections of L is infinite-dimensional. The homomorphism $\delta : \mathrm{Div}(s) \to \mathrm{Pic}(S)$ in the exact sequence* (2.1.4) *is surjective.*

Proof. The GAGA principle of Serre [180, Proposition 18] implies that L is algebraic according to the definition of Weil [209, p. 10]. This means that L admits trivializations $\tau : L|_{M_\tau} \to M_\tau \times \mathbb{C}^r$ over Zariski-open subsets M_τ of S, that is, complements of algebraic curves in S, with rational transition matrices. Here r denotes the rank of L. For each $\mathbb{C}^r$-valued rational function f on S_τ, $s \mapsto \tau^{-1}(s,\ f(s))$ defines a meromorphic section λ of L. According to Lemma 2.1.2, we have $[L] = \delta(\mathrm{Div}(\lambda))$ if $r = 1$ and $\lambda \not\equiv 0$. □

Proposition 6.2.5 *Let $\varphi : S \to C$ be a relatively minimal ruled surface. Then φ exhibits S as a complex analytic locally trivial $\mathbb{P}^1$-bundle over C. There exists a rank-two holomorphic vector bundle V over C such that S is isomorphic to $\mathrm{P}(V)$, the bundle of which the fiber over $c \in C$ is equal to $\mathrm{P}(V_c)$. If C is compact, then φ has infinitely many holomorphic sections.*

Proof. We expand the paragraph in Barth, Hulek, Peters, and van de Ven [11, pp. 190, 191]. If the fiber S_c is reducible and $\Theta \in S_c^{\mathrm{irr}}$, then (6.2.13) implies that $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta \geq 2\ \mathrm{vg}(\Theta) - 1 \geq -1$; hence $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta < 0$ if and only if $\Theta \cdot \Theta = -1$ and $\mathrm{g}(\widetilde{\Theta}) + c_\Theta = \mathrm{vg}(\Theta) = 0$, that is $c_\Theta = 0$ and $\mathrm{g}(\widetilde{\Theta}) = 0$, which means that Θ is nonsingular and $\mathrm{g}(\Theta) = \mathrm{g}(\widetilde{\Theta}) = 0$. Since this implies that Θ is a -1 curve, in contradiction to the assumption that the fibration is relatively minimal, we obtain that $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta \geq 0$ for every $\Theta \in S_c^{\mathrm{irr}}$, in contradiction to (6.2.14) for $g = 0$. The conclusion is that S_c is irreducible when $S_c = \mu_\Theta$, and $0 = S_c \cdot S_c = {\mu_\Theta}^2\, \Theta \cdot \Theta$ implies that $\Theta \cdot \Theta = 0$. This time (6.2.14) with $g = 0$ implies that $\mu_\Theta\,(\,\mathrm{vg}(\Theta) - 1) = -1$; hence $\mu_\Theta = 1$ and $\mathrm{vg}(\Theta) = 0$. That is, $S_c = \Theta$ is a smoothly embedded complex projective line. Since this holds for every $c \in C$, it follows that $C^{\mathrm{sing}} = \emptyset$, and Lemma 6.1.1 implies that $\varphi : S \to C$ is a real analytic locally trivial $\mathbb{P}^1$-bundle.

For every $c_0 \in C$ there exist an open neighborhood C_0 of c_0 in C and three disjoint holomorphic sections σ_0, σ_1, and σ_∞ of φ over C_0, holomorphic mappings from C_0 to S, such that for each $c \in C_0$, $\sigma_0(c)$, $\sigma_1(c)$, and $\sigma_\infty(c)$ are three distinct points in S_c. For each $c \in C_0$ there exists a unique complex analytic diffeomorphism A_c from $\mathbb{P}^1$ onto S_c such that $A_c([1 : 0]) = \sigma_0(c)$, $A_c([1 : 1]) = \sigma_1(c)$, and $A_c([0 : 1]) = \sigma_\infty(c)$. The mapping $A : (c,\ u) \mapsto A_c(u)$ is a complex analytic diffeomorphism from $C_0 \times \mathbb{P}^1$ onto $\varphi^{-1}(C_0)$ such that $\varphi \circ A$ is equal to the projection $(c,\ u) \mapsto c$. In other words, A^{-1} is a complex analytic local trivialization of φ over C_0. Therefore $\varphi : S \to C$ is a complex analytic locally trivial $\mathbb{P}^1$-bundle.

According to lemma 2.2.1, the automorphism group of $\mathbb{P}^1$ is the group $\mathrm{PGL}(\mathbb{C}^2) = \mathrm{GL}(\mathbb{C}^2)/\mathbb{C}^\times$, where the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers is identified with the group of all nonzero scalar multiplications in $\mathbb{C}^2$. Two local trivializations over open subsets C_α, C_β of C are related to each other over $C_\alpha \cap C_\beta$ by a holomorphic mapping $f_{\alpha\beta} : C_\alpha \cap C_\beta \to \mathrm{PGL}(\mathbb{C}^2)$. Refining the covering of C by the C_α's if necessary, one can find holomorphic mappings $g_{\alpha\beta} : C_\alpha \cap C_\beta \to \mathrm{GL}(\mathbb{C}^2)$ such that $f_{\alpha\beta} = \pi \circ g_{\alpha\beta}$, where π is the canonical homomorphism from $\mathrm{GL}(\mathbb{C}^2)$ onto $\mathrm{PGL}(\mathbb{C}^2)$. The cocycle condition $f_{\alpha\beta}\, f_{\beta\gamma}\, f_{\gamma\alpha} = 1$ on $C_\alpha \cap C_\beta \cap C_\gamma$ leads to a $\mathbb{C}^\times$-valued holomorphic function $h_{\alpha\beta\gamma} := g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha}$ on $C_\alpha \cap C_\beta \cap C_\gamma$. These functions define a two-cocycle h with cohomology class $[h] \in \mathrm{H}^2(C, \mathcal{O}^\times)$. Consider the part $\mathrm{H}^2(C, \mathcal{O}) \to \mathrm{H}^2(C, \mathcal{O}^\times) \xrightarrow{\delta} \mathrm{H}^3(C, \mathbb{Z})$ of the exact sequence (2.1.8) for $M = C$. It follows from (6.2.24) with $M = C$, $q = 2$, $p = 0$, L trivial when $\mathcal{F}^{(0,2)} = 0$, that $\mathrm{H}^2(C, \mathcal{O}) = 0$. On the other hand, for any real d-dimensional manifold M we have $\mathrm{H}^q(M, \mathbb{Z}) = 0$ when $q > d$, see for instance Spanier [190, p. 359], hence $\mathrm{H}^3(C, \mathbb{Z}) = 0$. It follows that $\mathrm{H}^2(C, \mathcal{O}^\times) = \{1\}$, $[h] = 1$, and the $g_{\alpha\beta}$ can be chosen to satisfy the cocycle condition $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$ on $C_\alpha \cap C_\beta \cap C_\gamma$. These $g_{\alpha\beta}$ are the transition functions for a rank two holomorphic vector bundle V over C, and $f_{\alpha\beta} = \pi \circ g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ implies that our $\mathbb{P}^1$-bundle $\varphi : S \to C$ is isomorphic to $\mathrm{P}(V)$.

Let v be a nonzero meromorphic section of V. The Laurent expansion of v at any pole of v shows that $c \mapsto \mathbb{C}\, v(c)$ extends to a holomorphic line subbundle of V, a holomorphic section of $\mathrm{P}(V)$. Because Lemma 6.2.4 implies that the space of all meromorphic sections of V is infinite-dimensional, it follows that φ has infinitely many holomorphic sections. □

Let $\varphi : S \to C$ be a relatively minimal ruled surface over the curve C. Let $b \in S$ and let $\pi : \widetilde{S} \to S$ be the blowup of S at b. If S_c denotes the fiber of φ over $c := \varphi(b)$, then the fiber over c of the ruled surface $\varphi \circ \pi : \widetilde{S} \to C$ is equal to the union $\pi^*(S_c)$ of the exceptional curve $E = \pi^{-1}(\{b\})$ and the strict transform $\pi'(S_c)$ of S_c. The restriction of π to $\pi'(S_c)$ is an isomorphism from $\pi'(S_c)$ onto $S_c \simeq \mathbb{P}^1$ and (3.2.8) yields that $\pi'(S_c) \cdot \pi'(S_c) = S_c \cdot S_c - 1 = -1$. Therefore $\pi'(S_c)$ is, next to E, another -1 curve in $\widetilde{S}$, and according to the Castelnuovo–Enriques criterion, Theorem 3.2.4, there is a surface T and a blowing up $\tau : \widetilde{S} \to T$ at a point $t \in T$ such that $\pi'(S_c) = \tau^{-1}(\{t\})$. There is a unique holomorphic mapping $\psi : T \to C$ such that $\psi \circ \tau = \varphi \circ \pi$. The ruled surface $\psi : T \to C$ over C is relatively minimal because the fiber of ψ over c is equal to $\tau(E)$, which is irreducible, and therefore does not contain a -1 curve. We call the bimeromorphic transformation $\tau \circ \pi^{-1} : S \setminus \{b\} \to T \setminus \{t\}$ the *zigzag* of the ruled surface $\varphi : S \to C$ at the point b. Note that $\tau \circ \pi^{-1}$ blows down the fiber S_c of φ in S to the point t in T and blows up the point b in S to the fiber $\tau(E) = T_c$ of ψ in T.

Theorem 6.2.6 *Let $\varphi : S \to C$ be a relatively minimal ruled surface over C. Assume that C is compact, or equivalently that S is compact. Then there is a finite sequence of zigzags that turns the ruling φ into the trivial ruling $C \times \mathbb{P}^1 \to C : (c, u) \mapsto c$. If e and f are the respective homology classes of any holomorphic section and any fiber of φ, then $\mathrm{H}_2(S, \mathbb{Z}) = \mathbb{Z}\, e \oplus \mathbb{Z}\, f$. Every compact ruled surface S is projective*

algebraic and birationally equivalent to $C \times \mathbb{P}^1$, where the genus of C is equal to the irregularity $\mathrm{q}(S) = \dim \mathrm{H}^0(S,\ \Omega^1)$ *of* S.

Proof. The last statement in Proposition 6.2.5 implies that there are two distinct holomorphic sections Σ and Σ' of φ, viewed as irreducible curves in S, and hence $\Sigma \cdot \Sigma' \in \mathbb{Z}_{\geq 0}$, where $\Sigma \cdot \Sigma' = 0$ if and only if $\Sigma \cap \Sigma' = \emptyset$. Let $b \in \Sigma \cap \Sigma'$. If $\tau \circ \pi^{-1}$ is the zigzag at b, then the proper transforms $\pi'(\Sigma)$ and $\pi'(\Sigma')$ of Σ and Σ' are both disjoint from the proper transform $\pi'(S_c)$ of the fiber S_c through b, and therefore $\tau(\pi'(\Sigma)) \cdot \tau(\pi'(\Sigma')) = \pi'(\Sigma) \cdot \pi'(\Sigma') = \Sigma \cdot \Sigma' - 1$ in view of (3.2.8), where $\tau(\pi'(\Sigma))$ and $\tau(\pi'(\Sigma'))$ are holomorphic sections of $\psi : T \to C$. It follows that after $\Sigma \cdot \Sigma'$ zigzags, each time at an intersection point of the transformed sections, we arrive at a relatively minimal ruled surface over C with two disjoint holomorphic sections Σ_0 and Σ_∞. A third holomorphic section has finite respective intersection numbers i_0 and i_∞ with Σ_0 and Σ_∞, and after $i_0 + i_\infty$ zigzags we arrive at a relatively minimal ruled surface $\psi : T \to C$ with three disjoint holomorphic sections Σ_0, Σ_1, and Σ_∞. For each $c \in C$ there is a unique isomorphism $A_c : \mathbb{P}^1 \to T_c$ such that $\Sigma_0 \cap T_c = \{A_c([1 : 0])\}$, $\Sigma_1 \cap T_c = \{A_c([1 : 1])\}$, and $\Sigma_\infty \cap T_c = \{A_c([0 : 1])\}$. The mapping $A : (c,\ u) \mapsto A_c(u)$ is a complex analytic diffeomorphism from $C \times \mathbb{P}^1$ onto T such that $\psi \circ A$ is equal to the projection $(c,\ u) \mapsto c$.

Let $\tau \circ \pi^{-1} : S \to T$ be the zigzag at a point $b \in S$, where $\pi : \widetilde{S} \to S$ is the blowing up of S at b and $\tau : \widetilde{S} \to T$ is the blowing down of the -1 curve $\pi'(S_{\varphi(b)})$ in $\widetilde{S}$. Because the image of $\pi^{-1}(S_{\varphi(b)})$ under π and that of $E = \pi^{-1}(\{b\})$ under τ are fibers of φ and ψ, respectively, and $\tau(\pi'(A))$ is a holomorphic section of ψ if A is a holomorphic section of φ, it follows from (3.2.4) that the statement about the homology group holds for T if it holds for S. In view of (2.4.4) it holds when $S = \mathbb{P}^1 \times C$, and therefore it holds for any relatively minimal ruled surface $\varphi : S \to C$.

By successively blowing down -1 curves in fibers, one obtains that every compact ruled surface is bimeromorphic to a relatively minimal ruled surface, which according to the first statement in Theorem 6.2.6 is bimeromorphic to $C \times \mathbb{P}^1$, a complex projective algebraic surface. It follows from Corollary 6.2.55 that the irregularity $\mathrm{q}(S) = \dim \mathrm{H}^0(S,\ \Omega^1)$ is a birational invariant, and therefore $\mathrm{q}(S) = \mathrm{q}(C \times \mathbb{P}^1)$. If ω is a holomorphic one-form on $C \times \mathbb{P}^1$ then, because every holomorphic one-form on $\mathbb{P}^1$ is equal to zero, the pullback of ω to every vertical axis is equal to zero. Therefore the pullback by means of the projection $C \times \mathbb{P}^1 \to C$ is a linear isomorphism from $\mathrm{H}^0(C,\ \Omega^1)$ onto $\mathrm{H}^0(C \times \mathbb{P}^1,\ \Omega^1)$; hence $\mathrm{g}(C) = \mathrm{q}(C \times \mathbb{P}^1)$. This proves the last statement in Theorem 6.2.6. □

Remark 6.2.7. For every compact Riemann surface C of genus $\mathrm{g}(C) > 0$ there exists a relatively minimal ruled surface $\varphi : S \to C$ that does not admit a pair of disjoint holomorphic sections.

According to Proposition 6.2.5 the statement is equivalent to the existence of a holomorphic rank-two vector bundle V over C that is not the direct sum of two holomorphic line subbundles. We begin with explaining footnote 4 in Grothendieck [76], attributed to Serre. Let M and Q be holomorphic line bundles over C. A *holomorphic extension of Q by M* is a rank-two holomorphic vector bundle V over C such

that M is a holomorphic line subbundle of V, and there exists a holomorphic bundle projection $\pi : V \to Q$ such that $M = \ker \pi$. There is a covering of C by open subsets C_α such that for each α there is an isomorphism $\pi \times \psi_\alpha : V \to Q \times M$ over C_α, where $\psi_\alpha : V \to M$ is a projection of holomorphic vector bundles over C_α. If $\psi_\beta : V|_{C_\beta} \to M|_{C_\beta}$, then, because both ψ_α and ψ_β are equal to the identity on $M|_{C_\alpha \cap C_\beta}$, we have $\psi_\beta - \psi_\alpha = 0$ on $M|_{C_\alpha \cap C_\beta}$, that is, $\psi_\beta - \psi_\alpha = \psi_{\beta\alpha} \circ \pi$ for a unique bundle endomorphism $\psi_{\beta\alpha} : Q \to M$ over $C_\alpha \cap C_\beta$. The $\psi_{\beta\alpha}$ define a cohomology class $[\psi] \in \mathrm{H}^1(C, \mathcal{O}(Q^* \otimes M))$, and this defines a bijection from the category of isomorphism classes of holomorphic extensions of Q by M onto $\mathrm{H}^1(C, \mathcal{O}(Q^* \otimes M))$. In particular, $[\psi] = 0$ if and only if $V = L \oplus M$ for a holomorphic line subbundle L of V.

We now apply the argument in "Remarque final 2" of Grothendieck [76]. Choose Q and M both trivial. Then $\dim \mathrm{H}^1(C, \mathcal{O}(Q^* \otimes M)) = \dim \mathrm{H}^1(C, \mathcal{O}) = \dim \mathrm{H}^0(C, \mathcal{O}(\mathrm{K}_C)) =: \mathrm{g}(C) > 0$, where in the second identity we have used (2.1.13). It follows that there exists a holomorphic extension V of Q by M that does not admit a holomorphic line subbundle L of V such that $V = L \oplus M$. If L is any holomorphic line subbundle of V not equal to M, and q is a holomorphic section of Q without zeros, then the equation $\pi \circ l = q$ defines a meromorphic section l of L without zeros, with poles at the $c \in C$ such that $L_c = M_c$. Because $V \neq L \oplus M$, such c exist, and it follows from (2.1.12) that $\deg(L) < 0$. If $V = L_1 \oplus L_2$ for some holomorphic line subbundles L_j of V, and m is a holomorphic section of M without zeros, then $m = l_1 + l_2$ for unique holomorphic sections l_j of L_j. If $l_1 \equiv 0$, then $M = L_2$, contradicting that M has no holomorphic linear complement in V. It follows that l_1 is a nonzero holomorphic section of L_1, and therefore (2.1.12) implies that $\deg(L_1) \geq 0$, which in view of the previous implies that $L_1 = M$, again contradicting that M has no holomorphic linear complement in V. Therefore V is not the direct sum of two holomorphic line subbundles. ⊘

The relatively minimal ruled surfaces over $\mathbb{P}^1$ are classified by the following theorem, which implies that every relatively minimal ruled surface over a compact Riemann surface of genus zero admits two disjoint holomorphic sections.

Theorem 6.2.8 *Every relatively minimal ruled surface over $\mathbb{P}^1$ is isomorphic to $\Sigma_n := \mathrm{P}(\mathbb{C}_{\mathbb{P}^1} \oplus \mathrm{O}(n))$, where $\mathbb{C}_{\mathbb{P}^1}$ denotes the trivial line bundle over $\mathbb{P}^1$ and n is a nonnegative integer. If C is an irreducible curve in Σ_n not equal to the section $T_n := \mathrm{P}(\{0\} \oplus \mathrm{O}(n))$ in Σ_n, then $C \cdot C \geq 0$, whereas $T_n \cdot T_n = -n$. Applying a zigzag at any point to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ we obtain Σ_1. If $n > 0$ then the application of the zigzag at a point on T_n or not on T_n leads to a relatively minimal ruled surface isomorphic to Σ_n or Σ_{n-1}, respectively. Blowing down the -1 curve T_1 in Σ_1 leads to a surface that is isomorphic to $\mathbb{P}^2$. If $\varphi : S \to C$ is a ruled surface, then S is rational if and only if C is rational, that is, $C \simeq \mathbb{P}^1$.*

Proof. According to Proposition 6.2.5 we may assume that the ruled surface is of the form $\mathrm{P}(V)$, where V is a holomorphic rank-two vector bundle over $\mathbb{P}^1$, with at least one holomorphic line subbundle. We begin with the proof of Grothendieck [76, pp. 127–129] that V is equal to the direct sum of two complementary holomorphic

line subbundles. In the sequel we use that any holomorphic line bundle L over $\mathbb{P}^1$ is isomorphic to $\mathrm{O}(d)$, where $d = \deg(L)$; see Remark 2.1.7.

Let L_0 and L be two distinct holomorphic line subbundles of V, s a meromorphic section of L, and $t = s + L$ the corresponding meromorphic section of the holomorphic line bundle V/L_0. Then $\mathrm{Div}(s) \leq \mathrm{Div}(t)$, hence $\deg(L) \leq \deg(V/L_0)$, and it follows that the degrees of the holomorphic line subbundles of V have a maximum m. Let M be a holomorphic line subbundle of V such that $\deg(M) = m$. We first prove that $\deg(V/M) \leq m$.

Assume that $\deg(V/M) > m$. Tensoring V and M with $\mathrm{O}(-m-1)$, we arrive at the situation that $\deg(M) = -1$ and $\deg(V/M) \geq 0$. The short exact sequence $0 \to \mathcal{O}(M) \to \mathcal{O}(V) \to \mathcal{O}(V/M) \to 0$ leads to the exact sequence $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(V)) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(V/M)) \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(M))$. The Serre duality (6.2.25) implies that $\dim \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(M)) = \dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(\mathrm{K}_{\mathbb{P}^1} \otimes M^*) = 0$, because $\deg(\mathrm{K}_{\mathbb{P}^1} \otimes M^*) = \deg(\mathrm{K}_{\mathbb{P}^1}) - \deg(M^*) = -2 - (-1) = -1 < 0$, where we have used (2.1.15) with $\mathrm{g}(\mathbb{P}^1) = 0$. Because $V/M \simeq \mathrm{O}(d)$ for some $d \geq 0$, V/M has a nonzero holomorphic section q, which in view of the exact sequence $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(V)) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(V/M)) \to 0$ is of the form $q = v + M$ for a nonzero holomorphic section v of V. The section v generates a holomorphic line subbundle L of V such that $\deg(L) \geq 0$. Returning to the original situation by tensoring with $\mathrm{O}(m+1)$ we obtain a holomorphic line subbundle of V of degree $\geq m+1 > m$, in contradiction to the definition of m.

Write $Q = V/M$ when $Q^* \otimes Q$ is the trivial line bundle over $\mathbb{P}^1$. The short exact sequence $0 \to \mathcal{O}(Q^* \otimes M) \to \mathcal{O}(Q^* \otimes V) \to \mathcal{O}(Q^* \otimes Q) \to 0$ leads to the exact sequence $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(Q^* \otimes V)) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(Q^* \otimes Q)) \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(Q^* \otimes M))$. Because $\deg(\mathrm{K}_{\mathbb{P}^1} \otimes (Q^* \otimes M)^* = -2 + \deg(Q) - \deg(M) \leq -2 < 0$, we obtain the exact sequence $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(Q^* \otimes V)) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(Q^* \otimes Q) \to 0$, which means that there exists a holomorphic section s of $Q^* \otimes V$ that by the projection $V \to Q$ is mapped to the identity section of $Q^* \otimes Q$. For each $p \in \mathbb{P}^1$, $s(p)$ is identified with a linear mapping from Q_p to V_p such that $\pi(p) \circ s(p)$ is equal to the identity on Q_p if $\pi(p)$ is the projection from V_p onto $Q_p = V_p/M_p$. It follows that the $L_p := s(p)(Q_p)$, $p \in \mathbb{P}^1$ form a holomorphic line subbundle L of V that is complementary to M, and $V = L \oplus M$. The definition of m implies that $l := \deg(L) \leq m = \deg(M)$, and we conclude that $S \simeq \mathrm{P}(V) \simeq \mathrm{P}(L^* \otimes V) = \mathrm{P}((L^* \otimes L) \oplus (L^* \otimes M)) \simeq \mathrm{P}(\mathbb{C}_{\mathbb{P}^1} \oplus \mathrm{O}(n))$, where $n = -l + m \geq 0$. This proves the first conclusion in Theorem 6.2.8.

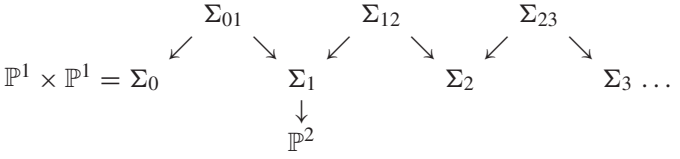
Let $V = L \oplus M$ be the direct sum of two one-dimensional linear subspaces L and M. Then every one-dimensional linear subspace M' of V that is complementary to L is of the form $M' = \{m + A(m) \mid m \in M\}$ for a unique linear mapping A from M to L. This leads to a canonical identification of the tangent space $\mathrm{T}_M \mathrm{P}(V)$ of $\mathrm{P}(V)$ at M with the space $M^* \otimes L$ of all linear mappings from M to L. We apply this, for every $p \in \mathbb{P}^1$, to $L = \mathbb{C}$, $M = \mathrm{O}(n)_p$, and $V = V_p = \mathbb{C} \oplus \mathrm{O}(n)_p$. It follows that the tangent space at $s := \mathrm{P}(\{0\} \oplus \mathrm{O}(n))_p \in T_n$ of the fiber of $S := \mathrm{P}(\mathbb{C}_{\mathbb{P}^1} \oplus \mathrm{O}(n))$ over p is canonically isomorphic to $\mathrm{O}(\mathrm{n})^*_p \simeq \mathrm{O}(-n)$. Because the tangent space at $s \in T_n$ of the fiber is canonically isomorphic to $\mathrm{T}_s S / \mathrm{T}_s(T_n)$, it follows that the holomorphic line bundle $\mathrm{O}(-n)$ over $\mathbb{P}^1$ is isomorphic to the normal bundle

$\mathrm{N}_S(T_n)$ of T_n in S, viewed as a holomorphic line bundle over $T_n \simeq \mathbb{P}^1$, and therefore $T_n \cdot T_n = \deg(\mathrm{N}_S(T_n)) = \deg(\mathrm{O}(-n)) = -n$; see (2.1.25).

Let C be an irreducible curve in Σ_n. It follows from the characterization of $\mathrm{H}_2(\Sigma_n, \mathbb{Z})$ in Theorem 6.2.6 that there are $a, b \in \mathbb{Z}$ such that C is homologous to the divisor $a\,T_n + b\,F$, where F is any fiber. Using the homology invariance and the bilinearity of the intersection form, $F \cdot F = 0$, $F \cdot T_n = 1$, and $T_n \cdot T_n = -n$, we have $C \cdot F = (a\,T_n + b\,F) \cdot F = a, C \cdot T_n = (a\,T_n + b\,F) \cdot T_n = -a\,n + b$, and therefore $C \cdot C = (a\,T_n + b\,F) \cdot (a\,T_n + b\,F) = -a^2\,n + 2\,a\,b = (C \cdot F)^2\,n + 2\,(C \cdot F)\,(C \cdot T_n)$. If $C = F$ then $C \cdot C = F \cdot F = 0$. If $C \neq F$ and $C \neq T_n$ then $C \cdot F \geq 0$ and $C \cdot T_n \geq 0$ because C, F, and T_n are irreducible, hence $C \cdot C \geq 0$. This completes the proof of the second statement in Theorem 6.2.8.

Applying a zigzag at b to the relatively minimal ruled surface Σ_n we obtain a relatively minimal ruled surface, which according to first statement in Theorem 6.2.8 is isomorphic to Σ_k for some $k \geq 0$. If $n > 0$ and $b \in T_n$ or $b \notin T_n$, then the zigzag maps the section T_n to a section T such that $T \cdot T = T_n \cdot T_n - 1 = -n - 1$ or $T \cdot T = T_n \cdot T_n + 1 = -n + 1$, and the second statement in Theorem 6.2.8 implies that $k = n + 1$ or $k = n - 1$, respectively. If $n = 0$ the horizontal axis through b is mapped to a section T such that $T \cdot T = -1$, and we obtain Σ_1. If we then blow down the -1 curve $T = T_1$ in Σ_1, we have performed the birational transformation from $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ as described in Section 4.1. If S is any ruled surface over $\mathbb{P}^1$, then successively blowing down -1 curves contained in fibers, we arrive at a relatively minimal ruled surface, which is isomorphic to a Σ_n. Preforming suitable zigzags, we arrive at Σ_1, and blowing down the unique -1 curve T_1 in Σ_1 leads to a surface that is isomorphic to $\mathbb{P}^2$. This shows that S is rational. Conversely, if the ruled surface S over the curve C is rational, then the irregularity of S, which according to Corollary 6.2.55 is a birational invariant, is equal to zero, while the last statement in Theorem 6.2.6 implies that $\mathrm{g}(C) = 0$, that is, $C \simeq \mathbb{P}^1$. □

The ruled surfaces Σ_n in Theorem 6.2.8 are called the *Hirzebruch surfaces*, after their first appearance in Hirzebruch [84], where it is proved that these surfaces are rational, and the sections T_n are used in order to prove that Σ_n and Σ_m are isomorphic complex analytic surfaces and homeomorphic topological spaces if and only if $n = m$ and $n - m \in 2\,\mathbb{Z}$, respectively. The Hirzebruch surface Σ_n is isomorphic to the compactification of $\mathrm{O}(n)$ as in the proof of Theorem 2.1.4, where T_n corresponds to the section "at infinity," added to $\mathrm{O}(n)$ in order to compactify the line bundle $\mathrm{O}(n)$. The statement that every relatively minimal ruled surface over $\mathbb{P}^1$ is isomorphic to such compactification of $\mathrm{O}(n)$ corresponds to Atiyah [8, Theorem 6.0]. In contrast with $T_n \cdot T_n = -n$, any holomorphic section disjoint from T_n, that is, any holomorphic section of $\mathrm{O}(n)$, has self-intersection number $+n$. In the following diagram, the arrows are blowdown morphisms:



Theorem 6.2.9 below yields a characterization of compact relatively minimal ruled surfaces. A surface is called *minimal* if it contains no -1 curves. (These are not the real two-dimensional surfaces in $\mathbb{R}^3$ that minimize area.) A holomorphic line bundle L over a compact surface M is called *numerically effective* or *nef*, if $L \cdot C \geq 0$ for every irreducible curve C in M. The implication (i) $\Rightarrow$ (ii) is due to Mori [140, Corollary (2.2)]. Since the proof is long, it may be skipped at a first reading of the book. The theorem of Mori will be used in the proof of (d) $\Rightarrow$ (a) in Theorem 9.1.3.

Theorem 6.2.9 *Let M be a compact and connected two-dimensional complex analytic manifold. Then the following conditions are equivalent:*

(i) *M is projective algebraic, M is minimal, and the canonical bundle K_M of M is not numerically effective.*
(ii) *M is isomorphic to the complex projective plane $\mathbb{P}^2$ or to a relatively minimal ruled surface other than the Hirzebruch surface Σ_1.*

Proof. The bulk of the proof concerns the implication (i) $\Rightarrow$ (ii). Our proof is inspired by Iskovskikh and Shafarevich [94, pp. 176–179] and Barth, Hulek, Peters, and van de Ven [11, pp. 249–252].

Let C be an irreducible curve in M such that $\mathrm{K}_M \cdot C < 0$. If $C \cdot C < 0$, then the numerical adjunction formula $2\,\mathrm{vg}(C) - 2 = C \cdot C + \mathrm{K}_M \cdot C$, see (6.2.8), can hold only if $C \cdot C = \mathrm{K}_M \cdot C = -1$ and hence $\mathrm{vg}(C) = 0$, because $\mathrm{vg}(C) \in \mathbb{Z}_{\geq 0}$. It follows that C is a -1 curve, in contradiction to the minimality of M. It follows that $C \cdot C \geq 0$. In other words, if E is an irreducible curve in M such that $E \cdot E < 0$, then $\mathrm{K}_M \cdot E \geq 0$.

Let ω be a nonzero holomorphic complex two-form on M. Then $\mathrm{Div}(\omega) = \sum_j n_j C_j$, where the C_j are finitely many distinct irreducible curves in M and $n_j \in \mathbb{Z}_{>0}$. For the irreducible curve C in M such that $\mathrm{K}_M \cdot C < 0$, we have $0 > \mathrm{K}_M \cdot C = \mathrm{Div}(\omega) \cdot C = \sum_j n_j C_j \cdot C$. Because $C_j \cdot C \geq 0$ if $C_j \neq C$, there exists exactly one j such that $C_j = C$ and $C_j \cdot C < 0$; hence $C \cdot C = C_j \cdot C < 0$, in contradiction to the previous paragraph. The conclusion is that $\mathrm{H}^0(M, \mathcal{O}(\mathrm{K}_M)) = 0$, which in view of the Serre duality (6.2.27) implies that $\mathrm{p_g}(M) := \dim \mathrm{H}^2(M, \mathcal{O}) = 0$. Therefore the exact sequence (2.1.8) yields that the Chern homomorphism $\mathrm{c} : \mathrm{Pic}^1(M) \to \mathrm{H}^2(M, \mathbb{Z})$ is surjective. Because M is projective algebraic, the homomorphism $\delta : \mathrm{Div}(M) \to \mathrm{Pic}(M)$ is surjective as well, see the second paragraph in Section 7.2, and hence $\mathrm{pd} \circ \mathrm{H} = \mathrm{c} \circ \delta$ is surjective. Here we have used (2.1.24). Since the Poincaré duality $\mathrm{pd} : \mathrm{H}_2(M, \mathbb{Z}) \to \mathrm{H}^2(S, \mathbb{Z})$ is an isomorphism, see (2.1.17) with $n = 4$ and $a = b = 2$, the homomorphism $\mathrm{H} : \mathrm{Div}(M) \to \mathrm{H}_2(M, \mathbb{Z})$, the mapping that assigns to each divisor its homology class, is surjective. That is, $\mathrm{NS}(M) := \mathrm{H}(\mathrm{Div}(S)) = \mathrm{H}_2(M, \mathbb{Z})$. The Poincaré duality implies that $\Lambda := \mathrm{NS}(M)/\mathrm{NS}(M)_{\mathrm{tor}}$ is a unimodular integral lattice, of rank ρ equal to the second Betti number $\mathrm{b}_2(M)$ of M. Because M is projective algebraic hence Kähler, $\mathrm{b}_1(M)$ is even, while Theorem 6.2.23(iii) with $\mathrm{p_g}(M) = 0$ implies that $\mathrm{b}_2^+(M) = 1$, and therefore the intersection form on Λ has the signature $(1, \rho - 1)$.

In Λ we have the image k of K_M under the homomorphism $\mathrm{pd}^{-1} \circ \mathrm{c}$ followed by the canonical projection $\pi : \mathrm{NS}(S) \to \mathrm{NS}(M)/\mathrm{NS}(M)_{\mathrm{tor}}$, and $c := \pi(\mathrm{H}(C))$

with C as above when $k \cdot c < 0$. We denote by $\Lambda^+ \subset \Lambda$ the semigroup of all homology classes modulo torsion of effective divisors, and Λ^{++} the semigroup of all $x \in \Lambda$ such that $x \cdot e \geq 0$ for every $e \in \Lambda^+$. Let h be the common homology class modulo torsion of the hyperplane sections H of a projective embeddding of M. If A is an irreducible curve in M and $a \in A$, then it follows from the first paragraph in Section 6.2.9 that there exists an irreducible hyperplane section H such that $a \in H$ and $H \neq A$, and hence $H \cdot C > 0$. Therefore $h \in \Lambda^+$ and $h \cdot e > 0$ for every image e in Λ of a nonzero effective divisor. In particular, $e \neq 0$ for every such e.

Let $\rho = \mathrm{b}_2(M) = 1$, and we will prove that M is isomorphic to $\mathbb{P}^2$. Because Λ is a positive definite rank-one unimodular lattice, there exists $l \in \Lambda$ such that $\Lambda = \mathbb{Z}\, l$ and $l \cdot l = 1$. There exist $\kappa,\ \gamma \in \mathbb{Z}$ such that $k = \kappa\, l$ and $c = \gamma\, l$. Because $0 > k \cdot c = (\kappa\, l) \cdot (\alpha\, l) = (\kappa\, \alpha)\, (l \cdot l) = \kappa\, \alpha$, both α and γ are nonzero and have opposite signs. Replacing l by $-l$ if necessary, we can arrange that $\gamma > 0$, and then $\kappa < 0$. It follows that $\mathrm{K}_M \cdot \mathrm{K}_M = (\kappa\, l) \cdot (\kappa\, l) = \kappa^2 > 0$. According to Theorem 6.2.23, where $\mathrm{b}_1(M)$ is even because M is projective algebraic hence Kähler, we have $\mathrm{b}_1(M) = 2\ \mathrm{q}(M)$ and $\dim \mathrm{H}^0(M,\ \Omega^1) =: \mathrm{h}^{1,\,0}(M) = \mathrm{q}(M)$. Because $\chi_{\mathrm{top}}(M) = 2 - 2\ \mathrm{b}_1(M) + \mathrm{b}_2(M) = 2 - 4\ \mathrm{q}(M) + 1$, it follows from (6.2.32) with $\chi(M,\ \mathcal{O}) = 1 - \mathrm{q}(M) + \mathrm{p}_{\mathrm{g}}(M) = 1 - \mathrm{q}(M)$ that $9 = \kappa^2 + 8\ \mathrm{q}(M)$. Because $\mathrm{q}(M) \in \mathbb{Z}_{\geq 0}$ and $\kappa^2 > 0$, the only possibilities are $\mathrm{q}(M) = 0$, $\kappa = -3$, and $\mathrm{q}(M) = 1$, $\kappa = -1$. We next prove that the second case does not occur.

Let $\mathrm{q}(M) = 1$. Then Theorem 6.2.23(iii) yields $\mathrm{h}^{1,\,0}(M) = 1$ and $\mathrm{b}_1(M) = 2$. There are a nonzero holomorphic complex one-form α on M such that $\mathrm{H}^0(M,\ \Omega^1) = \mathbb{C}\,\alpha$ and a $\mathbb{Z}$-basis $\gamma_1,\ \gamma_2$ of $\mathrm{H}_1(M,\ \mathbb{Z})/\,\mathrm{H}_1(M,\ \mathbb{Z})_{\mathrm{tor}}$, the images of which in $\mathrm{H}_1(M,\ \mathbb{R})$ form an $\mathbb{R}$-basis of $\mathrm{H}_1(S,\ \mathbb{R})$. The proof of Theorem 6.2.23 yields that the de Rham cohomology classes of $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\alpha$ are linearly independent. Hence the image P of $\mathrm{H}_1(M,\ \mathbb{Z})/\,\mathrm{H}_1(M,\ \mathbb{Z})_{\mathrm{tor}}$ under the homomorphism $\gamma \mapsto (\alpha \mapsto \int_\gamma \alpha)$ is a discrete subgroup of $\mathrm{H}^0(M,\ \Omega^1)^*$, and $\mathcal{A} := \mathrm{H}^0(M,\ \Omega^1)^*/P$ is a Hausdorff complex analytic manifold. Choose a base point $s^* \in S$. Then $s \mapsto (\alpha \mapsto \int_{s^*}^{s} \alpha)$ defines a holomorphic mapping $\mathfrak{a} : M \to \mathcal{A}$. Its derivative is equal to α, hence not identically zero, and therefore $\mathfrak{a}$ is not constant, hence a fibration. Let F be a fiber of $\mathfrak{a}$, a nonzero effective divisor such that $F \cdot F = 0$. Since the image f of F in Λ is nonzero, $f = \varphi\, l$ for a nonzero integer φ, while $0 = F \cdot F = f \cdot f = \varphi^2 > 0$, leads to a contradiction.

We therefore have $\mathrm{q}(M) = 0$ and $\kappa = -3$. Note that $\mathrm{q}(M) = \mathrm{p}_{\mathrm{g}}(M) = 0$ implies that $\chi(M,\ \mathcal{O}) = 1 - \mathrm{q}(M) + \mathrm{p}_{\mathrm{g}}(M) = 1$. Let L denote the holomorphic line bundle over M, unique up to isomorphism, such that $\mathrm{pd}^{-1} \circ \mathrm{c}(L)$ modulo torsion is equal to l, and write $\mathfrak{L} := \mathrm{H}^0(M,\ \mathcal{O}(L))$ for the space of all holomorphic sections of L. We have $h = \sigma\, l$ for some nonzero integer σ, and because $\sigma\, \gamma = H \cdot C > 0$ and $\gamma > 0$, we have $\sigma > 0$ and $\deg \iota_H^*(\mathrm{K}_M\ L^{-1}) = \sigma\, \kappa - \sigma = -4\, \sigma < 0$. Therefore the restriction to each such H of a holomorphic section of $\mathrm{K}_M\ L^{-1}$ is equal to zero, and because the union of these H's is dense in M it follows that $\mathrm{H}^0(M,\ \mathcal{O}(\mathrm{K}_M\ L^*)) = 0$, while (6.2.33) and (6.2.31) imply that $\dim \mathfrak{L} \geq \chi(M,\ \mathcal{O}) + (L \cdot L - \mathrm{K}_M \cdot L)/2 = 1 + (1 + 3)/2 = 3$. Let λ be a nonzero holomorphic section of L and $\mathfrak{l} = \mathrm{Div}(\lambda)$. There exist finitely many distinct irreducible curves C_j in M and $n_j \in \mathbb{Z}_{>0}$ such that $\mathfrak{l} = \sum_j n_j\, C_j$. For each j there is a unique $c_j \in \mathbb{Z}$ such that $\mathrm{H}(C_j) = c_j\, l$.

Since $H \cdot C_j > 0$, we have $c_j > 0$ for each j, while $l = \mathrm{H}(\mathfrak{l}) = \sum_j n_j\, \mathrm{H}(C_j) = (\sum_j n_j\, c_j)\, l$ modulo torsion implies $1 = \sum_j n_j\, c_j$, which in combination with $n_j,\ c_j \in \mathbb{Z}_{>0}$ implies that there is exactly one i, and $n_i = c_i = 1$. That is, $\mathfrak{l}$ is an irreducible curve. But then (6.2.8) yields $2\ \mathrm{vg}(\mathfrak{l}) - 2 = \mathfrak{l} \cdot \mathfrak{l} + \mathrm{K}_M \cdot \mathfrak{l} = 1 - 3 = -2$; hence $\mathrm{vg}(\mathfrak{l}) = 0$ and $\mathfrak{l}$ is an embedded complex projective line in M, and λ has only simple zeros. For each $p \in M$ the set of all $\lambda \in \mathfrak{L}$ such that $\lambda(p) = 0$ and $\mathrm{d}\lambda(p) = 0$ is a linear subspace of codimension ≤ 3 in $\mathfrak{L}$, and because this space is zero, it follows that $\dim \mathfrak{L} \leq 3$, which in combination with the previous inequality $\dim \mathfrak{L} \geq 3$ yields that $\dim \mathfrak{L} = 3$. Let λ and μ be nonzero holomorphic sections of $\mathfrak{L}$. If $\mathrm{Div}(\lambda) = \mathrm{Div}(\mu)$, then λ/μ extends to a holomorphic function c on M, while the maximum principle and the fact that M is compact and connected imply that c is a constant. Therefore, if λ and μ are linearly independent, we have $\mathrm{Div}(\lambda) \neq \mathrm{Div}(\mu)$, and because $\mathrm{Div}(\lambda) \cdot \mathrm{Div}(\mu) = L \cdot L = l \cdot l = 1$, it follows that $\mathrm{Div}(\lambda)$ and $\mathrm{Div}(\mu)$ intersect each other in a unique point, where the intersection is transversal. For each $p \in M$, $\kappa(p) := \{\lambda \in \mathfrak{L} \mid \lambda(p) = 0\}$ is a linear subspace of $\mathfrak{L}$ of codimension ≤ 1, and by the previous discussion not equal to $\mathfrak{L}$. This yields a holomorphic mapping κ from M to the space $\mathfrak{P}$ of all two-dimensional linear subspaces of $\mathfrak{L}$. We also have for each $P \in \mathfrak{P}$ a unique common zero $\zeta(P)$ of a basis $\lambda,\ \mu$ of P, which does not depend on the choice of the basis. This defines a holomorphic mapping $\zeta : \mathfrak{P} \to M$, $\zeta \circ \kappa$ is the identity in M, $\kappa \circ \zeta$ is the identity in $\mathfrak{P}$, and it follows that $\kappa : M \to \mathfrak{P}$ is a complex analytic diffeomorphism. Let θ be the mapping that assigns to each $P \in \mathfrak{P}$ the one-dimensional vector space of linear forms on $\mathfrak{L}$ that vanish on P. Then θ is an isomorphism from $\mathfrak{P}$ onto $\mathrm{P}(\mathfrak{L}^*) \simeq \mathbb{P}^2$. This completes the proof that M is complex analytic diffeomorphic to $\mathbb{P}^2$.

The remaining case is $\rho = \mathrm{b}_2(M) > 1$, and we will prove that in this case, M is a ruled surface. We will apply Lemma 6.1.6 in order to prove that M admits a fibration, for which we need a nonzero $f \in \Lambda^+ \cap \Lambda^{++}$ such that $f \cdot f = 0$. We will prove the existence of f such that in addition, $k \cdot f < 0$, and use such f in order to prove that M is a ruled surface. We begin by showing that Λ^+ is sufficiently rich. Let $x \in \Lambda$ satisfy $h \cdot x > h \cdot k$. There exists a divisor X in M and a holomorphic line bundle L over M such that $x = \mathrm{H}(X)$ and $[L] = \delta(X)$; see Lemma 2.1.2. Let μ be a holomorphic section of $\mathrm{K}_M \otimes L^*$. For every smooth hyperplane section H we have $\deg \iota_H{}^*(\mathrm{K}_M \otimes L^*) = h \cdot k - h \cdot x < 0$; hence the restriction to H of μ is equal to zero, and because the union of the H's is dense in M, it follows that $\mu = 0$. We have proved that $\mathrm{H}^0(M,\ \mathcal{O}(\mathrm{K}_M\ L^{-1})) = 0$, when (6.2.33) and (6.2.31) with $\chi(M,\ \mathcal{O}) = 1$ imply that $\dim \mathrm{H}^0(M,\ \mathcal{O}(L)) \geq \chi(M,\ \mathcal{O}) + (x \cdot x - k \cdot x)/2$, because $L \cdot L - \mathrm{K}_M \cdot L = x \cdot x - k \cdot x$. Since $x \in \Lambda^+$ if $\dim \mathrm{H}^0(M,\ \mathcal{O}(L)) > 0$, we obtain that $x \in \Lambda^+$ if $x \in \Lambda$, $h \cdot x > h \cdot k$ and $x \cdot x - k \cdot x > -2\,\chi(M,\ \mathcal{O})$.

The integral lattice Λ is embedded in the ρ-dimensional real vector space $V := \Lambda \otimes \mathbb{R} = \mathrm{H}_2(M,\ \mathbb{R})$, and the intersection form on Λ extends to a nondegenerate symmetric $\mathbb{R}$-bilinear form on V of signature $(1,\ \rho - 1)$. Because the intersection form on the ρ-dimensional vector space $V := \Lambda \otimes \mathbb{R}$ has signature $(1,\ \rho - 1)$, it is negative definite on the orthogonal complement $h^\perp = \{x \in V \mid h \cdot x = 0\}$ of h in V. It follows that $\{x \in V \mid x \cdot x \geq 0\}$ is the disjoint union of C^+, $\{0\}$, and $-C^+$, where C^+ is a cone over a solid ball in $h^\perp$ with respect to minus the intersection form. We

choose the component C^+ such that $h \in C^+$. Furthermore, for each nonzero $x \in V$ we have $x \in C^+$ if and only if $x \cdot y \geq 0$ for every $y \in C^+$. And $x \in C^+$ if and only if $x \cdot x \geq 0$ and $x \cdot y \geq 0$ for some $y \in C^+$. The last line of the previous paragraph implies that C^+ is contained in the closure in V of the set of all $(1/n)\, e$ such that $e \in \Lambda^+$ and $n \in \mathbb{Z}_{>0}$. If $x \in \Lambda^{++}$, then $x \cdot e \geq 0$; hence $x \cdot ((1/n)\, e) \geq 0$ for every $e \in \Lambda^+$ and $n \in \mathbb{Z}_{>0}$, while the continuity of the intersection form and the previous line imply that $x \cdot y \geq 0$ for every $y \in C^+$, hence $x \in C^+$. That is, $\Lambda^{++} \subset C^+$.

In the search for f, we will use an element $g \in \Lambda^+ \cap \Lambda^{++}$ such that $g \cdot g > 0$, g and k are linearly independent in V, and $k \cdot g < 0$. If h and c are not both proportional to k, then we can take $g = h + n\, c$ with n a sufficiently large integer, because $k \cdot c < 0$. In the rather special case that $k = \theta\, h = \gamma\, c$ for $\theta,\ \gamma \in \mathbb{R}$, we reason as follows. The inequalities $0 > k \cdot c = \gamma\, (c \cdot c)$ and $c \cdot c \geq 0$ imply $\gamma < 0$ and $c \cdot c > 0$, while $\theta\, (h \cdot c) = \gamma\, (c \cdot c)$ yields $\theta < 0$ because $h \cdot c > 0$. If there exists an irreducible curve E in M such that $E \cdot E < 0$, then the image e of E in M satisfies $k \cdot e = \theta\, (h \cdot e) \geq 0$, whereas $h \cdot e > 0$, a contradiction. Therefore the image in Λ of every irreducible curve in M belongs to C^+; hence $\Lambda \cap C^+ \subset \Lambda^{++}$. Because C^+ is contained in the closure in V of the set of all $(1/n)\, e$ such that $e \in \Lambda^+$ and $n \in \mathbb{Z}_{>0}$, there is a great abundance of $g \in \Lambda^+ \cap \Lambda^{++}$ such that $g \cdot g > 0$, g and k are linearly independent in V, and $k \cdot g < 0$.

For $\rho \in \mathbb{R}$, write $f(\rho) := (g + \rho\, k) \cdot (g + \rho\, k) = (k \cdot k)\, \rho^2 + 2\, (k \cdot g)\, \rho + g \cdot g$, where $g \cdot g > 0$ and $k \cdot g > 0$. If $k \cdot k \leq 0$, then f has a unique positive zero r, which is simple. If $k \cdot k > 0$, then the intersection form has signature $(1,\ 1)$ on $\mathbb{R}\, g + \mathbb{R}\, k$, hence the determinant $d = (g \cdot g)\, (k \cdot k) - (k \cdot g)^2$ of the intersection matrix of g and k is strictly negative, and it follows that f has two distinct real zeros, both positive because $f'(0) = 2\, (k \cdot g) < 0$. In this case we take r equal to the smallest zero of f. In both cases we write $n = g + r\, k$, when $n \cdot n = 0$ and $k \cdot n = (k \cdot k)\, r + (k \cdot g) = f'(r)/2 < 0$.

The next arguments will imply that $r \in \mathbb{Q}$, which is quite nontrivial in view of the square root of $-d$ in the formula for r when $k \cdot k \neq 0$. Using the continued fraction expansion of the real number r, see for instance Khinchin [104, §§1 and 2], we can find for each $\epsilon \in \mathbb{R}_{>0}$ a rational number p/q with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ arbitrarily large such that $r \leq p/q \leq r + \epsilon/q$. Consider $f := q\, g + p\, k \in \Lambda$, and let L be the holomorphic line bundle over M such that $[L] = \delta(F)$ for a divisor F in M with image in Λ equal to f. With the notation $\delta := p - q\, r$, we have $0 \leq \delta \leq \epsilon$ and $f = q\, g + p\, k = q\, n + \delta\, k$. Therefore $h \cdot f = q\, (h \cdot g) + \delta\, (h \cdot k)$, $k \cdot f = q\, (k \cdot n) + \delta\, (k \cdot k)$, $f \cdot f = 2\, q\, \delta\, (k \cdot n) + \delta^2\, (k \cdot k)$, and $f \cdot f - k \cdot f = q\, (1 - 2\, \delta)\, (-k \cdot n) - \delta\, (1 - \delta)\, (a \cdot a)$. If we take $\epsilon < 1/2$ and q sufficiently large, we have $h \cdot f > h \cdot k$ and $f \cdot f - k \cdot f \geq 4 - 2\, \chi(M,\ \mathcal{O})$, while the estimate of four paragraphs ago yields $\dim \mathrm{H}^0(M,\ \mathcal{O}(L)) \geq 2$. It follows that f is the image in Λ of an effective divsor F in M, when $F = \sum_i e_i\, E_i + \sum_j c_j\, C_j$, where $e_i,\ c_j \in \mathbb{Z}_{\geq 0}$, and the E_i and C_j are finitely many irreducible curves in M such that $E_i \cdot E_i < 0$ and $C_j \cdot C_j \geq 0$. Let D be any irreducible curve in M. If $D \cdot D \geq 0$, then $D \neq E_i$ hence $E_i \cdot D \geq 0$, $C_j \cdot D \geq 0$ if $C_j \neq D$ and $C_j \cdot D = D \cdot D \geq 0$ if $C_j = D$, and we conclude that $F \cdot D \geq 0$. On the other hand, if $D \cdot D < 0$, then $k \cdot \mathrm{H}(D) = \mathrm{K}_M \cdot D \geq 0$, $g \cdot \mathrm{H}(D) \geq 0$ because $g \in \Lambda^{++}$,

and therefore $f \cdot \mathrm{H}(D) = q\,(g \cdot \mathrm{H}(D)) + p\,(k \cdot \mathrm{H}(D)) \geq 0$, since $p \geq q\,r > 0$. It follows that $f \in \Lambda^{++}$, and because $\Lambda^{++} \subset C^+$ we conclude that $f \cdot f \geq 0$. Since $f \cdot f = \delta\,(2\,q\,(k \cdot n) + \delta\,(k \cdot k))$, where $k \cdot n < 0$ implies that $2\,q\,(k \cdot n) + \delta\,(k \cdot k) < 0$ if δ remains bounded and q is sufficiently large, we have $f \cdot f \leq 0$, hence $f \cdot f = 0$, $\delta = 0$, and $r = p/q$.

Since $L \cdot L = f \cdot f = 0$ and $\dim \mathrm{H}^0(M,\, \mathcal{O}(L)) \geq 2$, Lemma 6.1.6 implies that there is a two-dimensional linear subspace V of $\mathrm{H}^0(M,\, \mathcal{O}(L_{\mathrm{m}}))$ that defines a fibration $\kappa_V : M \to \mathrm{P}(V) \simeq \mathbb{P}^1$. Here $L_{\mathrm{m}} = L\,{L_{\mathrm{f}}}^{-1}$ denotes the moving part of L, where $L_{\mathrm{f}} \cdot L_{\mathrm{f}} = L \cdot L_{\mathrm{f}} = 0$ implies that $L \cdot L_{\mathrm{m}} = L_{\mathrm{m}} \cdot L_{\mathrm{m}} = 0$. Let f_{m} denote the image in Λ of $\mathrm{pd}^{-1} \circ \mathrm{c}(L_{\mathrm{m}})$. Then $f_{\mathrm{m}} \cdot f_{\mathrm{m}} = f \cdot f_{\mathrm{m}} = f \cdot f = 0$ implies that f_{m} and f are proportional. Because $\mathrm{H}^0(M,\, \mathcal{O}(L_{\mathrm{m}})) \neq \{0\}$, f_{m} is the image in Λ of a nonzero effective divisor D in M, when $h \cdot f_{\mathrm{m}} = H \cdot D > 0$. Because also $h \cdot f > 0$, we have $f_{\mathrm{m}} = \lambda\, f$, where $\lambda = (H \cdot D)/(h \cdot f) > 0$, which in turn implies $k \cdot f_{\mathrm{m}} = \lambda\,(k \cdot f) < 0$. Lemma 6.1.3 implies the existence of a fibration $\psi : M \to C$ with connected fibers A, while Lemma 6.1.2 implies that $A \cdot A = 0$, and $f_{\mathrm{m}} = d\,a$ if a denotes the image in Λ of the common homology class of the fibers of κ_V, and d is the number of connected components of each smnooth fiber of κ_V. It follows that $\mathrm{K}_M \cdot A = (k \cdot f_{\mathrm{m}})/d < 0$. If A is smooth hence irreducible, then (6.2.5) implies $2\,\mathrm{g}(A) - 2 = A \cdot A + \mathrm{K}_M \cdot A < 0$, hence $\mathrm{g}(A) = 0$ and A is an embedded complex projective line. That is, $\psi : M \to C$ exhibits M as a ruled surface. It is relatively minimal, since there are no -1 curves in M at all, which also excludes Σ_1, which has a -1 curve as a section.

(ii) $\Rightarrow$ (i). If $M = \mathbb{P}^2$ then $\mathrm{H}_2(M,\, \mathbb{Z}) = \mathbb{Z}\,l$, where l is the homology class of a complex projective line in $\mathbb{P}^1$ and $l \cdot l = 1$. See Griffiths and Harris [74, p. 60]. If $a \in \mathrm{H}_2(M,\, \mathbb{Z})$ then $a = \alpha\, l$ for some $\alpha \in \mathbb{Z}$; hence $a \cdot\ a = \alpha^2\,(l \cdot l) = \alpha^2 \geq 0$. Therefore there are no -1 curves in M. Lemma 4.1.1 implies that the anticanonical curves in $\mathbb{P}^2$ are the cubic curves in $\mathbb{P}^2$; hence $\mathrm{K}_M \cdot \mathfrak{l} = -\,\mathrm{K}_M^* \cdot \mathfrak{l} = -3$ for any complex projective line $\mathfrak{l}$ in M.

Let $\psi : M \to C$ be a relatively minimal ruled surface, in which case Theorem 6.2.6 implies that M is projective algebraic. If A a smooth fiber of ψ, then $\mathrm{g}(A) = 0$ and $A \cdot A = 0$, and therefore the adjunction formula (6.2.5) implies $\mathrm{K}_M \cdot A = -2$. If E is a -1 curve in M, then E is an embedded complex projective line not contained in a fiber, and $\psi|_E : E \to C$ is a branched covering. The Riemann–Hurwitz formula (2.3.18) yields that $0 \geq 1 + d\,(\mathrm{g}(C) - 1)$, where $d \in \mathbb{Z}_{>0}$ is the degree of the covering. Therefore $\mathrm{g}(C) = 0$ and $d = 1$, that is, $C \simeq \mathbb{P}^1$ and $\psi|_E : E \to \mathbb{P}^1$ is a complex analytic diffeomorphism, which implies that E is a holomorphic section of ψ. Therefore Theorem 6.2.8 implies that M is isomorphic to Σ_n with $n = -E \cdot E = 1$, hence $M \simeq \Sigma_1$. $\square$

6.2.4 Singular Fibers of an Elliptic Fibration

Let $\varphi : S \to C$ be a relatively minimal elliptic fibration. That is, we are in the situation of Section 6.2.2 with $g = 1$, with the right-hand side in (6.2.14) equal to zero.

Assume that S_c is singular and has only one irreducible component Θ, when $\Theta \cdot \Theta = 0$. In combination with (6.2.14) with $g = 1$ this yields that $\mathrm{vg}(\Theta) = 1$, hence $\mathrm{g}(\widetilde{\Theta}) = 0$ and $\widetilde{\Theta}$ is obtained from Θ by performing only one blowing up at the unique singular point of Θ, which is either a double point or a cusp. This leads to the case I_1 or II in Kodaira's list of singular fibers in Section 6.2.6.

Lemma 6.2.10 *If the fiber S_c has more than one irreducible component, then each irreducible component of S_c is a smoothly embedded complex projective line, with self-intersection number $\Theta\cdot\Theta = -2$. If S_c has two distinct irreducible components Θ, Θ', then $\mu_\Theta = \mu_{\Theta'}$ and $\Theta \cdot \Theta' = 2$. If S_c has more than two irreducible components, then these are disjoint, or intersect each other in only one point and transversally.*

Proof. Let $\Theta \in S_c^{\mathrm{irr}}$. For every $\Theta' \in S_c^{\mathrm{irr}}$ such that $\Theta' \neq \Theta$ we have $\Theta \cdot \Theta' \in \mathbb{Z}_{\geq 0}$, where $\Theta \cdot \Theta' \in \mathbb{Z}_{>0}$ if and only if Θ' intersects Θ. Because S_c is connected, see Lemma 3.2.1, there is at least one other irreducible component of S_c that intersects Θ, and it follows from (6.2.12) that $\Theta\cdot\Theta < 0$. If Θ is singular, then $\mathrm{vg}(\Theta) \in \mathbb{Z}_{>0}$, hence $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta > 0$. On the other hand, if Θ is smooth, then the assumption that the elliptic fibration is relatively minimal implies that we cannot have $\mathrm{g}(\Theta) = 0$ and $\Theta \cdot \Theta = -1$. Therefore, if Θ is smooth then either $\mathrm{g}(\Theta) > 0$, which is the case when $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta > 0$, or $\mathrm{g}(\Theta) = 0$ and $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta \geq 0$, with equality if and only if $\Theta \cdot \Theta = -2$. It now follows from (6.2.14) with $g = 1$ that the case $2\ \mathrm{vg}(\Theta) - 2 - \Theta \cdot \Theta > 0$ cannot occur for any $\Theta \in S_c^{\mathrm{irr}}$. That is, if $\Theta \in S_c^{\mathrm{irr}}$, then Θ is smooth, $\mathrm{g}(\Theta) = 0$, and $\Theta \cdot \Theta = -2$.

In the case of two irreducible components Θ, Θ', it follows from (6.2.12) that $2\mu_\Theta = \mu_{\Theta'}\, \Theta \cdot \Theta'$ and $2\mu_{\Theta'} = \mu_\Theta\, \Theta' \cdot \Theta$. Therefore $\mu_\Theta/\mu_{\Theta'} = \mu_{\Theta'}/\mu_\Theta$, and because the multiplicities are positive, it follows that $\mu_\Theta = \mu_{\Theta'}$, which in turn implies that $\Theta \cdot \Theta' = 2$.

Finally, assume that there are more than two irreducible components. If $\Theta,\ \Theta' \in S_c^{\mathrm{irr}}$, $\Theta \neq \Theta'$, and $\Theta \cdot \Theta' \geq 2$, then it follows from (6.2.12) that $\mu_{\Theta'} \leq \mu_\Theta$, and because of the symmetry of the intersection number we also have $\mu_\Theta \leq \mu_{\Theta'}$, that is, $\mu_\Theta = \mu_{\Theta'}$. However, then (6.2.12) implies that $\Theta \cdot \Theta'' = 0$ for every other irreducible component Θ'' of S_c, in contradiction to the fact that S_c is connected. Therefore $\Theta \cdot \Theta' \leq 1$ if $\Theta,\ \Theta' \in S_c^{\mathrm{irr}}$, and $\Theta \neq \Theta'$. □

In the sequel, let C^{red} denote the set of all $c \in C$ such that S_c has more than one irreducible component. Note that C^{red} is a subset of the finite set C^{sing} of all singular values of φ.

Let $r \in C^{\mathrm{red}}$. Then the matrix $\Theta \cdot \Theta'$, with $\Theta,\ \Theta' \in S_r^{\mathrm{irr}}$, is called the *intersection matrix* of S_r. The *intersection diagram* of S_r is the diagram whose vertices are the irreducible components of S_r, and two vertices are connected by an edge if the two irreducible components intersect each other. The fact that S_r is connected implies

that the intersection diagram is connected, which is equivalent to the fact that the intersection matrix is indecomposable. Using only the combinatorial information that the intersection matrix is an indecomposable symmetric matrix with integral coefficients, where all diagonal elements are equal to -2 and all off-diagonal elements are nonnegative, together with (6.2.12) where all μ_Θ's are strictly positive integers, Kodaira [109, pp. 567–571] proved that the diagrams $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(1)}$, $\mathrm{E}_l^{(1)}$ given below are the only possibilities for the intersection diagrams. The numbers attached to the vertices are the positive integers $\nu_\Theta := \mu_\Theta/m$, where m is the greatest common divisor of all the μ_Θ, $\Theta \in S_r^{\mathrm{irr}}$. See Section 6.2.5 for a discussion of the case that $m > 1$. In turn, the classification of the intersection diagrams leads to Kodaira's classification of the singular fibers, presented in Section 6.2.6.

Lemma 6.2.11 *Let $r \in C^{\mathrm{red}}$. Then the intersection matrix of S_r is negative semidefinite, with one-dimensional kernel spanned by the vector μ that assigns to each irreducible component Θ of S_r its multiplicity μ_Θ in S_r.*

Proof. Write $I = S_r^{\mathrm{irr}}$, the set of all vertices of the intersection diagram = indices of the intersection matrix. Because (6.2.12) means that $\Theta \mapsto \mu_\Theta$ belongs to the kernel of the intersection matrix, it suffices to exhibit a codimension-one linear subspace H of $\mathbb{R}^I$ on which the intersection form

$$\mathbb{R}^I \ni x \mapsto \sum_{\Theta,\,\Theta' \in I} x_\Theta\,(\Theta \cdot \Theta')\,x_{\Theta'}$$

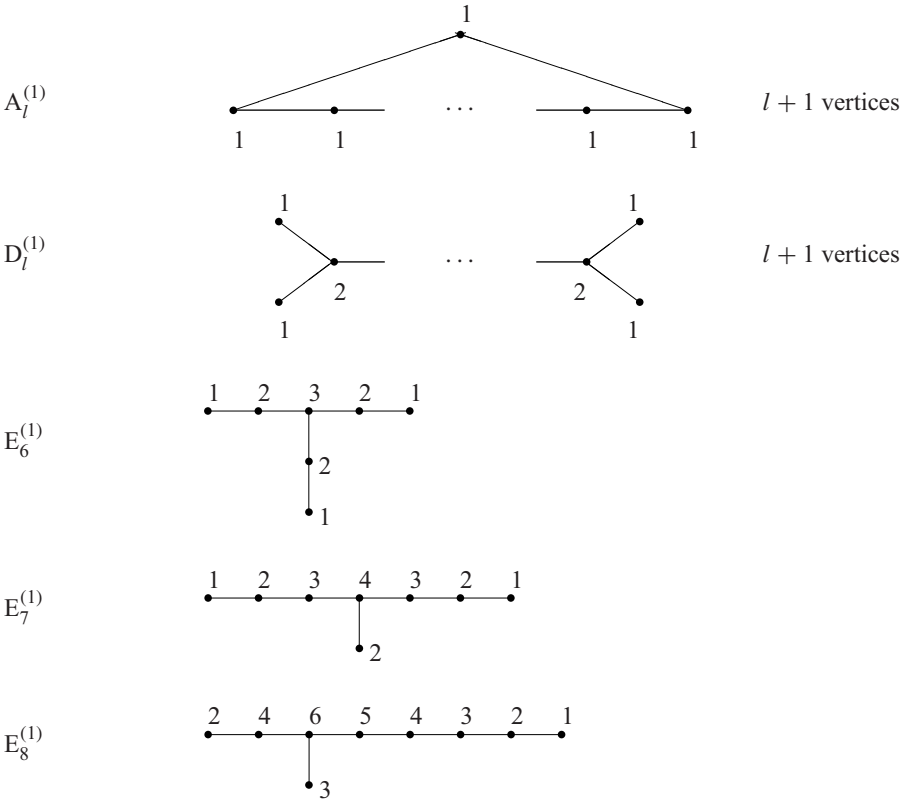
is negative definite. For each intersection diagram $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(l)}$, $\mathrm{E}_l^{(1)}$, we will choose one $\Theta \in I$ and take H equal to the subspace of all $x \in \mathbb{R}^I$ such that $x_\Theta = 0$. We then will prove that the intersection form is negative definite on H.

For the intersection diagram $\mathrm{A}_l^{(1)}$, we can remove any of the vertices, and then $-2\sum_{i=1}^{l} {x_i}^2 + 2\sum_{i=1}^{l-1} x_i\,x_{i+1} = -\sum_{i=1}^{l-1}(x_i - x_{i+1})^2 - {x_1}^2 - {x_l}^2$ shows that the intersection form is negative definite on H. The intersection diagram with one vertex removed is said to be of type A_l. For the intersection diagram $\mathrm{D}_l^{(1)}$, $l \geq 4$, we remove one of the end vertices, leading to an intersection diagram that is said to be of type D_l. In this case $-2\sum_{i=1}^{l} {x_i}^2 + 2\sum_{i=1}^{l-2} x_i\,x_{i+1} + 2\,x_{l-2}\,x_l = -\sum_{i=1}^{l-3}(x_i - x_{i+1})^2 - (x_{l-2} - 2\,x_{l-1})^2/2 - (x_{l-2} - 2\,x_l)^2/2 - {x_1}^2$ shows that the intersection form is negative definite on H. For the intersection diagrams $\mathrm{E}_l^{(1)}$, $l = 6,\ 7,\ 8$, we remove the vertex with three connections, resulting in a diagram that is equal to the union of three disjoint diagrams of type A_k. It follows that the intersection form on H is equal to the direct sum of three negative definite forms, and therefore is negative definite. □

Remark 6.2.12. A real matrix $A_{\Theta,\,\Theta'}$ is called a *generalized Cartan matrix* if all diagonal elements are equal to 2, all off-diagonal elements are ≤ 0, and $A_{\Theta\,\Theta'} = 0$ if and only if $A_{\Theta'\,\Theta} = 0$. See Kac [101, p. 1]. To the matrix A one assigns its *Dynkin diagram*, whose vertices are the indices, and two distinct vertices $\Theta,\ \Theta'$ are connected if and only if $A_{\Theta\,\Theta'} < 0$.

It follows from Lemma 6.2.10 that if $r \in C^{\mathrm{red}}$, then *minus* the intersection matrix of S_r is an indecomposable and symmetric generalized Cartan matrix A, where the intersection diagram of S_r is equal to the Dynkin diagram of A. Because A is positive semidefinite with one-dimensional kernel spanned by a vector with strictly positive coefficients, A is a generalized Cartan matrix of affine type; see Kac [101, Proposition 4.7]. According to the classification of Kac [100] and Moody [139] of the generalized Cartan matrices of affine type, see also Kac [101, Theorem 4.8(b) and Table Aff 1 on p. 54], the symmetric ones correspond to the Dynkin diagrams of type $\mathrm{A}_l^{(1)}$, $l \in \mathbb{Z}_{>0}$, $\mathrm{D}_l^{(1)}$, $l \in \mathbb{Z}_{>0}$, or $\mathrm{E}_l^{(1)}$, $l = 6, 7, 8$. We have attached the names of the Dynkin diagrams to our intersection diagrams.

In the enumeration in Section 6.2.6 of the irreducible components Θ_i of a reducible fiber, the vertices of the intersection diagram = Dynkin diagram, we have followed the enumeration of the roots α_i in Bourbaki [23, Planche I, IV, V, VI, VII]. If in the Dynkin diagram of type $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$ one deletes the vertex Θ_0, where $\nu_{\Theta_0} = 1$, and also deletes to edges connected to Θ_0, then one obtains the Dynkin diagram of the Cartan matrix of type A_l, D_l, E_l, respectively. Therefore the Dynkin diagrams of type $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$ are also called *extended Dynkin diagrams*.



6.2.5 *Multiple Singular Fibers*

Let $c_0 \in C^{\text{sing}}$, and let m denote the greatest common divisor of the multiplicities μ_Θ of the irreducible components Θ of S_{c_0}. If $m > 1$, then S_{c_0} is called a *multiple singular fiber*. As an intermezzo we discuss the elliptic fibration in a small open neighborhood of S_{c_0}. In particular, we describe an m-fold cover of this neighborhood that is an elliptic fibration without multiple singular fibers. This is a local version of the construction of Kodaira [109, pp. 571, 572].

If $s_0 \in S_{c_0}$ and $c \in C$ is close to c_0, then the connected smooth fiber S_c passes m times along S_{c_0} near s_0, and it follows that S_{c_0} admits an m-fold covering space. Because each singular fiber not of type I_b is simply connected, it follows that $S_{c_0} = m\,T$, where T is of type I_b. In this case the type of the multiple singular fiber S_{c_0} is denoted by ${}_m\,\mathrm{I}_b$. Note that T is an elliptic curve if $b = 0$.

Let z be a local coordinate in an open neighborhood C_0 of c_0 in C such that $z(c_0) = 0$. That is, replacing S, C, and φ by $\varphi^{-1}(C_0)$, $D = z(C_0)$, and $z \circ \varphi$, respectively, and shrinking D if necessary, we may assume that $\varphi : S \to D$ is a relatively minimal elliptic fibration over an open disk D around 0 in $\mathbb{C}$, where the fiber S_0 of φ over 0 is the only singular fiber, and is of type ${}_m\,\mathrm{I}_b$. Let E denote the set of all $\zeta \in \mathbb{C}$ such that $\mathrm{pow}(m)(\zeta) := \zeta^m \in D$. Then

$$R := \{(\zeta,\, s) \in E \times S \mid \varphi(s) = \zeta^m\}$$

is a complex analytic surface in $E \times S$, with $\{0\} \times S_0$ as its singular locus, where S_0 is the multiple singular fiber of type ${}_m\,\mathrm{I}_b$ of φ over the origin. See Section 6.2.13 for the definition and some basic facts about complex analytic spaces. Near any point $s_0 \in S_0$ the function φ is equal to the mth power of some holomorphic function h, and therefore near $(0,\, s_0)$ the complex analytic space R is equal to the union of the m smooth surfaces $h(s) = \omega\,\zeta$, where ω is an element of the multiplicative group $\mathrm{U}_m := \{\epsilon \in \mathbb{C} \mid \epsilon^m = 1\}$ of the complex mth roots of unity. It follows that there is a modification $\nu : N \to R$ in $\{0\} \times S_0$, where N is a smooth surface and the fiber of ν over each point of $\{0\} \times S_0$ consists of m points. In the terminology of Section 6.2.13, $\nu : N \to R$ is the normalization of R, and equal to the minimal resolution of singularities of R.

The restrictions to R of the projections $(\zeta,\, s) \mapsto \zeta$ and $(\zeta,\, s) \mapsto s$ are holomorphic mappings $p_1 : R \to E$ and $p_2 : R \to S$, respectively, such that $\varphi \circ p_2 = \mathrm{pow}(m) \circ p_1$. Write $\psi := p_1 \circ \nu : N \to E$. Furthermore, let N_0 and $R_0 = \{0\} \times S_0$ be the fibers in N and R of ψ and p_1 over 0, respectively. We have $\mathrm{pow}(m) \circ \psi = \varphi \circ p_2 \circ \nu$, and ν is a complex analytic diffeomorphism from $N \setminus N_0 = \nu^{-1}(R \setminus R_0)$ onto $R \setminus R_0$. It follows that $\Phi : p_2 \circ \nu : N \to S$ is a complex analytic m-fold covering map, which for each $\zeta \in E \setminus \{0\}$ defines a complex analytic diffeomorphism from the fiber N_ζ of ψ over ζ onto the fiber S_{ζ^m} of φ over ζ^m. This shows that $\psi : N \to E$ is an elliptic fibration, with $\mathrm{pow}(m) \circ \psi = \varphi \circ \Phi$.

Let $\gamma : N \to N$ be a deck transformation of the covering $\Phi : N \to S$, a complex analytic diffeomorphism of N such that $\Phi \circ \gamma = \Phi$. It follows that for every $\zeta \in E \setminus \{0\}$ there is a unique $\omega = \omega_\gamma(\zeta) \in \mathrm{U}_m$ such that γ maps N_ζ onto $N_{\omega\,\zeta}$. Because the function ω_γ is holomorphic and takes values in the discrete set U_m, it is a constant. Furthermore, $\Omega : \gamma \mapsto \omega_\gamma$ is a homomorphism from the group Γ of all deck transformations of Φ to $\mathrm{U}_m \simeq \mathbb{Z}/m\,\mathbb{Z}$. Let $\omega_\gamma = 1$. If $\zeta \in E \setminus \{0\}$, then γ preserves N_ζ, and it follows from $\Phi \circ \gamma = \Phi$ on N_γ and the injectivity of $\Phi|_{N_\gamma}$ that γ is the identity on N_γ. Because this holds for every $\zeta \in E \setminus \{0\}$, γ is the identity on the dense subset $\psi^{-1}(E \setminus \{0\})$ of N, and the continuity of γ implies that $\gamma = 1$. That is, the homomorphism Ω is injective. Because Γ acts transitively on each fiber of Φ, each of which contains m elements, the conclusion is that $\Omega : \Gamma \to \mathrm{U}_m$ is a

group isomorphism. In other words, $\Gamma \simeq \mathbb{Z}/m\,\mathbb{Z}$, and if γ is a generator of Γ, then ω_γ is a primitive mth root of unity, a generator of U_m.

The restriction Φ_0 of Φ to N_0 is a complex analytic m-fold covering map from the fiber N_0 of ψ over 0 onto the singular fiber S_0 of φ, and $\gamma_0 := \gamma|_{N_0}$ is an automorphism of N_0 of order m without fixed points. Note that N_0, because it is a fiber of the elliptic fibration $\psi : N \to E$, is connected. If $b = 0$, then N_0 is a smooth fiber of ψ, an elliptic curve, γ_0 is a nontrivial translation of order m on N_0, and Φ_0 induces an isomorphism from $N_0/\,\Gamma$ onto S_0. If $b > 0$, then S_0 is a cycle of b rational curves, and its fundamental group is generated by the homotopy class of a curve running along the cycle of the b rational curves. Its lift to N_0 closes only after running m times along S_0, and the conclusion is that N_0 is a cycle of $m\,b$ rational curves Θ_j, $j \in \mathbb{Z}/m\,b\,\mathbb{Z}$, where the corresponding deck transformation δ maps Θ_j to Θ_{j+b}. It follows that N_0 has Kodaira type equal to ${}_1\mathrm{I}_{m\,b} = \mathrm{I}_{m\,b}$. In particular, $\psi : N \to E$ is an elliptic fibration without multiple singular fibers. Because N can be contracted to N_0, the inclusion mapping from N_0 into N induces an isomorphism from the fundamental group of N_0 onto the fundamental group of N, and it follows that $\delta = \gamma_0$ for a generator γ of Γ as discussed above.

A global reduction of elliptic fibrations with multiple singular fibers to elliptic fibrations without multiple singular fibers has been given by Kodaira [109, Theorem 6.3]. For some more information about elliptic surfaces with multiple singular fibers, see Corollary 6.2.47. See Section 11.9.1 for an explicit example of Kodaira's reduction.

The aforementioned reduction is one of the reasons why one often assumes that the elliptic fibration has no multiple singular fibers. Another is that often the fibration admits a holomorphic section, when the following lemma implies that the fibration has no multiple singular fibers.

Lemma 6.2.13 *If the relatively minimal fibration $\varphi : S \to C$ admits a holomorphic section, then it has no multiple singular fibers.*

Proof. Let E be a holomorphic section of φ and let $c \in C$. Then E intersects S_c in one point $s \in S_c \cap S^{\mathrm{reg}}$ with multiplicity one. Therefore, if Θ is the irreducible component of S_c such that $s \in \Theta$, then $1 = S_c \cdot E = \mu_\Theta\,\Theta \cdot E = \mu_\Theta$. The equation $\mu_\Theta = 1$ implies that S_c is not a multiple fiber. □

6.2.6 Kodaira's Classification of Singular Fibers

Kodaira [109, II, Theorem 6.2] obtained the classification of the fibers of any relatively minimal elliptic fibration, listed below. The numbers are Kodaira's notations for the types. In comparison with the wide variety of singularities which arbitrary planar analytic curves can have, the singularities of the singular fibers of an elliptic fibration are remarkably simple.

I_0 Any smooth fiber, which is an elliptic curve. Its self-intersection number is 0.

I_1 An irreducible curve with a rational curve (= curve isomorphic to a complex projective line) as its desingularization, and an ordinary double point as its only singular point. Its self-intersection number is 0. The left-hand picture in Figure 2.3.2 shows two distinct affine real forms of a singular fiber of type I_1. One is the curve that has a hyperbolic singular point at $x = 1/2\sqrt{3}$, $y = 0$. The other has an isolated elliptic singular point at $x = -1/2\sqrt{3}$ and a second connected component that is smooth and intersects the x-axis at $x = 1/\sqrt{3}$.

I_b with $b \in \mathbb{Z}_{\geq 2}$. The fiber consists of a cycle of smooth rational curves Θ_i, $i \in \mathbb{Z}/b\,\mathbb{Z}$, each with self-intersection number -2. The fiber has b singular points s_i, $i \in \mathbb{Z}/b\,\mathbb{Z}$, where Θ_i intersects Θ_{i+1} transversally at s_i, and there are no other intersections. All the irreducible components occur with multiplicity 1. Intersection diagram: $\mathrm{A}_{b-1}^{(1)}$.

${}_m\mathrm{I}_b$ with $m \in \mathbb{Z}_{>1}$, $b \in \mathbb{Z}_{\geq 0}$, a fiber of type I_b with multiplicity m. See Section 6.2.5. Every point s is a singular point of the fibration φ in the sense that $\mathrm{T}_s\varphi = 0$, even if $b = 0$.

I_0^* The fiber is the union of one smooth rational curve Θ_2 that occurs with multiplicity 2 and four multiplicity 1 smooth rational curves Θ_0, Θ_1, Θ_3, Θ_4. All these irreducible components have self-intersection number -2. Each of the four Θ_i with $i \neq 2$ intersects α_2 at one point, where the four intersection points are distinct, and there are no other intersections. Intersection diagram: $\mathrm{D}_4^{(1)}$.

I_b^* with $b \in \mathbb{Z}_{>0}$. The fiber is the union of $b + 1$ multiplicity-2 smooth rational curves Θ_i, $2 \leq i \leq b + 2$, and four multiplicity-1 smooth rational curves Θ_0, Θ_1, Θ_{b+3}, Θ_{b+4}. All these irreducible components have self-intersection number -2. The Θ_i, $2 \leq i \leq b + 2$, form a chain in the sense that for each $2 \leq i \leq b + 1$ the curve Θ_i intersects Θ_{i+1} at one point and transversally. Θ_0 and Θ_1 both intersect Θ_2 at one point and transversally, whereas Θ_{b+3} and Θ_{b+4} intersect Θ_{b+2} at one point and transversally. There are no other intersection and all aforementioned intersection points are distinct. Intersection diagram: $\mathrm{D}_{b+4}^{(1)}$.

II An irreducible curve with one singular point p, which is an ordinary cusp point of the curve. The desingularization is a rational curve on which one point corresponds to the cusp point. The self-intersection number is equal to 0. The curve in the right-hand picture in Figure 2.3.2 that has a cusp singularity at the origin is the affine real form of a singular fiber of type II.

II* The union of nine smooth rational curves Θ_i, $0 \leq i \leq 8$, with multiplicities 1, 2, 3, 4, 6, 5, 4, 3, 2, respectively. All these irreducible components have self-intersection number -2. Θ_1 intersects Θ_3 intersects Θ_4 intersects Θ_5 intersects Θ_6 intersects Θ_7 intersects Θ_8 intersects Θ_0, whereas Θ_2 also intersects Θ_4. All these intersections have multiplicity 1, all the intersection points are distinct, and there are no other intersections. Intersection diagram: $\mathrm{E}_8^{(1)}$.

III The union of two smooth rational curves of multiplicity 1 that intersect each other at one point, with a second-order contact. The self-intersection number of each of the two irreducible components is equal to -2. Intersection diagram: $\mathrm{A}_1^{(1)}$.

III* The union of eight smooth rational curves Θ_i, $0 \le i \le 7$, with multiplicities 1, 2, 2, 3, 4, 3, 2, 1, respectively. Θ_0 intersects Θ_1 intersects Θ_3 intersects Θ_4 intersects Θ_5 intersects Θ_6 intersects Θ_7, whereas Θ_2 also intersects Θ_4. All these intersections have multiplicity 1, all the intersection points are distinct, and there are no other intersections. Intersection diagram: $\mathrm{E}_7^{(1)}$.

IV The union of three rational curves each of multiplicity 1, intersecting each other at a single point and with their tangent lines in general position. Each of the three irreducible components has self-intersection number -2, and there are no other intersections. Intersection diagram: $\mathrm{A}_2^{(1)}$.

IV* The union of seven smooth rational curves Θ_i, $0 \le i \le 6$, with multiplicities 1, 1, 2, 2, 3, 2, 1, respectively. All these irreducible components have self-intersection number -2. Θ_0 intersects Θ_2 intersects Θ_4, Θ_1 intersects Θ_3 intersects Θ_4, and Θ_6 intersects Θ_5 intersects Θ_4. All these intersections have multiplicity 1, all the intersection points are distinct, and there are no other intersections. Intersection diagram: $\mathrm{E}_6^{(1)}$.

The normal form of the monodromy around the singular fibers, see Table 6.2.40, the structure of the Lie group $\mathfrak{F}_c$ inherited from the translational group action on the elliptic fibers, see Theorem 6.3.29 below, and the behavior of the modulus function $J : C \to \mathbb{P}^1$ at the corresponding singular values of the fibration $\varphi : S \to C$ are listed in Kodaira [109, II, Table I after Theorem 9.1]. These appear in this book in Table 6.2.40 and the last part of Section 6.3.6.

6.2.7 The Bundle of Lie Algebras

In this subsection we assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration without multiple singular fibers.

For every $c \in C^{\mathrm{reg}}$, the space $\mathfrak{f}_c$ of all holomorphic vector fields on the elliptic curve S_c is complex one-dimensional, where its dual space $\mathfrak{f}_c^*$ is equal to the space of all holomorphic complex one-forms on S_c. Here the letter $\mathfrak{f}$ stands for "tangent to the fibers." The space $\mathfrak{f}_c$ is equal to the Lie algebra of the Lie group $\mathfrak{F}_c$ of all translations on the elliptic curve S_c, the identity component $\mathrm{Aut}(S_c)^{\mathrm{o}}$ of the group $\mathrm{Aut}(S_c)$ of all automorphisms of the fiber S_c over c. Note that for any $v \in \mathfrak{f}_c$ such that $v \neq 0$, the mapping $t \mapsto \mathrm{e}^{t\,v}$ induces an isomorphism from $\mathbb{C}/P$ onto G_c, where $P = P^{(v)}$ is the period group of the vector field v; see Section 2.3.1. The $\mathfrak{f}_c$, $c \in C^{\mathrm{reg}}$, form a complex line bundle over C^{reg}, which at this stage is only a set-theoretic bundle and defined only over C^{reg}. The next lemmas are a preparation for the definition in Theorem 6.2.18 of a holomorphic complex line bundle $\mathfrak{f}$ over C, called the *Lie algebra bundle of the elliptic fibration* $\varphi : S \to C$. For each $c \in C^{\mathrm{reg}}$, the fiber of $\mathfrak{f}$ over c is equal to $\mathfrak{f}_c$. That is, $\mathfrak{f}$ is an extension to C of the set-theoretic complex line bundle over C^{reg}, and provided with the natural structure of a holomorphic complex line bundle as described in Theorem 6.2.18.

The next lemma is based on the following *theorem of Kodaira and Spencer* [111, I, Theorem 2.1 and II, Theorem 18.1]. Let M and N be complex analytic manifolds and $\pi : M \to N$ a surjective proper complex analytic submersion with connected fibers. Let V be a holomorphic vector bundle over M. For each $n \in N$, the fiber $M_n := \pi^{-1}(\{n\})$ of π over n is a compact complex analytic manifold, the restriction $V_n := V|_{M_n}$ of V to M_n is a holomorphic vector bundle over M_n, and therefore the space $(\pi_* V)_n := \mathrm{H}^0(M_n, \mathcal{O}(V_n))$ of all holomorphic sections of V_n over M_n is a finite-dimensional vector space. The first statement is that the function $d : N \to \mathbb{Z}_{\geq 0} : n \mapsto d_n := \dim(\pi_* V)_n$ is upper semicontinuous in the sense that for every $n_0 \in N$ there is an open neighborhood N_0 of n_0 in N such that $d_n \leq d_{n_0}$ for every $n \in N_0$. Moreover, if the function d is constant, then there exist for each $n_0 \in N$ an open neighborhood N_0 of n_0 in N and d holomorphic sections v_j of V over $\pi^{-1}(N_0)$, $1 \leq j \leq d$, such that for each $n \in N$ the restrictions of the v_j to M_n form a basis of the vector space $(\pi_* V)_n$. Using the coordinates with respect to such bases as local trivializations, the vector spaces $(\pi_* V)_n$, $n \in N$, form a holomorphic vector bundle $\pi_* V$ over N, and $V \simeq \pi^*(\pi_* V)$.

The proof of the upper semicontinuity of the dimension and the statement that if d is constant, $\pi_* V$ is a C^∞ vector bundle over N of rank d, is based on the fiberwise ellipticity of the complex defined by the fiberwise Cauchy–Riemann operators. See Kodaira and Spencer [111, III]. Let d be constant. The proof in [111, II, Section 18, 19] of the existence of the holomorphic sections v_j of V over $\pi^{-1}(N_0)$ starts with the following observation. Let v be a C^∞ section v of $\pi_* V$, that is, a C^∞ section v of V such that for each $n \in N$ the restriction of v to M_n is holomorphic. Then the Cauchy–Riemann derivative $(\overline{\partial} v)_m$ of v at $m \in \pi^{-1}(N_0)$ is a well-defined complex antilinear mapping from $\mathrm{T}_m M$ to V_m, and equal to zero on the tangent space $\ker \mathrm{T}_m \pi$ of the fiber through m. It follows that for every $n \in N_0$ and $\nu \in \mathrm{T}_n N$ there is a unique $w \in (\pi_* V)_n$ such that $w_m = (\overline{\partial} v)_m(\mu)$ if $\mu \in \mathrm{T}_m M$ and $\mathrm{T}_m \pi(\mu) = \nu$. If we write $(\overline{\partial} v)_n(\nu) = w$, then this defines a Cauchy–Riemann operator $\overline{\partial}$ acting on C^∞ sections of $\pi_* V$. The proof of [111, II, Theorem 18.1] is completed by verifying that this Cauchy–Riemann operator is integrable, in the sense that for every $n_0 \in N$ and $v_0 \in (\pi_* V)_{n_0}$ there exist an open neighborhood N_0 of n_0 in N and a C^∞ section v of $\pi_* V$ over N_0 such that $\overline{\partial} v = 0$ and $v(n_0) = v_0$. Note that the theorem of Kodaira and Spencer implies that if d is constant, then the pullback to M of the holomorphic vector bundle $\pi_* V$ over N, see Section 2.1.4, is isomorphic to V. As a formula, $V \simeq \pi^*(\pi_* V)$.

Lemma 6.2.14 *Let C_0 be an open subset of C^{reg}, σ a holomorphic section over C_0 of φ, and w a holomorphic section of $\sigma^*(\ker \mathrm{T}\varphi)$. That is, for every $c \in C_0$, $w(c)$ is an element of the tangent space $\mathrm{T}_{\sigma(c)}(S_c) = \ker(\mathrm{T}_{\sigma(c)}\varphi)$ of S_c at $\sigma(c)$, depending holomorphically on c. Let v denote the unique vector field on $\varphi^{-1}(C_0)$ such that for each $c \in C_0$, $v|_{S_c} \in \mathfrak{f}_c$ and $v(\sigma(c)) = w(c)$. Then v is a holomorphic vector field on $\varphi^{-1}(C_0)$.*

Proof. The one-dimensional complex vector spaces $\ker \mathrm{T}_s \varphi$, $s \in S^{\mathrm{reg}}$, the tangent spaces of the fibers of φ, form a holomorphic line bundle $F := \ker \mathrm{T}\varphi$ over S^{reg}. The restriction of φ to $\pi^{-1}(C^{\mathrm{reg}})$ is a proper complex analytic surjective submersion

from $\pi^{-1}(C^{\mathrm{reg}})$ to C^{reg}. For each $c \in C^{\mathrm{reg}}$ the fiber S_c over c is an elliptic curve, which implies that the space $(\varphi_* F)_c$ of all holomorphic sections of $F|_{S_c}$, equal to the space of all holomorphic tangent vector fields of S_c, is one-dimensional. It therefore follows from the Kodaira–Spencer therorem that $\mathfrak{f} := \varphi_* F$ is a holomorphic line bundle over C^{reg}, and $F \simeq \varphi^* \mathfrak{f}$. Because $\varphi \circ \sigma$ is equal to the identity on C_0, it follows that $\sigma^* F \simeq \sigma^*(\varphi^* \mathfrak{f}) = (\varphi \circ \sigma)^* \mathfrak{f} = \mathfrak{f}|_{C_0}$. Lemma 2.1.3 therefore implies that φ^* defines a linear isomorphism from the space of holomorphic sections of $\sigma^* F \simeq \mathfrak{f}|_{C_0}$ onto the space of all holomorphic sections of F over $\varphi^{-1}(C_0)$. □

Lemma 6.2.15 *Let C_0 be an open subset of C^{reg} and let ζ be a holomorphic complex one-form on C_0 without zeros. For instance, $\zeta = \mathrm{d}z$ if $z : C_0 \to \mathbb{C}$ is a complex analytic coordinate function on C_0. Then the relation*

$$-\mathrm{i}_v\, \omega = \varphi^* \zeta \tag{6.2.15}$$

defines a bijection between the set of all holomorphic vector fields v on $\varphi^{-1}(C_0)$ without zeros and tangential to the fibers of φ, and the set of all holomorphic complex two-forms ω without zeros on $\varphi^{-1}(C_0)$.

Proof. Equation (6.2.15) means that

$$\omega_s(u,\, v(s)) = \zeta_{\varphi(s)}(\mathrm{T}_s \varphi(u)) \quad u \in \mathrm{T}_s\, S, \quad s \in \varphi^{-1}(C_0). \tag{6.2.16}$$

The mapping $(\omega,\, v) \mapsto (u \mapsto \omega(u,\, v))$ is a nondegenerate bilinear mapping from $\mathrm{K}_{S,\,s} \times F_s$ to ${N_s}^*$, where $\mathrm{K}_{S,\,s} = \Lambda^2\, (\mathrm{T}_s S)^*$ is the space of all antisymmetric complex bilinear forms on $\mathrm{T}_s S$, $F_s = \ker \mathrm{T}_s \varphi$, and ${N_s}^*$ is the space of all complex linear forms on $\mathrm{T}_s S$ that vanish on F_s. Note that each of the spaces $\mathrm{K}_{S,\,s}$, F_s, and ${N_s}^*$ is complex one-dimensional. Because $\zeta_s \neq 0$ and $\mathrm{T}_s \varphi \neq 0$, $\zeta_s \circ \mathrm{T}_s \varphi$ is a nonzero element of ${N_s}^*$. It follows that for each $s \in \varphi^{-1}(C_0)$, the equation in (6.2.16) induces a bijective inverse-linear relation between nonzero ω_s and nonzero $v(s)$, where the relation depends holomorphically on s. □

Remark 6.2.16. At first I thought that Lemma 6.2.14 was obvious. It is equivalent to the statement that for every $c \in C^{\mathrm{reg}}$ there are an open neighborhood C_0 of c in C^{reg} and a holomorphic vector field v without zeros on $\varphi^{-1}(C_0)$ and tangent to the fibers of φ. In view of Lemma 6.2.15 this is equivalent to the statement that for every $c \in C^{\mathrm{reg}}$ there is an open neighborhood C_0 of c in C^{reg} and a holomorphic complex two-form ω without zeros on $\varphi^{-1}(C_0)$. I found the latter statement in the first paragraph of the proof of Kodaira [109, II, Theorem 7.1], where the proof is based on the highly nontrivial formula [109, I, (2.16)], which in turn was proved under the assumption that S is compact. This made me realize that Lemma 6.2.14 needed a proof. ⊘

In the next lemma it is allowed that c_0 is a singular value of φ, and therefore its conclusion is much stronger. In the proof we will use the *direct image theorem of Grauert*, which we now explain. Let M be a complex analytic manifold and $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M. A sheaf $\mathcal{S}$ over M of $\mathcal{O}_M$-modules

is called *coherent* if it is locally finitely generated, and also, for each open subset M_0 of M and finitely many sections s_i, $1 \leq i \leq q$, of $\mathcal{S}$ over M_0, the sheaf of all $(f_1, \ldots, f_n) \in (\mathcal{O}_M)^q$ such that $f_1 s_1 + \cdots + f_q s_q = 0$ is locally finitely generated. It follows from Oka's theorem that for any holomorphic vector bundle K over M, the sheaf of germs of holomorphic sections of K is coherent. Let N be another complex analytic manifold and $\Phi : M \to N$ a complex analytic map. For every open subset N_0 of N, let $\Phi_*(\mathcal{S})_{N_0}$ denote the space of all holomorphic sections of $\mathcal{S}$ over $\Phi^{-1}(N_0)$. The $\Phi_*(\mathcal{S})_{N_0}$, where N_0 ranges over the open subsets of N, form a presheaf over N, and the corresponding sheaf over N is called the *direct image* $\Phi_*(\mathcal{S})$ of $\mathcal{S}$. It follows from the direct image theorem of Grauert [69, Theorem I], see also Bell and Narasimhan [14, Theorem 2.1] or Levy [120], that the sheaf $\Phi_*(\mathcal{S})$ over N is coherent if the sheaf $\mathcal{S}$ over M is coherent and the complex analytic mapping $\Phi : M \to N$ is proper.

Lemma 6.2.17 *For every $c_0 \in C$, there are an open neighborhood C_0 of c_0 in C and a holomorphic complex two-form ω on $\varphi^{-1}(C_0)$ such that ω has no zeros in $\varphi^{-1}(C_0)$.*

Proof. We first prove the existence of a holomorphic complex two-form ν on a sufficiently small open neighborhood of the fiber S_{c_0} of φ over c_0 such that ν is not identically equal to zero on any neighborhood of S_{c_0} in S.

The sheaf $\mathcal{O}(\mathrm{K}_S)$ of germs of holomorphic complex two-forms on S is equal to the sheaf of germs of holomorphic sections of the holomorphic line bundle K_S over S, and therefore coherent. Because $\varphi : S \to C$ is a proper complex analytic mapping, the direct image $\varphi_*(\mathcal{O}(\mathrm{K}_S))$ is a coherent sheaf over C. In particular, $\varphi_*(\mathcal{O}(\mathrm{K}_S))$ is locally finitely generated. Spelled out, this means that for every $c_0 \in C$ there are an open neighborhood C_0 of c_0 in C, which can be taken to be coneected, and finitely many holomorphic complex two-forms ω_i, $1 \leq i \leq q$, on $\varphi^{-1}(C_0)$ such that for every $c_1 \in C_0$, open neighborhood C_1 of c_1 in C_0, and holomorphic complex two-form ω on $\varphi^{-1}(C_1)$, there is an open neighborhood C_1' of c_1 in C_1 and there are holomorphic functions f_i, $1 \leq i \leq q$, on C_1' such that

$$\omega = \sum_{i=1}^{q} (f_i \circ \varphi)\, \omega_i \tag{6.2.17}$$

on $\varphi^{-1}(C_1')$. Because C^{reg} is dense in C, there exists $c_1 \in C_0 \cap C^{\mathrm{reg}}$, while it follows from Lemmas 6.2.14 and 6.2.15 that there exist an open neighborhood C_1 of c_1 in C_0 and a holomorphic complex two-form ω on $\varphi^{-1}(C_1)$ without zeros. Therefore (6.2.17) implies that for at least one i, ω_i is not identically equal to zero on $\varphi^{-1}(C_0)$. Because C_0 is connected and the fibers of φ are connected, $\varphi^{-1}(C_0)$ is connected, and it follows that the zero-set Z of ω_i is a nowhere dense closed analytic curve in $\varphi^{-1}(C_0)$. In particular, $\nu = \omega_i$ is the desired holomorphic complex area form.

Shrinking C_0 if necessary, there exists a holomorphic coordinate function $z : C_0 \to \mathbb{C}$. For each $c \in C^{\mathrm{reg}} \cap C_0$, the relative quotient $\vartheta = \nu / \mathrm{d}(z \circ \varphi)$ on $S_c = (z \circ \varphi)^{-1}(\{z(c)\})$ is a holomorphic complex one-form on the elliptic curve S_c, and therefore ϑ, hence, ν is either identically zero or has no zeros on S_c. It follows that the

zero-set Z of ν in $\varphi^{-1}(C^{\rm reg} \cap C_0)$ is a union of fibers. Because Z is an analytic curve, we can arrange, by shrinking C_0 if necessary, that $Z \subset S_{c_0}$. Therefore, viewed as a divisor, $Z = \sum_{\Theta \in S^{\rm irr}_{c_0}} \tau_\Theta \, \Theta$ for certain multiplicities $\tau_\Theta \in \mathbb{Z}_{\geq 0}$, where $Z = \tau \, S_{c_0}$ for some $\tau \in \mathbb{Z}_{\geq 0}$ if S_{c_0} is irreducible.

Assume that S_{c_0} is reducible, and let $\Theta' \in S^{\rm irr}$. In $\varphi^{-1}(C_0)$ we have $[{\rm K}_S] = \delta(Z)$, see Lemma 2.1.2, and it follows from (2.1.23) with $D = Z$ and $C = \Theta'$, and (6.2.8) with $A = \Theta'$, that

$$\sum_{\Theta \in S^{\rm irr}_{c_0}} \tau_\Theta \, \Theta \cdot \Theta' = Z \cdot \Theta' = {\rm K}_S \cdot \Theta' = 2 \, {\rm vg}(\Theta') - 2 - \Theta' \cdot \Theta'. \tag{6.2.18}$$

Lemma 6.2.10 implies that Θ' is a smoothly embedded $\mathbb{P}^1$, hence ${\rm vg}(\Theta') = 0$, and also that $\Theta' \cdot \Theta' = -2$. Therefore (6.2.18) is equal to zero, and because this holds for every $\Theta' \in S^{\rm irr}$, the vector $\Theta \mapsto \tau_\Theta$ belongs to the kernel of the intersection matrix of S_{c_0}. Therefore Lemma 6.2.11 implies the existence of an $m \in \mathbb{R}$ such that $\tau_\Theta = m \, \mu_\Theta$ for all $\Theta \in S^{\rm irr}_{c_0}$. The assumption that $\varphi : S \to C$ has no multiple singular fibers implies that there is a $\Theta \in S^{\rm irr}_{c_0}$ such that $\mu_\Theta = 1$, and therefore $m = \tau_\Theta \in \mathbb{Z}_{\geq 0}$. It follows that there exists an $m \in \mathbb{Z}_{\geq 0}$ such that $Z = m \, S_{c_0}$, whether S_{c_0} is irreducible or not. This means that $\nu = (z \circ \varphi)^m \, \omega$, where ω is a unit along S_{c_0}, that is, a holomorphic complex two-form without zeros on an open neighborhood of S_{c_0} in $\varphi^{-1}(C_0)$. Actually ω has no zeros in $\varphi^{-1}(C_0)$ because ν has no zeros in $\varphi^{-1}(C_0) \setminus S_{c_0}$. $\square$

Recall that $S^{\rm reg}$ is the set of all $s \in S$ such that ${\rm T}_s\varphi \neq 0$, when the fiber S_c of φ over $c = \varphi(s)$ is smooth at s and ${\rm T}_s S_c = \ker {\rm T}_s\varphi$. The $\ker {\rm T}_s\varphi$, $s \in S^{\rm reg}$, form a holomorphic complex line bundle $\ker {\rm T}\varphi$ over $S^{\rm reg}$ whose fiber over $s \in S^{\rm reg}$ is equal to the tangent space of the fiber of φ passing through the point s. For this reason a vector field v on S will be called *fiber-tangent* if $v(S) \subset \ker {\rm T}\varphi$. If $s \in S^{\rm sing}$ when ${\rm T}_s\varphi = 0$ and hence $\ker {\rm T}_s\varphi = {\rm T}_s S$, the condition $v \in \ker {\rm T}_s\varphi$ is no restriction. If C_0 is an open subset of C and $\sigma : C_0 \to S$ is a holomorphic section of φ over C_0, then the curve $\sigma(C_0)$ is contained in $S^{\rm reg}$ and $(\ker {\rm T}\varphi)|_{\sigma(C_0)}$ is a holomorphic complex line bundle over $\sigma(C_0)$.

Theorem 6.2.18 *The $\mathfrak{f}_c$, $c \in C^{\rm reg}$, extend to a unique holomorphic complex line bundle $\mathfrak{f}$ over C such that for each holomorphic section $\sigma : C_0 \to S$ of φ over an open subset C_0 of C, the mapping that assigns to each $v \in \mathfrak{f}_c$, $c \in C_0 \cap C^{\rm reg}$, its value $v(\sigma(c)) \in \ker {\rm T}_{\sigma(c)}\varphi$ extends to an isomorphism from $\mathfrak{f}|_{C_0}$ onto $(\ker {\rm T}\varphi)|_{\sigma(C_0)}$. For every open subset C_0 of C, the holomorphic sections v of $\mathfrak{f}$ over C_0 correspond bijectively to the fiber-tangent holomorphic vector fields on $\varphi^{-1}(C_0)$, which will be denoted by the same letter v.*

The mapping $(\omega, \, v) \mapsto \zeta$ defined by the equation $-{\rm i}_v \, \omega = \varphi^* \zeta$ induces an isomorphism of holomorphic complex line bundles from ${\rm K}_S \otimes \varphi^* \mathfrak{f}$ onto $\varphi^* \, {\rm K}_C$. The mapping $(\zeta, \, \xi) \mapsto \omega$ defined by

$$\omega(u,\, v) = \zeta(\mathrm{T}_s\varphi(u))\,\xi(v), \quad u \in \mathrm{T}_s\, S, \quad v \in \mathrm{T}_s(S_c), \tag{6.2.19}$$

induces an isomorphism $\iota : \varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*) \xrightarrow{\sim} \mathrm{K}_S$ *of holomorphic complex line bundles.*

Proof. If ω is a holomorphic complex two-form without zeros on $\varphi^{-1}(C_0)$ as in Lemma 6.2.17, and ζ is a holomorphic complex one-form without zeros on C_0, then the fiber-tangent holomorphic vector field v on $\varphi^{-1}(C_0)$ defined by $-\mathrm{i}_v\,\omega = \varphi^*\zeta$ corresponds to a trivialization of $\mathfrak{f}$ over C_0. These local trivializations define the holomorphic structure of $\mathfrak{f}$ with the desired properties.

Note that the zero-set of v is equal to $\varphi^{-1}(C_0) \cap S^{\mathrm{sing}}$. It is a bit paradoxical that this is exactly the set of all points s where the fiber-tangency condition $v(s) \in \ker \mathrm{T}_s\varphi$ is no restriction. □

If C is compact, or equivalently S is compact, then the degree of the holomorphic line bundle $\mathfrak{f}$ over the compact Riemann surface C is given in Theorem 6.2.31 below. The following remark will be used in the proof of Lemma 6.2.34.

Remark 6.2.19. If E is a nonsingular curve in S^{reg}, then the mappings

$$\ker \mathrm{T}_e\varphi \ni v \mapsto v + \mathrm{T}_e E \in \mathrm{T}_e S/\,\mathrm{T}_e\, E =: \mathrm{N}_S(E)_e, \quad e \in E,$$

induce an isomorphism of holomorphic complex line bundles over E from $(\ker \mathrm{T}\varphi)|_E$ onto $\mathrm{N}_S(E)$, the normal bundle of E in S. If $\sigma : C \to S$ is a holomorphic section of φ and $\sigma(C) = E$, then the isomorphism $\mathfrak{f} \xrightarrow{\sim} (\ker \mathrm{T}\varphi)|_E$ in Theorem 6.2.18, followed by the isomorphism $(\ker \mathrm{T}\varphi)|_E \xrightarrow{\sim} \mathrm{N}_S(E)$, is an isomorphism of holomorphic complex line bundles from $\mathfrak{f}$ onto $\mathrm{N}_S(E)$. ⊘

Remark 6.2.20. The holomorphic line bundle $\mathfrak{f}$ over C has been defined by Kodaira [109, III, p. 2] as the bundle of Lie algebras of the complex analytic system of Lie groups $B^{\#}$ defined in Kodaira [109, II, Theorem 9.1], where $B^{\#}$ is the bundle of Lie groups $\mathfrak{F}$ in Section 6.3.6. Since I found the statements about $\mathfrak{F}$ subtler than those about $\mathfrak{f}$, I have chosen to start with $\mathfrak{f}$ and use its properties to prove the statements about $\mathfrak{F}$.

The last statement in Theorem 6.2.18 is my understanding of Kodaira [109, III, Theorem 12.1], which says that $K = \Psi^*(\mathfrak{k} - \mathfrak{f})$ for a compact elliptic surface S with a holomorphic section. Here K, Ψ, and $\mathfrak{k}$ are our K_S, φ, and K_C, where Kodaira wrote sums instead of products of line bundles. Because the line bundles K_S and $\varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*)$ over S have different definitions, I interpreted Kodaira's equality sign as saying that K_S and $\varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*)$ are isomorphic as holomorphic complex line bundles, where the isomorphism is the mapping ι in Theorem 6.2.18. Note that two isomorphisms of holomorphic line bundles over S differ by a multiplicative factor that is a holomorphic function on S without zeros. If the connected surface S is compact, then it follows from the maximum principle that this factor is a nonzero constant, and there is not much freedom in the isomorphism between the holomorphic line bundles over S. ⊘

For any open subset C_0 of C, Lemma 2.1.3 together with the last statement in Theorem 6.2.18 imply that the mapping $\iota \circ \varphi^*$ is a linear isomorphism from the space of all holomorphic sections λ of $L = \mathrm{K}_C \otimes \mathfrak{f}^*$ over C_0 onto the space of all holomorphic complex two-forms on $\varphi^{-1}(C_0)$, where the zero-set of $\iota(\varphi^*(\lambda))$ in $\varphi^{-1}(C_0)$ is equal to the preimage under φ of the zero-set of λ in C_0. Because for every holomorphic line bundle L over C and every $c_0 \in C$ there are an open neighborhood C_0 of c_0 in C and a holomorphic section λ of L over C_0 without zeros, the last statement in Theorem 6.2.18 implies Lemma 6.2.17. Although presented here as a stepping stone to Theorem 6.2.18, I think that Lemma 6.2.17 is a useful observation in itself.

The following remarks discuss how elliptic surfaces can be viewed as integrable Hamiltonian systems, of one complex degree of freedom in Remark 6.2.21, and of two real degrees of freedom in Remark 6.2.22.

Remark 6.2.21. Let C_0 be an open subset of C, v a nowhere vanishing holomorphic section over C_0 of the Lie algebra bundle $\mathfrak{f}$, viewed as a holomorphic vector field on $\varphi^{-1}(C_0)$ as in Theorem 6.2.18, and $z : C_0 \to \mathbb{C}$ a coordinate function on C_0. Then $\mathrm{d}(z \circ \varphi) = \mathrm{d}(\varphi^* z) = \varphi^* \, \mathrm{d}z$, where $\zeta = \mathrm{d}z$ is a holomorphic complex one-form without zeros on C_0, and Theorem 6.2.18 implies that there is a unique holomorphic complex two-form ω without zeros on $\varphi^{-1}(C_0)$ such that

$$-\mathrm{i}_v \, \omega = \mathrm{d}(z \circ \varphi). \tag{6.2.20}$$

In other words, v is equal to the *complex Hamiltonian vector field* H_f^ω, defined by the holomorphic function $f := z \circ \varphi$, and with respect to the holomorphic complex symplectic form ω.

If w is an exterior two-vector field and α is a complex one-form, then the contraction $w \, \alpha$ of w and α can be defined as the vector field v such that $u \wedge v = \alpha(u) \, w$ for every vector field u. With this definition, equation (6.2.20) is equivalent to

$$v := w \, \mathrm{d}(z \circ \varphi) \tag{6.2.21}$$

if $w = 1/\omega$ is the meromorphic exterior two-vector field dual to the meromorphic complex two-form ω. If $c \in C \setminus C_0$, v has no pole at c, and ζ has a pole at c, then the meromorphic exterior two-vector field $w = 1/\omega$ is equal to zero along the fiber $\varphi^{-1}(\{c\})$, and the tangent vector field to Z^{reg} defined in Lemma 3.3.4 is equal to $\lambda \, v_c$, where λ is equal to the derivative of the vector field $1/\zeta$ on C at the point c where it is equal to zero.

The holomorphic section v of $\mathfrak{f}$ and the complex one-form $\mathrm{d}z$ can be chosen to have respective extensions to a meromorphic section v of $\mathfrak{f}$ and a meromorphic complex one-form ζ on C, with behavior of zeros and poles that is as simple as possible. Let $F \subset C$ denote the finite set of zeros and poles of v and ζ. Then the holomorphic complex area form ω on $\varphi^{-1}(C_0)$ has a meromorphic extension to S, which we denote also by ω, with zeros and poles only along the finitely many fibers over F. ⊘

Remark 6.2.22. The real and imaginary parts ω_1 and ω_2 of the holomorphic complex two-form ω on S_0 without zeros are both symplectic forms on the real four-dimensional manifold S_0. Furthermore, because the restriction of any complex two-form to any complex analytic curve is equal to zero, every complex analytic curve in S_0 is a real two-dimensional Lagrangian submanifold of S_0. These conclusions hold for any complex analytic surface. For our elliptic fibration $\varphi : S_0 \to C_0$, the nonsingular fibers, which are elliptic curves, are real two-dimensional Lagrangian tori.

The real and imaginary parts z_1 and z_2 of z are real coordinates on C_0, and $f_i = z_i \circ \varphi = \varphi^*(z_i)$, $i = 1, 2$, are real analytic functions on S_0 that define a real analytic fibration $f : S_0 \to \mathbb{R}^2$ of the real four-dimensional manifold, with respect to both symplectic forms ω_1 and ω_2 on S_0. If $\mathrm{H}_{f_j}^{\omega_i}$ denotes the Hamiltonian vector field on S_0 defined by the real symplectic form ω_i and the real-valued Hamiltonian function f_j on S_0, then we have

$$\mathrm{H}_f^{\omega} = \mathrm{H}_{f_1}^{\omega_1} = \mathrm{H}_{f_2}^{\omega_2}, \quad \text{and} \quad J\,\mathrm{H}_f^{\omega} = \mathrm{H}_{f_1}^{\omega_2} = -\,\mathrm{H}_{f_2}^{\omega_1}, \tag{6.2.22}$$

if J denotes the multiplication by i in the tangent spaces of S_0. Corresponding to the fact that the fibers of f are Lagrangian, (6.2.22) implies that the *Poisson brackets*

$$\{f_i,\ f_j\}^{\omega_k} := \omega_k(\mathrm{H}_{f_i}^{\omega_k},\ \mathrm{H}_{f_j}^{\omega_k}) = (\mathrm{d}f_j)(\mathrm{H}_{f_i}^{\omega_k})$$

of f_i and f_j with respect to the symplectic form ω_k are equal to zero, for every choice of i, j, and k. In this way the fibration $f : S_0 \to \mathbb{R}^2$ defines an *integrable bi-Hamiltonian system*.

For the spherical pendulum this occurs with $b = 1$ at the unstable equilibrium. In [48], not knowing about elliptic fibrations, I computed the monodromy matrix for the spherical pendulum to be conjugate to the monodromy matrix in Table 6.2.40 for type I_1.

For any $b \in \mathbb{Z}_{>0}$, Nguyên Tiên [153] studied general Lagrangian fibrations in a real 4-dimensional symplectic manifold near a cycle of b focus–focus singularities, where the singular fibration is similar to an elliptic fibration near a singular fiber of type I_b. He proved that the monodromy matrix is conjugate to the one in Table 6.2.40 for type I_b. Keeping the symplectic form fixed, Vũ Ngọc [201] gave a classification up to symplectic diffeomorphisms of Lagrangian fibrations in a real four-dimensional symplectic manifold near a cycle of b focus–focus singularities.

Elliptic fibrations provide a quite rich class of examples of integrable systems, where the list in Section 6.2.6 contains several types of singular fibers other than the I_b, $b > 0$, that is, the cycles of focus–focus singularities. However, it should be noted that the singularities of the fibrations into Lagrangian tori that arise in this way are quite special among the general fibrations into Lagrangian tori that can occur in real four-dimensional symplectic manifolds. For instance, near the stable equilibrium of the spherical pendulum, see for instance [48], the tori shrink to the equilibrium point, something that cannot happen in any complex analytic fibration with singular fibers. Also note that the real and imaginary parts of the holomorphic complex two-form ω

are symplectic forms that are very different from a Kähler form on S. With respect to a Kähler form each holomorphic curve in S is a real two-dimensional *symplectic* submanifold of S, instead of a Lagrangian submanifold.

If $\alpha \in \mathrm{Aut}(S)^+_\varphi$, then $\alpha_*(v) = v$, because α and $e^{t\,v}$ act as translations on each smooth fiber, and therefore commute. Because, moreover, $\varphi \circ \alpha = \alpha$, we have $\alpha^*(\mathrm{d}(z \circ \varphi)) = \mathrm{d}(z \circ \varphi)$, and because ω is defined in terms of v and $\mathrm{d}(z \circ \varphi)$, it follows that $\alpha^*(\omega) = \omega$. That is, α is a symplectic diffeomorphism with respect to both the real and the imaginary parts of ω. See Remark 9.2.9 for a choice of the vector field v and the affine coordinate z in the case of a rational elliptic surface, where the vector field has a simple pole along a single fiber, which, moreover, can be chosen at will.

In symplectic differential geometry, one is especially interested in real four-dimensional compact connected symplectic manifolds. In complex analytic geometry, examples of these are provided by compact connected nonsingular complex analytic complex two-dimensional surfaces S that carry a holomorphic complex two-form ω without zeros, where both the real and the imaginary parts of ω can be used to view S as a real four-dimensional compact symplectic manifold. These complex surfaces have been classified by Kodaira [110, I, Theorem 19]. For the subclass of the K3 surfaces, see Section 6.3.4. The K3 elliptic surfaces are explicitly constructed as the surfaces in Example 5 after Corollary 6.3.11 with $N = 2$. ⊘

6.2.8 Cohomology

For a complex vector space E of complex dimension n, let $E_{\mathbb{R}}$ denote E viewed as a real vector space of real dimension $2n$. Then, for any $r \in \mathbb{Z}_{\geq 0}$, we have the complex vector space $\Lambda^r E^*_{\mathbb{R}} \otimes \mathbb{C}$ of all complex-valued antisymmetric real r-linear forms ω on E, where the adjective "real" means that for each $1 \leq j \leq r$ the function $e_j \mapsto \omega(e_1, \ldots, e_{j-1}, e_j, e_{j+1}, \ldots, e_p)$ is a real linear mapping from $E_{\mathbb{R}}$ to $\mathbb{C} \simeq \mathbb{R}^2$. A complex-valued real linear form α on E, that is, $\alpha \in \Lambda^1 E^*_{\mathbb{R}} \otimes \mathbb{C} = E^*_{\mathbb{R}} \otimes \mathbb{C}$, is called *complex antilinear* if $\alpha(c\,e) = \overline{c}\,\alpha(e)$ for every $c \in \mathbb{C}$ and $e \in E$. The spaces of all complex linear and complex antilinear forms on E are denoted by $\Lambda^{(1,0)}E^* = E^*$ and $\Lambda^{(0,1)}E^*$, respectively, and it is easily verified that $E^*_{\mathbb{R}} \otimes \mathbb{C}$ is equal to the direct sum of its two complex linear subspaces $\Lambda^{(1,0)}E^*$ and $\Lambda^{(0,1)}E^*$. More generally, if $r = p + q$ with $p, q \in \mathbb{Z}_{\geq 0}$, then the space $\Lambda^{(p,q)}E^*$ of (p, q)-*forms* on E is defined as the complex linear subspace of $\Lambda^r E^*_{\mathbb{R}} \otimes \mathbb{C}$ spanned by all elements of the form $\alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \cdots \wedge \beta_q$, where $\alpha_i \in \Lambda^{(1,0)}E^*$ and $\beta_j \in \Lambda^{(0,1)}E^*$ for every $1 \leq i \leq p$ and $1 \leq j \leq q$, respectively. We have the direct sum decomposition

$$\Lambda^r E^*_{\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{(p,q)}E^*. \tag{6.2.23}$$

A section of the vector bundle over M of which $\Lambda^{(p,q)}(\mathrm{T}_m M)^*$ is the fiber over $m \in M$ is called a *differential form of type* (p, q), or a (p, q)-*form* for short, on

M. The direct sum decomposition (6.2.23) implies that every complex-valued real r-form has a unique sum decompostion in $(p,\ q)$-forms where p and q range over all combinations of nonnegative integers such that $r = p + q$.

Let M be a compact connected complex analytic manifold of complex dimension equal to n, and let L be a holomorphic vector bundle over M; see Section 2.1.3. Let $\mathcal{O}(L)$ denote the sheaf of germs of holomorphic sections of L. Let $\mathcal{D}^{(p,\,q)}$ and $(\mathcal{D}')^{(p,\,q)}$ denote the sheaf of germs of C^∞ and distributional $(p,\ q)$-forms on M. In the sequel $\mathcal{F} = \mathcal{D}$ or $\mathcal{F} = \mathcal{D}'$. If $\omega \in \mathcal{F}^{(p,\,q)}$, then $\mathrm{d}\,\omega = \partial\omega + \overline{\partial}\omega$, where $\partial\omega \in \mathcal{F}^{(p+1,\,q)}$ and $\overline{\partial}\omega \in \mathcal{F}^{(p,\,q+1)}$. This defines the *Cauchy–Riemann operator* $\overline{\partial}$, for which we have the exact sequence of sheaves

$$0 \to \Omega^p(L) \to \mathcal{F}^{(p,\,0)} \otimes \mathcal{O}(L) \overset{\overline{\partial}}{\to} \mathcal{F}^{(p,\,1)} \otimes \mathcal{O}(L) \overset{\overline{\partial}}{\to} \mathcal{F}^{(p,\,2)} \otimes \mathcal{O}(L) \overset{\overline{\partial}}{\to} \cdots,$$

called the *Dolbeault complex*. Here $\Omega^p(L) := \Omega^p \otimes \mathcal{O}(L)$ is the sheaf of germs of holomorphic L-valued complex p-forms on M, where Ω^p is the sheaf of germs of holomorphic complex p-forms on M. Because all the sheaves $\mathcal{F}^{(p,\,1)} \otimes \mathcal{O}(L)$ are fine, we have the canonical isomorphism

$$\mathrm{H}^q(M,\ \Omega^p(L)) \simeq \mathrm{H}_{\overline{\partial}}(M,\ \mathcal{F}^{(p,\,q)} \otimes \mathcal{O}(L)), \tag{6.2.24}$$

where the right-hand side is defined as the kernel of $\overline{\partial}$ in $\mathrm{H}^0(M,\ \mathcal{F}^{(p,\,q)} \otimes \mathcal{O}(L))$ modulo the image under $\overline{\partial}$ of $\mathrm{H}^0(M,\ \mathcal{F}^{(p,\,q-1)} \otimes \mathcal{O}(L))$. Because the Dolbeault complex is elliptic and M is compact, the image of $\overline{\partial}$ is a closed linear subspace of finite codimension in the kernel of $\overline{\partial}$, see Hörmander [91, Section 19.4]; hence all the cohomology groups $\mathrm{H}^q(M,\ \Omega^p(L))$ are finite-dimensional as well.

On the other hand, if $\alpha \in \mathcal{D}^{(p,\,q)}$, $\lambda \in \mathcal{O}(L)$ and $\beta \in (\mathcal{D}')^{(p',\,q')}$, $\mu \in \mathcal{O}(L^*)$, then

$$(\alpha \otimes \lambda,\ \beta \otimes \mu) \mapsto \langle \lambda,\ \mu \rangle\, \alpha \wedge \beta \in (\mathcal{D}')^{(p+p',\,q+q')}$$

defines a $(\mathcal{D}')^{(p+p',\,q+q')}$-valued $\mathcal{O}$-bilinear form on $(\mathcal{D}^{(p,\,q)} \otimes \mathcal{O}(L)) \times ((\mathcal{D}')^{(p',\,q')} \otimes \mathcal{O}(L^*))$. With $p' = n - p$, $q' = n - q$, integration of the distributional $(n,\ n)$-form over M yields a complex bilinear form on $\mathrm{H}^0(M, \mathcal{D}^{(p,\,q)} \otimes \mathcal{O}(L)) \times \mathrm{H}^0(M\, \mathcal{D}')^{(n-p,\,n-q)} \otimes \mathcal{O}(L^*))$, which is nondegenerate in the sense that it defines a canonical isomorphism from $\mathrm{H}^0(M,\ \mathcal{D}^{(p,\,q)} \otimes \mathcal{O}(L))$ onto the topological dual of $\mathrm{H}^0(M,\ \mathcal{D}')^{(n-p,\,n-q)} \otimes \mathcal{O}(L^*))$. Because the transpose of the continuous linear operator

$$\overline{\partial} : \mathrm{H}^0(M,\ \mathcal{F}^{(p,\,q)} \otimes \mathcal{O}(L)) \to \mathrm{H}^0(M,\ \mathcal{F}^{(p,\,q+1)} \otimes \mathcal{O}(L))$$

is equal to $(-1)^{p+q+1}\,\overline{\partial}$, this leads to the *Serre duality theorem*

$$\mathrm{H}^q(M,\ \Omega^p(L)) \simeq \mathrm{H}^{n-q}(M,\ \Omega^{n-p}(L^*))^*, \tag{6.2.25}$$

see Serre [179]. Griffiths and Harris [74, p. 102, 153] called it the *Kodaira–Serre duality theorem*. Here $\mathrm{H}^q(M,\ \Omega^p(L))$ is finite-dimensional for every $p,\ q \in \mathbb{Z}_{\geq 0}$, and equal to zero when $p > n$ or $q > n$. If $n = 2$ when $M = S$ is a compact complex analytic surface, and $p = 0$, $q = 2$, the Serre duality is

$$\mathrm{H}^2(S,\ \mathcal{O}(L)) \simeq \mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S \otimes L^*))^*. \tag{6.2.26}$$

If $L = \mathbb{C}$ is the trivial line bundle over S, then

$$\mathrm{H}^2(S,\ \mathcal{O}) \simeq \mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S))^* = \mathrm{H}^0(S,\ \Omega^2)^*, \tag{6.2.27}$$

where $\mathcal{O}$ and Ω^2 denote the sheaves of germs of holomorphic functions and complex two-forms on S, respectively. The exact sequence (2.1.8) leads in combination with the Serre duality (6.2.27) to

$$\mathrm{c}(\mathrm{Pic}(S)) = \{a \in \mathrm{H}^2(S,\ \mathbb{Z}) \mid \omega \cdot a = 0 \quad \forall \omega \in \mathrm{H}^0(S,\ \Omega^2)\}. \tag{6.2.28}$$

Here $\omega \cdot a = \int_S \omega \wedge a = \int_{\mathrm{i}^{-1}(a)} \omega$, where in the first integral a and ω are identified with their images in the de Rham cohomology group $\simeq \mathrm{H}^2(S,\ \mathbb{C})$, and $\mathrm{pd}^{-1}(a) \in \mathrm{H}_2(S,\ \mathbb{Z})$ is the homology class corresponding to a by means of the Poincaré duality (2.1.17). It is also used that every holomorphic (2, 0)-form ω is closed, because $\overline{\partial}\omega = 0$ and $\partial\omega \in \mathcal{F}^{(3,\,0)} = 0$ because $3 > 2 = \dim S$.

The alternating sum

$$\chi(M,\ \mathcal{O}(L)) := \sum_{q=0}^{n} (-1)^q\ \dim \mathrm{H}^q(M,\ \mathcal{O}(L)) \tag{6.2.29}$$

is called the *holomorphic Euler number of the vector bundle* L. If $L = \mathbb{C}$ is the trivial line bundle over M, then

$$\chi(M,\ \mathcal{O}) := \sum_{q=0}^{n} (-1)^q\ \dim \mathrm{H}^q(M,\ \mathcal{O}_M) \tag{6.2.30}$$

is called the *holomorphic Euler number of the manifold* M. A basic aspect of the holomorphic Euler number is that there is a formula for it in terms of characteristic classes of the vector bundle L and the tangent bundle $\mathrm{T}M$ of M, called the *Riemann–Roch formula for holomorphic vector bundles over compact complex analytic manifolds*. This formula is due to Hirzebruch [87, Theorem 25.3.1] when M is projective algebraic. It has been generalized to arbirary compact complex analytic manifolds by Atiyah and Singer [9, Section 4], by applying their *index formula for elliptic linear partial differential operators* acting on sections of vector bundles over compact manifolds to the Dolbeault complex. If L is a holomorphic line bundle over a compact complex analytic surface S when $n = 2$, the Riemann–Roch formula states that $\chi(\mathcal{O}(L))$ is equal to the integral over S of $(f \wedge f + f \wedge c_1)/2 + (c_1 \wedge c_1 + c_2)/12$, where $f = \mathrm{c}(L)$ is the first Chern class of L and c_i is the ith Chern class of $\mathrm{T}S$; see Hirzebruch [87, p. 5]. Note that $\chi(S,\ \mathcal{O}(L)) = (c_1 \wedge c_1 + c_2)/12 = \chi(S,\ \mathcal{O})$ if $\mathrm{c}(L) = 0$, that is, if L is topologically trivial. According to Griffiths and Harris [74, pp. 414, 416] we have $c_1 = \mathrm{c}(\Lambda^2\,\mathrm{T}S) = \mathrm{c}(\mathrm{K}_S^*)$; hence $\int_S c_1 \wedge c_1 = \mathrm{K}_S^* \cdot \mathrm{K}_S^* = \mathrm{K}_S \cdot \mathrm{K}_S$ and $c_2 = \chi_{\mathrm{top}}(S)$, the topological Euler number (6.2.38) of S. Therefore

$$\chi(S,\,\mathcal{O}(L)) = \chi(S,\,\mathcal{O}) + (L\cdot L - \mathrm{K}_S\cdot L)/2, \tag{6.2.31}$$

for a holomorphic line bundle L over a compact complex analytic surface S, where

$$\chi(S,\,\mathcal{O}) = (\mathrm{K}_S\cdot\mathrm{K}_S + \chi_{\mathrm{top}}(S))/12, \tag{6.2.32}$$

and we have used the definition (2.1.26) of the intersection number of line bundles. The proof of (6.2.31), called the *Riemann–Roch formula for surfaces*, is simpler if S is projective algebraic; see Griffiths and Harris [74, pp. 471, 472]. Equation (6.2.32) is called *Noether's formula*, a bit puzzling, since in Noether's time the holomorphic and topological Euler numbers did not yet exist. The relation between (6.2.32) and the work of Noether and his contempories is explained in Piene [159, Section 2]. For a proof of (6.2.32) for projective algebraic S that is close in spirit to Noether's work, see Piene [159, Section 1] or Griffiths and Harris [74, pp. 600–628]. The definition (6.2.29) for $M = S$ and $n = 2$, in combination with (6.2.26) and the observation that $\dim \mathrm{H}^1(S,\,\mathcal{O}) \geq 0$, leads to the inequality

$$\dim \mathrm{H}^0(S,\,\mathcal{O}(L)) + \dim \mathrm{H}^0(S,\,\mathcal{O}(\mathrm{K}_S\otimes L^*)) \geq \chi(S,\,\mathcal{O}(L)). \tag{6.2.33}$$

Kodaira [110, I, p.752] defined the *irregularity* and *geometric genus* of a compact complex analytic surface as $\mathrm{q}(S) := \dim \mathrm{H}^1(S,\,\mathcal{O})$ and $\mathrm{p_g}(S) := \dim \mathrm{H}^2(S,\,\mathcal{O})$, respectively. With this notation, Noether's formula (6.2.32) takes the form $\mathrm{K}_S\cdot\mathrm{K}_S + 2 - 2\,\mathrm{b}_1(S) + \mathrm{b}_2(S) = 12\,(1 - \mathrm{q}(S) + \mathrm{p_g}(S))$, where $\mathrm{b_k}(S) := \operatorname{rank} \mathrm{H}_k(S,\,\mathbb{Z})$ denotes the kth *Betti number* of S, and we have used that $\mathrm{b}_0(S) = \mathrm{b}_4(S) = 1$ and $\mathrm{b}_1(S) = \mathrm{b}_3(S)$. A compact complex analytic surface S is a real four-dimensional compact oriented manifold. It follows from Poincaré duality that the intersection form is a nondegenerate symmetric bilinear form on the $\mathrm{b}_2(S)$-dimensional real vector space $\mathrm{H}_2(S,\,\mathbb{R})$; hence $\mathrm{b}_2(S) = \mathrm{b}_2^+(S) + \mathrm{b}_2^-(S)$, where $\mathrm{b}_2^+(S)$ and $\mathrm{b}_2^-(S)$ denote the maximal dimensions of a linear subspace on which the intersection form is positive and negative definite, respectively. For any real four-dimensional compact oriented manifold S, Hirzebruch [87, Theorem 8.2.2 and pp. 12, 11] obtained a formula

$$3\,(\mathrm{b}_2^+(S) - \mathrm{b}_2^-(S)) = \mathrm{K}_S\cdot\mathrm{K}_S - 2\,\chi_{\mathrm{top}}(S) \tag{6.2.34}$$

for the index $\mathrm{b}_2^+(S) - \mathrm{b}_2^-(S)$, where $\mathrm{K}_S\cdot\mathrm{K}_S = \mathrm{K}_S^*\cdot\mathrm{K}_S^* = \int_S c_1\wedge c_1$ and c_1 is the first Chern class of the tangent bundle of S. This leads to the following *theorem of Kodaira on numerical invariants of compact complex analytic surfaces*, where $\mathrm{h}^{1,\,0}(S) := \dim \mathrm{H}^0(S,\,\Omega^1)$. It implies that the complex analytic invariants $\mathrm{q}(S)$ and $\mathrm{h}^{1,\,0}(S)$ are determined by the homological invariant $\mathrm{b}_1(S)$, and the complex analytic invariant $\mathrm{p_g}(S)$ is determined by the homological invariants $\mathrm{b}_1(S)$ and $\mathrm{b}_2^+(S)$ of the surface S.

Theorem 6.2.23 *Let S be a compact complex analytic surface. Then*

(i) *The Hermitian form $(\omega,\,\nu) \mapsto \omega\cdot\overline{\nu}$ is positive definite on $\mathrm{H}^0(S,\,\mathcal{F}^{(2,\,0)} \oplus \mathcal{F}^{(0,\,2)})$.*

(ii) *Every holomorphic one-form on S is closed, and we have the exact sequence* $0 \to \mathrm{H}^0(S,\, \Omega^1) \to \mathrm{H}^1(S,\, \mathbb{C}) \to \mathrm{H}^1(S,\, \mathcal{O}) \to 0$.

(iii) *If* $\mathrm{b}_1(S)$ *is even, then* $\mathrm{b}_1(S) = 2\ \mathrm{q}(S)$, $\mathrm{h}^{1,\,0}(S) = \mathrm{q}(S)$, *and* $\mathrm{b}_2^+(S) = 2\ \mathrm{p_g}(S) + 1$.

(iv) *If* $\mathrm{b}_1(S)$ *is odd, then* $\mathrm{b}_1(S) = 2\ \mathrm{q}(S) - 1$, $\mathrm{h}^{1,\,0}(S) = \mathrm{q}(S) - 1$, *and* $\mathrm{b}_2^+(S) = 2\ \mathrm{p_g}(S)$.

Proof. The complex structure on S defines, for each $s \in S$, a canonical orientation of the real part of the complex one-dimensional vector space $\Lambda^{(2,\,2)}\,\mathrm{T}_s^*\,S$, with respect to which the Hermitian form $(\omega,\ \nu) \mapsto \omega \wedge \overline{\nu}$ is positive definite on $\Lambda^{(2,\,0)}\,\mathrm{T}_s^*\,S \oplus \Lambda^{(0,\,2)}\,\mathrm{T}_s^*\,S$. Therefore, if $\omega \in \mathrm{H}^0(S,\, \mathcal{F}^{(2,\,0)} \oplus \mathcal{F}^{(0,\,2)})$, then $\omega \cdot \overline{\omega} = \int_S \omega \wedge \overline{\omega} \geq 0$, with equality if and only if $\omega = 0$. This proves (i). If α is a holomorphic one-form on S then $\omega := \mathrm{d}\alpha = \partial\alpha$ is a $(2,\ 0)$-form, $\int_S \omega \wedge \overline{\omega} = \int_S \mathrm{d}\alpha \wedge \mathrm{d}\,\overline{\alpha} = \int_S \mathrm{d}(\alpha \wedge \mathrm{d}\overline{\alpha}) = 0$, and hence $\omega = 0$ in view of (i). This proves the first statement in (ii).

Substracting (6.2.34) from Noether's formula $12\,(1 - \mathrm{q}(S) + \mathrm{p_g}(S)) = \mathrm{K}_S \cdot \mathrm{K}_S + \chi_{\mathrm{top}}(S)$ leads to $2\ \mathrm{q}(S) - \mathrm{b}_1(S) + \mathrm{b}_2^+(S) - 2\ \mathrm{p_g}(S) = 1$. Let Ω^1_{cl} denote the sheaf of germs of closed holomorphic one-forms on S. The short exact sequence $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{\mathrm{d}} \Omega^1_{\mathrm{cl}} \to 0$, in combination with the observation that the canonical homomorphism $\mathrm{H}^0(S,\, \mathbb{C}) \to \mathrm{H}^0(S,\, \mathcal{O})$ is an isomorphism of complex one-dimensional vector spaces, leads to the long exact sequence $0 \to \mathrm{H}^0(S,\, \Omega^1) = \mathrm{H}^0(S,\, \Omega^1_{\mathrm{cl}}) \to \mathrm{H}^1(S,\, \mathbb{C}) \to \mathrm{H}^1(S,\, \mathcal{O})$, and therefore $\mathrm{b}_1(S) \leq \mathrm{h}^{1,\,0}(S) + \mathrm{q}(S)$. If $\alpha,\ \beta \in \mathrm{H}^0(S,\, \Omega^1) = \mathrm{H}^0(S,\, \Omega^1_{\mathrm{cl}})$ satisfy $\alpha + \overline{\beta} = \mathrm{d}f$ for a C^∞ complex-valued function f on S, then $\alpha = \partial f$, $\overline{\beta} = \overline{\partial}\, f$, hence $\overline{\partial}\partial f = \overline{\partial}\alpha = 0$, while the maximum principle for harmonic functions implies that f is a constant, $\alpha + \overline{\beta} = 0$, and therefore $\alpha = 0$ and $\beta = 0$. It follows that the mapping that assigns to $(\alpha,\ \beta)$ the de Rham cohomology class of $\alpha + \overline{\beta}$ is injective; hence $2\ \mathrm{h}^{1,\,0}(S) \leq \dim \mathrm{H}^1(S,\, \mathbb{C}) = \mathrm{b}_1(S)$. Combining of the two inequalities yields $2\ \mathrm{q}(S) - \mathrm{b}_1(S) \geq \mathrm{b}_1(S) - 2\ \mathrm{h}^{1,\,0}(S) \geq 0$.

The Serre duality (6.2.27) implies that $\mathrm{p_g}(S) = \dim \mathrm{H}^0(S,\, \Omega^2)$. If $\omega \in \mathrm{H}^0(S,\, \Omega^2)$, then ω and $\overline{\omega}$ are closed two-forms on S, and then $\omega \cdot \overline{\omega} = \int_S \omega \wedge \overline{\omega} \geq 0$, with equality if and only if $\omega = 0$. It follows that there exists a basis ω_j, $1 \leq j \leq \mathrm{p_g}(S)$, of $\mathrm{H}^0(S,\, \Omega^2)$ such that $\omega_j \cdot \overline{\omega_k} = \delta_{jk}$. If $\mu,\ \nu \in \mathrm{H}^0(S,\, \Omega^2)$, then $\mu \wedge \nu = 0$ and $\overline{\mu} \wedge \overline{\nu} = 0$. Therefore, if $\mu_j := (\omega_j + \overline{\omega_j})/2$ and $\nu_j := (\omega_j - \overline{\omega_j})/2\,\mathrm{i}$ are the respective real and imaginary parts of ω_j, we have $\mu_j \cdot \mu_k = \delta_{jk}/2$, $\nu_j \cdot \nu_k = \delta_{jk}/2$, and $\mu_j \cdot \nu_k = 0$, and the μ_j, ν_j span a linear subspace of $\mathrm{H}^2(S,\, \mathbb{R})$ of dimension $2\ \mathrm{p_g}(S)$ on which the intersection form is positive definite; hence $2\ \mathrm{p_g}(S) \leq \mathrm{b}_2^+(S)$. The equation $(2\ \mathrm{q}(S) - \mathrm{b}_1(S)) + (\mathrm{b}_2^+(S) - 2\ \mathrm{p_g}(S)) = 1$ together with the inequalities $2\ \mathrm{q}(S) - \mathrm{b}_1(S) \geq \mathrm{b}_1(S) - 2\ \mathrm{h}^{1,\,0}(S) \geq 0$ and $2\ \mathrm{p_g}(S) - \mathrm{b}_2^+(S) \geq 0$ allow only the cases (iii) and (iv). The second statement in (ii) follows because in both cases, $\dim \mathrm{H}^1(S,\, \mathbb{C}) = \mathrm{b}_1(S) = \mathrm{h}^{1,\,0}(S) + \mathrm{q}(S)$. □

The Hermitian form $(\omega,\ \nu) \mapsto \omega \wedge \overline{\nu}$ has signature $(1,\ 3)$ on $\Lambda^{(1,\,1)}\,\mathrm{T}_s^*\,S$, which explains the restriction to $\mathrm{H}^0(S,\, \mathcal{F}^{(2,\,0)} \oplus \mathcal{F}^{(2,\,0)})$ in (i). The images in $\mathrm{H}^2(S,\, \mathbb{R})$ of the real and imaginary parts of the elements of $\mathrm{H}^0(S,\, \Omega^2)$ span a real $2\ \mathrm{p_g}(S)$-dimensional vector space V on which the intersection form is positive definite; hence the intersection form is nondegenerate on the orthogonal complement $V^\perp$ of V in

$\mathrm{H}^2(S,\,\mathbb{R})$ with respect to the intersection form. Since the kernel of the canonical homomorphism $\mathrm{H}^2(S,\,\mathbb{Z}) \to \mathrm{H}^2(S,\,\mathbb{R}) \simeq \mathrm{H}^2(S,\,\mathbb{Z}) \otimes \mathbb{R}$ is equal to the torsion subgroup $\mathrm{H}^2(S,\,\mathbb{Z})_{\mathrm{tor}}$ of $\mathrm{H}^2(S,\,\mathbb{Z})$, we view $\Lambda := \mathrm{H}^2(S,\,\mathbb{Z})/\mathrm{H}^2(S,\,\mathbb{Z})_{\mathrm{tor}}$ as a subgroup of $\mathrm{H}^2(S,\,\mathbb{R})$. The Poincaré duality (2.1.17) implies that the intersection form is nondegenerate on Λ; hence rank $\Lambda = \mathrm{b}_2(S)$, that implies that Λ has a $\mathbb{Z}$-basis which is an $\mathbb{R}$-basis of $\mathrm{H}^2(S,\,\mathbb{R})$. In this situation, (6.2.28) implies that the image of $\mathrm{c}(\mathrm{Pic}(S))$ in Λ is equal to $\Lambda_0 := \Lambda \cap V^\perp$, a subgroup of Λ of rank $\leq \dim V^\perp = \mathrm{b}_2(S) - 2\ \mathrm{p_g}(S)$. Theorem 6.2.23 implies that on $V^\perp = \mathrm{H}^2(S,\,\mathbb{R}) \cap \mathrm{H}^0(S,\,\Omega)^\perp$ the intersection form has signature $(1,\,\mathrm{b}_2^-(S))$ or $(0,\,\mathrm{b}_2^-(S))$ if $\mathrm{b}_1(S)$ is even or odd, respectively.

The compact manifold M is called *Kähler* if it admits a Kähler metric, a Hermitian metric whose imaginary part is a closed real-valued two-form. Because the Fubini–Study metric on the complex projective space is a Kähler metric, and every complex analytic submanifold of a Kähler manifold is Kähler, every compact complex analytic manifold that can be embedded in a complex projective space is Kähler; see Griffiths and Harris [74, p. 109]. The cohomology group $\mathrm{H}^r(M,\,\mathbb{C})$ is isomorphic to the space $\mathfrak{H}^r(M)$ of harmonic complex-valued r-forms. If M is Kähler, then the $(p,\,q)$-component of a harmonic r-form is harmonic, and the space $\mathfrak{H}^{p,\,q}(M)$ of all harmonic $(p,\,q)$-forms is isomorphic to the qth cohomology group with values in the sheaf Ω^p of locally defined holomorphic $(p,\,0)$-forms. Therefore the direct sum decomposition (6.2.23) induces the *Hodge decomposition*

$$\mathrm{H}^r(M,\,\mathbb{C}) \simeq \bigoplus_{p+q=r} \mathfrak{H}^{p,\,q}(M), \quad \mathfrak{H}^{p,\,q}(M) \simeq \mathrm{H}^q(M,\,\Omega^p). \tag{6.2.35}$$

Furthermore, because the complex conjugation is a bijective complex antilinear map from $\mathfrak{H}^{p,\,q}(M)$ onto $\mathfrak{H}^{q,\,p}(M)$, one has a canonical isomorphism

$$\mathrm{H}^p(M,\,\Omega^q) \simeq \overline{\mathrm{H}^q(M,\,\Omega^p)}. \tag{6.2.36}$$

See Griffiths and Harris [74, p. 116]. If r is odd and $p + q = r$ then $p \neq q$, and it follows from (6.2.35) and (6.2.36) that $\mathrm{b}_r(M) = \dim_{\mathbb{C}} \mathrm{H}^r(M,\,\mathbb{C}) = 2\sum \dim \mathfrak{H}^{p,\,q}$, where the sum is over all p, q with $p = q = r$ and $p < q$. It follows that

$$M \text{ is Kähler and } r \text{ is odd} \Rightarrow \mathrm{b}_r(M) \text{ is even.} \tag{6.2.37}$$

If S is a compact Kähler surface, with $V^\perp$ as above, the Hodge decomposition (6.2.35) implies that $V^\perp \otimes \mathbb{C} = \mathfrak{H}^{1,\,1}(S)$. Because $\mathrm{b}_1(S)$ is even, the cup product on $\mathfrak{H}^{1,\,1}(S)$ is nondegenerate and has signature $(1,\,\mathrm{b}_2^-(S))$. This is called the *index theorem for Kähler surfaces*; see Griffiths and Harris [74, p. 126]. Furthermore, (6.2.28) becomes the statement that $\mathrm{c}(\mathrm{Pic}(S))$ is equal to the set of all $a \in \mathrm{H}^2(S,\,\mathbb{Z})$ whose image in $\mathrm{H}^2(S,\,\mathbb{C})$ belongs to $\mathfrak{H}^{1,\,1}(S)$. If S is projective algebraic, then the GAGA principle implies that every holomorphic line bundle L over S is rational and has many rational sections λ, while Lemma 2.1.2 implies that $[L] = \delta(D)$ if $D = \mathrm{Div}(\lambda)$. Therefore, if the image of $a \in \mathrm{H}^2(S,\,\mathbb{Z})$ in $\mathrm{H}^2(S,\,\mathbb{C})$ belongs to

$\mathfrak{H}^{1,1}(S)$, then there is a divisor D in S such that $\mathrm{pd}(D)$ is cohomologous to a. This is called the *Lefschetz theorem on* (1, 1) *classes*; see Griffiths and Harris [74, p. 163].

Remark 6.2.24. For nonsingular complex projective algebraic surfaces S the definitions in Castelnuovo and Enriques [29] of the irregularity and geometric genus are different from Kodaira's, as cohomology had not yet been invented. In [29, No. 11] the geometric genus $p_g'(S)$ of S is defined as the dimension of the space of all rational sections without poles of the canonical bundle K_S. Here the prime stands for "classical." Another integer defined there is the *arithmetic genus* $p_a'(S)$. In [29, No. 28] it is stated that the difference $q'(S) := p_g'(S) - p_a'(S)$, called the irregularity of S, is equal to the dimension of the space of all rational complex one-forms without poles on S. In view of the GAGA principle, $p_g'(S) = \dim H^0(S, \Omega^2)$ and $q'(S) = \dim H^0(S, \Omega^1) =: h^{1,0}(S)$. The Serre duality (6.2.27) implies that $\dim H^0(S, \Omega^2) = \dim H^2(S, \mathcal{O}) =: p_g(S)$, whereas the fact that S is Kähler implies that $\dim H^0(S, \Omega^1) = \dim H^1(S, \mathcal{O}) =: q(S)$. Therefore $q(S) = q'(S)$ and $p_g(S) = p_g'(S)$ if the surface S is projective algebraic. Kodaira has chosen his definition of the irregularity such that the formula $\chi(S, \mathcal{O}) = 1 - q(S) + p_g(S) = 1 + p_a(S)$ remains valid for every compact complex analytic surface S, where $\chi(S) = 1 + p_a(S)$ if $p_a(S) := p_g(S) - q(S)$. It is one of the points of [29] that the irregularity and the geometric genus are birational invariants of nonsingular complex projective surfaces. This statement is generalized to arbitrary compact complex analytic surfaces in Corollary 6.2.55.

Siu [188] proved that for every compact connected complex analytic surface S the first Betti number is even if and only if S is Kähler. See also Remark 6.3.19. If $b_1(S)$ is even, then the conclusion $q(S) = h^{1,0}(S)$ in Theorem 6.2.23 confirms (6.2.36) for Kähler $M = S$, $p = 1$, and $q = 0$. If $b_1(S)$ is odd when S is not Kähler, then Theorem 6.2.23 yields $h^{1,0}(S) = q(S) - 1$. In Remark 6.2.27 and Example 3 after Corollary 6.2.28 there appear S with $b_1(S) = 1$ and $b_1(S) = 3$, respectively.

Another historical note: Lefschetz [117, pp. 18, 30, 33, 8] proved that for a nonsingular complex projective algebraic surface the first Betti number is equal to twice the irregularity and mentioned that Volterra constructed, for any $k \in \mathbb{Z}_{\geq 0}$, a real four-dimensional compact manifold M with $b_1(M) = k$, when M cannot be a projective algebraic surface if k is odd. ⊘

6.2.9 Nonalgebraic Compact Surfaces

If S is a nonsingular complex projective algebraic surface, then it has a very rich geometry of complex analytic curves. For instance, for every $s \in S$ and every one-dimensional linear subspace l of $T_s S$ there exists a complex analytic curve C in S such that s is a smooth point of C and $T_s C = l$. For this it suffices to take a codimension-one projective subspace H of the surrounding projective space $\mathbb{P}^n$ such that $s \in H$ and $T_s H \cap T_s S = l$. It follows that H is transversal to S at s, while the implicit function theorem implies that the *hyperplane section* $S \cap H$ near s is a

nonsingular complex analytic curve such that $\mathrm{T}_s(S \cap H) = \mathrm{T}_s H \cap \mathrm{T}_s S = l$. Also, the proposition in Shafarevich [182, p. 401] with $n = 2$ and $r = 0$ says that each hyperplane section is connected. Let $\mathcal{H} \simeq \mathbb{P}^n$ denote the space of all hyperplanes in $\mathbb{P}^n$, $\Sigma := \{(s, H) \in S \times \mathcal{H} \mid s \in H \text{ and } \mathrm{T}_s S \subset \mathrm{T}_s H\}$, and π_i the projection from Σ to the ith component. Then every fiber of π_1 is isomorphic to $\mathbb{P}^{n-3}$; hence $\dim \Sigma \leq 2 + (n-3) = n-1$. It follows that $\pi_2(\Sigma)$ is an algebraic subset of $\mathcal{H}$ of dimension $\leq n-1$, where $S \cap H$ is smooth if and only if $H \notin \pi_2(\Sigma)$. Therefore the generic hyperplane section is a nonsingular irreducible curve in S.

In contrast with this, Theorems 3.1, 4.1, 4.2, 4.3, and 5.1 of Kodaira [109, I] imply the very strong conclusions of Theorem 6.2.25 below if the surface is not projective algebraic. If there are only finitely many irreducible curves in S, then the proof provides more information about the configuration of these than stated in the theorem. For a simple example of a compact elliptic surface that is not even Kähler, see Example 3 after Corollary 6.2.28.

The proof of Theorem 6.2.25 is based on the *theorem of Chow and Kodaira*, see [109, I, Theorem 3.1], which states that S allows a complex analytic embedding into a complex projective space if there exist at least two algebraically independent meromorphic functions on S. An *algebraic variety* X over $\mathbb{C}$ is defined by gluing together finitely many affine varieties A_i over $\mathbb{C}$ by means of rational coordinate transformations. If X is irreducible, the fields of rational functions on the A_i are canonically isomorphic, and define the field $\mathbb{C}(X)$ of rational functions on X. The dimension of X is defined as the transcendence degree of $\mathbb{C}(X)$, the largest number of algebraically independent elements in it. See Shafarevich [182, p. 263]. X has the structure of a complex analytic set of dimension equal to the algebraic dimension, and each rational function on X is a meromorphic function. Therefore, if X is a smooth algebraic surface that as a complex analytic space is compact and connected, then it has two algebraically independent meromorphic functions, and the theorem of Chow and Kodaira implies that X is projective algebraic. In other words, the adjective "nonalgebraic" in the title of Section 6.2.9 is equivalent to "not projective algebraic."

Theorem 6.2.25 *Assume that S is a compact smooth complex analytic surface that does not allow a holomorphic embedding into a complex projective space. Then either there are only finitely many irreducible complex analytic curves in S, or there exists an elliptic fibration $\varphi : S \to C$ such that each irreducible curve in S is contained in a fiber of φ. In the first or second case every meromorphic function on S is a constant or of the form $\varphi^* h$ for a meromorphic function h on C, respectively. For every holomorphic line bundle L over S we have $L \cdot L \leq 0$, and in the first case* $\dim \mathrm{H}^0(S, \mathcal{O}(L)) \leq 1$.

Proof. We first assume that every meromorphic function on S is a constant, and we will prove in this case that there are only finitely many irreducible curves in S. Let L be a holomorphic line bundle over S. Then for every nonzero $s, s' \in \mathrm{H}^0(S, \mathcal{O}(L))$ the quotient s'/s is a meromorphic function on S, hence a constant, and therefore $\dim \mathrm{H}^0(S, \mathcal{O}(L)) \leq 1$. Therefore (6.2.33) and (6.2.31) yield $2 \geq \dim \mathrm{H}^0(S, \mathcal{O}(L)) + \dim \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S\, L^*)) \geq \chi(S, \mathcal{O}) + (L \cdot L - \mathrm{K}_S \cdot L)/2$. If

$L \cdot L > 0$ then we arrive at a contradiction if we replace L by L^k with a sufficently large k. It follows that $L \cdot L \leq 0$ for every holomorphic line bundle L over S. If $L \cdot L = 0$ then $0 \geq (L^k\, L') \cdot (L^k\, L') = 2\, k\, L \cdot L' + L' \cdot L'$ for all $k \in \mathbb{Z}$ implies that $L \cdot L' = 0$ for every other holomorphic line bundle L'. If D is a divisor in S, then application of the above to a holomorphic line bundle L such that $[L] = \delta(D)$ yields that $D \cdot D \leq 0$. If $D \cdot D = 0$, then $D \cdot L' = D \cdot D' = 0$ for every other holomorphic line bundle L' and divisor D'. If C is an irreducible curve in S such that $C \cdot C = 0$, and C' is another irreducible curve, then $C \cdot C' = 0$; hence $C \cap C' = \emptyset$ if $C \neq C'$.

Let Γ be the graph of all irreducible curves in S, where $C,\ C' \in \Gamma$ are connected by a vertex if and only if $C \cap C' \neq \emptyset$. Let $\mathcal{C}$ be a connected component of Γ. By induction on i one finds distinct $C_j \in \mathcal{C}$, $1 \leq j \leq i$, such that $C_i \cap C_j \neq \emptyset$; hence $C_i \cdot C_j > 0$ for some $j < i$, and $m_{jk} \in \mathbb{Z}_{>0}$ such that the divisors D_j equal $\sum_{k \leq j} m_{jk}\, C_k$ satisfy $D_j \cdot D_j < 0$ and $D_j \cdot D_k = 0$ if $j \neq k$. For each j the coefficients m_{jk}, $1 \leq k \leq j$, are made unique by requiring that they not have a factor in $\mathbb{Z}_{>1}$. It follows that the intersection matrix $C_j \cdot C_k$, $1 \leq j,\ k \leq i$, is negative definite. Let $\mathrm{NS}(S)$ denote the Néron–Severi group of S, the subgroup of $\mathrm{H}_2(S,\ \mathbb{Z})$ of all homology classes of divisors in S. If the divisor $D = \sum_j a_j\, C_j$ is homologous to zero, then, because the intersection number is a homology invariant, we have $0 = D \cdot D = \sum_{jk} a_j\, C_j \cdot C_k\, a_k$, hence $a_j = 0$ for every j. Therefore the homology classes of the C_j generate a subgroup of $\mathrm{NS}(S)$ of rank i; hence $i \leq \mathrm{rank\ NS}(S) \leq \mathrm{rank\ H}_2(S,\ \mathbb{Z}) < \infty$, and the orthogonalization procedure has to stop after finitely many steps. If it stops at stage i then either $i = \#(\mathcal{C})$, or $i < \#(\mathcal{C})$ and $D_{i+1} \cdot D_{i+1} = 0$. In the second case $0 = D_{i+1} \cdot C = \sum_k m_{i+1,\,k}\, C_k \cdot C$; hence $C_k \cdot C = 0$ and therefore $C_k \cap C = \emptyset$ for every irreducible curve C not equal to one of the C_j. We conclude that $i+1 = \#(\mathcal{C})$, and the intersection matrix $(C \cdot C')_{C,\,C' \in \mathcal{C}}$ of $\mathcal{C}$ is negative semidefinite with a one-dimensional kernel. Here it is allowed that $i = 0$, which happens if $\mathcal{C} = \{C\}$, $C \cdot C = 0$, and C is disjoint from every other irreducible curve in S. In the sequel a connected component of Γ will be called of negative or zero type if its intersection matrix is negative definite or negative semidefinite, respectively.

Let $D = \sum_{j=1}^{m} D^{(j)}$, where the $D^{(j)}$ are nonzero divisors with disjoint supports $|D^{(j)}|$. Let L and s respectively be a holomorphic line bundle over S and a holomorphic section of L^{-1} such that $[L] = \delta(D)^{-1}$ and $\mathrm{Div}(s) = D$ as in Lemma 2.1.2. Then multiplication by s leads to a short exact sequence $0 \to \mathcal{O}(L) \to \mathcal{O} \to Q \to 0$; the quotient sheaf $Q := \mathcal{O}/s\,\mathcal{O}(L)$ is supported by the union of the $|D^{(j)}|$. We have the ensuing long exact sequence $0 \to \mathrm{H}^0(S,\ \mathcal{O}(L)) \to \mathrm{H}^0(S,\ \mathcal{O}) \to \mathrm{H}^0(S,\ Q) \to I \to 0$ in cohomology, where I is the image in $\mathrm{H}^1(S,\ \mathcal{O}(L))$ of $\mathrm{H}^0(S,\ Q)$ under the coboundary operator. Let U_j be disjoint open neighborhoods of the $|D^{(j)}|$, and U the union of the U_j. For any $c \in \mathbb{C}^m$, let $f(c) \in \mathrm{H}^0(U,\ \mathcal{O})$ be equal to the constant function c_j on U_j, and let $q(c)$ be the image of $f(c)$ in $\mathrm{H}^0(S,\ Q) = \mathrm{H}^0(U,\ Q)$ under the homomorphism $\mathrm{H}^0(U,\ \mathcal{O}) \to \mathrm{H}^0(U,\ Q)$. Then q is a linear mapping $\mathbb{C}^m \to \mathrm{H}^0(S,\ Q)$. If $q(c) = 0$ then the fact that s vanishes on $|D^{(j)}|$ implies that $c_j = 0$ for every j. Hence q is injective and we have $m \leq \dim \mathrm{H}^0(S,\ Q) = -\dim \mathrm{H}^0(S,\ \mathcal{O}(L)) + \dim \mathrm{H}^0(S,\ \mathcal{O}) + \dim I \leq -\dim \mathrm{H}^0(S,\ \mathcal{O}(L)) + 1 + \dim \mathrm{H}^1(S,\ \mathcal{O}(L)) = 1 - \chi(S,\ \mathcal{O}(L)) + \dim \mathrm{H}^2(S,\ \mathcal{O}(L)) \leq 2 - \chi(S,\ \mathcal{O}(L)) =$

$2-\chi(S,\,\mathcal{O})-(L\cdot L-\mathrm{K}_S\cdot L)/2$, where in the second and third identities we have used (6.2.29) for $n=2$, the Serre duality (6.2.26) yields $\dim \mathrm{H}^2(S,\,\mathcal{O}(L)) = \dim \mathrm{H}^0(S,\,\mathcal{O}(\mathrm{K}_S\,L^*)) \le 1$, and we have used (6.2.31) in the last identity. If we use (6.2.8) as the definition of the virtual genus of an arbitrary divisor A, then $(L\cdot L-\mathrm{K}_S\cdot L)/2 = (D\cdot D+\mathrm{K}_S\cdot D)/2 = \sum_j (D^{(j)}\cdot D^{(j)}+\mathrm{K}_S\cdot D^{(j)})/2 = \sum_j \mathrm{vg}(D^{(j)})-m$. Therefore $\sum_j \mathrm{vg}(D^{(j)}) \le 2-\chi(S,\,\mathcal{O})$, which generalizes Kodaira's estimate [109, I, (5.4)]. If each of the $D^{(j)}$ is one of the above D_{i+1}, then $D^{(j)}\cdot D^{(j)}=0$ hence $\mathrm{K}_S\cdot D^{(j)}=0$, $\mathrm{vg}(D^{(j)})=1$, and we conclude that there are at most $2-\chi(S,\,\mathcal{O})$ connected components of Γ of zero type.

Let $\mathrm{Div}(S)\simeq\mathbb{Z}^\Gamma$ denote the group of all divisors in S and $\mathrm{H}:\mathrm{Div}(S)\to\mathrm{NS}(S)$ the surjective homomorphism that assigns to each divisor its homology class. If $D=\sum_{C\in\Gamma} a_C\,C\in\ker\mathrm{H}$, then $0=D\cdot C'=\sum_{C\in\Gamma} a_C\,C\cdot C'$ for every $C'\in\Gamma$; hence D is an integral linear combination of the above D_{i+1} in the connected components of zero type. It follows that $\mathrm{rank}(\ker\mathrm{H})\le$ the number of connected components of zero type; hence $\#(\Gamma)=\mathrm{rank}(\mathrm{Div}(S))=\mathrm{rank}(\mathrm{NS}(S))+\mathrm{rank}(\ker\mathrm{H})$ is finite, and there are at most $\mathrm{rank}(\mathrm{NS}(S))+2-\chi(S,\,\mathcal{O})$ irreducible curves in S.

The other alternative is that there exists a nonconstant meromorphic function f on S. Blowing up points of indeterminacy and applying the Stein factorization lemma, Lemma 6.1.3, we arrive at a modification $\pi:R\to S$, a branched covering $p:C\to\mathbb{P}^1$, and a fibration $\varphi:R\to C$ with connected fibers such that $p\circ\varphi=f\circ\pi$, where the left-hand side is a meromorphic function on R without points of indeterminacy. Lemma 3.2.3 implies that R is not projective algebraic, while the theorem of Chow and Kodaira implies that every meromorphic function on R is algebraically dependent of $p\circ\varphi$. If g is a meromorphic function on R that is functionally dependent on $p\circ\varphi$, then g is locally constant, hence constant, on the generic connected and smooth fiber of φ, and it follows that $g=\varphi^*h$ for a meromorphic function h on C.

We will prove that $L\cdot L\le 0$ for any holomorphic line bundle L over R. Assume that L has a nonzero meromorphic section s, and write $D=\mathrm{Div}(s)$. Since the pullback φ^* by $\varphi:R\to C$ is a linear isomorphism from the space of all meromorphic functions on C onto the space of all meromorphic functions on R, $f\mapsto\varphi^*(f)\,s$ is a linear isomorphism from the space of all meromorphic functions f on C with $\mathrm{Div}(\varphi^*(f))+D\ge 0$ onto the space $\mathrm{H}^0(R,\,\mathcal{O}(L))$ of all holomorphic sections of L. In the sequel we will denote the fiber of φ over a point $c\in C$, viewed as a divisor in R, by R_c. Then $\mathrm{Div}(s)$ has a decomposition $\sum_i\sum_j m_{ij}\,\Theta_{ij}+D'$ with $m_{ij}\in\mathbb{Z}$, the Θ_{ij} are the irreducible components of distinct fibers R_{c_i}, and the irreducible curves in the divisor D' are not contained in a fiber of φ. Writing $\mathrm{Div}(f)=\sum_i n_i\,c_i$ with $n_i\in\mathbb{Z}$, $c_i\in C$, we have $\mathrm{Div}(\varphi^*(f))+D\ge 0$ if and only if $D'\ge 0$ and $n_i\,\mu_{ij}+m_{ij}\ge 0$ for every j if $R_{c_i}=\sum_j \mu_{ij}\,\Theta_{ij}$, where $\mu_{ij}\in\mathbb{Z}_{>0}$. Let ν_i denote the smallest integer ν such that $\nu\ge -m_{ij}/\mu_{ij}$ for all j, and write $\mathfrak{d}:=\sum_i \nu_i\,c_i$, when $\deg(\mathfrak{d})=\sum_i \nu_i$. We then have $\mathrm{Div}(\varphi^*(f))+D\ge 0$ if and only if $D'\ge 0$ and $\mathrm{Div}(f)+\mathfrak{d}\ge 0$. Note that replacing s by $\varphi^*(f)\,s$ for a nonzero meromorphic function f on C does not change D' and changes m_{ij} into $n_i\,\mu_{ij}+m_{ij}$, hence ν_i into $n_i+\nu_i$, hence $\deg(\mathfrak{d})$ into $\deg(f)+\deg(\mathfrak{d})=\deg(\mathfrak{d})$. Therefore D' and $\deg(\mathfrak{d})$ do not depend on the choice of s. If $\mathrm{H}^0(R,\,\mathcal{O}(L))\ne 0$, then s can be chosen to be holomorphic, hence $D'\ge 0$, and it follows that $\mathrm{H}^0(R,\,\mathcal{O}(L))$ is

isomorphic to the space $\mathcal{M}(C)_{\mathfrak{d}}$ of all meromorphic functions g on C such that $\mathrm{Div}(g) + \mathfrak{d} \geq 0$. If $\mathfrak{L}$ is a holomorphic line bundle over C such that $[\mathfrak{L}] = \delta(\mathfrak{d})$, and $\mathfrak{s}$ a holomorphic section of $\mathfrak{L}$ such that $\mathrm{Div}(\mathfrak{s}) = \mathfrak{d}$, see Lemma 2.1.2, then $g \mapsto g\,\mathfrak{s}$ is a linear isomorphism from $\mathcal{M}(C)_{\mathfrak{d}}$ onto $\mathrm{H}^0(C,\, \mathcal{O}(\mathfrak{L}))$. If $\mathrm{H}^0(C,\, \mathcal{O}(\mathrm{K}_C\ \mathcal{L}^*))$ contains a nonzero t, then multiplication by t defines an injective linear mapping from $\mathrm{H}^0(C,\, \mathcal{O}(\mathfrak{L}))$ into the $\mathrm{g}(C)$-dimensional vector space of all holomorphic one-forms on C. If $\mathrm{H}^0(C,\, \mathcal{O}(\mathrm{K}_C\ \mathcal{L}^*)) = 0$, then the Riemann–Roch formula (2.1.14) for compact curves yields $\dim \mathrm{H}^0(C,\, \mathcal{O}(\mathfrak{L})) = 1 - \mathrm{g}(C) + \deg(\mathfrak{d})$. We conclude that $\dim \mathrm{H}^0(R,\, \mathcal{O}(L)) = 1 - \mathrm{g}(C) + \deg(\mathfrak{d})$ if $\dim \mathrm{H}^0(R,\, \mathcal{O}(L)) > \mathrm{g}(C)$.

Assume that $L \cdot L > 0$. Then the Riemann–Roch formula (6.2.31) for compact surfaces, with L replaced by L^k, $k \in \mathbb{Z}$, yields that $\chi(\mathcal{O}(L^k)) \sim (L \cdot L)\, k^2/2$ as $k \to \infty$. Because $\dim \mathrm{H}^1(R,\, \mathcal{O}(L^k)) \geq 0$ it follows from (6.2.29) with $n = 2$ and the Serre duality (6.2.26) that $\dim \mathrm{H}^0(R,\, \mathcal{O}(L^k))$ or $\dim \mathrm{H}^0(R,\, \mathcal{O}(\mathrm{K}_R\ L^{-k}))$ grows at least quadratically in k as $k \to \infty$. In the first case a quotient λ of nonzero holomorphic sections of L^{k+1} and L^k is a nonzero meromorphic section of L. If a component of $\mathrm{Div}(\lambda)$ has an effective component D in a fiber F such that $k\, D \geq l\, F$, then $l = \mathrm{O}(k)$ as $k \to \infty$. In any case it follows, in the notation of the previous paragraph, that $\dim \mathrm{H}^0(R,\, \mathcal{O}(L^k)) \sim \deg(\mathfrak{d}(\,\mathrm{Div}(\lambda^k))) = \deg(\mathfrak{d}(k\ \mathrm{Div}(\lambda))) = \mathrm{O}(k)$ as $k \to \infty$, in contradiction to the quadratic growth of $\dim \mathrm{H}^0(R,\, \mathcal{O}(L^k))$. In the second case a quotient λ of nonzero holomorphic sections of $\mathrm{K}_R\ L^{-k}$ and $\mathrm{K}_R\ L^{-k-1}$ is a nonzero meromorphic section of L, in which case the product of λ^k and the section of $\mathrm{K}_R\ L^{-k}$ is a nonzero meromorphic section κ of K_R. A similar argument as above yields that $\dim \mathrm{H}^0(R,\, \mathcal{O}(\,\mathrm{K}_R\ L^{-k})) \sim \deg(\mathfrak{d}(\,\mathrm{Div}(\kappa\, \lambda^{-k}))) = \deg(\mathfrak{d}(\,\mathrm{Div}(\kappa) - k\ \mathrm{Div}(\lambda))) = \mathrm{O}(k)$ as $k \to \infty$, in contradiction to the quadratic growth of $\dim \mathrm{H}^0(R,\, \mathcal{O}(\,\mathrm{K}_R\ L^{-k}))$. Therefore $L \cdot L \leq 0$ for every holomorphic line bundle L over R.

Let E be an irreducible curve in R that is not contained in a fiber of φ. If F is an irreducible fiber of φ, then $E \cdot F \geq 0$. Assume that $E \cdot F = 0$. If F' is another fiber of φ, then $E \cdot F' = E \cdot F = 0$, because F' is homologous to F, see Lemma 6.1.2, and the intersection number is a homology invariant; see Section 2.1.6. If F' is irreducible, then $E \cdot F' = 0$ implies that $E \cap F' = \emptyset$, and it follows that E is contained in the union of the finitely many reducible fibers of φ, hence in one of these because E is irreducible, in contradiction to the assumption. Therefore $E \cdot F > 0$. If $n \in \mathbb{Z}_{>0}$ and L is a holomorphic line bundle over R such that $[L] = \delta(E + n\, F)$, then $L \cdot L = (E + n\, F) \cdot (E + n\, F) = E \cdot E + 2\, n\, E \cdot F + n^2\, F \cdot F = E \cdot E + 2\, n\, E \cdot F > 0$ if we choose $n > -(E \cdot E)/2\,(E \cdot F)$. Since this contradicts the conclusion of the previous paragraph, it follows that each irreducible curve in R is contained in a fiber of φ.

If E is the -1 curve that appears in the last blowup of the modification $\pi : R \to S$ in order to remove the indeterminacy of f, then E is not contained in a fiber of φ, in contradiction to the previous paragraph. It follows that f has no points of indeterminacy, $R = S$, π is the identity, $f = p \circ \varphi$, and $\varphi : S \to C$ is a fibration such that each irreducible curve in S is contained in a fiber of φ. If F is a fiber, then $F \cdot F = 0$. If L is a holomorphic line bundle such that $[L] = \delta(F)$, then $L \cdot L = 0$, and $0 \geq (\mathrm{K}_S\ L^n) \cdot (\mathrm{K}_S\ L^n) = \mathrm{K}_S \cdot \mathrm{K}_S + 2\, n\ \mathrm{K}_S \cdot L$ for all $n \in \mathbb{Z}$ implies

that $K_S \cdot F = K_S \cdot L = 0$. Therefore, if F is smooth, the adjunction formula (6.2.5) with $A = F$ implies that $g(F) = 1$; hence the fibration $\varphi : S \to C$ is elliptic. □

Corollary 6.2.26 *A compact surface S is projective algebraic if and only if $L \cdot L > 0$ for some holomorphic line bundle L over S.*

Proof. Theorem 6.2.25 implies that $L \cdot L \leq 0$ for every holomorphic line bundle L over S if S is not projective algebraic. Therefore, if $\iota : S \to \mathbb{P}^n$ is a holomorphic embedding, it suffices to find a holomorphic line bundle L over S such that $L \cdot L > 0$.

Let $O(1)$ denote the line bundle over $\mathbb{P}^d$ such that for each $l \in \mathbb{P}^n$, a one-dimensional linear subspace of $\mathbb{C}^{n+1}$, the fiber of $O(1)$ over l is equal to the space of all linear forms on l. Define $L := \iota^* O(1)$. The divisors $\mathrm{Div}(\lambda)$ with λ nonzero holomorphic sections of L are the hyperplane sections used in the beginning of Section 6.2.9 in order to show that a projective algebraic surface has many curves. We have $\mathrm{Div}(\lambda) \cdot \mathrm{Div}(\lambda') = L \cdot L$ for any nonzero $\lambda,\ \lambda' \in H^0(S,\ \mathcal{O}(L))$. Because the set of nonzero λ such that $\mathrm{Div}(\lambda)$ is a smooth irreducible curve is nonempty and open, and for any point s in such $\mathrm{Div}(\lambda)$ there exist nearby λ' such that $s \in \mathrm{Div}(\lambda')$ and $T_s(\mathrm{Div}(\lambda)) \neq T_s(\mathrm{Div}(\lambda'))$, it follows that $L \cdot L > 0$. In fact, $L \cdot L$ is equal to the degree of $\iota(S)$ in $\mathbb{P}^n$; see Griffiths and Harris [74, p. 171]. □

The following long remark discusses the surfaces that admit only finitely many irreducible curves, and can be skipped if one is mainly interested in elliptic surfaces.

Remark 6.2.27. Let S be a compact smooth complex analytic surface without nonconstant meromorphic functions. Blowing down -1 curves, successively if necessary, we can arrange that there are no -1 curves in S, keeping the property that S is smooth, compact, and has no nonconstant meromorphic functions. Kodaira [110, I, Theorem 11] proved that either (a) S is isomorphic to a complex torus, or (b) $q(S) = 1$, $b_1(S) = 1$, and $p_g(S) = 0$, or (c) S is a K3 surface. In the next paragraphs we discuss these three cases in more detail, including some of the developments after Kodaira [110].

(a) A *complex torus* is a quotient $S = \mathbb{C}^2/\Gamma$ of $\mathbb{C}^2$ by a cocompact discrete additive subgroup Γ of $\mathbb{C}^2$. It is called an *abelian variety* if S is projective algebraic. Riemann's necessary and sufficient conditions for S to be an abelian variety, see Griffiths and Harris [74, pp. 300–307], show that S is not projective algebraic for the generic lattice Γ in $\mathbb{C}^2$. Assume that S is not projective algebraic. Then Theorem 6.2.25 implies that either S admits only finitely many irreducible curves, or S is an elliptic surface and each irreducible curve is contained in a fiber. Since each translate of an irreducible curve in S is an irreducible curve, it follows that in the first case S does not admit any irreducible curve, whereas in the second case there is a unique one-dimensional linear subspace A of $\mathbb{C}^2$ such that $A \cap \Gamma$ is a rank-two sublattice of Γ, the irreducible curves in S are the translates of the elliptic curve $A/(A \cap \Gamma)$, and the quotient of S by the elliptic fibration is the elliptic curve $(\mathbb{C}^2/A)/(\Gamma/(A \cap \Gamma))$. The case that S does not admit any irreducible curve is the generic one.

For any complex torus S, every translation-invariant Kähler structure, holomorphic one-form, and nonvanishing holomorphic two-form on $\mathbb{C}^2$ defines a corresponding object on S, and therefore S is Kähler, $h^{1,0}(S) \geq 2$, and K_S is trivial, and hence $p_g(S) = 1$. Since $b_1(S) = 4$ and $b_2(S) = 6$, it follows from Theorem 6.2.23(iii) that $q(S) = h^{1,0}(S) = 2$; hence every holomorphic one-form on S is translation invariant, and $b_2^+(S) = b_2^-(S) = 3$.

(b) S belongs to the class VII_0 in Kodaira's classification [110, I, Table I on p. 790], with a question mark regarding the structure of these surfaces. Because $b_1(S) = 1$ is odd, (6.2.37) implies that S is not Kähler. Because $H^2(S, \mathcal{O}) = 0$ and the mapping $\epsilon := e^{2\pi i} : \mathbb{C} \simeq H^0(S, \mathcal{O}) \to H^0(S, \mathcal{O}^\times) \simeq \mathbb{C}^\times$ is surjective, (2.1.8) leads to the exact sequence $0 \to H^1(S, \mathbb{Z}) \to H^1(S, \mathcal{O}) \to \mathrm{Pic}(S) \to H^2(S, \mathbb{Z}) \to 0$, where the third mapping is induced by ϵ and the fourth mapping is the Chern homomorphism. Because the additive group $H^1(S, \mathcal{O})$ has no torsion, the injectivity of the second map implies that $H^1(S, \mathbb{Z})$ has no torsion, in which case $b_1(S) = 1$ implies that $H^1(S, \mathbb{Z}) \simeq \mathbb{Z}$. In combination with $\dim H^1(S, \mathcal{O}) =: q(S) = 1$ and $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^\times$, it follows that the kernel of c, = the image of ϵ, is isomorphic to the group $\mathbb{C}^\times$. Since $\mathrm{Div}(S) \simeq \mathbb{Z}^\Gamma$, where Γ is the finite set of all irreducible curves in S, $\mathrm{Div}(S)$ is countable, and because $\mathrm{Pic}(S)$ contains a subgroup isomorphic to $\mathbb{C}^\times$, it follows that the homomorphism $\delta : \mathrm{Div}(S) \to \mathrm{Pic}(S)$ in (2.1.4) is not surjective. That is, there exist holomorphic line bundles L over S without nonzero meromorphic sections. Note that $H^0(S, \mathcal{M}^\times) \simeq \mathbb{C}^\times$, hence the embedding $H^0(S, \mathcal{O}^\times) \to H^0(S, \mathcal{M}^\times)$ is an isomorphism, and therefore (2.1.4) yields that the mapping $\delta : \mathrm{Div}(S) \to \mathrm{Pic}(S)$ is injective. It follows from the paragraph after Theorem 6.2.23 that the intersection form is negative definite on $V^\perp$, which space contains the image of $c(\mathrm{Pic}(S))$ in $H^2(S, \mathbb{R})$ in view of (6.2.28). Therefore, if L is a holomorphic line bundle over S such that $L \cdot L = 0$, then $c(L)$ belongs to the kernel $H^2(S, \mathbb{Z})_{\mathrm{tor}}$ of the canonical homomorphism $H^2(S, \mathbb{Z}) \mapsto H^2(S, \mathbb{R})$; hence there exists $m \in \mathbb{Z}_{>0}$ such that $[L^m] \in \ker c \simeq \mathbb{C}^\times$. Because $\chi(S, \mathcal{O}) = 0$ and $\chi_{\mathrm{top}}(S) = 2 - 2\, b_1(S) + b_2(S) = b_2(S) =: \mathrm{rank}\, H_2(S, \mathbb{Z})$, see (6.2.38), Noether's formula (6.2.32) implies that $K_S \cdot K_S = -\, b_2(S)$.

Since $\chi(S, \mathcal{O}) = 1 - 1 + 0 = 0$, the proof of Theorem 6.2.25 yields that there are at most two connected components $\mathcal{C}$ of Γ of zero type. Enoki [59, Proposition 4.12] proved that if there are two such $\mathcal{C}$, then both are smooth elliptic curves C such that $C \cdot C = 0$. Furthermore, if there is a smooth elliptic curve C in S such that $C \cdot C = 0$, then $b_2(S) = 0$. In this situation, Kodaira [110, II, pp. 699, 700] proved that S is a Hopf surface, that is, its universal covering is the complement of the origin in $\mathbb{C}^2$. He gave explicit constructions of the surfaces that can occur, from which it follows that there are Hopf surfaces with two disjoint elliptic curves C such that $C \cdot C = 0$ and Hopf surfaces with only one elliptic curve C with $C \cdot C = 0$. Since $b_2(S) = 0$, there are no connected components of Γ of negative type; hence there are no irreducible curves in S other than the elliptic curves C with $C \cdot C = 0$.

When there is a connected component $\mathcal{C}$ of Γ of zero type and $b_2(S) > 0$, Enoki [59, Main Theorem and Propositions 3.4, 3.5] proved that there is exactly one such $\mathcal{C}$, and it has the structure of a divisor of type I_b for some $b > 0$, as in

Section 6.2.6. S is isomorphic to one of Enoki's surfaces $S_{b,\alpha,t}$, and $\mathrm{b}_2(S) = b$. It can be proved that there is another irreducible curve C in S if and only if $t = 0$, in which case C is a smooth elliptic curve in S such that $C \cdot C = -b < 0$.

The case that is left is that there is no connected component of Γ of zero type when the intersection matrix of each connected component $\mathcal{C}$ of Γ is negative definite. Then Grauert's criterion, Lemma 6.2.49, implies that there exist a compact complex analytic space X and a modification $f : S \to X$ in a finite subset F of X such that f blows down the unions of the connected components of γ to the points of F. The space X with singularities in F does not carry any complex analytic curve. Inoue [92] gave a classification when $\mathrm{b}_2(S) = 0$, and there exists a holomorphic line bundle L over S such that there is a nonzero holomorphic section of $\Omega^1 \otimes L$. It follows from Bogomolov [19, Theorem 1], see also Teleman [195], that $\mathrm{b}_2(S) = 0$ implies the latter condition. For the case that there are $\mathrm{b}_2(S)$ rational curves in S, see Dloussky, Oeljeklaus, and Toma [46].

(c) S is a K3 surface, that is, $\mathrm{q}(S) = 0$ and K_S is trivial. See Section 6.3.4 for the properties of K3 surfaces that we will use in the next two paragraphs. Since K_S is trivial, we have $\chi(S, \mathcal{O}) = 1 + 0 + 1 = 2$. The proof of Theorem 6.2.25 implies that there are no connected components of zero type, and the intersection form is negative definite on the group of divisors. For each irreducible curve C in S we have $C \cdot C < 0$, and the triviality of K_S implies that $\mathrm{K}_S \cdot C = 0$. Therefore (6.2.8) implies that the nonnegative integer $\mathrm{vg}(C)$ is equal to zero; hence C is an embedded $\mathbb{P}^1$ and $C \cdot C = -2$. Since minus the intersection matrix of each connected component $\mathcal{C}$ of Γ is positive definite, it therefore is a Cartan matrix of type A_l, D_l, or E_l. According to Grauert's criterion, Lemma 6.2.49, there are a two-dimensional complex analytic space X and a modification $\pi : S \to X$ of X in a finite subset F such that for each $x \in F$ the fiber of π over x is the union of a connected component $\mathcal{C}_x$ of Γ. The compact analytic space X does not admit any complex analytic curve, $X \setminus F$ is smooth, and for each $x \in F$ the singularity of X at x is a simple singularity, as discussed in Remark 6.3.8, of type A_l, D_l, or E_l of the intersection matrix of $\mathcal{C}_x$.

Let S be an arbitrary K3 surface. Corollary 6.2.26 implies that S is not projective algebraic if and only if $L \cdot L \leq 0$ for every holomorphic line bundle over S. If $L \cdot L = 0$ and L is nontrivial, then the second paragraph of Section 6.3.4 yields that $\dim \mathrm{H}^0(S, \mathcal{O}(L)) \geq 2$ or $\dim \mathrm{H}^0(S, \mathcal{O}(L^{-1})) \geq 2$, and Theorem 6.2.25 implies that S admits an elliptic fibration. It therefore follows from (6.2.28) and the paragraph following Theorem 6.2.23 that the K3 surface S contains only finitely many irreducible curves if and only if the intersection form is negative definite on the sublattice $\Lambda_0 := \Lambda \cap V^\perp$ of Λ. The surjectivity of the period mapping for K3 surfaces implies that V can be any two-dimensional linear subspace of $\mathrm{H}^2(S, \mathbb{R})$ on which the intersection form is positive definite. Since generically $\Lambda_0 = 0$, there exist K3 surfaces without irreducible curves. The surjectivity of the period mapping may be used to classify the K3 surfaces with finitely many irreducible curves and determine which configurations of connected components $\mathcal{C}$ of Γ of type A_l, D_l, and E_l can occur. ⊘

Corollary 6.2.28 *If the elliptic fibration $\varphi : S \to C$ over a compact curve C admits a holomorphic section $\sigma : C \to S$, then S is projective algebraic, hence admits a Kähler structure.*

Proof. The curve $E = \sigma(C)$ is irreducible, and $\varphi(E) = \varphi \circ \sigma(C) = C$ shows that E is not contained in a fiber of φ. Furthemore, if $\iota : S \to \mathbb{P}^N$ is a holomorphic embedding, then the pullback by ι of the Fubini–Study metric on $\mathbb{P}^N$, the standard Kähler structure on $\mathbb{P}^N$, is a Kähler structure on S. See Griffiths and Harris [74, p. 31, 109]. □

If the fibration is relatively minimal and admits a section, then one can use the Weierstrass model to show that the surface is projective algebraic; see Remark 6.3.9.

Example 3. The third case in Kodaira [110, I, Theorem 19] is a surface $S = \mathbb{C}^2/G$, where G is the group of affine linear transformations on $\mathbb{C}^2$ generated by $g_j : (w_1,\ w_2) \mapsto (w_1 + \alpha_j,\ w_2 + \overline{\alpha_j}\, w_1 + \beta_j)$ for $j = 1,\ 2,\ 3,\ 4$, where $\alpha_1 = \alpha_2 = 0$, $\overline{\alpha_3}\,\alpha_4 - \overline{\alpha_4}\,\alpha_3 = m\,\beta_2 \neq 0$, $m \in \mathbb{Z}_{>0}$, and $\beta_1,\ \beta_2$ form an $\mathbb{R}$-basis of $\mathbb{C}$. Using that $g_3\, g_4 = {g_2}^m\, g_4\, g_3$ one can show that the action of G on $\mathbb{C}^2$ is proper, free, and cocompact, in which case the orbit space $S = \mathbb{C}^2/G$ is a compact complex analytic surface. The projection $(w_1,\ w_2) \mapsto w_1$ induces a mapping $\varphi : S \to C := \mathbb{C}/(\mathbb{Z}\,\alpha_3 + \mathbb{Z}\,\alpha_4)$, which exhibits S as a principal H-bundle over C, where both C and $H := \mathbb{C}/(\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2)$ are elliptic curves and H is viewed as a complex one-dimensional compact complex analytic Lie group. In particular, S is an elliptic surface, where the fact that it is a principal H-bundle implies that the modulus function J is constant and the monodromy representation M of Section 6.2.11 is trivial. On the other hand, $\mathrm{H}_1(S,\ \mathbb{Z}) \simeq \mathbb{Z}^3 \times (\mathbb{Z}/m\,\mathbb{Z})$, which implies that the first Betti number of S is equal to 3, hence odd. Therefore (6.2.37) implies that S does not admit any Kähler structure. In particular, S is not projective algebraic, and Theorem 6.2.25 implies that the fibers of $\varphi : S \to C$, the H-orbits, are the only irreducible complex analytic curves in S. ⊘

6.2.10 Euler Numbers of Elliptic Surfaces

In this subsection we assume that S is compact, and that $\varphi : S \to C$ is a relatively minimal elliptic fibration without multiple singular fibers, which allows us to apply the results of Section 6.2.7.

The *topological Euler number* of any real n-dimensional simplicial complex M is defined as the alternating sum

$$\chi_{\mathrm{top}}(M) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{R}} \mathrm{H}_k(M,\ \mathbb{R}) = \sum_{k=0}^{n} (-1)^k\, N_k, \tag{6.2.38}$$

where N_k is the number of simplices of dimension k. Note that $\dim_{\mathbb{R}} \mathrm{H}_k(M,\ \mathbb{R}) = \operatorname{rank} \mathrm{H}_k(M,\ \mathbb{Z})$, because the universal coefficient theorem implies that $\mathrm{H}_k(M,\ \mathbb{R}) \simeq$

$\mathrm{H}_2(M, \mathbb{Z}) \otimes \mathbb{R}$. If M is an oriented compact smooth manifold, then $\chi_{\mathrm{top}}(M)$ is also equal to the number of zeros of any smooth vector field v on M, defined as the topological intersection number of v with the zero section of the tangent bundle $\mathrm{T}M$ of M. In this case $\chi_{\mathrm{top}}(M)$ is also called the *Euler–Poincaré characteristic* of M. The next two lemmas are contained in Kodaira [109, III, p. 14] and [109, II, Table I]. In the determination in Lemma 6.2.29 of the topological Euler number of a singular fiber S_c, each irreducible component Θ of S_c is counted once, that is, not with its multiplicity μ_Θ if $\mu_\Theta > 1$.

Lemma 6.2.29 *The topological Euler number $\chi_{\mathrm{top}}(S)$ of the elliptic surface S is equal to the sum of the topological Euler numbers of the singular fibers. The topological Euler number of a singular fiber of type* I_b, I_b^*, II, II*, III, III*, IV, *and* IV* *is equal to* b, $b+6$, 2, 10, 3, 9, 4, *and* 8, *respectively. Therefore*

$$\begin{aligned}\chi_{\mathrm{top}}(S) = \sum_{b>0} b\,\#(\mathrm{I}_b) + \sum_{b\geq 0} (b+6)\,\#(\mathrm{I}_b^*) + 2\,\#(\mathrm{II}) + 10\,\#(\mathrm{II}^*) \\ + 3\,\#(\mathrm{III}) + 9\,\#(\mathrm{III}^*) + 4\,\#(\mathrm{IV}) + 8\,\#(\mathrm{IV}^*),\end{aligned} \tag{6.2.39}$$

where $\#(T)$ denotes the number of singular fibers of Kodaira type equal to T.

Proof. Let $c \in C$ and let $z : C_0 \to \mathbb{C}$ be a complex analytic coordinate function on an open neighborhood of c in C such that $z(c) = 0$. Shrinking C_0 if necessary, we can arrange that $C_0 \cap C^{\mathrm{sing}} \subset \{c\}$ and z is a complex analytic diffeomorphism from C_0 onto an open disk centered at the origin in $\mathbb{C}$, of radius $r > 0$. Then $f = |z \circ \varphi|^2$ is a nonnegative real analytic function on $\varphi^{-1}(C_0)$ with zero-set equal to the fiber S_c of φ over c. Also, the gradient vector field $\operatorname{grad} f$ of f with respect to a real analytic Riemannian structure on $\varphi^{-1}(C_0)$ has S_c as its zero-set. The vector field $g = -|\operatorname{grad} f|^{-2} \operatorname{grad} f$ is real analytic in $\varphi^{-1}(C_0) \setminus S_c$ and $\mathrm{d}f(v) = -1$, and it follows that $f(\mathrm{e}^{t\,g}(s)) \searrow 0$ as $t \nearrow f(s)$. Even stronger, a theorem of Łojasiewicz [124] implies the following. For each $s \in \varphi^{-1}(C_0) \setminus S_c$, $\mathrm{e}^{t\,g}(s)$ has a unique limit point $\pi(s)$ on the zero-set S_c of f as $t \nearrow f(s)$. For $s \in \varphi^{-1}(C_0) \setminus S_c$, define $\rho_\epsilon(s) = s \mapsto \mathrm{e}^{\epsilon\, f(s)\, g}(s))$ if $0 \leq \epsilon < 1$ and $\rho_1(s) := \pi(s)$. Furthermore, $\rho_\epsilon(s) = s$ if $s \in S_c$. Then the mappings $\rho_\epsilon : \varphi^{-1}(C_0) \to \varphi^{-1}(C_0)$ for $0 \leq \epsilon \leq 1$ form a deformation retraction of $\varphi^{-1}(C_0)$ onto the zero-set S_c of f. This implies that $\chi_{\mathrm{top}}(\varphi^{-1}(C_0)) = \chi_{\mathrm{top}}(S_c)$, where the conclusion remains valid for any contractible open neighborhood C_0 of c in C such that $C_0 \cap C^{\mathrm{sing}} \subset \{c\}$. If $c \in C^{\mathrm{reg}}$ then S_c is an elliptic curve, which carries a holomorphic vector field without zeros, see Section 2.3.1, hence $\chi_{\mathrm{top}}(S_c) = 0$. It follows that $\chi_{\mathrm{top}}(\varphi^{-1}(C_0)) = 0$ for any contractible open subset C_0 of C^{reg}.

For any open subsets A and B of a topological space, where all homology groups of $A \cap B$, A, B, and $A \cup B$ are finite-dimensional, one has a long exact sequence

$$\cdots \to \mathrm{H}\,q(A \cap B) \to \mathrm{H}_q(A) \oplus \mathrm{H}_q(B) \to \mathrm{H}_q(A \cup B) \to \mathrm{H}_{q-1}(A \cap B) \to \cdots,$$

called the *homology Mayer–Vietoris sequence*; see for instance Bott and Tu [22, Corollary 15.6]. Because the alternating sum of the dimensions of a finite exact

sequence of finitedimensional vector spaces is equal to zero, it follows that $\chi_{\text{top}}(A \cup B) = \chi_{\text{top}}(A) + \chi_{\text{top}}(B) - \chi_{\text{top}}(A \cap B)$. Iterating this identity we obtain, if the U_i, $1 \leq i \leq N$, form a finite open covering of S such that all homology groups of all intersections are finite-dimensional, that

$$\chi_{\text{top}}(S) = \sum (-1)^k \, \chi_{\text{top}}(U_{i_0} \cap \cdots \cap U_{i_k}),$$

where the sum is over all increasing sequences $1 \leq i_0 < \cdots < i_k \leq N$ such that $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$. There exists a covering of C with open subsets C_i, $1 \leq i \leq N$, such that for each i the set C_i contains at most one singular value of φ, $C_i \cap C_j \cap C^{\text{sing}} = \emptyset$ if $i \neq j$, and all intersections, when nonempty, are contractible. Let $U_i := \varphi^{-1}(C_i)$. Then $\chi_{\text{top}}(U_{i_0} \cap \cdots \cap U_{i_k}) = 0$ if $k > 0$, $\chi_{\text{top}}(U_i) = 0$ if $U_i \cap C^{\text{sing}} = \emptyset$, and $\chi_{\text{top}}(U_i) = \chi_{\text{top}}(S_c)$ if $U_i \cap C^{\text{sing}} = \{c\}$. This proves that the topological Euler number of S is equal to the sum of the topological Euler numbers of the singular fibers of φ.

The description in Section 6.2.6 of the singular fibers, provided with a triangulation as in Remark 6.2.3, leads to the following topological Euler numbers. The smooth model $\widetilde{S}_c$ of a singular fiber of Kodaira type I_1 is a complex projective line = Riemann sphere, which has $\chi = 2$, whereas the normal crossing corresponds to two points on $\widetilde{A}$, and therefore $\chi_{\text{top}}(S_c) = 2 - 1 = 1$. For the Kodaira type II we have $\widetilde{S}_c \simeq \mathbb{P}^1$ and only one point on it corresponding to the singular point of S_c; hence $\chi_{\text{top}}(S_c) = 2$. The Kodaira type III consists of two Riemann spheres meeting at one point, hence $\chi = 2 \cdot 2 - 1 = 3$. The Kodaira type IV, where three Riemann spheres meet at one point, has $\chi = 3 \cdot 2 - 2 = 4$. In all other cases we have Riemann spheres with intersection diagrams as indicated in Section 6.2.6, where at each intersection point only two Riemann spheres meet. Therefore $\chi_{\text{top}}(S_c)$ is equal to two times the number of vertices minus the number of edges in the intersection diagram. This leads to the following Kodaira type, intersection diagram, and topological Euler number χ: I_b with $b > 1$, $\mathrm{A}^{(1)}_{b-1}$, and $\chi = 2\,b - b = b$; I^*_b, $\mathrm{D}^{(1)}_{b+4}$, and $\chi = 2\,(b+5) - (b+4) = b+6$; II^*, $\mathrm{E}^{(1)}_8$, and $\chi = 2 \cdot 9 - 8 = 10$; III^*, $\mathrm{E}^{(1)}_7$, and $\chi = 2 \cdot 8 - 7 = 9$. IV^*, $\mathrm{E}^{(1)}_6$, and $\chi = 2 \cdot 7 - 6 = 8$. □

The following lemma is Kodaira's [109, III, (12.6) on p. 14]. The fact that the Euler number of each fiber is equal to the order of the zero of the discriminant $\Delta = {g_2}^3 - 27\,{g_3}^2$ allows another proof if $\varphi : S \to C$ admits a holomorphic section. See Corollary 6.3.3 and the subsequent paragraph.

Lemma 6.2.30 *We have* $\mathrm{K}_S \cdot \mathrm{K}_S = 0$ *and* $\chi_{\text{top}}(S) = 12\,\chi(S, \mathcal{O})$.

Proof. It follows from Theorem 2.1.4 that the holomorphic line bundle $\mathrm{K}_C \otimes \mathfrak{f}^*$ over C has a nonzero meromorphic section s, with divisor $\mathrm{Div}(s) = \sum_{\gamma \in \Gamma} m_\gamma \, \gamma$, where Γ is a finite subset of C and $m_\gamma \in \mathbb{Z}$ for every $\gamma \in \Gamma$. In view of Theorem 6.2.18, $\omega := \iota(\varphi^*(s))$ is a meromorphic section of K_S, with divisor

$$\mathrm{Div}(\omega) = \varphi^*(\,\mathrm{Div}(s)) = \varphi^*(\sum_{\gamma \in \Gamma} m_\gamma \, \gamma) = \sum_{\gamma \in \Gamma} m_\gamma \, \varphi^*(\gamma) = \sum_{\gamma \in \Gamma} m_\gamma \, S_\gamma,$$

where the fibers S_γ of φ over γ are viewed as divisors; see Section 6.1. Lemma 2.1.2 and (2.1.24) imply that $\mathrm{pd}^{-1} \circ \mathrm{c}(\mathrm{K}_S) = \mathrm{pd}^{-1} \circ \mathrm{c}(\delta(\mathrm{Div}(\omega)) = [\mathrm{Div}(\omega)]$; hence

$$\mathrm{K}_S \cdot \mathrm{K}_S = \mathrm{Div}(\omega) \cdot \mathrm{Div}(\omega) = \sum_{\gamma,\, \gamma' \in \Gamma} m_\gamma \, m_{\gamma'} \, S_\gamma \cdot S_{\gamma'} = 0,$$

because $S_\gamma \cdot S_{\gamma'} = 0$ for all $\gamma,\ \gamma' \in \Gamma$; see Lemma 6.1.2.

The formula $\chi_{\mathrm{top}}(S) = 12\, \chi(S,\, \mathcal{O})$ follows from (6.2.32) and $\mathrm{K}_S \cdot \mathrm{K}_S = 0$. □

The following theorem is due to Kodaira [109, III, Theorem 12.3] and [110, I, Theorem 12].

Theorem 6.2.31 *The holomorphic Euler number of S is*

$$\chi(S,\, \mathcal{O}) = -\ \deg(\mathfrak{f}), \tag{6.2.40}$$

where $\mathfrak{f}$ is the Lie algebra bundle over C of the elliptic fibration $\varphi : S \to C$.

Proof. We spell out the arguments of Kodaira [109, III, p. 15], [109, I, Theorems 2.2, 2.3 and Appendix II], and [110, p. 773]. The proof of Barth, Hulek, Peters, and van de Ven [11, p. 214] is shorter, but uses concepts that we have not introduced in this book.

Let $D = \varphi^{-1}(\mathfrak{D})$ for a subset $\mathfrak{D}$ of C^{reg} of $m > 0$ elements. The set $\mathfrak{D}$ can and will be chosen such that if v and ξ are holomorphic sections over C of $\mathfrak{f}$ and $\mathfrak{f}^*$, then $v|_{\mathfrak{D}} = 0$ and $\xi|_{\mathfrak{D}} = 0$ implies that $v = 0$ and $\xi = 0$, respectively. The minimal such sets $\mathfrak{D}$ are obtained as follows. If neither $\mathfrak{f}$ nor $\mathfrak{f}^*$ admits a nonzero holomorphic section, then we can take $\mathfrak{D} = \emptyset$. If both $\mathfrak{f}$ and $\mathfrak{f}^*$ admit respective nonzero holomorphic sections v and ξ, then $\langle v,\, \xi \rangle$ is a nonzero holomorphic function on C, hence a nonzero constant, the holomorphic line bundles $\mathfrak{f}$ and $\mathfrak{f}^*$ are trivial, and $\mathfrak{D} = \{c\}$ for any $c \in C^{\mathrm{reg}}$ will do. In the remaining case the space V of all holomorphic sections of $\mathfrak{f}$ or $\mathfrak{f}^*$ has a positive dimension, where $\mathfrak{f}^*$ or $\mathfrak{f}$ has no nonzero holomorphic sections, respectively. For any nonzero $v \in V$, the zero-set Z of V is a finite subset of C, and because also $C^{\mathrm{sing}} = C \setminus C^{\mathrm{reg}}$ is a finite subset of C, it follows that $C^{\mathrm{reg}} \setminus Z = C \setminus (C^{\mathrm{sing}} \cup Z)$ is not empty. Using this, one can inductively find $v_i \in V$, $c_i \in C^{\mathrm{reg}}$, $1 \le i \le \dim V$, such that $v_i(c_i) \neq 0$ and $v_i(c_j) = 0$ whenever $j < i$. The invertibility of the matrix $v_i(c_j)$ implies that if $\lambda_i \in \mathbb{C}$ and $\sum_i \lambda_i\, v_i(c_j) = 0$ for all j, then $\lambda_i = 0$ for all i. It follows that the v_i are linearly independent, hence form a basis of V, and $v \in V$, $v(c_j) = 0$ for every j imply $v = 0$, and we can take $\mathfrak{D}$ equal to the set of the c_i.

Let $\mathfrak{L}$ denote a holomorphic line bundle over C such that $[\mathfrak{L}] = \delta(\mathfrak{D})^{-1}$, when $L := \varphi^*(\mathfrak{L})$ is a holomorphic line bundle over S such that $[L] = \delta(D)^{-1}$. Let s be a holomorphic section of L^{-1} such that $\mathrm{Div}(s) = D$, see Lemma 2.1.2. Let $\iota_D : D \to S$ denote the inclusion mapping, viewed as a holomorphic mapping from the curve D to S. Then multiplication by s, and the restriction operator ι_D^* yield a short exact sequence $0 \to \mathcal{O}(L) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$, which induces the long exact sequence $0 \to \mathrm{H}^0(S,\, \mathcal{O}(L)) \to \mathrm{H}^0(S,\, \mathcal{O}_S) \to \mathrm{H}^0(D,\, \mathcal{O}_D) \to \mathrm{H}^1(S,\, \mathcal{O}(L)) \to \mathrm{H}^1(S,\, \mathcal{O}_S) \to \mathrm{H}^1(D,\, \mathcal{O}_D) \to \mathrm{H}^2(S,\, \mathcal{O}(L)) \to \mathrm{H}^2(S,\, \mathcal{O}_S) \to$

0 in cohomology. Because the alternating sum of the dimensions in a finite exact sequence of finite-dimensional vector spaces is equal to zero, it follows that $\chi(S,\,\mathcal{O}(L)) - \chi(S,\,\mathcal{O}) + \chi(D,\,\mathcal{O}) = 0$. The holomorphic Euler number of D is equal to the sum of the holomorphic Euler numbers $\chi(F,\,\mathcal{O}) = 1 - \mathrm{g}(F)$ of each of its connected components F. Because F is a smooth fiber of φ, it is an elliptic curve, hence $\mathrm{g}(F) = 0$, and it follows that $\chi(S,\,\mathcal{O}) = \chi(S,\,\mathcal{O}(L)) = \dim \mathrm{H}^0(S,\,\mathcal{O}(L)) - \dim \mathrm{H}^1(S,\,\mathcal{O}(L)) + \dim \mathrm{H}^2(S,\,\mathcal{O}(L))$. Because multiplication by s is an injective linear mapping from $\mathrm{H}^0(S,\,\mathcal{O}(L))$ onto the space of all holomorphic functions on S that vanish on D, we have $\mathrm{H}^0(S,\,\mathcal{O}(L)) = 0$. In order to avoid the computation of $\dim \mathrm{H}^1(S,\,\mathcal{O}(L))$, we replace $\mathrm{H}^1(S,\,\mathcal{O}_S) \to \mathrm{H}^1(D,\,\mathcal{O}_D)$ in the long exact sequence by $N \to 0$, where N is the kernel of the restriction mapping $\iota_D^* : \mathrm{H}^1(S,\,\mathcal{O}_S) \to \mathrm{H}^1(D,\,\mathcal{O}_D)$. The maximum principle implies that each holomorphic function on S and D is constant on each connected component; hence $\dim \mathrm{H}^0(S,\,\mathcal{O}_S) = 1$ and $\dim \mathrm{H}^0(D,\,\mathcal{O}_D) = m$. Since the alternating sum of the dimensions in a finite exact sequence of finite-dimensional vector spaces is equal to zero, we obtain in view of (6.2.29) for $n = 2$ that

$$\chi(S,\,\mathcal{O}) = 1 - m - \dim N + \dim \mathrm{H}^2(S,\,\mathcal{O}(L)). \tag{6.2.41}$$

Assume that S admits a Kähler structure. In order to compute the dimension of $N = \ker \iota_D^* : \mathrm{H}^1(S,\,\mathcal{O}_S) \to \mathrm{H}^1(D,\,\mathcal{O}_D)$, we use the canonical isomorphism of (6.2.24) with $M = S$, $q = 1$, $p = 0$, and $L = \mathbb{C}$, in order to obtain a canonical isomorphism of $\mathrm{H}^1(S,\,\mathcal{O}_S)$ with $\mathrm{H}_{\overline{\partial}}(\mathcal{F}_S^{(0,\,1)})$, where complex conjugation yields a canonical complex antilinear isomorphism with $\mathrm{H}_{\partial}(\mathcal{F}_S^{(1,\,0)})$. The same holds with S replaced by D, and the word "canonical" means that the isomorphisms commute with the restriction operator ι_D^*. The assumption that S is Kähler implies that $\alpha \mapsto \alpha + \partial(\mathrm{H}^0(S,\,\mathcal{F}^{(0,\,0)}))$ is a linear isomorphism from the space $\mathrm{H}^0(S,\,\Omega^1)$ of all holomorphic complex one-forms on S onto $\mathrm{H}_{\partial}(\mathcal{F}_S^{(1,\,0}$), and it follows that $\dim N$ is equal to the dimension of the space of all $\alpha \in \mathrm{H}^0(S,\,\Omega^1)$ such that $\iota_D^*\alpha \in \partial(\mathrm{H}^0(D,\,\mathcal{F}^{(0,\,0)}))$. Assume that $\alpha \in \mathrm{H}^0(S,\,\Omega^1)$, in which case α is closed, $\overline{\partial}\alpha = \partial\alpha = 0$, and $\iota_D^*\,\alpha = \partial f$ for a C^∞ function f on D. Then $\overline{\partial} \circ \partial f = \overline{\partial} \circ \iota_D^*\alpha = \iota_D^* \circ \overline{\partial}\alpha = 0$, that is, f is a harmonic function on the compact complex analytic curve D. The maximum principle applied to the real and imaginary parts of f implies that f is constant on each connected component of f, which in turn implies that $\partial f = 0$, hence $\iota_D^*\,\alpha = 0$. We conclude that $\dim N$ is equal to the dimension of the kernel of the pullback operator $\iota_D^* : \mathrm{H}^0(S,\,\Omega^1) \to \mathrm{H}^0(D,\,\Omega^1)$ acting on holomorphic complex one-forms. For each $c \in C^{\mathrm{reg}}$, $\beta_c := \iota_{S_c}^*\,\alpha$ is a holomorphic complex one-form on S_c, and therefore an element of $\mathfrak{f}_c^*$; see the beginning of Section 6.2.7. This defines a holomorphic section β of $\mathfrak{f}^*$ over C^{reg}. If C_0 is an open subset of C and $v : C_0 \to \mathfrak{f}$ a holomorphic section without zeros, a trivialization of $\mathfrak{f}$ over C_0, then $\mathrm{i}_v\,\alpha = \varphi^*\langle v,\,\beta\rangle$ on $C_0 \cap C^{\mathrm{reg}}$, and this shows that β extends to a unique holomorphic section of $\mathfrak{f}^*$ over C, which we denote by $\iota_{\mathrm{fiber}}^*\alpha$. Because $\iota_D^* = \iota_{\mathfrak{D}}^* \circ \iota_{\mathrm{fiber}}^*$, and every holomorphic section of $\mathfrak{f}^*$ that vanishes on $\mathfrak{D}$ is equal to zero, it follows that $\ker \iota_D^* = \ker \iota_{\mathrm{fiber}}^*$.

Let α be a holomorphic complex one-form on S such that $\iota_{\text{fiber}}^*\alpha = 0$. Let $c \in C^{\text{reg}}$ and $u \in \mathrm{T}_c C$. For every $s \in S_c$ the linear mapping $\mathrm{T}_s\varphi : \mathrm{T}_s S \to \mathrm{T}_c C$ is surjective; hence there exists $v \in \mathrm{T}_s S$ such that $\mathrm{T}_s\varphi(v) = u$. Because two such v differ by an element of $\ker \mathrm{T}_s\varphi$, on which α_s is equal to zero, the complex number $f(s) = \alpha_s(v)$ does not depend on the choice of v. This defines a holomorphic function f on S_c, and because S_c is compact and connected, it follows from the maximum principle that f is a constant. f depends linearly on $u \in \mathrm{T}_c C$ and holomorphically on $c \in C^{\text{reg}}$. That is, there is a unique holomorphic complex one-form ζ on C^{reg} such that the restriction to $\varphi^{-1}(C^{\text{reg}})$ of α is equal to $\varphi^*\zeta$. Let $c \in C^{\text{sing}}$. Then there exists an open neighborhood C_0 of c in C and a holomorphic section $\sigma : C_0 \to S$ of φ over C_0, and $\zeta = (\varphi \circ \sigma)^*\zeta = \sigma^* \circ \varphi^*\zeta = \sigma^*\alpha$ over $C_0 \setminus C^{\text{sing}}$ shows that ζ has a holomorphic extension to C. Because conversely $\iota_{\text{fiber}}^*(\varphi^*\zeta) = 0$ for every complex one-form ζ on C, we have proved that the kernel of ι_{fiber}^* is equal to the image of φ^*. Because φ^* is injective, it follows that the kernel of ι_{fiber}^* is isomorphic to the space of all holomorphic complex one-forms on C, and therefore its dimension is equal to the genus $\mathrm{g}(C)$ of the base curve C; hence $\dim N = \dim \ker \iota_{\text{fiber}}^* = \mathrm{g}(C)$.

It follows from the Serre duality (6.2.26) that $\dim \mathrm{H}^2(S, \mathcal{O}(L))$ is equal to the dimension of the space of all holomorphic sections of

$$\mathrm{K}_S \otimes L^* \simeq \varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*) \otimes \varphi^*(\mathfrak{L}^*) = \varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^* \otimes \mathfrak{L}^*),$$

where the first identity follows from Theorem 6.2.18. It follows from Lemma 2.1.3 that the space of all holomorphic sections over S of $\varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^* \otimes \mathfrak{L}^*)$ is isomorphic to the space H of all holomorphic sections over C of $\mathrm{K}_C \otimes \mathfrak{f}^* \otimes \mathfrak{L}^*$. The formula of Riemann–Roch (2.1.14) with L replaced by $M = \mathfrak{f} \otimes \mathfrak{L}$, when $\deg(M) = \deg(\mathfrak{f}) - m$, implies that the dimension of H is equal to $\dim \mathrm{H}^0(C, \mathcal{O}(M)) - 1 + \mathrm{g}(C) - \deg(\mathfrak{f}) + m$. Because the space of all holomorphic sections of $M = \mathfrak{f} \otimes \mathfrak{L}$ is isomorphic to the space of all holomorphic sections of $\mathfrak{f}$ that vanish on $\mathfrak{D}$, and the latter space is equal to zero because of the choice of the set $\mathfrak{D}$ at the beginning of the proof, we arrive at $\dim \mathrm{H}^2(S, \mathcal{O}(L^*)) = -1 + \mathrm{g}(C) - \deg(\mathfrak{f}) + m$. Combining this with (6.2.41) and $\dim N = \mathrm{g}(C)$, we obtain the desired identity (6.2.40) if S is Kähler.

In the general case we use the basic member $\beta : B \to C$ of the family $\mathcal{F}(J, M)$ to which $\varphi : S \to C$ belongs; see Definition 6.4.5 and Theorem 6.4.3. Because β admits a holomorphic section, Corollary 6.2.28 yields that B is Kähler, and hence $\chi(B, \mathcal{O}) = -\deg(\mathfrak{f}_B)$. It follows from the description of the members of $\mathcal{F}(J, M)$ in Theorem 6.4.8 that $\mathfrak{f}_S = \mathfrak{f}_B$ and S and B have the same configuration of singular fibers. Therefore (6.2.39) implies that $\chi(S) = \chi(B)$, hence $\chi(S, \mathcal{O}) = \chi(B, \mathcal{O})$ in view of Lemma 6.2.30. We therefore conclude that $\chi(S, \mathcal{O}) = \chi(B, \mathcal{O}) = -\deg(\mathfrak{f}_B) = -\deg(\mathfrak{f}_S)$. □

Corollary 6.2.32 *Let* $\iota_{\text{fiber}}^* : \mathrm{H}^0(S, \Omega^1) \to \mathrm{H}^0(C, \mathcal{O}(\mathfrak{f}^*))$ *be defined as in the proof of Theorem* 6.2.31. *Then either* $\iota_{\text{fiber}}^* = 0$ *and* $\varphi^* : \mathrm{H}^0(C, \Omega^1) \to \mathrm{H}^0(S, \Omega^1)$ *is a linear isomorphism, or* $\mathrm{b}_1(S)$ *is even and* $\mathfrak{f}$ *is trivial. In the first case* $\mathrm{h}^{1,0}(S) = \mathrm{g}(C)$. *In the second case* $\mathrm{h}^{1,0}(S) = \mathrm{g}(C) + 1$, *or more precisely,* $\iota_{\text{fiber}}^* : \mathrm{H}^0(S, \Omega^1) \to \mathrm{H}^0(C, \mathcal{O}(\mathfrak{f}^*)) \simeq \mathbb{C}$ *is surjective and* φ^* *is an injective linear mapping from the* $\mathrm{g}(C)$-

dimensional vector space $\mathrm{H}^0(C,\,\Omega^1)$ *onto the codimension-one linear subspace* $\ker \iota^*_{\mathrm{fiber}}$ *of* $\mathrm{H}^0(S,\,\Omega^1)$.

Proof. We have

$$\mathrm{H}^0(S,\,\mathcal{O}(\mathrm{K}_S)) = \mathrm{H}^0(S,\,\mathcal{O}(\varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*))) = \varphi^*(\mathrm{H}^0(C,\,\mathcal{O}(\mathrm{K}_C \otimes \mathfrak{f}^*))) \simeq \mathrm{H}^0(C,\,\mathcal{O}(\mathrm{K}_C \otimes \mathfrak{f}^*)),$$

where the second identity and the last isomorphism follow from Theorem 6.2.18 and Lemma 2.1.3, respectively. The formula of Riemann–Roch (2.1.14) with $L = \mathfrak{f}$ implies that

$$\dim\,\mathrm{H}^0(C,\,\mathcal{O}(\mathrm{K}_C \otimes \mathfrak{f}^*)) = \dim\,\mathrm{H}^0(C,\,\mathcal{O}(\mathfrak{f})) - 1 + \mathrm{g}(C) - \deg(\mathfrak{f}).$$

Since the dimension of $\mathrm{H}^0(S,\,\mathcal{O}(\mathrm{K}_S)) = \mathrm{H}^0(S,\,\Omega^2) \simeq \mathrm{H}^2(S,\,\mathcal{O})$ is equal to $\mathrm{p_g}(S)$, and (6.2.40) leads to $-\deg(\mathfrak{f}) = \chi(S,\,\mathcal{O}) = 1 - \mathrm{q}(S) + \mathrm{p_g}(S)$, a comparison of the two displayed formulas leads to $\mathrm{q}(S) = \dim\,\mathrm{H}^0(C,\,\mathcal{O}(\mathfrak{f})) + \mathrm{g}(C)$. Theorem 6.2.23(iii) yields that $\mathrm{q}(S) = \mathrm{h}^{1,\,0}(S) := \dim \mathrm{H}^0(S,\,\Omega^1)$ if $\mathrm{b}_1(S)$ is even, and $\mathrm{q}(S) = \mathrm{h}^{1,\,0}(S) + 1$ if $\mathrm{b}_1(S)$ is odd.

The proof of Theorem 6.2.31 yields the exact sequence $0 \to \mathrm{H}^0(C,\,\Omega^1) \to \mathrm{H}^0(S,\,\Omega^1) \to \mathrm{H}^0(C,\,\mathcal{O}(\mathfrak{f}^*))$, where the second and third arrows are φ^* and ι^*_{fiber}, respectively. Therefore $\mathrm{h}^{1,\,0}(S) = \mathrm{g}(C) + d$, if where d denotes the dimension of the image of ι^*_{fiber} in $\mathrm{H}^0(C,\,\mathcal{O}(\mathfrak{f}^*))$. It follows that the dimension of $\mathrm{H}^0(C,\,\mathcal{O}(\mathfrak{f}))$ is equal to d or $d+1$ if $\mathrm{b}_1(S)$ is even or odd, respectively. Note that $d = 0$ if and only if $\iota^*_{\mathrm{fiber}} = 0$ and φ^* is an isomorphism. If both $\mathfrak{f}$ and $\mathfrak{f}^*$ have nonzero holomorphic sections v and ξ, the nonzero holomorphic function $\langle v,\,\xi\rangle$ on C is a nonzero constant; hence v and ξ have no zeros, and the conclusion is that $\mathfrak{f}$ and $\mathfrak{f}^*$ are trivial. Therefore, if $\mathrm{b}_1(S)$ is even and $d > 0$, then $\mathfrak{f}$ is trivial and $d = 1$, whereas $d > 0$ leads to a contradiction if $\mathrm{b}_1(S)$ is odd. □

Corollary 6.2.33 *We have* $-\deg(\mathfrak{f}) = \chi(S,\,\mathcal{O}) \geq 0$*, with equality if and only if* $\deg(\mathfrak{f}) = 0$ *if and only if* $\chi_{\mathrm{top}}(S) = 0$ *if and only if there are no singular fibers if and only if* $\varphi : S \to C$ *is a locally trivial complex analytic fiber bundle.*

Proof. Equation (6.2.39) implies that $\chi_{\mathrm{top}}(S) \geq 0$ with equality if and only if there are no singular fibers, whereas Lemma 6.2.30 implies that $\chi(S,\,\mathcal{O})$ has the same sign as $\chi_{\mathrm{top}}(S)$. Assume that there are no singular fibers. Then the modulus function J, which is a meromorphic function on S, has no poles on C, and it follows from the maximum principle that J is constant. Therefore Lemma 6.4.10 with $C^{\mathrm{reg}} = C$ implies that $\varphi : S \to C$ is a locally trivial complex analytic fiber bundle. Conversely, the latter implies that φ has no singular fibers. □

For the existence of nontrivial degree-zero Lie algebra bundles $\mathfrak{f}$, see Remark 6.4.14.

Lemma 6.2.34 *Let* E *be a holomorphic section of the elliptic fibration* $\varphi : S \to C$*. Then*

$$-E \cdot E = -\deg(\mathfrak{f}) = \chi(S,\,\mathcal{O}) = \chi_{\mathrm{top}}(S)/12, \tag{6.2.42}$$

where $\chi(S, \mathcal{O})$ and $\chi_{\text{top}}(S)$ are the holomorphic and topological Euler numbers of S as defined in (6.2.30) and (6.2.38), respectively. The number in (6.2.42) is nonnegative, and equal to zero if and only if $\varphi : S \to C$ is a locally trivial complex analytic fiber bundle with constant modulus function J. Some (every) holomorphic section is a -1 curve in $S \Leftrightarrow C \simeq \mathbb{P}^1$ and the number in (6.2.42) is equal to one $\Leftrightarrow$ S is a rational elliptic surface.

Proof. The adjunction formula, see (6.2.3), implies that for any smooth curve E in S we have $(\mathrm{K}_S)_E \simeq \mathrm{K}_E \otimes \mathrm{N}(E)^*$. If $E = \sigma(C)$ for a holomorphic section $\sigma : C \to S$ of φ, then applying σ^* to $\mathrm{K}_S \simeq \varphi^*(\mathrm{K}_C \otimes \mathfrak{f}^*)$, see Theorem 6.2.18, and using $\sigma^* \circ \varphi^* = (\varphi \circ \sigma)^* = 1^* = 1$ and Remark 6.2.19, we obtain that $\mathrm{K}_C \otimes \mathfrak{f}^* \simeq \mathrm{K}_C \otimes \sigma^*(\mathrm{N}_S(E)^*)$, or equivalently $\mathfrak{f} \simeq \sigma^*(\mathrm{N}(E))$. This implies that the Chern number of $\mathfrak{f}$ is equal to $\deg(\mathfrak{f}) = \deg(N(E)) = E \cdot E$, which implies the first identity in (6.2.42).

It follows from Corollary 6.2.28 that S is projective algebraic, hence Kähler, and therefore the second and third identities follow from (6.2.40) and Lemma 6.2.30, respectively. If $\chi_{\text{top}}(S) = 0$ then it follows from (6.2.39) that φ has no singular fibers, and the conclusions of Corollary 6.2.33 hold. See also Remark 6.4.14. The first equivalence in the last statement follows from the observations that $C \simeq E$, and E is a -1 curve if and only if $E \simeq \mathbb{P}^1$ and $E \cdot E = -1$. The second equivalence follows from Definition 9.1.4 with the choice of (c) in Theorem 9.1.3. □

Remark 6.2.35. Matsumoto [132] has proved the following theorem. Let $\varphi_1 : S_1 \to C_1$ and $\varphi_2 : S_2 \to C_2$ be relatively minimal elliptic fibrations of compact surfaces, each with at least one singular fiber and without multiple singular fibers. Then there exists an orientation-preserving real diffeomorphism Φ from S_1 onto S_2 if and only if $\mathrm{g}(C_1) = \mathrm{g}(C_2)$ and $\chi_{\text{top}}(S_1) = \chi_{\text{top}}(S_2)$, that is, if and only if the topological Euler numbers of both the base curves and the surfaces are the same. Here the complex analytic surfaces S_1 and S_2 are viewed as real four-dimensional smooth manifolds, and the diffeomorphism Φ in general will not be complex analytic. This generalizes the theorem of Kas [103], which deals with the case that $\mathrm{g}(C_1) = \mathrm{g}(C_2) = 0$. ⊘

6.2.11 Monodromy of the Period Lattices

In this subsection we retain the assumption that $\varphi : S \to C$ is a relatively minimal elliptic fibration without multiple singular fibers. We recall from Section 6.2.7 that for each $c \in C^{\text{reg}}$ we have the complex one-dimensional vector space $\mathfrak{f}_c$ of all holomorphic vector fields on the elliptic curve S_c. Let $P_c \subset \mathfrak{f}_c$ be the period group as in Section 2.3.3, such that the exponential mapping defines an isomorphism from the elliptic curve $\mathfrak{f}_c / P_c$ onto the group $\mathfrak{F}_c = \mathrm{Aut}(S_c)^{\mathrm{o}}$ of all translations on S_c. Because $\mathfrak{F}_c$ acts freely and transitively on S_c, we have for each choice of an element $\sigma(c) \in S_c$ that the mapping $g \mapsto g \cdot \sigma(c)$ is an isomorphism from $\mathfrak{F}_c$ onto the smooth fiber S_c. Recall that because S_c is compact, P_c has a $\mathbb{Z}$-basis that is an $\mathbb{R}$-basis of $\mathfrak{f}_c$, and

therefore P_c is isomorphic to $\mathbb{Z}^2$ as a group. We also recall from Section 2.3.3 that P_c is canonically isomorphic to the one-dimensional homology group $\mathrm{H}_1(S_c, \mathbb{Z})$ of S_c.

Lemma 6.2.36 *The P_c, $c \in C^{\mathrm{reg}}$, form a holomorphic subbundle P of $\mathfrak{f}$ over C^{reg} with discrete fibers. The modulus function $J : C^{\mathrm{reg}} \to \mathbb{C} : c \mapsto \mathrm{J}(S_c) = \mathrm{J}(\mathfrak{f}_c/P_c)$, see* (2.3.10)*, is holomorphic.*

Proof. Let $c_0 \in C^{\mathrm{reg}}$, C_0 an open neighborhood of $c_0 \in C^{\mathrm{reg}}$, and $v : C_0 \to \mathfrak{f}$ a holomorphic section of $\mathfrak{f}$ over C_0 without zeros. As in Theorem 6.2.18, we view v as a holomorphic fiber-tangent vector field without zeros on the open subset $\varphi^{-1}(C_0)$ of S. We write $v_c \in \mathfrak{f}_c$ but $v(s) \in \mathrm{T}_s S$ when v is viewed as a vector field on S. Also note that $C_0 \times \mathbb{C} \ni (c, t) : t\, v_c \in \mathfrak{f}$ is the inverse of a trivialization of $\mathfrak{f}$ over C_0. Shrinking C_0 if necessary, there exists a holomorphic section $\sigma : C_0 \to S$ of φ over C_0. If we write $s_0 = \sigma(c_0)$, then, because $\mathrm{T}_{s_0}\varphi \neq 0$, there exists a holomorphic function ψ on an open neighborhood S_0 of s_0 in S with $\psi(s_0) = 0$ such that (φ, ψ) is a complex analytic diffeomorphism from S_0 onto an open neighborhood of $(c_0, 0)$ in $C_0 \times \mathbb{C}$.

If $v_0 \in P_{c_0} \subset \mathfrak{f}_{c_0}$, that is, e^{v_0} is equal to the identity on S_{c_0}, then there is a unique $t_0 \in \mathbb{C}$ such that $v_0 = t_0\, v_{c_0}$. An application of the implicit function theorem to the equation $\psi(\mathrm{e}^{t\,v}(\sigma(c))) = \psi(\sigma(c)))$ with t near t_0 as the unknown and c near c_0 as the parameter yields for every c near c_0 a unique solution $t = t(c)$ near t_0 that depends holomorphically on c. Because $\varphi(\mathrm{e}^{t\,v}(\sigma(c))) = \varphi(\sigma(c))) = c$ for every $t \in \mathbb{C}$, it follows that $\mathrm{e}^{t(c)\,v}(\sigma(c))) = \sigma(c)$, hence $t(c)\, v_c \in P_c$, and we have proved the first statement in the lemma.

Let v_0^1, v_0^2 be a $\mathbb{Z}$-basis of $\mathfrak{f}_{c_0}$, with unique t_0^i such that $v_0^i = t_0^i\, v(c_0)$, where it can be arranged, by switching these elements if necessary, that $q := t_0^2/t_0^1 = v_0^2/v_0^1$ is in the complex upper half-plane. Then the above yields for every c near c_0 unique complex numbers $t^i(c)$ near t_0^i such that $v_c^i := t^i(c)\, v_c \in P_c$, where $t^i(c)$ depends holomorphically on c and v_v^1, v_c^2 is a $\mathbb{Z}$-basis of $\mathfrak{f}_c$. It follows that $q(c) := t^2(c)/t^1(c) = v_c^2/v_c^1$ is a holomorphic function of c, and, in the notation of Definition 2.3.4, $J(c) = \mathrm{J}(\mathfrak{f}/P_c) = \mathcal{J}(q(c))$. Because the modular function $\mathcal{J}$ is holomorphic on the complex upper half-plane, it follows that the function J is holomorphic in a neighborhood of c_0 in C. Since this holds for every $c_0 \in C^{\mathrm{reg}}$, the function $J : C^{\mathrm{reg}} \to \mathbb{C}$ is holomorphic. □

Let $\mathrm{SF}(P_c)$ be the set of all $\mathbb{Z}$-bases (v^1, v^2) of P_c that are positively oriented with respect to the orientation defined by the complex structure on $\mathfrak{f}_c$. Here the positivity of the orientation is equivalent to $\mathrm{Im}(q) > 0$ if $q = v^2/v^1 \in \mathbb{C}$. Let $\mathrm{SF}(P)$ denote the locally trivial complex analytic bundle over C^{reg} whose fiber over $c \in C^{\mathrm{reg}}$ is equal to the discrete set $\mathrm{SF}(P_c)$. The formula (2.3.11) defines a free and transitive right action of the group $\mathrm{SL}(2, \mathbb{Z})$ on $\mathrm{SF}(P_c)$, which exhibits $\mathrm{SF}(P)$ as a principal $\mathrm{SL}(2, \mathbb{Z})$-bundle over C^{reg}. Because the fiber is discrete, we have a well-defined monodromy representation, a homomorphism M from the fundamental group of C^{reg} to $\mathrm{SL}(2, \mathbb{Z})$, which we now describe in more detail.

For each continuous mapping $\gamma : [0, 1] \to C^{\mathrm{reg}}$, called a *path in* C^{reg}, and every $v_0 \in P_{\gamma(0)}$, there is a unique path v in P such that $v(0) = v_0$ and $v(t) \in P_{\gamma(t)}$ for

every $0 \le t \le 1$. The "parallel transport" $v_0 \mapsto v(1)$ is an orientation-preserving isomorphism T from $P_{\gamma(0)}$ onto $P_{\gamma(1)}$. Let c_* be a chosen point in C^{reg}, called a *base point*, and γ a *loop in* C^{reg} *based at* c_*, a path in C^{reg} such that $\gamma(1) = \gamma(0) = c_*$. Then T is an orientation preserving automorphism of P_{c_*}. If $(v_*^1,\ v_*^2) \in \mathrm{SF}(P_{c_*})$ and $v^i = T_\gamma(v_*^i)$, then there is a unique matrix $M \in \mathrm{SL}(2,\ \mathbb{Z})$ such that $v^i = \sum_j M_j^i\, v_*^j$, called *the monodromy matrix defined by the loop* γ. This is (2.3.11) in a somewhat changed notation.

The matrix M does not change under a homotopy of the loop γ in C^{reg} based at c_*, and therefore we can write $M = M([\gamma])$, where $[\gamma]$ denotes the homotopy class of γ. The set of all homotopy classes of loops in C^{reg} based at c_*, provided with the group structure of concatenation of loops, is called the *fundamental group* $\pi_1(C^{\mathrm{reg}},\ c_*)$ *of* C^{reg} *with respect to the base point* c_*. The mapping $M : [\gamma] \mapsto M([\gamma])$ is a homomorphism from $\pi_1(C^{\mathrm{reg}},\ c_*)$ to the group $\mathrm{SL}(2,\ \mathbb{Z})$, called the *monodromy representation of the elliptic surface* $\varphi : S \to C$. The subgroup $\mathfrak{M} := M(\pi_1(C^{\mathrm{reg}},\ c_*))$ of $\mathrm{SL}(2,\ \mathbb{Z})$ is called the *monodromy group*. Note that the homomorphism M induces an isomorphism of groups from $\pi_1(C^{\mathrm{reg}},\ c_*)/\ker M$ onto $\mathfrak{M}$.

In general $\pi_1(C^{\mathrm{reg}},\ c_*)$ is highly noncommutative, with the result that the fundamental groups for different base points are not isomorphic to each other in a unique way, since two isomorphisms differ by a conjugation. Note also that another choice of $(v_*^1,\ v_*^2) \in \mathrm{SF}(P_{c_*})$ corresponds to a conjugation in $\mathrm{SL}(2,\ \mathbb{Z})$. Therefore the noncommutativity of $\mathrm{SL}(2,\ \mathbb{Z})$ results in, although this is not expressed by the notation, the homomorphism M also depending in general on the choice of the oriented $\mathbb{Z}$-basis $(v_*^1,\ v_*^2)$ of P_{c_*}.

Example 4. In Example 2 with $z_0 = 1$ and $z_1 = -z$ we have the family of Weierstrass curves $y^2 - 4\,x^3 + 3\,x + z^6 = 0$. The fiber over z in the elliptic surface S is singular if and only if $z = \omega^k$, $k \in \mathbb{Z}/12\,\mathbb{Z}$, where $\omega = \mathrm{e}^{2\pi\,\mathrm{i}/12}$. Let, for each $k \in \mathbb{Z}/12\,\mathbb{Z}$, $M(\gamma_k)$ be the monodromy matrix along the loop γ_k in the complex z-plane that starts at $z = 0$, then runs radially outward to a point close to $z = \omega^k$, then runs around $z = \omega^k$ along a small circle, and finally returns to $z = 0$. Write $M_+ := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $M_- := \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Each singular fiber S_{ω^k} in the elliptic surface S of Theorem 6.3.10 is of type I_1, and therefore Table 6.2.40 yields that M_k is conjugate in $\mathrm{SL}(2,\ \mathbb{Z})$ to M_+. If we substitute $z = t^{1/6}\,\omega^k$, where $0 \le t \le 1$, then we have the Weierstrass curve $y^2 - 4\,x^3 + 3\,x + \tau = 0$, where $\tau = t$ if k is even and $\tau = -t$ if k is odd. If $-1 < \tau < 1$, let α_τ be the cycle in the Weierstrass curve in the real $(x,\ y)$-plane over the bounded x-interval where $4\,x^3 - 3\,x \ge \tau$, and β_τ the cycle in the Weierstrass curve when y is purely imaginary and x varies in the bounded real x-interval where $4\,x^3 - 3\,x \le \tau$. These cycles form a $\mathbb{Z}$-basis of the homology group of real one-dimensional cycles of the Weierstrass curve, because they have intersection number ± 1. If $\tau \uparrow 1$ and $\tau \downarrow -1$, the respective curves α_τ and β_τ shrink to a point. That is, the curves α_0 and β_0 correspond to a $\mathbb{Z}$-basis of the homology group of the fiber over $z = 0$ where γ_τ and δ_τ for $\tau \uparrow 1$ and $\tau \downarrow -1$ correspond to a vanishing cycle in the fiber over $z = \omega^k$ when k is even and k is odd, respectively. Because M_- is the only element that is conjugate in $\mathrm{SL}(2,\ \mathbb{Z})$ to M_+ and has second column equal

to the second standard basis vector, it follows that $M(\gamma_k) = M_+$ if k is even and $M(\gamma_k) = M_-$ if k is odd.

A rational elliptic surface has the generic configuration of singular fibers 12 I_1 if and only if the discriminant $\delta(\Delta)$ of $\Delta := {g_2}^3 - 27\,{g_3}^2$ is not equal to zero. Because the set of pairs of homogeneous polynomials g_2, g_3 in $(z_0,\ z_1)$ of the respective degrees 4 and 6 with this property is connected, it follows that for every rational elliptic surface with twelve singular fibers, the monodromy representation is isomorphic to the above one in the following sense. If along a path in the $(g_2,\ g_3)$-space in the complement of $\delta(\Delta) = 0$ the loops γ_k are deformed along to the loops γ'_k around the twelve singular values, then $M(\gamma'_k) = M_+$ if k is even and $M(\gamma'_k) = M_-$ if k is odd. Note that the deformation can force the loops γ'_k to wander around the singular values before returning in quite complicated ways. The proof of Magnus [127, Lemma 3.1] shows that the group $\mathrm{SL}(2,\ \mathbb{Z})$ is generated by M_+ and M_-. Therefore the image of the monodromy representation, the monodromy group $\mathfrak{M}$, is the whole group $\mathrm{SL}(2,\ \mathbb{Z})$.

This is not true for every rational elliptic surface. Let $P_{c_*} \simeq \mathbb{Z}^2$ be the period lattice over a regular value c_* of φ, and let $\mathcal{M}$ be the subgroup of $\mathbb{Z}^2$ generated by the elements $M(\gamma)(p) - p$, with $p \in \mathbb{Z}^2$ and $M(\gamma)$ the monodromy matrix of an element $\gamma \in \pi_1(C^{\mathrm{reg}},\ c_*)$. Proposition 9.2.10 implies that $\mathbb{Z}^2/\mathcal{M} \simeq \mathrm{MW}_{\mathrm{tor}}$, and there are many rational elliptic surfaces for which $\mathrm{MW}_{\mathrm{tor}}$ is nontrivial. Clearly $\mathcal{M} = \mathbb{Z}^2$ if $\mathfrak{M} = \mathrm{SL}(2,\ \mathbb{Z})$. ⊘

Remark 6.2.37. Schmickler-Hirzebruch [176, §3], see also Remark 6.3.12, determined the monodromy representation in the case that $C \simeq \mathbb{P}^1$, and the elliptic fibration has at most three singular fibers. If there is at most one singular fiber then C^{reg} is simply connected, and the monodromy representation is trivial. If there are two singular fibers, then the fundamental group of C^{reg} is isomorphic to $\mathbb{Z}$, and the monodromy representation is of the form $\mathbb{Z} \ni p \mapsto A^p$ for some $A \in \mathrm{SL}(2,\ \mathbb{Z})$, where A is conjugate in $\mathrm{SL}(2,\ \mathbb{Z})$ to one of the monodromy matrices in Table 6.2.40. If there are three singular fibers, then the fundamental group of C^{reg} is a free group with two generators $[\gamma]$ and $[\delta]$, with corresponding monodromy matrices $A,\ B \in \mathrm{SL}(2,\ \mathbb{Z})$. The monodromy group $\mathfrak{M} = M(\pi_1(C^{\mathrm{reg}},\ c_*)$ is the subgroup of $\mathrm{SL}(2,\ \mathbb{Z})$ generated by A and B. For the eight configurations of singular fibers II IV I_0^*, IV* II* I_0^*, 2 III I_0^*, 2 III* I_0^*, 2 II IV*, 3 IV*, 3 IV, and 2 II* IV, the group $\mathfrak{M}$ is isomorphic to $\mathbb{Z}/6\,\mathbb{Z}$, $\mathbb{Z}/4\,\mathbb{Z}$ or $\mathbb{Z}/3\,\mathbb{Z}$, see [176, p. 78]. For the remaining 23 configurations of three singular fibers, the monodromy reresentation is almost surjective, in the sense that $\mathfrak{M}$ is a subgroup of finite index in $\mathrm{SL}(2,\ \mathbb{Z})$, see [176, Table 2 on p. 67]. For the full list of the monodromy groups in all cases, see [176, p. 78, 79]. In seven cases $\mathfrak{M} = \mathrm{SL}(2,\ \mathbb{Z})$. ⊘

6.2.12 Behaviour of the Periods near Singular Values

Let $c_0 \in C^{\mathrm{sing}}$ be a singular value of the elliptic fibration $\varphi : S \to \mathbb{C}$, with the singular fiber S_{c_0} of φ over c_0. Let $z : C_0 \to \mathbb{C}$ be a complex analytic coordinate

function on an open neighborhood C_0 of c_0 in C such that $z(c_0) = 0$. Shrinking C_0 if necessary, we may assume that z is a complex analytic diffeomorphism from C_0 onto $D = \{z \in \mathbb{C} \mid |z| < \delta\}$, where $\delta \in \mathbb{R}_{>0}$. In the sequel we will write φ, S, and S_0 instead of $z \circ \varphi$, $\varphi^{-1}(C_0) = (z \circ \varphi)^{-1}(D)$, and S_{c_0}, respectively, in which case we have the elliptic fibration $\varphi : S \to D$ with the singular fiber S_0 over the origin in D.

Shrinking D if necessary, it can be arranged that S_0 is the only singular fiber of φ, and that there is a holomorphic section v of $\mathfrak{f}$ over D without zeros, where v is viewed as a holomorphic vector field on S; see Theorem 6.2.18. For each $z \in D$, let $P(z)$ denote the set of all $t \in \mathbb{C}$ such that $\mathrm{e}^{t\,v}(s) = s$ when $\varphi(s) = z$. For $z \in D \setminus \{0\}$ the period group $P(z)$ in $\mathbb{C}$ is mapped onto the period group P_z in $\mathfrak{f}_z$ by means of the linear isomorphism $\mathbb{C} \to \mathfrak{f}_z : t \mapsto t\,v_z$. Because the fundamental group $D^{\mathrm{reg}} = D \setminus \{0\}$ is isomorphic to $\mathbb{Z}$, with the loop around 0 in the positive direction as a generator, the monodromy representation is determined by the image $M \in \mathrm{SL}(2, \mathbb{Z})$ of the generator, called the *monodromy matrix around the singular value of* φ. Since the monodromy representation depends on the choice of a $\mathbb{Z}$-basis of $P(z_*)$, where $z_* \in D \setminus \{0\}$ is a chosen base point for the fundamental group, and every other choice of such a basis corresponds to conjugation of the monodromy represenation by an arbitrary element of $\mathrm{SL}(2, \mathbb{Z})$, it is only the conjugacy class of M in $\mathrm{SL}(2, \mathbb{Z})$ that is invariantly defined. Therefore in the sequel "the monodromy matrix" will mean a suitable element in the conjugacy class.

A given $\mathbb{Z}$-basis $p_1(z_*)$, $p_2(z_*)$ of $P(z_*)$ extends in a multivalued holomorphic fashion to $\mathbb{Z}$-bases $p_1(z)$, $p_2(z)$ of $P(z)$, $z \in D \setminus \{0\}$. The analytic continuation of $p_i(z)$ along a loop around the origin in the positive direction is equal to

$$p_i^{\circlearrowleft}(z) = \sum_{j=1}^{2} M_i^j\, p_j(z), \quad \text{where} \quad M = \begin{pmatrix} M_1^1 & M_2^1 \\ M_1^2 & M_2^2 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \tag{6.2.43}$$

is the monodromy matrix. Any other holomorphic section of $\mathfrak{f}$ over an open neighborhood of $z = 0$ is of the form $u\,v$, where $u(z)$ is a unit, a holomorphic function on an open neighborhood of $z = 0$ such that $u(0) \neq 0$. Replacing v by $u\,v$ amounts to replacing $p_1(z)$ and $p_2(z)$ by $p_1(z)/u(z)$ and $p_2(z)/u(z)$, respectively, which leaves the quotient $q(z) = p_2(z)/p_1(z)$ unchanged. Note that $J(z) = \mathcal{J}(q(z))$ is the modulus of the fiber over z. With this notation we have the following lemma. The proof is long, since it is given separately for each type of the singular fiber. For multiple singular fibers, see Corollary 6.2.47.

The statement that the modulus function extends to a holomorphic mapping $J : C \to \mathbb{P}^1$ is due to Kodaira [109, II, Theorem 7.3]. Note that the properties of J mentioned in Table 6.2.40 do not distinguish between types I_b and I_b^*, the types II and IV*, types IV and II*, and types III and III*, in which case for each of these pairs the monodromy matrices are each other's opposites. See Theorems 6.4.3 and 6.4.11 for the relation between the meromorphic functions J on C and the homomorphisms $M : \pi_1(C^{\mathrm{reg}}, c_*) \to \mathrm{SL}(2, \mathbb{Z})$ that can appear as the respective modulus functions and monodromy representations of an elliptic surface over the curve C.

Lemma 6.2.38 *There exist holomorphic functions $a(z)$ and $b(z)$ on an open neighborhood of $z = 0$ such that $a(0) \neq 0$ and for every $z \neq 0$ and $|z|$ sufficiently small, the complex numbers $p_1(z)$ and $p_2(z)$ in Table 6.2.39 below form a $\mathbb{Z}$-basis of $P(z)$. Multiplying the vector field v by a meromorphic function $u(z)$ corresponds to dividing both $p_1(z)$ and $p_2(z)$ by $u(z)$. The modulus function $J : C^{\mathrm{reg}} \to \mathbb{C}$ is determined by the ratio $r(z) = bz)/a(z)$ and extends to a holomorphic mapping $J : C \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. In Table 6.2.40, the value and degree at $z = 0$ of $J(z)$ are given in the second and third columns, respectively, where infinite degree means that J is constant.*

The matrix M in the fourth column is the monodromy matrix around the singular fiber as in (6.2.43)*, where* $\mathrm{o}(M)$ *denotes the order of M. In the last column we have added the Euler number χ of the singular fiber, as given in Lemma 6.2.29.*

Table 6.2.39

Type	$p_1(z)$	$p_2(z)$
I_b	$a(z)$	$(b/2\pi\,\mathrm{i})\,(\log z)\,a(z) + b(z)$
I_b^*	$z^{-1/2}\,a(z)$	$z^{-1/2}\,(b/2\pi\,\mathrm{i})\,(\log z)\,a(z) + z^{-1/2}\,b(z)$
II	$z^{-1/6}\,a(z) + z^{1/6}\,b(z)$	$z^{-1/6}\,\mathrm{e}^{2\pi\mathrm{i}/6}\,a(z) + z^{1/6}\,\mathrm{e}^{-2\pi\mathrm{i}/6}\,b(z)$
II*	$z^{-5/6}\,a(z) + z^{-1/6}\,b(z)$	$z^{-5/6}\,\mathrm{e}^{2\pi\mathrm{i}/6}\,a(z) + z^{-1/6}\,\mathrm{e}^{-2\pi\mathrm{i}/6}\,b(z)$
III	$z^{-1/4}\,a(z) + z^{1/4}\,b(z)$	$z^{-1/4}\,\mathrm{e}^{2\pi\mathrm{i}/4}\,a(z) + z^{1/4}\,\mathrm{e}^{-2\pi\mathrm{i}/4}\,b(z)$
III*	$z^{-3/4}\,a(z) + z^{-1/4}\,b(z)$	$z^{-3/4}\,\mathrm{e}^{2\pi\mathrm{i}/4}\,a(z) + z^{-1/4}\,\mathrm{e}^{-2\pi\mathrm{i}/4}\,b(z)$
IV	$z^{-1/3}\,a(z) + z^{1/3}\,b(z)$	$z^{-1/3}\,\mathrm{e}^{2\pi\mathrm{i}/6}\,a(z) + z^{1/3}\,\mathrm{e}^{-2\pi\mathrm{i}/6}\,b(z)$
IV*	$z^{-2/3}\,a(z) + z^{-1/3}\,b(z)$	$z^{-2/3}\,\mathrm{e}^{2\pi\mathrm{i}/6}\,a(z) + z^{-1/3}\,\mathrm{e}^{-2\pi\mathrm{i}/6}\,b(z)$

Table 6.2.40

Type	$J(0)$	$\deg_0 J$	M	$\mathrm{o}(M)$	χ
I_0	$\notin \{0,\ 1,\ \infty\}$	*Any*	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	0
I_0	0	$\in 3\,\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	0
I_0	1	$\in 2\,\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	0

I_b, $b \in \mathbb{Z}_{>0}$	∞	b	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	∞	b
I_0^*	$\notin \{0, 1, \infty\}$	*Any*	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	6
I_0^*	0	$\in 3\mathbb{Z}$ *or* ∞	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	6
I_0^*	1	$\in 2\mathbb{Z}$ *or* ∞	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	6
I_b^*, $b \in \mathbb{Z}_{>0}$	∞	b	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	∞	$b+6$
II	0	$\in 1+3\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	2
IV*	0	$\in 1+3\mathbb{Z}$ *or* ∞	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	8
IV	0	$\in 2+3\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	4
II*	0	$\in 2+3\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	10
III	1	$\in 1+2\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	3
III*	1	$\in 1+2\mathbb{Z}$ *or* ∞	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	9

Proof. Let ω be the holomorphic and nowhere-vanishing complex two-form on $\varphi^{-1}(C_0)$ such that v is equal to the Hamiltonian vector field, defined by the holomorphic function φ, and with respect to the symplectic form ω, as in Remark 6.2.21. The known behavior of the function φ near the singular fiber leads to sufficiently detailed information about the vector field v near the singular fiber in order to determine the behavior of the period functions $p_1(z)$ and $p_2(z)$ for small $|z|$. We will write $D = \{z \in \mathbb{C} \mid |z| < \delta\}$, where δ is a sufficiently small positive real number.

For the different types of the singular fiber, see Section 6.2.6, the constructions are different. In many cases each irreducible component of the singular fiber is a rational curve = a curve isomorphic to the complex projective line $\mathbb{P}^1$, which is holomorphically embedded into the elliptic surface S. Although the description in

each case is not too complicated, all the cases together make the proof look a bit long. When a certain argument in the proof occurs for the first time, we have tried to give it in sufficient detail, whereas at later occurrences of the argument we will abbreviate the proof by referring to the first occurrence. For the properties of the modular function $\mathcal{J}$ that we use in the proof, see Section 2.3.3.

Type $\mathbf{I_0}$
The fiber S_0 is regular, an elliptic curve, and $p_1(z)$, $p_2(z)$ are holomorphic functions on D, without zeros and such that $\operatorname{Im} q(z) > 0$ if $q(z) = p_2(z)/p_1(z)$. Because $\deg_q \mathcal{J} = 3$ if $\mathcal{J}(q) = 0$, $\deg_q \mathcal{J} = 2$ if $\mathcal{J}(q) = 1$, and $\deg_q \mathcal{J} = 1$ if $\mathcal{J}(q) \notin \{0, 1\}$, the behavior of the modulus function $J(z) = \mathcal{J}(q(z))$ is as listed in Table-6.2.40 for type I_0.

Type$\mathbf{I}_b$, $\boldsymbol{b \in \mathbb{Z}_{>0}}$
Near each singular point s of S_0, which is an ordinary double point, we have a system x, y of complex analytic coordinates in S such that the singular point corresponds to the origin $(x, y) = (0, 0)$ of the coordinate system and $\varphi(x, y) = x\,y$. In these local coordinates we have

$$\omega = u(x, y)\, \mathrm{d}y \wedge \mathrm{d}x, \tag{6.2.44}$$

where u is a holomorphic function on an open neighborhood of $(0, 0)$ such that $u(0, 0) \neq 0$. It follows that the Hamiltonian vector field v near $(0, 0)$ takes the form

$$u(x, y)\,\dot{x} = \partial\varphi(x, y)/\partial y = x, \quad u(x, y)\,\dot{y} = -\partial\varphi(x, y)/\partial x = -y. \tag{6.2.45}$$

Because $x\,y = z$ is a constant of motion, we can substitute $y = z/x$ in the equation for $\dot{x}$ and obtain $u(x, z/x)\,\dot{x} = x$. That is, along a solution curve we have $\mathrm{d}t = \omega/\,\mathrm{d}\varphi = u(x, z/x)\,x^{-1}\,\mathrm{d}x$, where $\omega/\,\mathrm{d}\varphi$ denotes the relative quotient of ω by $\mathrm{d}\varphi$ on the level curve S_z of φ for the level z. Substituting $x = r\,\mathrm{e}^{\mathrm{i}\,\theta}$ with $0 < r \ll 1$ and $\theta \in \mathbb{R}$, we obtain a period

$$p_1(z) = \int_{\lambda_z} \omega/\,\mathrm{d}\varphi = \mathrm{i} \int_0^{2\pi} u(r\,\mathrm{e}^{\mathrm{i}\,\theta},\, z\,\mathrm{e}^{\mathrm{i}\,\theta}/r)\,\mathrm{d}\theta, \tag{6.2.46}$$

which is a holomorphic function of z for $|z| \ll r^2$ such that $p_1(0) = 2\pi\,\mathrm{i}\,u(0, 0) \neq 0$. Here λ_z is any loop in S_z such that x runs over a small circle around the origin in the positive direction when $y = z/x$ runs over a small circle around the origin in the negative direction if $0 < |z| << 1$.

Choose $0 < a \ll 1$. The time needed to go from the point $(a, z/a) \in S_z$ to the point $(z/a, a) \in S_z$ is equal to

$$T(z) = \int_a^{z/a} u(x, z/x)\,x^{-1}\,\mathrm{d}x,$$

where for small positive z we integrate over the path γ_z such that x runs over a path from a to z/a avoiding the origin, and $y = z/x$. When z moves in $D \setminus \{0\}$ we keep the

paths γ_z homotopic to each other. This leads to a multivalued holomorphic function T on $D \setminus \{0\}$. If z runs along a loop around the origin, the final γ_z is homotopic to the initial one followed by λ_z. Because the integral over λ_z is equal $p_1(z)$, it follows that $T^{\circlearrowleft}(z) = T(z) + p_1(z)$. Therefore $S(z) := T(z) - (1/2\pi\,\mathrm{i})\,(\log z)\,p_1(z)$ is a single-valued holomorphic function of z on $D \setminus \{0\}$. Because $|T(z)| = \mathrm{O}(\log(1/|z|))$ when $z \to 0$ and $\arg(z)$ remains bounded, we have $|S(z)| = \mathrm{O}(\log(1/|z|))$ as $z \to 0$, and it follows from Riemann's theorem on removable singularities that S extends to a holomorphic function on D, again denoted by S. In other words, $T(z) = (1/2\pi\,\mathrm{i})\,(\log z)\,p_1(z) + S(z)$, where S is a holomorphic function on D. Note that if z is a positive real number and $z/a < r < a$ then $\lambda_z \cdot \gamma_z = 1$, which remains the case for the continuations as long as the initial and final point of γ_z stay away from the loop λ_z.

If $b = 1$, then there is a holomorphic immersion $\iota : \mathbb{P}^1 \to S$ such that $\iota(0) = \iota(\infty) = s$, the images of $\mathrm{T}_0\iota$ and $\mathrm{T}_\infty\iota$ are the two distinct tangent spaces of the local components of S_0 at s, and the restriction to $\mathbb{P}^1 \setminus \{0,\ \infty\}$ of ι is a complex analytic diffeomorphism from $\mathbb{P}^1 \setminus \{0,\ \infty\}$ onto $S_0 \setminus \{s\}$. Let ξ be the affine coordinate on $\mathbb{P}^1$ and let B be a small open ball in S around s. Slightly deforming B if necessary, we can arrange that $\iota^{-1}(S_0 \cap B)$ is the union of the disks $|\xi| < \rho$ and $|1/\xi| < \rho$ around $\xi = 0$ and $\xi = \infty$ in $\mathbb{P}^1$, respectively, when $\iota^{-1}(S_0 \setminus B)$ is the real two-dimensional cylinder $R = \{\xi \in \mathbb{C} \mid \rho \le |\xi| \le 1/\rho\}$. Here $0 < \rho \ll 1$. If D is sufficiently small, then there is a real analytic diffeomorphism Φ from $\varphi^{-1}(D) \setminus B$ onto $D \times R$ such that φ corresponds to the projection $(z,\ \xi) \mapsto z$. The aforementioned coordinates $(x,\ y)$ in S around s can be chosen such that for $|\xi| \ll 1$ we have $\iota(\xi) = (X(\xi),\ 0)$, where X is a holomorphic function such that $X(0) = 0$ and $X'(0) \neq 0$, and for $|1/\xi| \ll 1$ we have $\iota(\xi) = (0,\ Y(1/\xi))$, where Y is a holomorphic function such that $Y(0) = 0$ and $Y'(0) \neq 0$. If ρ is chosen sufficiently small, then for $|z|$ sufficiently small the points $(a,\ z/a)$ and $(z/a,\ a)$ lie in $\varphi^{-1}(D) \setminus B$; hence $\Phi(a,\ z/a) = (z,\ \xi_1(z))$ and $\Phi(z/a,\ a) = (z,\ \xi_2(z))$, where $\xi_1(z)$ and $\xi_2(z)$ depend in a real analytic way on z, $\xi_1(0) = X^{-1}(a)$, and $\xi_2(0) = 1/Y^{-1}(a)$. There exists a path $\tau \mapsto \xi_z(\tau)$ in R running from $\xi_2(z)$ and $\xi_1(z)$ that depends in a real analytic fashion on z. Then $\delta_z : \tau \mapsto \Phi^{-1}(z,\ \xi_z(\tau))$ is a path in $\varphi^{-1}(D) \setminus B$, depending in a real analytic fashion on $z \in D$, and running in S_z from the terminal point $(z/a,\ a)$ of γ_z to the initial point $(a,\ z/a)$ of γ_z. Because the initial and terminal points of δ_z depend in a complex analytic way on z and the curves δ_z in S_z depend in a continuous way on $z \in D$ while staying away from the singular point z, the integral $I(z)$ of the relative quotient $\omega/\,\mathrm{d}\varphi$ over δ_z is a holomorphic function of $z \in D$. For $0 < |z| \ll 1$ the curve ϵ_z equal to γ_z followed by δ_z is a loop in S_z depending in a continuous fashion on z, and $p_2(z) := \int_{\epsilon_z} \omega/\,\mathrm{d}\varphi = (1/2\pi\,\mathrm{i})\,(\log z)\,p_1(z) + b(z)$, where $b(z) = S(z) + I(z)$ is a holomorphic function of $z \in D$. The loop λ_z can be chosen such that $\lambda_z \cdot \gamma_z = 1$, where δ_z is disjoint from λ_z. Therefore $\lambda_z \cdot \gamma_z = 1$; hence $[\lambda_z]$ and $[\epsilon_z]$ form a positively oriented $\mathbb{Z}$-basis of $\mathrm{H}_1(S_z,\ \mathbb{Z})$. It follows that $p_1(z)$ and $p_2(z)$ form a positively oriented $\mathbb{Z}$-basis of the period lattice $P(z)$.

Let $b > 1$. Then S_0 is a cycle of b holomorphically embedded complex projective lines Θ_i, $i \in \mathbb{Z}/b\,\mathbb{Z}$, where each Θ_i is an irreducible component of S_0 of multiplicity one. S_0 has b singular points s_i, $i \in \mathbb{Z}/b\,\mathbb{Z}$, where Θ_i intersects Θ_{i+1} transversally at

s_i and there are no other intersections. Note that s_{i-1} and s_i are the singular points of S_0 on Θ_i. Let ξ_i be an affine coordinate on Θ_i such that $\xi_i(s_{i-1}) = 0$ and $\xi_i(s_i) = \infty$. The complement in $\varphi^{-1}(D)$ of sufficiently small open balls B_i around the singular points s_i has b connected components, where the one that contains $\Theta_i \setminus (B_{i-1} \cup B_i)$ is C^∞ diffeomorphic to $D \times \{\xi_i \in \mathbb{C} \mid \rho \le \xi_i \le 1/\rho\}$, $\varphi : (z, \xi_i) \mapsto z$, and ξ_i is the affine coordinate on Θ_i when $z = 0$. Let (x_i, y_i) be the previously introduced local coordinates (x, y) near the singular point s_i of S_0, where the x_i-axis $y_i = 0$ corresponds to small $|\xi_i|$ and the y_i-axis $x_i = 0$ corresponds to small $|1/\xi_{i+1}|$. If x_i runs over a small circle about the origin in the positive direction, then $y_i = z/x_i$ runs over a small circle about the origin in the negative direction, $1/\xi_{i+1}$ runs over a small circle about the origin in the negative direction, and ξ_{i+1} runs over a large circle about the origin in the positive direction. This loop is homotopic to ξ_{i+1} running over a small circle about the origin in the positive direction, and therefore to x_{i+1} running over a small circle about the origin in the positive direction. It follows that, if λ_z^i denotes the loop in S_z near s_i, the loops λ_z^i are homotopic to each other, and the period function $p_1(z)$ does not depend on the choice of the singular point s_i. Let γ_z^i denote the path γ_z in S_z near s_i and δ_z^i the path δ_z in S_z near Θ_i, defined as for $b = 1$. If ϵ_z^i is γ_z^i followed by δ_z^{i+1}, then the integral of $\omega/\mathrm{d}\varphi$ over ϵ_z^i is equal to $p_1(z)\ (1/2\pi\ \mathrm{i})\ \log z$ plus a holomorphic function of $z \in D$. The cyclic concatenation ϵ_z of the paths ϵ_z^i is a loop in in S_z, where $\lambda_z^i \cdot \epsilon_z = 1$, hence $[\lambda_z^i]$ and $[\epsilon_z]$ form a positively oriented $\mathbb{Z}$-basis of $\mathrm{H}_1(S_z, \mathbb{Z})$. The integral $p_2(z)$ of $\omega/\mathrm{d}\varphi$ over ϵ_z is equal to $p_1(z)\ (b/2\pi\ \mathrm{i})\ \log z$ plus a holomorphic function of $z \in D$. Because $p_1(z)$ and $p_2(z)$ form a positively oriented $\mathbb{Z}$-basis of $P(z)$, we have proved Table 6.2.39 for I_b, $b \in \mathbb{Z}_{>1}$.

Because $p_1^{\circlearrowleft}(z) = p_1(z)$ and $p_2^{\circlearrowleft}(z) = p_2(z) + b\, p_1(z)$, the monodromy matrix is as in Table 6.2.40 for the type I_b. The modulus function is equal to $J(z) = \mathcal{J}(q(z))$, where $q(z) := p_2(z)/p_1(z) = (b/2\pi\ \mathrm{i})\ \log z + r(z)$, and r is a holomorphic function on D. Combining this with (2.3.14) with $\epsilon = \mathrm{e}^{2\pi\,\mathrm{i}\,q(z)}$, we conclude that J has a pole of order b at c_0.

Type I_0^*

The singular fiber S_0 has one multiplicity-two component Θ_2 and four multiplicity-one components $\Theta_0, \Theta_1, \Theta_3, \Theta_4$. Each of these irreducible components is a rational curve. Each of the curves $\Theta_0, \Theta_1, \Theta_3, \Theta_4$ intersects Θ_2 once and transversally at a point b_0, b_1, b_3, b_4, respectively, where these are four distinct points on Θ_2, and there are no other intersections.

Near each point $s \in \Theta_2 \setminus \{b_0, b_1, b_3, b_4\}$ we have holomorphic local coordinates x, y in S such that $x(s) = y(s) = 0$, $z = \varphi(x, y) = y^2$, and $\omega = u(x, y)\ \mathrm{d}x \wedge \mathrm{d}y$ for a unit u. We have $u(x, y)\, \dot{x} = \partial\varphi(x, y)/\partial y = 2\, y$, hence $u(x, y)^2\, \dot{x}^2 = 4\, y^2 = 4\, z$. That is, substituting $z = \zeta^2$, multiplying the vector field by the factor $1/\zeta$, and taking the limit for $z \to 0$, we obtain two vector fields on Θ_2 near p. These limit vector fields are holomorphic and have no zeros, and define a twofold unbranched covering of Θ_2 near s.

Near a_j we have holomorphic local coordinates x, y in S such that $x(b_j) = y(b_j) = 0$, $z = \varphi(x, y) = x\, y^2$, and $\omega = u(x, y)\ \mathrm{d}x \wedge \mathrm{d}y$ for a unit u. With

the substitution $x = \xi^2$ the equation $z = \varphi(x,\ y)$ is equivalent to $\zeta = \xi\, y$, where $\zeta^2 = z$. The differential equation $2\,u\,\xi\,\dot{\xi} = u\,\dot{x} = \partial\varphi/\partial y = 2\,x\,y = 2\,\xi^2\,y = 2\,\xi\,\zeta$ leads to the time-rescaled limit differential equation $u(\xi^2,\ 0)\xi' = \pm 1$. That is, if we parametrize the smooth curve $P := x - \xi^2 = 0$ in the $(x,\ \xi)$-plane by ξ, then the restriction to P of the projection $(x,\ \xi) \mapsto x$ defines a twofold branched covering with $x = 0$ as the brach point, where a holomorphic vector field on P without zeros is mapped to the two-times rescaled limit vector fields on Θ_2. Gluing the local coverings together, we obtain a twofold branched covering $\pi : \Theta_2^{\sim} \to \Theta_2$, branching over the points b_j, such that the limit for $z \to 0$ of the rescaled vector field $\zeta^{-1}\, v$ on the fiber S_z of φ over z corresponds to a holomorphic vector field $v^{\sim}$ on $\Theta_2^{\sim}$ without zeros. It follows that $\Theta_2^{\sim}$ is an elliptic curve, where (2.3.23) yields its modulus $J := \mathrm{J}(\Theta_2^{\sim})$, see (2.3.10), in terms of the configuration of the four branch points b_j on $\Theta_2 \simeq \mathbb{P}^1$.

Along the multiplicity-one irreducible components Θ_0, Θ_1, Θ_3, Θ_4, the vector field v on S_z is of order 1 if we stay away from the intersection points b_j with Θ_2. Near Θ_2 but away from the b_j, it is of order $|z|^{1/2}$, whereas near the b_j it is small but large compared to $|z|^{1/2}$. This means that compared to the time spent along Θ_2, the complex curves with velocity v spend relatively little time running along the other irreducible components of the singular fiber. Also note that the limit flow along each of the complex projective lines Θ_0, Θ_1, Θ_3, Θ_4 is a one-parameter group of automorphisms leaving only the intersection point with Θ_2 fixed, and therefore it is a one-parameter group of translations in the affine coordinate for which the intersection point lies at infinity.

Let c_1, c_2 be a $\mathbb{Z}$-basis of the period group P of the vector field $v^{\sim}$ on $\Theta_2^{\sim}$, where $c_1,\ c_2 \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Im}(q_0) > 0$ if $q_0 = c_2/c_1$. Note that $\mathcal{J}(q_0) = J$. Then for each $\theta \in \Theta_2^{\sim}$, the real curves $[0,\ 1] \ni t_j \mapsto \mathrm{e}^{t_j\,\pi_j\,v^{\sim}}(\theta)$ are closed curves on $\Theta_2^{\sim}$, and for an open dense subset of points $\theta \in \Theta_2^{\sim}$ these curves avoid the ramification points, the four points r_j in $\Theta_2^{\sim}$ such that $\pi(r_j) = b_j$. For such θ the images under the branched covering map $\pi : \Theta_2^{\sim} \to \Theta_2$ of these curves are closed curves in Θ_2 that miss the branch points b_j, and for z close to 0 these curves in Θ_2 are shadowed by nearby solution curves of $\zeta^{-1}\, v$ on S_z that stay away from the points b_j. It follows that after a small complex time each of these two curves can be closed in S_z, and we obtain periods $\pi_1(\zeta)$ and $\pi_2(\zeta)$ of $\zeta^{-1}\, v$ on S_z that depend holomorphically on ζ and satisfy $\pi_i(0) = c_i$.

There exists a holomorphic mapping $\chi : D \to S$ such that $\chi(0) \in \Theta_2 \cap S^{\mathrm{reg}}$ and $\varphi(\chi(\zeta)) = \zeta^2 = z$. For $0 < |z| \ll 1$ the mapping $t \mapsto \mathrm{e}^{(t/\zeta)\,v}(\chi(\zeta))$ induces a holomorphic covering map from the elliptic curve $\mathbb{C}/(\mathbb{Z}\,\pi_1(\zeta) + \mathbb{Z}\,\pi_2(\zeta))$ onto S_z. In the coordinate $t \in \mathbb{C}/(\mathbb{Z}\,\pi_1(\zeta) + \mathbb{Z}\,\pi_2(\zeta))$, the parts of S_z close to the multiplicity-one components of S_0 are small disks around the points in $\mathbb{C}/(\mathbb{Z}\,c_1 + \mathbb{Z}\,c_2)$ that correspond to the ramification points r_j under the isomorphism from $\mathbb{C}/(\mathbb{Z}\,c_1 + \mathbb{Z}\,c_2)$ onto $\Theta_2^{\sim}$. The part of S_z near Θ_2 away from the points b_j is diffeomorphic to the complement of these small disks. It follows that the covering map is an isomorphism, and therefore the periods $\pi_i(\zeta)$ form a $\mathbb{Z}$-basis of the period lattice of the vector field $\zeta^{-1}\, v$ on S_z.

In other words, $p_1(z) = z^{-1/2}\,\pi_1(z^{1/2})$ and $p_2(z) = z^{-1/2}\,\pi_2(z^{1/2})$ form a $\mathbb{Z}$-basis of the period lattice of the vector field v on S_z. Substituting this into the equation for the monodromy matrix M, we obtain, after multiplication by $z^{1/2}$, using that $z^{1/2}$

changes to $-z^{1/2}$ if z runs once around the origin, and taking the limit for $z \to 0$, that $M = -1$. That is, the monodromy matrix is as in Table 6.2.40 for type I_0^*. This in turn implies that $\varpi_i : z \mapsto z^{1/2}\, p_i(z)$ is a single-valued holomorphic function on $D \setminus \{0\}$, converging to π_i as $z \to 0$. It therefore follows from Riemann's theorem on removable singularities that the ϖ_i extend to a holomorphic function on D, denoted by the same letters, with $\varpi_i(0) = c_i$. This proves Table 6.2.39 for type I_b^* with $b = 0$, where $a(z) = \varpi_1(z)$, $b(z) = \varpi_2(z)$, and $\mathrm{Im}(b(0)/a(0)) = \mathrm{Im}(q_0) > 0$. The proof is concluded by the observation that $q(z) := p_2(z)/p_1(z) = b(z)/a(z)$ is a holomorphic function, and therefore, as for type I_0, the modulus function $J(z) = \mathcal{J}(q(z))$ is as in Table 6.2.40 for type I_0^*.

Type $\mathbf{I}_b^*$, $b \in \mathbb{Z}_{>0}$

The singular fiber consists of $b+1$ rational curves Θ_i, $2 \le i \le b+2$, of multiplicity two and four rational curves Θ_0, Θ_1, Θ_{b+3}, Θ_{b+4} of multiplicity one. For each $2 \le i \le b+1$, the curve Θ_i intersects Θ_{i+1} at one point s_i and transversally. The curves Θ_0 and Θ_2 intersect Θ_2 at the distinct points b_0 and b_1 and transversally, whereas Θ_{b+3} and Θ_{b+4} intersect Θ_{b+2} at the distinct points b_{b+3} and b_{b+4} and transversally. There are no other intersections.

Near s_i we have holomorphic local coordinates x, y such that $x(s_i) = y(s_i) = 0$, $z = \varphi(x,\ y) = x^2\, y^2$, and the Hamiltonian vector field is given by

$$u\,\dot{x} = \partial\varphi/\partial y = 2\,x^2\,y = 2\,z^{1/2}\,x, \quad u\,\dot{y} = -\partial\varphi/\partial x = 2\,x\,y^2 = -2\,z^{1/2}\,y.$$

Here $y = 0$ and $x = 0$ correspond to Θ_i and Θ_{i+1} with the local coordinates x and y, respectively.

Near b_j we have holomorphic local coordinates x, y such that $x(b_j) = y(b_j) = 0$, $z = \varphi(x,\ y) = x\,y^2$, and the Hamiltonian vector field is given by

$$u\,\dot{x} = \partial\varphi/\partial y = 2\,x\,y = 2\,z^{1/2}\,x^{1/2}, \quad u\,\dot{y} = -\partial\varphi/\partial x = -y^2.$$

Here $y = 0$ corresponds to the multiplicity-two component Θ_2 or Θ_{b+2}, with x as its local coordinate, whereas $x = 0$ corresponds to the multiplicity-one component through b_j, with y as its local coordinate.

Corresponding to the two square roots $z^{1/2}$ and $x^{1/2}$, respectively, we pass to the twofold covering of the union of the multiplicity-two components minus the intersection points s_i, near which v is small of order $z^{1/2}$ on the fiber S_z of φ over z, without branching over the intersection points s_i, but with branching over the intersection points b_j. On this twofold covering we have a single-valued holomorphic limit vector field $v^\sim$, for $z \to 0$ of the time rescaled vector field $z^{-1/2}\,v$ on S_z.

The twofold coverings $\Theta_2^\sim$ and $\Theta_{b+2}^\sim$ of Θ_2 and Θ_{b+2}, branching over b_0, b_1 and b_{b+3}, b_{b+4}, respectively, have genus zero according to the Riemann–Hurwitz formula (2.3.18), and therefore are rational curves. For $3 \le i \le b+1$, the twofold covering of Θ_i has two connected components $\Theta_i^\pm$ such that the $2\,b$ rational curves $\Theta_2^\sim$, Θ_3^+, $\ldots$, Θ_{b+1}^+, $\Theta_{b+2}^\sim$, Θ_{b+1}^-, $\ldots$, Θ_3^- intersect each other successively, and Θ_3^- intersects $\Theta_2^\sim$, thus forming a cycle. The limit vector field $v^\sim$ on this cycle has the same form as a vector field v on a singular fiber of type $\mathrm{I}_{2\,b}$. On the complex

projective line $\Theta_2^\sim$ the deck transformation that interchanges the two points in $\Theta_2^\sim$ over $s_2 \in \Theta_2$ leaves the two ramification points over the branch points b_0 and b_1 fixed. If we use an affine coordinate x in $\Theta_2^\sim$ such that the two points over s_2 correspond to $x = 0$ and $x = \infty$, then the deck transformation is of the form $x \mapsto 1/x$, with $x = \pm 1$ as its only fixed points. Therefore b_0 and b_1 correspond to $x = \pm 1$. A similiar observation holds for the configuration of the two points in $\Theta_{b+2}^\sim$ over the intersection point s_{b+1} of Θ_{b+1} with Θ_{b+2}, together with the two ramification points over the branch points b_{b+3} and b_{b+4}.

With the same reasoning as for the type I_b, with b and v replaced by $2\,b$ and $z^{-1/2}\,v$, respectively, we arrive at a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of periods for the vector field v on S_z, such that, asymptotically for $z \to 0$, $p_1(z) \sim z^{-1/2}\,c_1$ with $c_1 \neq 0$, and

$$p_2(z) \sim z^{-1/2}\,(2\,b/2\pi\,\mathrm{i})\,\log(z^{1/2}) = z^{-1/2}\,(b/2\pi\,\mathrm{i})\,\log z.$$

In the proof it is also used that the complex time spent by the curve on S_z with velocity $z^{-1/2}\,v$ near the multiplicity one components Θ_0, Θ_1, Θ_{b+3}, Θ_{b+4} form small disks around the points corresponding to the branch points q_j; see the arguments for type I_0^* about the time spent near the multiplicity-one components. This leads to a monodromy matrix that is equal to the opposite of the monodromy matrix in Table 6.2.40 for type I_b, which in turn implies that the period functions are as in Table 6.2.39 for type I_b^*. The same argument as for type I_b implies that the modulus function $J(z) = \mathcal{J}(q(z))$ has a pole of order b at $z = 0$.

TypeII

This is an irreducible curve S_0 with a cusp, where its normalization is a rational curve = a complex projective line, with one point corresponding to the cusp point. After blowing up the cusp point we obtain a -1 curve E_1 of multiplicity 2, whereas the proper transform S_0' of S_0 is a rational curve that has a contact of order two with E_1 at a single point e. After blowing up e, we obtain a -1 curve E_2 of multiplicity 3, where E_2 and the respective proper transforms E_1' and S_0'' of E_1 and S_0' intersect each other at a single point e', with three distinct tangent lines. After blowing up e', we obtain a -1 curve E_3 of multiplicity 6, transversally intersected by the respective proper transforms S_0''', E_1'', E_2' of S_0'', E_1', E_2, which are rational curves of multiplicity 1, 2, 3, where the intersection points b_0, b_1, b_2 with E_3 are distinct, the intersections are transversal, and there are no other intersections.

Near a point $a \in E_3 \backslash \{b_0,\ b_1,\ b_2\}$ we have holomorphic local coordinates $(x,\ y)$ in the blowup such that $x(a) = y(a) = 0$, $z = \varphi(x,\ y) = y^6$, and $\omega = u(x,\ y)\,y^4\ \mathrm{d}x \wedge \mathrm{d}y$, where u is a unit. Then $u(x,\ y)\,y^4\,\dot{x} = \partial\varphi(x,\ y)/\partial y = 6\,y^5$; hence $u\,\dot{x} = 6\,y$, or $(u\,\dot{x}/6)^6 = y^6 = z$. That is, substituting $z = \zeta^6$, multiplying v by ζ^{-1} and taking the limit for $z \to 0$, we obtain six limit vector fields on E_3, corresponding to the ordinary differential equations $u(x,\ 0)\,x' = 6\,\epsilon$, where ϵ runs over the sixth roots of unity. We will view these six vector fields as the images of a holomorphic vector field witout zeros on a sixfold unbranched covering of E_3 near a.

Near b_0 we have holomorphic local coordinates $(x,\ y)$ in the blowup such that $x(b_0) = y(b_0) = 0$, $z = \varphi(x,\ y) = x\,y^6$, and $\omega = u(x,\ y)\,y^4\ \mathrm{d}x \wedge \mathrm{d}y$, where u is a unit. This leads to $u\,\dot{x} = 6\,x\,y$, hence $(u(x,\ 0)\,x'/6)^6 = x^5$. Running with x around

the origin over a small circle, x' passes each of the six decks of the covering, which shows that the covering is a connected curve. With the substitution $x = \xi^6$ we have $(u(\xi^6,\ 0)\,\xi')^6 = 1$.

Near b_1 we have holomorphic local coordinates $(x,\ y)$ in the blowup such that $x(b_1) = y(b_1) = 0$, $z = \varphi(x,\ y) = x^2\,y^6$, and $\omega = u(x,\ y)\,x\,y^4\ \mathrm{d}x \wedge \mathrm{d}y$, where u is a unit. This leads to $u\,\dot{x} = 6\,x\,y$, hence $(u(x,\ 0)\,x'/6)^6 = x^4$. This defines a curve in the $(x,\ x')$-plane with two irreducible components $(u(x,\ 0)\,x'/6)^3 = \pm\,x^2$, which are separated in the normalization. With the substitution $x = \xi^3$ we have $(u(\xi^3,\ 0)\,\xi'/2)^6 = 1$.

Near b_2 we have holomorphic local coordinates $(x,\ y)$ in the blowup such that $x(b_2) = y(b_2) = 0$, $z = \varphi(x,\ y) = x^3\,y^6$, and $\omega = u(x,\ y)\,x^2\,y^4\ \mathrm{d}x \wedge \mathrm{d}y$, where u is a unit. This leads to $u\,\dot{x} = 6\,x\,y$, hence $(u(x,\ 0)\,x'/6)^6 = x^3$. This defines a curve in the $(x,\ x')$-plane with three irreducible components $(u(x,\ 0)\,x'/6)^3 = \delta\,x^2$, $\delta^3 = 1$, which are separated in the normalization. With the substitution $x = \xi^3$ we have $(u(\xi^3,\ 0)\,\xi'/3)^6 = 1$.

It follows that there are a sixfold branched covering $\pi : E_3^{\sim} \to E_3$ and a single-valued holomorphic vector field $v^{\sim}$ on $E_3^{\sim}$ without zeros that is mapped by π to the six-time rescaled limit vector fields on E_3. According to Section 2.3, this implies that $E_3^{\sim}$ is an elliptic curve. Over small neighborhoods of the branch points b_0, b_1, and b_2 the six branches form 1, 2, and 3 connected cycles of 6, 3, or 2 branches each, respectively, where the branches in each cycle come together when the point in E_3 approaches the branch point. The Riemann–Hurwitz formula (2.3.18) therefore reconfirms that $E_3^{\sim}$ is an elliptic curve.

Near the irreducible components E_2', E_1'', and C''', the vector field v on the fiber S_z of φ over z, where $0 < |z| << 1$, is of order 1 if we stay away from the intersection points with E_3, whereas it is still of larger order than $|z|^{1/6}$ near the intersection points. In the same way as for type I_0^*, we obtain a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of $P(z)$, depending in a multivalued holomorphic manner on $z \in D \setminus \{0\}$, such that, asymptotically for $z \to 0$, we have $p_i(z) \sim z^{-1/6}\,c_i$, where c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if we write $q_0 = c_2/c_1$. The times spent near the multiplicity-one components E_2', E_1'', and C''' form small disks in the elliptic curve near $E_3^{\sim}$ around the finitely many points that are mapped by π to the branch points in E_3 = the intersection points of E_2', E_1'', and C''' with E_3.

Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ in (6.2.43), we obtain, after multiplication by $z^{1/6}$ and taking the limit for $z \to 0$, that

$$\begin{aligned} \omega\,c_1 &= c_1\,M_1^1 + c_2\,M_1^2, \\ \omega\,c_2 &= c_1\,M_2^1 + c_2\,M_2^2, \end{aligned} \tag{6.2.47}$$

where $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}/6}$. It follows that M has the eigenvalue $\mathrm{e}^{-2\pi\,\mathrm{i}/6}$. Because $\det M = 1$, the other eigenvalue is equal to $\mathrm{e}^{2\pi\,\mathrm{i}/6} \neq \mathrm{e}^{-2\pi\,\mathrm{i}/6}$; hence $M^6 = 1$. Dividing both equations in (6.2.47) by c_1, we obtain that the complex number $q_0 = c_2/c_1$ in the complex upper half-plane H satisfies $q_0 = (M_2^1 + q_0\,M_2^2)/(M_1^1 + q_0\,M_1^2)$, that is, q_0 is a fixed point of the action of $\pm M \in \mathrm{PSL}(2,\ \mathbb{Z})$ on H, which has order 3. It follows from Section 2.3.3 that $\mathcal{J}(q_0) = 0$ and q_0 is in the $\mathrm{PSL}(2,\ \mathbb{Z})$-orbit

of $e^{2\pi i/6}$. That is, by applying an element of SL(2, $\mathbb{Z}$) to the basic periods, we can arrange the $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period group $P_v(z)$ in such a way that $q_0 = e^{2\pi i/6}$. Then the equations in (6.2.47) imply that $e^{-2\pi i/6} = M_1^1 + e^{2\pi i/6}\, M_1^2$ and $1 = M_2^1 + e^{2\pi i/6}\, M_2^2$, whose imaginary parts yield $M_1^2 = -1$, $M_2^2 = 0$, after which the real parts yield $M_1^1 = 1$ and $M_2^1 = 1$. That is, the monodromy matrix M is as in Table 6.2.40 for type II, and (6.2.43) becomes $p_1^{\circlearrowleft}(z) = p_1(z) - p_2(z)$ and $p_2^{\circlearrowleft}(z) = p_1(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type II are equivalent to $a(z) = z^{1/6}\,(p_1(z) - e^{2\pi i/6}\, p_2(z))/(1 - e^{2\pi i/3})$ and $b(z) = z^{-1/6}\,(p_1(z) - e^{-2\pi i/6}\, p_2(z))/(1 - e^{2\pi i/3})$, where $p_1^{\circlearrowleft}(z) = p_1(z) - p_2(z)$ and $p_2^{\circlearrowleft}(z) = p_1(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic functions on $D \setminus \{0\}$. Furthermore $p_i(z) \sim z^{-1/6}\, c_i$ and (6.2.47) with $M_2^2 =$, $M_2^1 = 1$ imply that $a(z) \to c_1 \neq 0$ and $b(z) = O(|z|^{-1/3})$ as $z \to 0$. It follows from Riemann's theorem on removable singularities that a and b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$.

We have $q(z) := p_2(z)/p_1(z) = (e^{2\pi i/6} + z^{1/3}\, e^{-2\pi i/6}\, r(z))/(1 + z^{1/3}\, r(z))$, where $r(z) := b(z)/a(z)$. The modulus function $J(z) = \mathcal{J}(q(z))$ is single-valued on $D \setminus \{0\}$ and has the limit value $\mathcal{J}(q(0)) = \mathcal{J}(e^{2\pi i/6}) = 0$ when $z \to 0$. Therefore Riemann's theorem on removable singularities implies that J extends to a holomorphic function on D, again denoted by J, such that $J(0) = 0$. Because $\mathcal{J}$ has degree 3 at points where it is equal to 0, it follows that the degree of J at $z = 0$ is equal to $1 + 3\,m$, where m is the degree of $r(z)$ at $z = 0$. Note that the modulus function is constant, see Section 6.4.4, and then identically equal to its limit value 0 for $z \to 0$, if and only if $q(z) \equiv e^{2\pi i/6}$ if and only if $r(z) \equiv 0$ if and only if $b(z) \equiv 0$.

TypeII*

In contrast to type II, the singular fiber S_0 is already in a normal crossing situation; see Kodaira's list in Section 6.2.6. Therefore $\omega = u(x,\ y)\ dx \wedge dy$ in holomorphic local loordinates $(x,\ y)$, where u is a unit. The vector field v on the fiber S_z of φ over z is of order $z^{(m-1)/m}$ near an irreducible component of multiplicity m, and therefore its flow is the slowest near Θ_4, where $v \sim z^{5/6}$. Substituting $z = \zeta^6$, the six-time rescaled vector fields $\zeta^{-5}\, v$ have six limit vector fields for $z \to 0$ on Θ_4, where the index ϵ runs over the sixth roots of unity. The component Θ_4 is intersected by Θ_5, Θ_3, and Θ_2, which have multiplicity $k = 5, 4$, and 3, respectively. Near the respective intersection points b_5, b_3, and b_2 we have $z = \varphi = x^5\, y^6$, $z = x^4\, y^6$, and $z = x^3\, y^6$, in which case the substitutions $x = \xi^6$, $x = \xi^3$, and $x = \xi^2$ regularize the time-rescaled limit vector fields. This leads to a sixfold branched covering $\pi : \Theta_4^{\sim} \to \Theta_4$ and a holomorphic vector field $v^{\sim}$ without zeros on $\Theta_4^{\sim}$ that is mapped by π to the six-times rescaled limit vector fields on Θ_4. The branching behavior is the same as for type II with the branch points b_0, b_1, and b_2 on E_3 replaced by the respective branch points b_5, b_3, and b_2 on Θ_4, and therefore in the same way the Riemann–Hurwitz formula (2.3.18) reconfirms that $\Theta_4^{\sim}$ is an elliptic curve.

The complex time spent with the flow of $\zeta^{-5}\, v$ on S_z near Θ_1 is a small disk within a small disk corresponding to the time spent near Θ_3. Similarly, the time spent near Θ_0 is a small disk within a small disk of the time spent near Θ_8 within a small disk

of the time spent near Θ_7 within a small disk of the time spent near Θ_6 within a small disk of the time spent near Θ_5. Finally, the time spent near Θ_2 is a small disk. With a similar argument as for the type I_0^*, we arrive at a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period group of v on S_z, where asymptotically $p_i(z) \sim z^{-5/6}\, c_i$ as $z \to 0$, and c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if $q_0 = c_2/c_1$.

Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ in (6.2.43), we obtain, after multiplication by $z^{5/6}$ and taking the limit for $z \to 0$, that (6.2.47) holds with $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}\,5/6}$. It follows that M has the eigenvalue $\mathrm{e}^{-2\pi\,\mathrm{i}\,5/6}$, the other eigenvalue is equal to $\mathrm{e}^{2\pi\,\mathrm{i}\,5/6} \neq \mathrm{e}^{-2\pi\,\mathrm{i}\,5/6}$, and $M^6 = 1$. As for type II, we conclude that we can arrange the limit value $q_0 = c_2/c_1$ of p_2/p_1 to be $\mathrm{e}^{2\pi\,\mathrm{i}/6}$, in which case $\mathcal{J}(q_0) = 0$. Dividing both equations in (6.2.47) by c_1 and comparing the real and imaginary parts, we this time obtain $M_1^1 = 0$, $M_1^2 = 1$, $M_2^1 = -1$, and $M_2^2 = 1$, which is the monodromy matrix M as in Table 6.2.40 for type II^*, and (6.2.43) becomes $p_1^{\circlearrowleft}(z) = p_2(z)$ and $p_2^{\circlearrowleft}(z) = -p_1(z) + p_2(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type II^* are equivalent to $a(z) = z^{5/6}\,(p_1(z) - \mathrm{e}^{2\pi\,\mathrm{i}/6}\; p_2(z))/(1 - \mathrm{e}^{2\pi\,\mathrm{i}/3})$ and $b(z) = z^{1/6}\,(p_1(z) - \mathrm{e}^{-2\pi\,\mathrm{i}/6}\; p_2(z))/(1 - \mathrm{e}^{2\pi\,\mathrm{i}/3})$, where $p_1^{\circlearrowleft}(z) = p_2(z)$ and $p_2^{\circlearrowleft}(z) = -p_1(z) + p_2(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic functions on $D \setminus \{0\}$. Furthermore, $p_i(z) \sim z^{-5/6}\, c_i$ and (6.2.47) with ω and M_i^j as above imply that $a(z) \to c_1 \neq 0$ and $b(z) = \mathrm{O}(|z|^{-2/3})$ as $z \to 0$. It follows from Riemann's theorem on removable singularities that a an b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$.

We have $q(z) := p_2(z)/p_1(z) = (\mathrm{e}^{2\pi\,\mathrm{i}/6} + z^{2/3}\; \mathrm{e}^{-2\pi\,\mathrm{i}/6}\; r(z))/(1 + z^{2/3}\, r(z))$, where $r(z) := b(z)/a(z)$. As for type II, we conclude that J extends to a holomorphic function on D, again denoted by J, where $J(0) = 0$, but this time the degree of J at $z = 0$ is equal to $2 + 3\,m$, where m is the degree of $r(z)$ at $z = 0$.

TypeIII

The singular fiber $\varphi = 0$ is the union of two smooth rational curves Θ_+, Θ_- that intersect each other at one point s, with a second-order contact. Blowing up s, we obtain a -1 curve E_1 of multiplicity 2, where the proper transforms $\Theta'_\pm$ of $\Theta_\pm$ are such that E_1, Θ'_+, and Θ'_- intersect each other at a single point e, with different tangent lines. After blowing up e, we obtain a -1 curve E_2 of multiplicity 4, transversally intersected by the respective proper transforms E'_1, Θ''_+, and Θ''_- of E_1, Θ'_+, and Θ'_-, where the intersection points b_1, $_+$, and b_- in E_2 are distinct. There are no other intersections.

Near $a \in E_2 \setminus \{b_1,\ b_+,\ b_-\}$ we have holomorphic local coordinates $(x,\ y)$ in the blowup such that $x(a) = y(a) = 0$, $z = \varphi(x,\ y) = y^4$, and $\omega = u(x,\ y)\, y^2\; \mathrm{d}x \wedge \mathrm{d}y$, where u is a unit. With the substitution $z = \zeta^4$ the differential equation $u\, y^2\, \dot{x} = \partial\varphi(x,\ y)/\partial y$ is equivalent to $u\, \dot{x} = 4\, \zeta$, hence the vector field $\zeta^{-1}\, v$ converges to four holomorphic vector fields without zeros on $E_2 \setminus \{b_1,\ b_+,\ b_-\}$. Near b_1 we have $z = \varphi(x,\ y) = x^2\, y^4$, and $\omega = u(x,\ y)\, x\, y^2\; \mathrm{d}x \wedge \mathrm{d}y$, $u\, \dot{x} = 4\, x\, y$, $(u\, x'/4)^4 = x^2$, and the curve in the $(x,\ x')$-plane has two irreducible components $(u\, x'/4)^2 = \pm x$, where the substitution $x = \xi^2$ leads to $(u(\xi^2,\ 0)\xi'/2)^4 = 1$. Near $b_\pm$ we have $z = \varphi(x,\ y) = x\, y^4$, and $\omega = u(x,\ y)\, y^2\; \mathrm{d}x \wedge \mathrm{d}y$, $u\, \dot{x} = 4\, x\, y$, $(u\, x'/4)^4 = x^3$, the curve in the $(x,\ x')$-plane is irreducible, and the substitution $x = \xi^4$ leads to

$(u(\xi^4,\,0)\xi')^4 = 1$. This leads to a fourfold branched covering $\pi : E_2^{\sim} \to E_2$ and a holomorphic vector field $v^{\sim}$ without zeros on $E_2^{\sim}$ that is mapped by π to the four-times rescaled limit vector fields on E_2. Over a small neighborhood of the branch point a_1 the four branches form two connected cycles of two branches each, where the branches in each cycle come together over a_1. Over the other two branch points $a_\pm$ all four branches come together, and the Riemann–Hurwitz formula (2.3.18) reconfirms that $E_2^{\sim}$ is an elliptic curve.

Because the vector field v on the fiber S_z of φ over z is of order $z^{1/2}$ and 1 near E_1' and $C_\pm''$, respectively, its flow is fast since compared to the flow near E_2. With a similar argument as for type I_0^*, we arrive at a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period lattice of v on S_z, where asymptotically $p_i(z) \sim z^{-1/4}\, c_i$ as $z \to 0$, and c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if $q_0 = c_2/c_1$. Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ in (6.2.43), we obtain, after multiplication by $z^{1/4}$ and taking the limit for $z \to 0$, that (6.2.47) holds with $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}/4} = -\,\mathrm{i}$. It follows that M has the eigenvalue $-\,\mathrm{i}$, the other eigenvalue is equal to $\mathrm{i} \neq -\,\mathrm{i}$, and hence $M^4 = 1$. It follows from Section 2.3.3 that $\mathcal{J}(q_0) = 1$ if $q_0 := c_2/c_1$, and it can be arranged that $q_0 = \mathrm{i}$. With this choice, division of (6.2.47) by c_1 and consideration of the real and imaginary parts leads to $M_1^1 = 0$, $M_1^2 = -1$, $M_2^1 = 1$, and $M_2^2 = 0$. That is, M is as in Table 6.2.40 for type III, and $p_1^{\circlearrowleft}(z) = -p_2(z)$ and $p_2^{\circlearrowleft}(z) = p_1(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type III are equivalent to $a(z) = z^{1/4}\,(p_1(z) - \mathrm{i}\;p_2(z))/2$ and $b(z) = z^{-1/4}\,(p_1(z) + \mathrm{i}\;p_2(z))/2$, where $p_1^{\circlearrowleft}(z) = -p_2(z)$ and $p_2^{\circlearrowleft}(z) = p_1(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic functions on $D\setminus\{0\}$. Furthermore, $p_i(z) \sim z^{-1/4}\, c_i$ and (6.2.47) with ω and M_i^j as above imply that $a(z) \to c_1 \neq 0$ and $b(z) = \mathrm{O}(|z|^{-1/2})$ as $z \to 0$. It follows from Riemann's theorem on removable singularities that a and b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$. We have $q(z) := p_2(z)/p_1(z) = \mathrm{i}\,(1 - z^{1/2}\,r(z))/(1 + z^{1/2}\,r(z))$, where $r(z) := b(z)/a(z)$. The modulus function $J(z) = \mathcal{J}(q(z))$ is single-valued and has the limit value $\mathcal{J}(\mathrm{i}) = 1$ as $z \to 0$. Because $\mathcal{J}$ has degree 2 at points where $\mathcal{J} = 1$, the degree of $J(z)$ at $z = 0$ is equal to $1 + 2\,m$, where m is the degree of $r(z)$ at $z = 0$.

TypeIII*

In contrast to type III, the singular fiber S_0 is already in a normal crossing situation, see Kodaira's list in Section 6.2.6; when $\omega = u(x,\,y)\;\mathrm{d}x \wedge \mathrm{d}y$ for a unit u in any holomorphic system of local coordinates $(x,\,y)$. The vector field v on the fiber S_z of φ over z is of the smallest order $z^{3/4}$ near the irreducible component Θ_4 of the maximal multiplicity 4. The component Θ_4 is intersected by Θ_3, Θ_5, and Θ_2, which have multiplicities 3, 3, and 2, respectively. Let b_3, b_5 and b_2 denote the respective intersection points. With the substitution $z = \zeta^4$, the rescaled vector field $\zeta^{-3}\,v$ on S_z converges for $z \to 0$ to four holomorphic vector fields without zeros on $\Theta_4 \setminus \{b_3,\,b_5,\,b_2\}$. Near b_3, b_5, and b_2 we have $z = x^3\,y^4$, $z = x^3\,y^4$, and $z = x^2\,y^4$, when the time rescaled vector limit vector fields are regularized by the substitutions $x = \xi^4$, $x = \xi^4$, and $x = \xi^2$, respectively. This leads to a fourfold branched covering $\pi : \Theta_4^{\sim} \to \Theta_4$ and a holomorphic vector field $v^{\sim}$ without zeros on $\Theta_4^{\sim}$ that is mapped by π to the four-times rescaled limit vector fields on Θ_4. Since the branching behavior

is the same as for type III, the Riemann–Hurwitz formula (2.3.18) reconfirms in the same way that $\Theta_4^\sim$ is an elliptic curve.

In the same way as for type II* we arrive at a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period lattice of v on S_z, where asymptotically $p_i(z) \sim z^{-3/4}\, c_i$ as $z \to 0$, and c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if $q_0 = c_2/c_1$. Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ into (6.2.43,) we obtain, after multiplication by $z^{3/4}$ and taking the limit for $z \to 0$, that (6.2.47) holds with $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}\,3/4} = \mathrm{i}$. As for type II, we conclude that $M^4 = 1$, in which case $\mathcal{J}(q_0) = 1$ if $q_0 := c_2/c_1$, and it can be arranged that $q_0 = \mathrm{i}$. With this choice, division of (6.2.47) by c_1 and consideration of the real and imaginary parts leads to $M_1^1 = 0$, $M_1^2 = 1$, $M_2^1 = -1$, and $M_2^2 = 0$. That is, M is as in Table 6.2.40 for type III*, and $p_1^\circlearrowleft(z) = p_2(z)$ and $p_2^\circlearrowleft(z) = -p_1(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type III* are equivalent to $a(z) = z^{3/4}\,(p_1(z) - \mathrm{i}\; p_2(z))/2$ and $b(z) = z^{1/4}\,(p_1(z) + \mathrm{i}\; p_2(z))/2$, where $p_1^\circlearrowleft(z) = p_2(z)$ and $p_2^\circlearrowleft(z) = -p_1(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic functions on $D\backslash\{0\}$. Furthermore, $p_i(z) \sim z^{-3/4}\, c_i$ and (6.2.47), with ω and M_i^j as above imply that $a(z) \to c_1 \neq 0$ and $b(z) = \mathrm{O}(|z|^{-1/2})$ as $z \to 0$; hence a and b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$. The quotient $q(z) = p_2(z)/p_1(z)$, and hence the modulus function $J(z) = \mathcal{J}(q(z))$, is the same as for type III*.

TypeIV

The singular fiber $\varphi = 0$ is the union of three smooth rational curves Θ_i, $i = 1,\ 2,\ 3$, that intersect each other at one point p, with different tangent lines at p. Blowing up p we obtain a -1 curve E of multiplicity 3, where each of the proper transforms Θ_i' of Θ_i intersects E at one point and transversally. The three intersection points b_i in E are distinct, and there are no other intersections. With the substitution $z = \zeta^3$ the vector field $\zeta^{-1}\, v$ on the fiber S_z of φ over z near E converges to three holomorphic vector fields without zeros on $E \setminus \{b_1,\ b_2,\ b_3\}$. Furthermore, there exists a threefold branched covering $\pi : E^\sim \to E$ and a holomorphic vector field $v^\sim$ without zeros on $E^\sim$ that is mapped by π to the three-times rescaled limit vector fields on E. Over each of the branch points b_i all the three branches come together, and the Riemann–Hurwitz formula (2.3.18) reconfirms that $E^\sim$ is an elliptic curve.

In the same way as for type I_0^* this leads to a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period lattice of v on S_z, where asymptotically $p_i(z) \sim z^{-1/3}\, c_i$ as $z \to 0$, and c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if $q_0 = c_2/c_1$. Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ into (6.2.43), we obtain, after multiplication by $z^{1/3}$ and taking the limit for $z \to 0$, that (6.2.47) holds with $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}\,/3}$. It follows that $M^3 = 1$, Section 2.3.3 yields that $\mathcal{J}(q_0) = 0$ if $q_0 := c_2/c_1$, and it can be arranged that $q_0 = \mathrm{e}^{2\pi\,\mathrm{i}\,/6}$. With this choice, division of (6.2.47) by c_1 and consideration of the real and imaginary parts leads to $M_1^1 = 0$, $M_1^2 = -1$, $M_2^1 = 1$, and $M_2^2 = -1$. That is, M is as in Table 6.2.40 for type IV, and $p_1^\circlearrowleft(z) = -p_2(z)$ and $p_2^\circlearrowleft(z) = p_1(z) - p_2(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type IV are equivalent to $a(z) = z^{1/3}\,(p_1(z) - \mathrm{e}^{2\pi\,\mathrm{i}\,/6}\; p_2(z))/(1 - \mathrm{e}^{2\pi\,\mathrm{i}\,/3})$ and $b(z) = z^{-1/3}\,(p_1(z) - \mathrm{e}^{-2\pi\,\mathrm{i}\,/6}\; p_2(z))/(1 - \mathrm{e}^{-2\pi\,\mathrm{i}\,/3})$, where $p_1^\circlearrowleft(z) = -p_2(z)$ and $p_2^\circlearrowleft(z) = p_1(z) - p_2(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic

functions on $D \setminus \{0\}$. Furthermore, $p_i(z) \sim z^{-1/3}\, c_i$ and (6.2.47) with ω and M_i^j as above imply that $a(z) \to c_1 \neq 0$ and $b(z) = \mathrm{O}(|z|^{-2/3})$ as $z \to 0$; hence a and b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$. The quotient $q(z) = p_2(z)/p_1(z)$ and hence the modulus function $J(z) = \mathcal{J}(q(z))$ is the same as for type II*.

TypeIV* In contrast to type IV, the singular fiber is already in a normal crossing situation; see Kodaira's list in Section 6.2.6. The vector field of v on the fiber S_z of φ over z is of the smallest order $z^{2/3}$ near the component Θ_4, which is intersected by the multiplicity-2 components Θ_2, Θ_3, and Θ_5 at the points b_2, b_3, and b_5, respectively. With the substitution $z = \zeta^3$ the vector field $\zeta^{-2}\, v$ on the fiber S_z of φ over z near Θ_4 converges to three holomorphic vector fields without zeros on $\Theta_4 \setminus \{b_2,\ b_3,\ b_5\}$. Furthermore, there exist a threefold branched covering $\pi : \Theta_4^{\sim} \to \Theta_4$ and a holomorphic vector field $v^{\sim}$ without zeros on $\Theta_4^{\sim}$ that is mapped by π to the three rescaled limit vector fields on Θ_4. Since the configuration of the branches is the same as for type IV, the Riemann–Hurwitz formula (2.3.18) reconfirms that $E^{\sim}$ is an elliptic curve.

In the same way as for type II*, this leads to a $\mathbb{Z}$-basis $p_1(z)$, $p_2(z)$ of the period lattice of v on S_z, where asymptotically $p_i(z) \sim z^{-2/3}\, c_i$ as $z \to 0$, and c_1 and c_2 are nonzero complex numbers such that $\operatorname{Im} q_0 > 0$ if $q_0 = c_2/c_1$. Substituting this asymptotic behavior of $p_1(z)$, $p_2(z)$ into (6.2.43), we obtain, after multiplication by $z^{2/3}$ and taking the limit for $z \to 0$, that (6.2.47) holds with $\omega = \mathrm{e}^{-2\pi\,\mathrm{i}\,2/3}$. It follows that $M^3 = 1$, Section 2.3.3 yields that $\mathcal{J}(q_0) = 0$ if $q_0 := c_2/c_1$, and it can be arranged that $q_0 = \mathrm{e}^{2\pi\,\mathrm{i}/6}$. With this choice, division of (6.2.47) by c_1 and consideration of the real and imaginary parts leads to $M_1^1 = -1$, $M_1^2 = 1$, $M_2^1 = -1$, and $M_2^2 = 0$. That is, M is as in Table 6.2.40 for type IV*, and $p_1^{\circlearrowleft}(z) = -p_1(z) + p_2(z)$ and $p_2^{\circlearrowleft}(z) = -p_1(z)$. The equations for $p_1(z)$ and $p_2(z)$ in Table 6.2.39 for type IV* are equivalent to $a(z) = z^{2/3}\,(p_1(z) - \mathrm{e}^{2\pi\,\mathrm{i}/6}\ p_2(z))/(1 - \mathrm{e}^{2\pi\,\mathrm{i}/3})$ and $b(z) = z^{1/3}\,(p_1(z) - \mathrm{e}^{-2\pi\,\mathrm{i}/6}\ p_2(z))/(1 - \mathrm{e}^{-2\pi\,\mathrm{i}/3})$, where $p_1^{\circlearrowleft}(z) = -p_1(z) + p_2(z)$ and $p_2^{\circlearrowleft}(z) = -p_1(z)$ imply that $a(z)$ and $b(z)$ are single-valued holomorphic functions on $D \setminus \{0\}$. Furthermore, $p_i(z) \sim z^{-1/3}\, c_i$ and (6.2.47) with ω and M_i^j as above imply that $a(z) \to c_1 \neq 0$ and $b(z) = \mathrm{O}(|z|^{-1/3})$ as $z \to 0$; hence a and b extend to holomorphic functions on D, denoted by the same letters, where $a(0) \neq 0$. The quotient $q(z) = p_2(z)/p_1(z)$, and hence the modulus function $J(z) = \mathcal{J}(q(z))$, is the same as for the type II.

This completes the determination of the behavior of the periods of the vector field v on the regular fiber of φ near the singular fiber for each of the Kodaira types. □

Inspection of Table 6.2.40 leads to the following observations.

Corollary 6.2.41 *Two singular fibers of elliptic surfaces have the same Kodaira type if and only if the monodromy matrices around these fibers are conjugate by an element of* $\mathrm{SL}(2,\ \mathbb{Z})$.

Corollary 6.2.42 *The modulus function $J : C^{\mathrm{reg}} \to \mathbb{C}$ has an extension to a unique holomorphic mapping $J : C \to \mathbb{P}^1$. If C is compact, then $J : C \to \mathbb{P}^1$ has a mapping of degree j, equal to*

$$j = \sum_{k>0} k\,\#(I_k) + \sum_{l>0} l\,\#(I_l^*), \tag{6.2.48}$$

where $\#(T)$ denotes the number of singular fibers of Kodaira type T. We have $j \leq \chi_{\mathrm{top}}(S)$, with equality if and only if each singular fiber is of type I_k for some $k > 0$. The modulus function is a finite constant if and only if $j = 0$ if and only if there are no singular fibers of Kodaira type I_k, $k > 0$, or I_l^*, $l \geq 0$.

Proof. The degree j of J is equal to the number of points c such that $J(c) = \infty$, counted with multiplicities. It follows from Table 6.2.40 that J has a pole of order b at each $c \in C$ such that S_c is a singular fiber of type I_b or I_b^*, and J has no other poles. This proves (6.2.48). The second statement follows from the combination of (6.2.48) with (6.2.39). □

Remark 6.2.43. If S_0 is a singular fiber of Kodaira type I_b, $b > 0$, then the fundamental group of S_0 is isomorphic to $\mathbb{Z}$, and therefore $\mathrm{H}_1(S_0, \mathbb{Z}) \simeq \mathbb{Z}$. Let $\lambda_z = \lambda_z^i$ and ϵ_z be the loops in S_z as in the proof of Lemma 6.2.38, where for $0 < |z| \ll 1$ the homology classes $[\lambda_z]$ and $[\epsilon_z]$ form a positively oriented $\mathbb{Z}$-basis of $\mathrm{H}_1(S_z, \mathbb{Z})$, and $[\epsilon_0]$ generates $\mathrm{H}_1(S_0, \mathbb{Z})$. If, for $0 < |z| \ll 1$, $r_z : S_z \to S_0$ is a retraction as in the first paragraph of the proof of Lemma 6.2.29, then $(r_z)_* : \mathrm{H}_1(S_z, \mathbb{Z}) \to \mathrm{H}_1(S_0, \mathbb{Z})$ maps $[\lambda_z]$ to zero and $[\epsilon_z]$ to ϵ_0. That is, $(r_z)_*$ is surjective and $\ker(r_z)_* = \mathbb{Z}\,[\lambda_z]$. For this reason λ_z is called the *vanishing cycle* in the regular fiber S_z near the singular fiber S_0. The kernel of $(r_z)_*$ has two generators, $\pm\,[\lambda_z]$, where the sign depends on the direction in which the retraction to one of the two local branches of $S_0 \setminus \{s\}$ at the singular point s runs around s. If in one of the local branches the orientation is positive, it is negative in the other local branch.

For all the other Kodaira types the singular fiber is simply connected, hence $(r_z)_* = 0$, and all cycles in the nearby regular fibers are vanishing. Also note that the type I_b, $b > 0$, is the only one for which there is a single-valued nonzero period function, where the basic one is obtained by integrating $\omega/\,\mathrm{d}\varphi$ over the vanishing cycle. ⊘

Remark 6.2.44. The branched coverings in the proof of Lemma 6.2.38 for the types other than I_b and I_b^* for $b > 0$, the types for which the monodromy matrix has finite order, extend to the elliptic fibrations $\varphi^\sim : S^\sim \to D^\sim$ without singular fibers described in Remark 6.3.21. This extension is the construction of Kodaira [109, pp. 582–596] in the reverse order, where we did not need the extension in order to prove Table 6.2.39.

Remark 6.2.45. Kodaira [109, II, Section 7] proved directly that the quotient $q(z)$ of two basic periods defines a holomorphic function from $D \setminus \{0\}$ to the complex upper half-plane H, such that $J(z) := \mathcal{J}(q(z))$ extends to a meromorphic function on D, with a pole at $z = 0$ if and only if the monodromy of the function J, see Section 6.4.1,

has inifite order. In [109, II, Section 8] he then determined the monodromy matrices M with lift equal to the monodromy of J, and constructed for each meromorphic function J and compatible M an elliptic fibration over a small disk around the singular value $z = 0$, and in the construction he identified the type of the singular fiber. In [109, II, Section 9] he proved that any relative elliptic fibration over a small disk around $z = 0$ with a holomorphic section, modulus function equal to J, and monodromy matrix equal to M is isomorphic to the constructed one. This implies Table 6.2.40. I have not found Table 6.2.39 explicitly in the literature. The construction of Kodaira [109, pp. 582–596] uses a multivalued function $q : D\backslash\{0\} \to H$ such that $J(z) = \mathcal{J}(q(z))$, where inspection of the formulas in [109, pp. 582–596] shows that $q(z) = p_2(z)/p_1(z)$ with $p_1(z)$ and $p_2(z)$ as in Table 6.2.39. However, $p_1(z)$ and $p_2(z)$ differ from the denominators and numerators in [109, pp. 582–596] by a common factor of the form z^m, where m is equal to the aymptotically leading exponents $-1/6$, $-1/4$, $-1/3$, $-1/2$, $-2/3$, $-3/4$, and $-5/6$ for types II, III, IV, I_0^*, IV*, III*, and II*, respectively. ⊘

Remark 6.2.46. If the Lie algebra line bundle $\mathfrak{f}$ over C is not trivial, as is often the case, it is not possible to find a holomorphic section v of $\mathfrak{f}$ over the whole curve C without zeros, as required in Lemma 6.2.38. If the vector field v corresponds to a meromorphic section of $\mathfrak{f}$ over C with a zero or pole at $c_0 \in C$ of order $m \in \mathbb{Z}_{>0}$, then the exponents at the point $c_0 \in C$ are equal to the exponents in Table 6.2.39 minus or plus m, respectively. The exponents agree at all points of C where v has neither zeros nor poles. In the case of a rational elliptic surface one can arrange that v has a simple pole at the point at infinity on the complex projective line that parametrizes the fibers of the elliptic fibration; see Remark 9.2.9.

Schmickler–Hirzebruch [176] determined the elliptic fibrations $\varphi : S \to \mathbb{P}^1$ with at most three singular fibers; see also Remark 6.3.12. She always positioned one of the singular values at ∞ and, for the Weierstrass normal form of the fibration, gave the hypergeometric differential equation $\mathcal{L}\, p = 0$ for the periods as in Section 2.5.3. Analyzing the regular singular points of the differential equation, she determined the exponents in the period functions for each singular fiber; see [176, pp. 154–157]. These agree with the exponents in Table 6.2.39, except for the fibers over ∞, where her exponents are equal to ours plus 1 or 2 if the elliptic surface is rational or a K3 elliptic surface, respectively. This is explained by the fact that the section v of $\mathfrak{f}$ that corresponds to the differential equation $\mathcal{L}\, p = 0$ has no zeros or poles in the finite domain $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$. Because $\mathfrak{f} \simeq \mathrm{O}(-N)$ with $N = 1$ or $N = 2$ if the elliptic surface is rational or K3, see Example 5, it follows that v has a pole of order N at ∞, and therefore the exponents are equal to those in Table 6.2.39 plus N.

As observed in Remark 6.2.43, one of the period functions for a singular fiber of type I_b can be chosen to be a single-valued holomorphic function without zeros in an open neighborhood of the singular value of the fibration. This leads to the following conclusions, where the last statement corresponds to the description in Kodaira [110, I, Theorem 12 on p. 772] of the canonical bundle of S in the presence of multiple singular fibers.

Corollary 6.2.47 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration, $c_0 \in C$, and S_{c_0} a singular fiber of type ${}_m\,\mathrm{I}_b$, where it is allowed that $m > 1$, that is, S_{c_0} is a multiple singular fiber. Then*

(i) *There are an open neighborhood C_0 of c_0 in C and a holomorphic vector field v on $\varphi^{-1}(C_0)$ that is tangent to the fibers of φ and has the property that $\mathbb{R}/\mathbb{Z} \ni t \mapsto \mathrm{e}^{t\,v}$ defines an action of the circle group $\mathbb{R}/\mathbb{Z}$ on $\varphi^{-1}(C_0)$ by means of complex analytic diffeomorphisms, where this action is free on the complement of the intersection points of the distinct irreducible components of S_{c_0}, when these exist.*

(ii) *The basic period functions defined by v are the same as given in Lemma* 6.2.38 *for a singular fiber of Kodaira type $\mathrm{I}_b =_1 \mathrm{I}_b$, with $a(z) \equiv 1$, and therefore the monodromy around c_0 is the same as for a singular fiber of type I_b.*

(iii) *If z is a holomorphic coordinate on C_0, then the holomorphic complex two-form ω on $\varphi^{-1}(C_0)$ such that $\mathrm{i}_v\,\omega = -\,\mathrm{d}(z \circ \varphi)$ has a zero of order $m-1$ along S_{c_0}.*

Proof. We use the description of the elliptic fibration near a multiple singular fiber in Section 6.2.5. By shrinking D, hence E, if necessary, we have a holomorphic section u over E without zeros of the Lie algebra bundle of the elliptic fibration $\psi : N \to E$. It follows from Lemma 6.2.38 in the case I_b, with $\varphi : S \to D$ and I_b replaced by $\psi : N \to E$ and $\mathrm{I}_{m\,b}$, respectively, that one of the basic period functions is a single-valued holomorphic function without zeros on E, and by multiplying u by this function we can arrange that this basic period function is identically equal to 1. That is, $\mathbb{R}/\mathbb{Z} \ni t \mapsto \mathrm{e}^{t\,u}$ defines an action on N of the circle group $\mathbb{R}/\mathbb{Z}$ by means of complex analytic diffeomorphisms of N, where the action is free on the complement of the set I of all intersection points of the distinct irreducible components of the fiber N_0 of ψ over 0, when these exist.

The generator $\gamma : N \to N$ of the group of deck transformations of the m-fold covering $\Phi : N \to S$ maps u to a holomorphic vector field u' on N that is also tangent to the fibers of ψ and generates an $\mathbb{R}/\mathbb{Z}$-action on N that is free on $N \setminus I$. It follows that $u' = (p \circ \psi)\,u$, where p is a single-valued primitive period function. That is, if $p_1 = 1$ and p_2 are basic period functions, where p_2 is multivalued if $b > 0$, then $p = k_1\,p_1 + k_2\,p_2$ for $k_1,\ k_2 \in \mathbb{Z}$ such that $\gcd(k_1,\ k_2) = 1$. If $b = 0$, then γ is a translation on N_0 that commutes with the translations $\mathrm{e}^{t\,u}$, and it follows that $u' = u$ on N_0, hence $k_1 = 1$, $k_2 = 0$, and therefore $u' = u$. If $b > 0$, the single-valuedness of p and the multivaluedness of p_2 imply that $k_2 = 0$, hence $k_1 = \pm 1$ in view of $\gcd(k_1,\ k_2) = 1$, and we conclude that $u' = \pm\,u$. It follows from the proof of Lemma 6.2.38 for type I_b that the concatenation of the curves $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}\,s\,u}(n_j)$, $0 \le j \le m\,b - 1$, is a loop in N_0 whose homotopy class generates the fundamental group $\simeq \mathbb{Z}$ of N_0. Because the action of γ on the fundamental group of N_0 preserves this generator, see the text preceding Definition 6.1.4, and we obtain the inverse of this generator if we replace u by $-u$, it follows again that $u' = u$. The invariance of u under γ implies that the covering map $\Phi : N \to S$ intertwines u with a unique vector field v on S, which satisfies (i).

It follows from Lemma 6.2.38, for type $\mathrm{I}_{m\,b}$ and with v replaced by u, that there is a holomorphic function r on E such that $p_{2\,u}(\zeta) = (m\,b/2\pi\,\mathrm{i})\,\log\zeta + r(\zeta)$. Then

$p_{2,v}(z) = p_{2,u}(\zeta) = (b/2\pi\,\mathrm{i})\,\log z + r(z^{1/m})$, $z = \zeta^m$, defines a second basic period function for the vector field v. If $p^{\circlearrowleft}_{2,v}(z)$ denotes the analytic continuation of $p_{2,v}(z)$ when z runs once around the origin, then the integrality of the monodromy matrix implies that there exist $l_1,\ l_2 \in \mathbb{Z}$ such that

$$\begin{aligned} l_1 + l_2\,((b/2\pi\,\mathrm{i})\,\log z + r(z^{1/m})) &= l_1\,p_{1,v}(z) + l_2\,p_{2,v}(z) = p^{\circlearrowleft}_{2,v}(z) \\ &= (b/2\pi\,\mathrm{i})\,\log z + b + r(\mathrm{e}^{2\pi\,\mathrm{i}/m}\ z^{1/m}). \end{aligned}$$

Comparing the asymptotic behavior of the left- and right-hand sides as z tends radially to 0, we conclude subsequently that $l_2 = 1$, $l_1 = b$, and $r(\zeta) = r(\mathrm{e}^{2\pi\,\mathrm{i}/m}\ \zeta)$. The last identity implies that there is an analytic function s on D such that $r(\zeta) = s(\zeta^m)$; hence $p_{2,v}(z) = (b/2\pi\,\mathrm{i})\,\log z + s(z)$. This proves (ii).

Finally, (iii) follows from the observation that $z \circ \varphi - z(c_0)$ has a zero of order m along S_{c_0}, hence $\mathrm{d}(z \circ \varphi)$ has a zero of order $m-1$ along S_{c_0}. □

6.2.13 Resolution of Singularities of Surfaces

In this subsection we review some basic general facts about resolution of singularities of complex analytic surfaces, that will be used later on.

A *complex analytic space* is a Hausdorff topological space X provided with a family of homeomorphisms $\varphi_\iota : X_\iota \to V_\iota$, called charts, where the X_ι are open subsets of X with union equal to X, V_ι open subsets of locally analytic sets in $\mathbb{C}^{n_\iota}$, such that for each $\kappa,\ \iota$, the mapping $\varphi_\kappa \circ {\varphi_\iota}^{-1}$ is holomorphic on $\varphi_\iota(X_\iota \cap X_\kappa)$. Here a subset V of $\mathbb{C}^n$ is called locally analytic if for each $v \in V$ there exists an open neighborhood U of v in $\mathbb{C}^n$ such that $V \cap U$ is equal to the common zero-set of a collection of holomorphic functions on U. A mapping $\psi : V \to \mathbb{C}^m$ is called holomorphic if for every $v \in V$ there is an open neighborhood U of v in $\mathbb{C}^n$ such that $\psi|_{V\cap U}$ extends to a holomorphic mapping from U to $\mathbb{C}^m$.

We refer to Łojasiewicz [125, Chapter V, §4] for the basic properties of complex analytic spaces. For instance, $x \in X$ is called a *smooth point* or *nonsingular point* of X of dimension n if there is a biholomorphic mapping from an open neighborhood U of x in X onto an open subset of $\mathbb{C}^n$. The set of all nonsingular points of X form an open dense subset X° of X. Its complement X^*, the *singular locus* of X, is a closed analytic subset of X.

A complex analytic space X is called *irreducible* if X is not equal to the union of two closed analytic subspaces, both not equal to X. Any complex analytic space X uniquely decomposes into a locally finite union of irreducible closed analytic subspaces, called the *irreducible components* of X. If X is irreducible, then X is connected, and the nonsingular part $X \setminus X^*$ of X is a connected complex analytic manifold of some complex dimension n. Furthermore, the smooth part of each irreducible component of the singular locus X^* of X is a connected complex analytic manifold of complex dimension $< n$. Continuing this process of taking the smooth parts of irreducible components of singular loci, one obtains a locally finite decomposition of an arbitrary complex analytic space into connected complex analytic

manifolds, called the strata, where the closure of a stratum consists of the stratum itself and lower-dimensional strata. The dimension of an irreducible complex analytic space is defined as the dimension of its smooth part. The dimension $\dim X$ of any complex analytic space X is defined as the supremum of the dimensions of its irreducible components.

A complex analytic space X is called a complex analytic curve or surface if each of its irreducible components has complex dimension one or two, respectively. If X is a complex analytic surface, then the singular locus of each irreducible component of X is a locally finite union of complex analytic curves and isolated singular points.

A *modification* of an analytic space X is a surjective proper holomorphic map f from an analytic space Y onto X such that there is a nowhere dense closed analytic subset S of X whose inverse image $f^{-1}(S)$ is nowehere dense in Y and such that the restriction of f to $Y \setminus f^{-1}(S)$ is a biholomorphic mapping from $Y \setminus f^{-1}(S)$ onto $X \setminus S$. That is, there exists a holomorphic mapping $g = g_S : X \setminus S \to Y \setminus f^{-1}(S)$ such that $f \circ g$ and $g \circ f$ are equal to the identity on $X \setminus S$ and $Y \setminus f^{-1}(S)$, respectively. If we have the same properties with S and g replaced by S' and g', then the surjectivity of f implies that $f(Y \setminus f^{-1}(S \cup S')) = X \setminus (S \cup S') = (X \setminus S) \cap (X \setminus S')$; hence $g \circ f = 1 = g' \circ f$ on $(Y \setminus f^{-1}(S)) \cap (Y \setminus f^{-1}(S')) = Y \setminus f^{-1}(S \cup S')$ implies that $g = g'$ on $(X \setminus S) \cap (X \setminus S')$. Therefore, if S_0 denotes the intersection of all S's, then the g_S have a common extension g_0 to the union $X \setminus S_0$ of all $X \setminus S$'s, where g_0 is a holomorphic map from $X \setminus S_0$ to $Y \setminus f^{-1}(S_0)$, and $f \circ g_0$, $g_0 \circ f$ are equal to the identity on $X \setminus S_0$ and $Y \setminus f^{-1}(S_0)$, respectively. In this situation, f is called a *modification of the space X in the subspace S_0*. See Łojasiewicz [125, Chapter V, §4, No. 13].

Let X_i, $i \in I$, be the irreducible components of X. Let Y be the set of all $(x,\ i) \in X \times I$ such that $x \in X_i$, and let f be the restriction to Y of the mapping $(x,\ i) \mapsto x$. Then Y has a unique structure of a complex analytic space such that the subsets $X_i \times \{i\}$ are the connected components of Y, and $f : Y \to X$ is a modification. The complex analytic space Y is called the disjoint union of the irreducible components of X, with the identification mapping $f : Y \to X$. This modification allows us to restrict the discussion of many properties of complex analytic spaces to irreducible ones.

The point $x \in X$ is called a *normal point* if for every open neigborhood U of x in X and every bounded holomorphic function f on $U \setminus X^* = U \cap X^\circ$ there is an open neighborhood U' of x in U such that $f|_{U' \setminus X^*}$ has a holomorphic extension to U'. It follows from Riemann's theorem on removable singularities that every smooth point of X is a normal point of X. Also, the point $x \in X$ is a normal point of X if and only if the ring $\mathcal{O}_x$ of germs of holomorphic functions at x is integrally closed in its ring of quotients. This means that if $f,\ g \in \mathcal{O}_x$, $g \neq 0$, and $q := f/g$ satisfies a polynomial equation of the form

$$\sum_{k=0}^{m} c_k\, q^{m-k} = 0,$$

with $c_k \in \mathcal{O}_x$ and $c_0 = 1$, then $q \in \mathcal{O}_x$.

The analytic space X is called *normal* if every $x \in X$ is a normal point of X. The set of nonnormal points of X is a closed analytic subspace of X. If X is irreducible and normal, then $\dim X^* \leq \dim X - 2$. In particular, if X is a normal surface, then its singular locus consists of isolated points in X. A *normalization* of an analytic space X is a modification $f : Y \to X$ in a closed complex analytic subspace of the singular locus X^* of X, where Y is a normal analytic space and the mapping f is finite, in the sense that for each $x \in X$, the fiber $f^{-1}(\{x\})$ of f over x is a finite subset of Y. Every analytic space X has a normalization. See Łojasiewicz [125, Chapter VI, §§2–4] for these basic facts about normal spaces. The following theorem on uniqueness of normalizations is inspired by Łojasiewicz [125, Chapter VI, §4, No. 1].

Lemma 6.2.48 *Let X, Y, Z be complex analytic spaces, $f : Y \to X$ a finite modification of X in the subspace S of X^*, and $g : Z \to X$ a holomorphic map such that $g(Z_i)$ is not contained in S for any irreducible component Z_i of Z. If Z is normal, then the holomorphic mapping $f^{-1} \circ g$ from $g^{-1}(X \setminus S)$ to Y extends to a holomorphic mapping h from Z to Y, and h is the unique continuous mapping from Z to Y such that $g = f \circ h$. If f and g are both normalizations of X, then the mapping h is biholomorphic.*

Proof. The uniqueness of the continuous extension h of $f^{-1} \circ g$ follows from the fact that $g^{-1}(X \setminus S)$ is dense in Z.

Let $z \in Z$. Then $f^{-1}(\{g(z)\})$ is finite, and therefore it has a neighborhood K in Y that is relatively compact in a chart. It follows from the properness of f that there exists a neighborhood U of $g(z)$ in X such that $f^{-1}(U) \subset K$, and because g is continuous, there is a neighborhood W of z in Z such that $g(W) \subset U$. Therefore $(f^{-1} \circ g)(W \setminus g^{-1}(S)) \subset B$. If $z \in Z^\circ = Z \setminus Z^*$, then it follows from Riemann's theorem on removable singularities that $f^{-1} \circ g$ has a holomorphic extension to an open neighborhood of z in Z, and the uniqueness of continuous extensions implies that these local holomorphic extensions piece together to a global holomorphic extension $h : Z^\circ \to Y$ of $f^{-1} \circ g$. But now the boundedness of h on $W \setminus Z^*$ and the assumption that z is a normal point of Z imply that h has a holomorphic extension to an open neighborhood of z in Z.

If both f and g are normalizations, then $f^{-1} \circ g$ and $g^{-1} \circ f$ have holomorphic extensions $h : Z \to Y$ and $k : Y \to Z$, respectively, where $h \circ k$ and $k \circ h$ are equal to the identity on a dense subset of Y and Z, and therefore equal to the identity on Y and Z, respectively. That is, h is bijective from Z onto Y and its inverse is holomorphic. □

Note that if Y is the disjoint union of the irreducible components of X and $f : Y \to X$ the identification mapping, then $f : Y \to X$ satisfies the conditions in Lemma 6.2.48. It follows that the normalization of X consists in first passing to the disjoint union of the irreducible components of X, and then taking the normalization of each irreducible component. This observation also shows that every normal space is locally irreducible.

Let $C = \{(x, y) \in \mathbb{C}^2 \mid y^2 - x^3 = 0\}$ be the ordinary cusp, viewed as a complex analytic subspace of $\mathbb{C}^2$. Then $\gamma : \mathbb{C} \to C : t \mapsto (t^2, t^3)$ is a normalization of C by the smooth curve $\mathbb{C}$. Since $\gamma^{-1} : C \to \mathbb{C}$ is not holomorphic at $(0, 0)$, C is not

normal at $(0, 0)$, whereas on the other hand, C is locally irreducible. This example also shows that normalization maps can be bijective without having a holomorphic inverse.

A modification $f : Y \to X$ of X with Y nonsingular is called a *resolution of singularities* of the complex analytic space X. Resolutions of singularities are far from unique, because if $f : Y \to X$ is a resolution of singularities of X, and $g : Z \to Y$ is a blowup map with Z a nonsingular complex analytic space, then $f \circ g : Z \to X$ is also a resolution of singularities of X.

In the sequel we will restrict the discussion to surfaces. Note that g maps a -1 curve in Z to a point in Y, and therefore $f \circ g$ maps the same -1 curve in Z to a point in X. Note that $g : Z \to Y$ is a resolution of singularities of Y, despite the fact that Y is nonsingular. A resolution of singularities $f : Y \to X$ of X will be called *minimal* if f does not map a -1 curve in Y to a point in X.

If Y is a nonsingular complex analytic surface, then a complex analytic curve E is called *exceptional* if there exist an open neighborhood Y_0 of E in Y, a complex analytic space X with a point $x \in X$, and a modification $f : Y_0 \to X$ of X in $\{x\}$ such that $E = f^{-1}(\{x\})$. Lemma 3.2.1 yields that E is compact and connected. *Grauert's criterion*, see Grauert [70, p. 367], is the content of the following lemma:

Lemma 6.2.49 *A compact connected curve E in a nonsingular complex analytic surface is exceptional if and only if the intersection matrix $E_i \cdot E_j$ is negative definite, where the E_i are the irreducible components of E.*

In the setting of complex projective algebraic surfaces, this criterion is due to Mumford [144, pp. 6 and 18]. The "only if" part in Grauert's criterion is an important tool in understanding the structure of the possible desingularizations of a given complex analytic surface. According to Brieskorn [25, p. 78], in the algebraic setting the "only if" part is due to Du Val [52, §4]. For the "only if" part when X is nonsingular, see Remark 6.2.51.

Lemma 6.2.50 *Let X, Y be nonsingular complex analytic surfaces and $f : Y \to X$ a modification of X. Then there is a discrete subset D of X such that f is a modification of X in D. For each $x \in D$ the restriction to $f^{-1}(X \setminus (D \setminus \{x\}))$ of f is the composition of finitely many blowing up transformations, where the number of blowups is equal to the number of irreducible components of the fiber of f over x.*

Proof. For every connected component A of X we may replace Y by $B = f^{-1}(A)$, which is a union of connected components of Y, and restrict the discussion to $f : B \to A$. That is, we may assume that X is connected. Let S be the nowhere dense closed complex analytic subspace of X such that f is a modification of X in S, where $T := f^{-1}(S)$ is the nowhere dense closed complex analytic subspace of Y such that f is a complex analytic diffeomorphism from $Y \setminus T$ onto $X \setminus S$. Because $X \setminus S$ is connected it follows that $Y \setminus T = f^{-1}(X \setminus S)$ is connected, and therefore Y is connected. The properness of f implies that $f(Y)$ is a closed subset of X that contains the dense subset $x \setminus S$; hence $f(Y) = X$.

It follows from Lemma 3.2.1 that every fiber of f is connected. Let $x \in X$, $y \in Y$, and $f^{-1}(\{x\}) = \{y\}$. The second paragraph in the proof of Lemma 3.2.1 yields for every open neighborhood Y_0 of y in Y a compact neighborhood K of x in X such that $f^{-1}(K) \subset Y_0$. Let X_0 be the interior of K. If we choose for Y_0 a coordinate neighborhood, and $\chi : Y_0 \to \mathbb{C}^2$ a complex analytic system of coordinates on Y_0, then the compactness of $f^{-1}(K)$ implies that $\chi \circ f^{-1}$ is a bounded complex analytic mapping from $X_0 \setminus S$ to $\mathbb{C}^2$, and it follows from Riemann's theorem on removable singularties that it has a complex analytic extension to X_0. That is, f^{-1} has an extension to a complex analytic mapping $g : X_0 \to Y$, where the identities $f \circ g = \mathrm{id}$ on $X_0 \setminus S$ and $g \circ f = \mathrm{id}$ on $f^{-1}(X_0) \setminus T$ extend by continuity to $f \circ g = \mathrm{id}$ on X_0 and $g \circ f = \mathrm{id}$ on $f^{-1}(X_0)$. That is, the restriction of f to $f^{-1}(X_0)$ is a complex analytic diffeomorphism from $f^{-1}(X_0)$ onto X_0. If D denotes the set of all $x \in X$ such that the fiber of f over x consists of more than one point, then D is a closed subset of S and the restriction of f to $f^{-1}(X \setminus D)$ is a complex analytic diffeomorphism from $f^{-1}(X \setminus D)$ onto $X \setminus D$.

Write $E := f^{-1}(D)$, and let Y_y^{sing} denote the set of all $y \in Y$ such that the tangent map $\mathrm{T}_y f$ of f at y is not bijective, where I apologize for using the same notation as for the entirely different concept of the singular set with respect to a fibration. If $y \in E$ then $x := f(y) \in D$; hence $f^{-1}(\{x\})$ consists of more than one point. Lemma 3.2.1 implies that $f^{-1}(\{x\})$ is connected; hence it is a compact complex analytic subset of Y of strictly positive dimension at each of its points. Because $f^{-1}(\{x\}) \subset f^{-1}(D) \subset f^{-1}(S) = T$ where T has strictly positive codimension at each of its points, and $\dim Y = 2$, it follows that $f^{-1}(\{x\})$ is a compact connected complex analytic curve in Y. Because f is constant, equal to x on the nonsingular part A of $f^{-1}(\{x\})$, we have for each $a \in A$ that $\mathrm{T}_a f = 0$ on the one-dimensional linear subspace $\mathrm{T}_a A$ of $\mathrm{T}_a Y$, hence $A \subset Y_f^{\mathrm{sing}}$. Because Y_f^{sing} is a closed complex analytic subset of Y and A is dense in $f^{-1}(\{x\}$, it follows that $y \in f^{-1}(\{x\}) \subset Y^{\mathrm{sing}}$. Because this holds for every $y \in E$, we have $E \subset Y_f^{\mathrm{sing}}$. Because $f|_{Y \setminus E}$ is a complex analytic diffeomorphism, $Y \setminus E \subset Y \setminus Y_f^{\mathrm{sing}}$, and therefore $E = Y^{\mathrm{sing}}$. It follows that E is a closed complex analytic subset of Y, contained in T and of strictly positive dimension at each of its points. Hence E is a compact complex analytic curve if $E \neq \emptyset$. For each compact subset K of X, $E \cap f^{-1}(K)$ is equal to the union of the disjoint curves $f^{-1}(\{x\})$, $x \in D \cap K$, whose irreducible components are the irreducible components C of E such that $C \subset f^{-1}(K)$. Since any complex analytic set has locally finitely many irreducible components, it follows that $D \cap K$ is finite. Because every point of X has a compact neighborhood, the subset D of X is locally finite. We have proved that there exists a discrete subset D of X such that f is a modification of X in D and for each $x \in D$, the fiber of f over x is a compact connected complex analytic curve in Y. The above proof is Remmert's proof of Remark 2.2.3, which is much simpler for modifications between smooth surfaces.

Let $x \in D$, where $f^{-1}(\{x\})$ is a compact connected complex analytic curve in Y. Replacing Y by $f^{-1}(X \setminus (D \setminus \{x\}))$, we may assume in the sequel that f is a modification of X in $\{x\}$ and $E = f^{-1}(\{x\})$. The first two paragraphs in the proof of the uniqueness in Theorem 3.2.2 imply that if $\pi : X_1 \to X$ is a blowup of X

at the point x, then there is a unique modification $\psi : Y \to X_1$ of X_1 in $E_1 := \pi^{-1}(\{x\}) \simeq \mathbb{P}^1$ such that $f = \pi \circ \psi$. Note that $E = \psi^{-1}(E_1)$ and $\psi(E) = E_1$. The previous paragraph implies that ψ actually is a modification in a discrete subset F of E_1, where the compactness of E_1 implies that F is finite. For each $f \in F$, the fiber of ψ over f is a compact connected complex analytic curve in Y, contained in E, and therefore a union of irreducible components of E. Since the fibers of ψ are disjoint, and $E \neq \psi^{-1}(F)$ because $\psi(E) = E_1 \neq F = \psi(\psi^{-1}(F))$, it follows that the number of irreducible components of $\psi^{-1}(F)$ is strictly smaller than the number of irreducible components of E. It therefore follows by induction on the number of irreducible components of the fiber that over each $f \in F$ the modification ψ is a sequence of n_f blowups, where n_f is equal to the number of irreducible components of $\psi^{-1}(\{f\})$. The proper transform $\psi'(E_1)$ of E_1 is isomorphic to $E_1 \simeq \mathbb{P}^1$, and therefore $\psi'(E_1)$ is irreducible, hence an irreducible component of E. Because ψ maps $\psi'(E_1) \setminus \Psi^{-1}(F)$ bijectively onto $E_1 \setminus F$, E is equal to the union of $\psi'(E_1)$ and $\psi^{-1}(F)$, hence the irreducible components of E are $\psi'(E_1)$ and the irreducible components of $\psi^{-1}(F)$. Therefore the number of irreducible components of E is equal to $1 + \sum_{f \in F} n_f =$ the number of blowups over x in the modification f. □

Remark 6.2.51. When f is a modification of X in one point x, Lemma 6.2.50 is due to Hopf [90]. An analogous statement in the algebraic category follows from Zariski [213, lemma on p. 538].

Hopf [90, §2, No. 6] also made the following observations. Each irreducible component E_i of the exceptional fiber $E := f^{-1}(\{x\})$ is a holomorphically embedded complex projective line. Here E_i is equal to, or the proper transform of, the $\mathbb{P}^1$ that appears at the ith blowup, where $1 \leq i \leq n$ if there are n blowups. If $E_i \neq E_j$ and $E_i \cap E_j \neq \emptyset$, then E_i and E_j intersect each other at a single point, and the intersection is transversal. The intersection diagram T, whose vertices are the E_i, and the vertices E_i and E_j are connected by an edge if and only if $E_i \cap E_j \neq \emptyset$, is a tree, with root E_1. If at any intermediate stage the blowup is at the intersection point of E_i and E_j, then the new vertex is inserted between the proper transforms of E_i and E_j. Hopf called E a *tree of spheres*. He also noted that $E_n \cdot E_n = -1$.

It follows from (3.2.7) and (3.2.8) with $\mathrm{ord}_b(C) = \mathrm{ord}_b(C') = 1$ that if E_i is equal to, or the proper transform of, the projective line E_i^0 that appears at the ith blowup, then $E_i \cdot E_i = -1 - k_i$ if there are k_i later blowups at points in the proper transforms of E_i^0. Together with the tree structure of T, this determines the intersection matrix $M = (E_i \cdot E_j)|_{1 \leq i,\, j \leq n}$. If E_i', $1 \leq i \leq n+1$ is the configuration after one more blowup at b with intersection matrix M', then $x\,M'\,x = x\,M\,x - (x_p - x_{n+1})^2$ if $b \in E_p$ is not an intersection point, whereas $x\,M'\,x = x\,M\,x - (x_p + x_q - x_{n+1})^2$ if $b \in E_p \cap E_q$, $p \neq q$. It follows by induction on n that the intersection matrix is negative definite, confirming the "only if" part of Lemma 6.2.49 in the case that X is nonsingular.

In the special case of successive blowups at base points of pencils of curves with at least one smooth member, the trees have no branches. This follows from the uniqueness of the base point of $\pi'(\mathfrak{P})$ over b in Lemma 3.2.8. ⊘

Corollary 6.2.52 *If X is a complex analytic surface and $f : Y \to X$ is a minimal resolution of singularties of X, then f is a modification of X in the singular locus X^* of X.*

Proof. We apply Lemma 6.2.50 with X and Y replaced by $X^\circ = X \setminus X^*$ and $f^{-1}(X \setminus X^*)$, respectively. The exceptional curve E that appears in the last blowing up over a point $x \in D$ is a -1 curve in $f^{-1}(X^\circ)$, and $f(E) = \{x\}$. The minimality of the resolution f therefore implies that $D = \emptyset$. That is, f is a complex analytic diffeomorphism from $f^{-1}(X^\circ)$ onto X°. □

Lemma 6.2.53 *Every complex analytic surface admits a minimal resolution of singularities.*

Proof. Let X be a complex analytic surface and $n : N \to X$ a normalization of X. The irreducible components of N are the connected components of N. Every irreducible normal complex analytic space has a resolution of singularities, see Barth, Hulek, Peters and van de Ven [11, Theorem 6 on p. 106], where a modification is called a "bimeromorphic map." The disjoint union of the resolutions of singularities of the connected components of N is a resolution of singularities of N, and the composition with n defines a resolution of singularities of X. Because n is finite mapping, the resolution of singularities of X is minimal if and only if the resolution of singularities of N is minimal, and for this reason we may assume in the sequel of the proof that X is normal.

We now expand the existence part of the proof of Barth, Hulek, Peters, and van de Ven [11, Theorem 6.2 on p. 106]. Let $f : Y \to X$ be a resolution of singularities of the normal complex analytic space X. Then the restriction f° of f to $f^{-1}(X^\circ)$ is a resolution of singularities of the nonsingular complex analytic surface $X^\circ = X \setminus X^*$, and it follows from Lemma 6.2.50 that f° is a sequence of blowing-up transformations over a discrete subset D of X°. Let $x \in X$ be a limit point of a sequence $x_j \in D$. Let φ be a coordinate chart on an open neighborhood U of x in X, viewed as a holomorphic map from U to $\mathbb{C}^n$. Then $\varphi \circ f$ is a holomorphic map from the open subset $V := f^{-1}(U)$ of Y to $\mathbb{C}^n$, and we have, for every $y \in f^{-1}(U \cap X^\circ)$ that $f(y) \in D$ if and only if the rank of $\mathrm{T}_y(\varphi \circ f)$ is < 2. The set R of all $y \in f^{-1}(U)$ such that the rank of $\mathrm{T}_y(\varphi \circ f)$ is < 2 is a closed complex analytic subspace of $f^{-1}(U)$, and $R \cap f^{-1}(U \cap X^\circ) = f^{-1}(D)$, which is the union of infinitely many disjoint closed curves, which in view of the properness of f have an accumulation point in Y. This leads to a contradiction to the local finiteness of the number of irreducible components of R, and we conclude that D has no accumulation points in X, that is, D is discrete as a subset of X.

Because the set X^* of singular points of the normal surface X is also discrete, we obtain that f is a modification of X in the discrete subset $X^* \cup D$ of X. Each fiber of f over each point of $X^* \cup D$ has finitely many irreducible components, and by successively blowing down all -1 curves over the points of $X^* \cup D$ we arrive at a minimal resolution of singularities of X. □

According to Brieskorn [25, Satz on p. 78], the existence of a resolution of singularities of any normal two-dimensional complex analytic space was proved first by

Hirzebruch [85], where Grauert and Remmert [71, Satz 31] showed that the two-dimensional complex analytic spaces considered in [85] are precisely the normal ones.

Lemmas 6.2.50 and 6.2.53 lead to the following characterization of bimeromorphic transformations between nonsingular complex analytic surfaces. This characterization was used in the proof of Lemma 3.4.1.

Corollary 6.2.54 *Let X and Y be complex two-dimensional complex analytic manifolds. Then f is a bimeromorphic transformation from X to Y if and only if there exists a complex two-dimensional complex analytic manifold H and compositions $\pi : H \to X$ and $\psi : H \to Y$ of blowups, as in Lemma 6.2.50, such that $f \circ \psi = \pi$ where defined.*

Proof. The "if" part follows from the facts that blowups are bimeromorphic transformations, and compositions and inverses of bimeromorphic transformations are bimeromorphic transformations.

Assume that f is a bimeromorphic transformation. Let G be the closure in $X \times Y$ of the graph of f, which according to Definition 2.2.2 is an irreducible complex analytic surface in $X \times Y$ such that the restrictions π_1 and π_2 to G of the projections $(x,\ y) \mapsto x$ and $(x,\ y) \mapsto y$ are modifications $G \to X$ and $G \to Y$ of X and Y, respectively. We have $\pi_2 = f \circ \pi_1$ where defined. Let $h : H \to G$ be a resolution of singularities of G, a modification of G where H is a complex two-dimensional complex analytic manifold. Then $\pi := \pi_1 \circ h : H \to X$ and $\psi := \pi_2 \circ h : H \to Y$ are modifications as in Lemma 6.2.50, and $f \circ \psi = f \circ \pi_2 \circ h = \pi_1 \circ h = \pi$ where defined. □

I found the following corollary in Barth, Hulek, Peters, and van de Ven [11, Chapter III, Corollary 6.4].

Corollary 6.2.55 *For nonsingular compact complex analytic surfaces S, the irregularity $\mathrm{q}(S) := \dim \mathrm{H}^1(S,\ \mathcal{O})$, the number $\mathrm{h}^{1,\,0}(S) := \dim \mathrm{H}^0(S,\ \Omega^1)$, the geometric genus $\mathrm{p_g}(S) := \dim \mathrm{H}^2(S,\ \mathcal{O})$, and the plurigenera $\mathrm{P}_k(S) := \dim \mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S{}^k))$, $k \in \mathbb{Z}_{\geq 0}$, are bimeromorphic invariants, in the sense that these numbers remain unchanged if S is replaced by a nonsingular complex complex analytic surface S' such that there exists a bimeromorphic transformation $\varphi : S \supset\!\!\to S'$.*

Proof. Theorem 6.2.23 implies that $\mathrm{q}(S)$ and $\mathrm{h}^{1,\,0}(S)$ are determined by the first Betti number $\mathrm{b}_1(S)$, which in view of Lemma 3.2.5 does not change under blowups. The Serre duality (6.2.27) implies that $\mathrm{p_g}(S) = \mathrm{P}_1(S)$, and the last statement in Lemma 3.2.5 implies that the plurigenera do not change under blowups. Therefore Corollary 6.2.55 follows from Corollary 6.2.54. □

The following lemma is a somewhat strengthened version of the uniqueness statement in Barth, Hulek, Peters, and van de Ven [11, Theorem 6.2 on p. 106].

Lemma 6.2.56 *Let X be a complex analytic surface and $f : Y \to X$, $g : Z \to X$ resolutions of singularities of X, with $f : Y \to X$ minimal. Then there is a unique holomorphic mapping $h : Z \to Y$ such that $g = f \circ h$. If $g : Z \to X$ is also minimal, then h is a complex analytic diffeomorphism from Z onto Y.*

Proof. Suppose that K and L are the nowehere dense closed analytic subspaces of X such that $f : Y \to X$ and $g : Z \to X$ are modifications of X in K and L, respectively. The uniqueness of h follows from the continuity of h and the fact that $h = f^{-1} \circ g$ on the dense subset $g^{-1}(K \cup L)$ of Z.

Let R denote the union of the irreducible components of

$$\{(y, z) \in Y \times Z \mid f(y) = g(z)\}$$

that contain points $(y, z) \in Y \times Z$ such that $f(y) = g(z) \notin K \cup L$. Let $r : M \to R$ be a minimal resolution of singularities of R. Note that if $\pi_Y : R \to Y$ and $\pi_Z : R \to Z$ are defined by $\pi_Y(y, z) = y$ and $\pi_Z(y, z) = z$ for every $(y, z) \in R$, then

$$f \circ \pi_Y \circ r(m) = f(y) = g(z) = g \circ \pi_Z \circ r(m)$$

if $m \in M$ and $r(m) = (y, z)$, which shows that $f \circ \pi_Y \circ r = g \circ \pi_Z \circ r$.

Assume that $\pi_Z \circ r : M \to Z$ is not a complex analytic diffeomorphism from M onto Z. Because it is a modification of Z with M and Z nonsingular, it follows from Lemma 6.2.50 that it is a composition of blowing-up transformations, and the last blowing up produces a -1 curve E in M such that $\pi_Z \circ r(E) = \{z\}$ for some $z \in Z$. If $y \in Y$ and $\pi_Y \circ r(E) = \{y\}$, then $r(E) = \{(y, z)\}$, in contradiction to the minimality of $r : M \to R$. It follows that $C := \pi_Y \circ r(E)$ is an irreducible curve in Y, of which E is the proper transform. Because

$$f(C) = f \circ \pi_Y \circ r(E) = g \circ \pi_Z \circ r(E) = g(\{z\}) = \{g(z)\},$$

it follows from the "only if" part of Grauert's criterion Lemma 6.2.49, that $C \cdot C < 0$.

Because $\pi_Y \circ r : M \to Y$ is a modification of Y with M and Y nonsingular, $\pi_Y \circ r$ is a composition of blowing-up transformations. If $\pi_Y \circ r$ is not a diffeomorphism on a neighborhood of E, then it follows from (3.2.8) that $-1 = E \cdot E < C \cdot C < 0$, which leads to a contradiction. Therefore $\pi_Y \circ r$ is a diffeomorphism on a neighborhood of E, which implies that $C = \pi_Y \circ r(E)$ is a -1 curve in Y, in contradiction to the minimality of $f : Y \to X$.

The conclusion is that $\pi_Z \circ r$ is a complex analytic diffeomorphism from M onto Z. The mapping

$$h := (\pi_Y \circ r) \circ (\pi_Z \circ r)^{-1}$$

is holomorphic from Z to Y, and $f \circ (\pi_Y \circ r) = g \circ (\pi_Z \circ r)$ implies that $f \circ h = g$. □

Corollary 6.2.57 *If $f : Y \to X$ is a minimal resolution of singularities of a complex projective algebraic surface X, then Y is projective algebraic in the sense that there is a complex analytic diffeomorphism from Y onto a nonsingular complex projective algebraic surface.*

Proof. The existence of complex projective algebraic resolutions of singularities of complex projective algebraic surfaces goes back to the Italian geometers, the first proof meeting modern standards of completeness having been given by Walker [205].

Because blowing down -1 curves in nonsingular complex projective algebraic surfaces leads to nonsingular complex projective algebraic surfaces, this leads to the existence of a minimal resolution $g : Z \to X$ of singularities of X where Z is a nonsingular complex projective algebraic surface. It follows from Lemma 6.2.56 that there is a complex analytic diffeomorphism h from Y onto Z such that $f = g \circ h$.

□

6.3 The Weierstrass Model

In this section, we assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration with a holomorphic section $\sigma : C \to S$, where it is allowed that S and C are not compact.

Theorem 6.3.6 below states that the Weierstrass $\wp$-function and its derivative, see Section 2.3.2, exhibit S as a family W of Weierstrass curves. The family W is a complex analytic surface that may have singularities, in which case it can be viewed as obtained from S by contracting, for every reducible fiber S_c of φ, all irreducible components of fibers of S_c that do not intersect $\sigma(C)$. Such a contraction of all but one irreducible component of each reducible fiber is necessary, because each Weierstrass curve, whether singular or not, is irreducible; see Remark 2.3.3.

The complex analytic surface W is defined in terms of holomorphic sections g_2 and g_3 of L^4 and L^6, respectively, where the holomorphic line bundle L over C is the dual of the Lie algebra bundle $\mathfrak{f}$ of $\varphi : S \to C$. Theorem 6.3.10 below implies conversely that if L is any holomorphic line bundle over C, and g_2 and g_3 are holomorphic sections of L^4 and L^6, respectively, such that the discriminant $\Delta := {g_2}^3 - 27\,{g_3}^2$, a holomorphic section of L^{12}, is not identically equal to zero, and there is no point $c \in C$ at which g_2 has a zero of order ≥ 4 and g_3 has a zero of order ≥ 6, then there is a relatively minimal elliptic fibration $\varphi : S \to C$, unique up to isomorphism, such that $L \simeq \mathfrak{f}^*$ and g_2 and g_3 are the holomorphic sections that define the Weierstrass model W of $\varphi : S \to C$. A very useful application is the fact that the fiber S_{c_0} of φ over $c_0 \in C$ is singular if and only if $\Delta(c_0) = 0$, while Table 6.3.2 yields a bijective correspondence between the Kodaira type of S_{c_0} and the orders of zeros at c_0 of g_2, g_3, and Δ.

Finally, we use Theorem 6.3.10 in order to prove Proposition 6.3.20, which is a converse to Lemma 6.2.38.

6.3.1 The Weierstrass Model of an Elliptic Surface

We begin this subsection with the definition of the sections g_2 and g_3 of L^4 and L^6, respectively, where $L = \mathfrak{f}^*$. For every $c \in C^{\mathrm{reg}}$ and $v \in \mathfrak{f}_c \setminus \{0\}$, the mapping $\mathbb{C} \ni t \mapsto \mathrm{e}^{t\,v}(\sigma(c))$ induces an isomorphism from the elliptic curve $\mathbb{C}/P_{c,\,v}$ onto the fiber S_c of φ over c. Note that this isomorphism depends on the choice of both

elements v and $\sigma(c)$ in $\mathfrak{f}_c$ and S_c, respectively. Let $t_{c,v} : S_c \to \mathbb{C}/P_{c\,v}$ denote the inverse of this isomorphism. Here $P_{c,v}$ is the period lattice in $\mathbb{C}$, where the linear isomorphism $\mathbb{C} \ni t \mapsto t\,v \in \mathfrak{f}$ induces an isomorphism from $P_{c,v}$ onto the period lattice P_c in $\mathfrak{f}_c$.

We now apply, for each $c \in C^{\mathrm{reg}}$ and $v \in \mathfrak{f}_c \setminus \{0\}$, the constructions of Section 2.3.2, with P replaced by $P_{c,v}$. This leads to the meromorphic functions $x_{c,v}$ and $y_{c,v}$ on S_c defined by

$$x_{c,v}(s) := \wp_{P_{c,v}}(t_{c,v}(s)) \quad \text{and} \quad y_{c,v}(s) := \wp'_{P_{c,v}}(t_{c,v}(s)), \quad s \in S_c,$$

and the complex numbers $g_2(c,\ v) := g_2(P_{c,v})$ and $g_3(P_{c,v})$.

For each $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\mathrm{e}^{(\lambda^{-1}t)\,(\lambda\,v)} = \mathrm{e}^{t\,v}$ and $P_{c,\lambda\,v} = \lambda^{-1}\,P_{c,v}$, and it follows from (2.3.3) that

$$x_{c,\lambda\,v}(s) = \lambda^2\,x_{c,v}(s), \quad y_{c,\lambda\,v}(s) = \lambda^3\,y_{c,v}(s), \tag{6.3.1}$$

and from (2.3.4) that

$$g_2(c,\ \lambda\,v) = \lambda^4\,g_2(c,\ v), \quad g_3(c,\ \lambda\,v) = \lambda^6\,g_3(c,\ v). \tag{6.3.2}$$

The space of all homogeneous functions of degree k on $\mathfrak{f}_c \setminus \{0\}$ is isomorphic to the kth tensor power $(\mathfrak{f}_c^*)^k$ of $\mathfrak{f}_c^*$. This is a complex one-dimensional vector space, which will be denoted by ${L_c}^k$, where $L_c := (\mathfrak{f}_c)^*$. Then (6.3.1) shows that $v \mapsto x_{c,v}(s)$ and $y \mapsto y_{c,v}(s)$ are elements $x(s)$ and $y(s)$ of ${L_c}^2$ and ${L_c}^3$, respectively. Similarly (6.3.2) shows that $v \mapsto g_2(c,\ v)$ and $v \mapsto g_3(c,\ v)$ are elements of $g_2(c)$ and $g_3(c)$ of ${L_c}^4$ and ${L_c}^6$, respectively.

We have the Weierstrass equation (2.3.5) with x, y, g_2, and g_3 replaced by $x(s)$, $y(s)$, $g_2(c)$, and $g_3(c)$, respectively, which is viewed as an equation between elements of the complex one-dimensional vector space ${L_c}^6$. Because $\wp(t)$ and $\wp'(t)$ each have a pole, of orders 2 and 3, respectively, as $t \to 0$, where $t = t_{c,v}(s) = 0$ modulo $P_{c,v}$ if and only if $s = \sigma(c)$, we pass to the projective Weierstrass equation (2.3.6), viewed, for each given c, as a homogeneous equation of degree three in $(x_0,\ x_1,\ x_2) \in {L_c}^0 \times {L_c}^2 \times {L_c}^3$, where ${L_c}^0 := \mathbb{C}$, whose solution set is a cubic curve in the complex projective plane $\mathrm{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3)$.

Because $L := \mathfrak{f}^*$ is a holomorphic complex line bundle over C, the ${L_c}^k$, $c \in C$, form a holomorphic complex line bundle over C, which is denoted by L^k. We have the corresponding holomorphic bundle

$$\mathrm{P}(L^0 \oplus L^2 \oplus L^3) := \mathrm{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3), \quad c \in C,$$

of complex projective planes over C, where $\mathrm{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3)$ denotes the space of all one-dimensional linear subspaces of the three-dimensional vector space ${L_c}^0 \times {L_c}^2 \times {L_c}^3$.

Lemma 6.3.1 *The mappings $x : \varphi^{-1}(C^{\mathrm{reg}}) \to L^2$ and $y : \varphi^{-1}(C^{\mathrm{reg}}) \to L^3$ extend to holomorphic mappings $x : S \to L^2$ and $y : S \to L^3$, respectively, with $p_2 \circ x = \varphi = p_3 \circ y$ if $p_k : L^k \to C$ denotes the canonical projection. The sections*

$g_2 : C^{\mathrm{reg}} \to L^4$ and $g_3 : C^{\mathrm{reg}} \to L^6$ extend to holomorphic sections $g_2 : C \to L^4$ and $g_3 : C \to L^6$ of L^4 and L^6, respectively.

If $c_0 \in C^{\mathrm{sing}}$, then the orders of zeros at c_0 of g_2, g_3, and the holomorphic section $\Delta = {g_2}^3 - 27\,{g_3}^2$ of L^{12} are given in Table 6.3.2, with the Kodaira type of the singular fiber S_{c_0} in the first column.

Table 6.3.2

Type	g_2	g_3	Δ
I_0	≥ 0	≥ 0	0
I_b, $b \geq 1$	0	0	b
I_0^*	≥ 2	≥ 3	6
I_b^*, $b \geq 1$	2	3	$b+6$
II	≥ 1	1	2
II*	≥ 4	5	10
III	1	≥ 2	3
III*	3	≥ 5	9
IV	≥ 2	2	4
IV*	≥ 3	4	8

The Kodaira type of the singular fiber S_{c_0} is uniquely determined by the conditions in Table 6.3.2 on the orders of zeros at c_0 of g_2, g_3, and Δ.

Proof. Let v be a holomorphic section of $\mathfrak{f}$ without zeros over a coordinate neighborhood of c_0 in C, where z is a local holomorphic coordinate function such that $z(c_0) = 0$. Let $p_1(z)$, $p_2(z)$, for $z \in D \setminus \{0\}$, be a $\mathbb{Z}$-basis of the period lattice $P(z)$ of v as in Table 6.2.39. With v defining a trivialization of $L = \mathfrak{f}^*$, hence of the line bundles L^k over D, we have $x(s) = \wp(t) = \wp(z,\, t)$, $y(s) = \wp'(t) = \wp'(z,\, t)$, $g_2 = g_2(z)$, and $g_3 = g_3(z)$ as in (2.3.3), (2.3.4), with P replaced by $P(z)$, $z \in D \setminus \{0\}$. Note that although the functions $p_1(z)$, $p_2(z)$ on $D \setminus \{0\}$ are not single-valued, the fact that the monodromy matrix for these functions has integral coefficients implies that the lattice $P(z) = \mathbb{Z}\, p_1(z) + \mathbb{Z}\, p_2(z)$, and therefore all the functions defined in terms of $P(z)$, are single-valued.

After the substitution of $P = P(z)$ in (2.3.3), (2.3.4), an asymptotic analysis of these series for $z \to 0$ shows that the functions $\wp(t,\, z)$, $\wp'(t,\, z)$, $g_2(z)$, and $g_3(z)$ remain bounded as $z \to 0$, and therefore extend to holomorphic functions of $z \in D$ in view of Riemann's theorem on removable singularities of holomorphic functions of one complex variable. This proves the statements about g_2 and g_3, where a closer scrutiny of the asymptotics for $z \to 0$ leads to Table 6.3.2. For the order of the zeros at $z = 0$ of Δ for types I_b and I_b^* for $b > 0$, we use that the modulus function $J = {g_2}^3/\Delta = 1 + 27\,{g_3}^2/\Delta$ has a pole of order b at $z = 0$; see Table 6.2.40.

Let A denote the holomorphic mapping $(z,\, t) \mapsto \mathrm{e}^{t\,v}(\sigma(z))$ from $D \times \mathbb{C}$ to S. Then $A(D \times \mathbb{C}) = S' = \varphi^{-1}(D) \setminus K$, where $K = S_{c_0} \setminus (\Theta_0 \cap S^{\mathrm{reg}})$, if Θ_0 denotes the irreducible component of S_{c_0} that intersects $\sigma(D)$, as in the second half of the proof of Theorem 6.3.23. We have $x = \wp \circ A$ and $y = \wp' \circ A$, which shows that the functions x and y have holomorphic extensions to S'. If $s = \mathrm{e}^{t\,v}(\sigma(z)) \in S'$ converges to a point $k \in K$, then $t \to \infty$ modulo $P(z)$, and $x(s)$ and $y(s)$ converge to values that are independent of k. Therefore the functions x and y extend to continuous, hence holomorphic, functions on S, which actually are constant on K. □

In view of $J = {g_2}^3/\Delta = 1 + 27\,{g_3}^2/\Delta$, Table 6.3.2 implies the statements in Table 6.2.40 about the behavior of the modulus function J near c_0. A comparison of Tables 6.3.2 and 6.2.40 leads to the following conclusion.

Corollary 6.3.3 *For each $c \in C$ the order of the zero of the discriminant Δ at c is equal to the topological Euler number $\chi_{\mathrm{top}}(S_c)$ of the fiber S_c of φ over c.*

If C is compact, then it follows, in combination with Lemma 6.2.29, that

$$\chi_{\mathrm{top}}(S) = \sum_{c_0 \in C^{\mathrm{sing}}} \chi_{\mathrm{top}}(S_{c_0}) = \deg L^{12} = 12 \deg \mathfrak{f}^*, \tag{6.3.3}$$

an integer that is always nonnegative, and equal to zero if and only if there are no singular fibers. This leads to another proof of the fact that the Euler number of every compact relatively minimal elliptic surface with a holomorphic section is a multiple of 12, which we have seen before in Lemma 6.2.30 as a consequence of Noether's formula. Note that Lemma 6.2.30 in combination with (6.3.3) implies that $\deg \mathfrak{f}^* = \chi(S, \mathcal{O})$, and we recover the formula $\deg \mathfrak{f} = -\deg \mathfrak{f}^* = -\chi(S, \mathcal{O})$ in Lemma 6.2.34.

Remark 6.3.4. In a continuous family of elliptic fibrations, any confluence of singular fibers has to respect the orders of zeros at the singular values of g_2, g_3, and Δ in Table 6.3.2, where the order of the zero of Δ is equal to the Euler number of the singular fiber; see Corollary 6.3.3. This leads to the following restrictions. Order the types as I_1, I_2, …, II, III, IV, I_0^*, I_1^*, …, IV*, III*, II*. According to Table 6.2.39, this order corresponds to the increase in the asymptotic growth order of the period functions. Then any type must be a confluence of previous types in the list. The sum of the topological Euler numbers of the merging fibers is equal to the topological Euler number of the limit fiber. For $i = 2$ and for $i = 3$ the sum of the orders of zeros of g_i of the merging fiber is at most the order of the zero of g_i of the limit fiber. For instance, a singular fiber of type I_b can be only at most, a confluence of singular fibers of type I_{b_i} with $\sum_i b_i = b$, and a singular fiber of type II can be only a confluence of two I_1's. Two "starred" types, that is, from $\{\mathrm{I}_0^*, \mathrm{I}_1^*, \ldots, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*\}$, cannot merge.

Another restriction is that for a suitable choice of loops around the nearby singular values, the product of the monodromy matrices is equal to the monodromy matrix around the limit fiber. For instance, if the limit fiber is of type I_b, then the nearby fibers of type I_{b_i} have a common vanishing cycle, and their monodromy matrices commute, whereas if two I_1's come together to a II, the monodromy matrices of the I_1's will be conjugate to the M_+ and M_- in Example 4. Some additional information on this theme can be found in Naruki [150]. Section 5 in [150] contains the statement that three I_2's cannot merge to a I_0^*. Mathijs Wintraecken, in his 2009 master's thesis, found that the Weierstrass model $\mathrm{g}_2(z) = 3\,(2\,z + \epsilon)\,(z + 4\,\epsilon)$, $\mathrm{g}_3(z) = ((2\,z + \epsilon)^3 + (z + 4\,\epsilon)^3)/2$ exhibits such a confluence when $\epsilon \to 0$.

In Section 9.2.5 a bifurcation is analyzed where two I_1's merge to a I_2. For the Lyness map, see Section 11.4; there are several bifurcations as the parameter a varies. For $a \notin \{-1/4,\ 0,\ 3/4,\ 1,\ 2\}$, the configuration of the singular fibers is

I_5 I_3 I_2 2 I_1. When $a \to -1/4$, the two I_1's merge to a II. When $a \to 0$, the I_5 merges with one of the I_1's to a I_6. When $a \to 3/4$, the I_2 merges with one of the I_1's to a III. When $a \to 1$, the I_3 and I_2 merge to a I_5. Finally, if $a \to 2$, then the I_3 merges with one of the I_1's to a I_4. ⊘

Remark 6.3.5. If C, hence S, is compact, and g_i is not identically zero, which is the case if and only if the modulus function is not constant, equal to 0 or 1, then the number of zeros of g_i, counted with multiplicities, is equal to $\deg L^{2\,i} = 2\,i \deg \mathfrak{f}^* = 2\,i\,\chi(S, \mathcal{O})$. This leads to further global restrictions on the orders of the zeros of g_2 for types I, II*, IV, IV* and of the zeros of g_3 for types III, III*. We obtain the corresponding restrictions on the degrees of the modulus function J at the points where $J = 0$ or $J = 1$, respectively. ⊘

It follows from Lemma 6.3.1 that the complex analytic mapping $s \mapsto (\varphi(s), [1 : x(s) : y(s)])$ from $\varphi^{-1}(C^{\mathrm{reg}}) \setminus \sigma(C)$ to the holomorphic bundle $\mathrm{P}(L^0 \oplus L^2 \oplus L^3)$ extends to a complex analytic mapping from S to $\mathrm{P}(L^0 \oplus L^2 \oplus L^3)$, which we also denote by f. The image $f(S)$ of S under f is equal to the complex analytic surface

$$W := \{(c, [x]) \in \mathrm{P}(L^0 \oplus L^2 \oplus L^3) \mid x_0\,{x_2}^2 - 4\,{x_1}^3 + g_2(c)\,{x_0}^2\,x_1 + g_3(c)\,{x_0}^3 = 0\} \tag{6.3.4}$$

in $\mathrm{P}(L^0 \oplus L^2 \oplus L^3)$. Here, for any nonzero element $x = (x_0, x_1, x_2)$ of the three-dimensional vector space ${L_c}^0 \times {L_c}^2 \times {L_c}^3$, we write $[x] = [x_0 : x_1 : x_2]$ for the one-dimensional linear subspace spanned by x. Because ${L_c}^0 = \mathbb{C}$, we have $[x_0 : x_1 : x_2] = [1 : x : y]$ with $x = {x_0}^{-1} x_1$ and $y = {x_0}^{-1} x_2$ if $x_0 \neq 0$, in which case the equation in (6.3.4) takes the familiar Weierstrass form $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$, as in (2.3.5).

If $p : W \to C$ denotes the restriction to W of the projection $(c, [x]) \to c$, then p is a complex analytic mapping from W onto C, and $\varphi = p \circ f$. Furthermore, because $\wp(0) = \wp'(0) = \infty$, we have that $f(\sigma(c)) = (c, \{0\} \times \{0\} \times {L_c}^3)$ for every $c \in C$.

The surface W in (6.3.4) is equal to the set of all solutions of the projective Weierstrass equation 2.3.6, with g_2 and g_3 replaced by $g_2(c)$ and $g_3(c)$, respectively, and $c \in C$ is added as a tag in order to make the projective Weierstrass curves for different values of c disjoint from each other. The projection $p : W \to C$ is an elliptic fibration with $f \circ \sigma : C \to W$ as a holomorphic section, where it is allowed for the complex analytic surface W to have singular points.

In order to determine the singular points of W, we use a local trivialization of the line bundle $\mathfrak{f}^*$, leading to a corresponding local trivialization of its powers. If $(c, [x_0 : x_1 : x_2]) \in W$ and $x_0 = 0$, then $x_1 = 0$, and $(c, [0 : 0 : 1])$ is a nonsingular point of W. That is, the section $\infty(C) = f(\sigma(C))$ is contained in the nonsingular part W° of W.

It also follows that for the further determination of the singular points of W we can use the affine coordinates $x_0 = 1, x_1 = x, x_2 = y$. The point $(c, [1 : x : y])$ then is a singular point of W if and only if $w(c, x, y) := y^2 - 4\,x^3 + g_2(c)\,x + g_3(c) = 0$, $\partial w(c, x, y)/\partial y = 0$, $\partial w(c, x, y)/\partial x = 0$, and $\partial w(c, x, y)/\partial c = 0$. The second equation yields $y = 0$. Substituting this into the first equation and combining the result with the third equation yields that $\Delta(c) = g_2(c)^3 - 27\,g_3(c)^2 = 0$, which

implies that $c \in C^{\text{sing}}$. Furthermore, either (i) $g_2(c) \neq 0$, $g_3(c) \neq 0$, and $x = -3\,g_3(c)/2\,g_2(c)$, in which case the curve in the $[x_0 : x_1 : x_2]$-plane has a normal crossing, a singular curve of type I_1, or (ii) $g_2(c) = g_3(c) = x = 0$, when the planar cubic has a cusp, a singular curve of type II. Finally the fourth equation in case (i) is equivalent to $g_3'(c) = g_2'(c)\,3\,g_3(c)/2\,g_2(c)$, which implies that Δ has a zero of order at least two at c, whereas in case (ii) the fourth equation is equivalent to $g_3'(c) = 0$, and it follows that Δ has a zero of order at least three at c. It follows from Table 6.3.2 that if the singular fiber S_c in S of φ is of type I_1, then Δ has a simple zero at c, and it follows that W has no singular points over c. Furthermore, if S_c is of type II, then $g_3(c) = 0$ and $g_3'(c) \neq 0$, and again the conclusion is that W has no singular points of W over c. Because the singular fibers of φ of types I_1 and II are the only irreducible ones, it follows that if W has a singular point over c, then the fiber S_c is reducible.

The mapping $f : S \to W$ is a modification of W in the discrete set W^* of all singular points of W. That is, f is a proper complex analytic mapping and its restriction to $f^{-1}(W \setminus W^*) = f^{-1}(W^\circ)$ is a complex analytic diffeomorphism from $f^{-1}(W^\circ)$ onto $W^\circ = W \setminus W^*$.

Now let $w \in W^*$, which implies that $c := p(w) \in C^{\text{sing}}$ and the fiber S_c of φ is reducible. Let Θ_0 be the irreducible component of S_c such that $\sigma(c) \in \Theta_0$, where we note that $\sigma(c) \in \Theta \cap S^{\text{reg}}$. If Θ_j, $j \geq 1$, are the other irreducible components of S_c, then $\Theta_0 \cap S^{\text{reg}}$ is equal to the complement K in S_c of the union of the Θ_j with $j \geq 1$, and we have seen in the proof of Lemma 6.3.1 that f contracts K to a point, which is equal to the singular point w of W. Because the fibration φ was assumed to be relatively minimal, none of the irreducible components of S_c is a -1 curve, hence none of the irreducible components of $f^{-1}(\{w\})$ is a -1 curve, and we conclude that $f : S \to W$ is a minimal resolution of singularities of W.

Because $W_c := p^{-1}(\{c\})$ is a Weierstrass curve, it is irreducible, and therefore its proper transform under f is an irreducible component of S_c, actually the unique irreducible component Θ of S_c that is not contracted by f to the point w. Because $f(\sigma(c)) = \infty(c) \notin W^*$ and $\sigma(c) \in \Theta_0$, we conclude that the proper transform under f of W_c is equal to Θ_0. It follows that f is a complex analytic diffeomorphism from the complement of K onto $W^\circ = W \setminus W^*$. Because the fibers of f are connected, the surface W is normal. We summarize the results in the following theorem.

Theorem 6.3.6 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with a holomorphic section $\sigma : C \to S$. Let $\mathfrak{f}$ denote the Lie algebra line bundle over C, with dual bundle $L := \mathfrak{f}^*$. Then the formulas* (2.3.4) *define holomorphic sections g_2 and g_3 of the line bundles L^4 and L^6 over C. The discriminant $\Delta := {g_2}^3 - 27\,{g_3}^2$ is a holomorphic section of the line bundle L^{12} over C, and $\Delta(c) = 0$ if and only if $c \in C^{\text{sing}}$, where the order of the zero of Δ at c is equal to the topological Euler number $\chi_{\text{top}}(S_c)$ of the singular fiber S_c. For the orders of the zeros of g_2 and g_3, see Table 6.3.2. In particular, at any $c \in C$ the order of the zero of g_2 at c is < 4 or the order of the zero of g_3 at c is < 6. The modulus function of $\varphi : S \to C$ is equal to the meromorphic function $J = {g_2}^3/\Delta$ on C.*

Let W be the complex analytic subset of the complex projective plane bundle

$$\pi : \mathrm{P}(L^0 \oplus L^2 \oplus L^3) \to C$$

over C, defined by the projective Weierstrass equation as in (6.3.4). *Then the mapping* $s \mapsto (\varphi(s), [1 : \wp(s) : \wp'(s)])$ *defined by the Weierstrass* $\wp$*-function and its derivative extends to a proper holomorphic mapping* $f : S \to W$ *such that* $p \circ f = \varphi$ *if* $p : W \to C$ *is the restriction to* W *of* π, *and* $\infty := f \circ \sigma$ *is the section* $c \mapsto (c, \{0\} \times \{0\} \times {L_c}^3)$ *at infinity of* $p : W \to C$, *where* $\infty(C)$ *is contained in the nonsingular part of* W. *The set* W^* *of all singular points of* W *is a discrete subset of* W *and the mapping* $f : S \to W$ *is a minimal resolution of singularities of the normal surface* W. *The map* $p|_{W^*}$ *is bijective from* W^* *onto the set of all* $c \in C$ *such that the fiber* S_c *of* φ *over* c *is reducible. If* $w \in W^*$ *and* $c = p(w)$, *then* $f^{-1}(\{w\})$ *is equal to the union of the irreducible components of* S_c *that do not intersect* $\sigma(C)$.

The surface W, together with its minimal resolution of singularities $f : S \to W$, is called the *Weierstrass model* of the elliptic fibration $\varphi : S \to C$. Theorem 6.3.6 is due to Kas [102, Theorem 1], where we have added the explicit description of the line bundle $L := \mathfrak{f}^*$, the holomorphic sections g_2 and g_3, and the mapping $f : S \to W$.

Remark 6.3.7. The following conditions are equivalent:

(i) The complex analytic surface W defined in (6.3.4) is nonsingular.
(ii) The mapping $f : S \to W$ in Theorem 6.3.6 is an isomorphism.
(iii) Every fiber of $\varphi : S \to C$ is irreducible.
(iv) The singular fibers of $\varphi : S \to C$ are of Kodaira type I_1 or II.
(v) Let $\Delta := {g_2}^3 - 27\,{g_3}^2$ and $\Delta(c_0) = 0$. Then $\Delta'(c_0) \neq 0$, or $g_3(c_0) = 0$ and $g_3'(c_0) \neq 0$.

(i) $\Leftrightarrow$ (ii) follows from the fact that f is a minimal resolution of singularities. (ii) $\Leftrightarrow$ (iii) follows from the paragraph preceding Theorem 6.3.6. (iii) $\Leftrightarrow$ (iv) follows from the description of Kodaira's types of singular fibers in Section 6.2.6. (iv) $\Leftrightarrow$ (v) follows from Table 6.3.2. ⊘

Remark 6.3.8. Let w be a singular point of W. According to Theorem 6.3.6 there is exactly one reducible fiber S_c of φ such that the exceptional curve $f^{-1}(\{w\})$ over w in S for the minimal resolution of singularities $f : S \to W$ is equal to the union of the irreducible components of S_c not equal to the unique multiplicity-one component of S_c that intersects $\sigma(C)$. The intersection diagram of $f^{-1}(\{w\})$ therefore is equal to the intersection diagram of S_c, as described in Remark 6.2.12, from which one of the multiplicity-one vertices and its edge(s) are deleted. That is, if $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(1)}$, or $\mathrm{E}_l^{(1)}$ is the intersection diagram of S_c, then A_l for $l \geq 1$, D_l for $l \geq 4$, or E_l for $l \in \{6, 7, 8\}$ is the intersection diagram of the exceptional curve $f^{-1}(\{w\})$. In terms of the Kodaira type of S_c, see Section 6.2.6, the intersection diagram of $f^{-1}(\{w\})$ is equal to I_b, $b \geq 2 \Rightarrow \mathrm{A}_{b-1}$, III $\Rightarrow \mathrm{A}_1$, IV $\Rightarrow \mathrm{A}_2$, I_b^*, $b \geq 0 \Rightarrow \mathrm{D}_{b+4}$, IV* $\Rightarrow \mathrm{E}_6$, III* $\Rightarrow \mathrm{E}_7$, II* $\Rightarrow \mathrm{E}_8$.

Let w be an isolated singular point of a complex analytic surface W, and let C_i be the irreducible components of the exceptional curve over w in the minimal resolution of singularities of W. Du Val [51, I] proved that the isolated singularity

"does not affect the conditions of adjunction," a term that I cannot briefly explain here, if and only if the C_i are rational curves, embedded $\mathbb{P}^1$'s, with intersection matrix equal to minus a Cartan matrix of type A_n, D_n, or E_n. In Du Val [52, II, III] such singularities are related to the Weyl groups of the corresponding types. Kirby [105] proved that W has such a singularity at w if and only if there is a complex analytic system of local coordinates (ξ, η, ζ) in an open neighborhood of w in W such that $(\xi, \eta, \zeta)(w) = (0, 0, 0)$ and near w the surface W is given by the equation A_n, $n \geq 1 \Leftrightarrow \xi^2 + \eta^2 + \zeta^{n+1} = 0$, D_n, $n \geq 4 \Leftrightarrow \xi^2 + \eta^2 \zeta + \zeta^{n-1} = 0$, $E_6 \Leftrightarrow \xi^2 + \eta^3 + \zeta^4 = 0$, $E_7 \Leftrightarrow \xi^2 + \eta^3 + \eta\,\zeta^3 = 0$, and $E_8 \Leftrightarrow \xi^3 + \eta^3 + \zeta^5 = 0$. There are many names in the literature for these surface singularities. They are called the *Du Val singularities* because of their classification in [52, I]. In the classical terminology for surface singularities as employed by Du Val and Kirby, a singularity of type A_n, D_n, E_6, E_7, E_8 is called a *binode of order* $n + 1$, *regular unode of order* $n + 2$, *exceptional unode of order* 8, 9, 10, respectively. Du Val mentioned in [53, Preface] that Cayley introduced the terms unode and binode, and in [53, Section 39, 40] that his singularities appear as the singularities of the orbit space $\mathbb{C}^2/G$, where G is a finite subgroup of $SL(2, \mathbb{C})$, where the orbit space is defined as the surface defined by the relation between the three basic G-invariant polynomials. Since these orbit spaces $\mathbb{C}^2/G$ have been described by Klein [107, Kap. II, §§9–13], the A - D - E singularities are also called the *Klein singularities*. Brieskorn [25] showed that a normal singularity W at w is an A - D - E singularity if and only if w is a so-called *rational double point* of W, defined as follows. The cohomology groups $H^k(f^{-1}(W_0), \mathcal{O}_S)$, where W_0 ranges over the open subsets of W, form a presheaf over W, the corresponding sheaf being denoted by $f_{*k}\mathcal{O}_S$. Note that for $k = 0$ this is the direct image of $\mathcal{O}_S$ under f as defined before of Lemma 6.2.17. One says that w is a "rational singularity" if $f_{*1}\mathcal{O}_S = 0$ in a neighborhood of w in W, which I think is a quite sophisticated definition. The singularity is called a "double point" if in the local equation $g = 0$ for W the second-order derivative at w of g is not equal to zero, which corresponds to the term ξ^2 in the local normal form. Finally, the A - D - E singularities are the *simple singularities* of Arnol'd [3] in the case of surfaces. For the appearance of simple singularities in simple algebraic groups, see Slodowy [189].

For the surface W defined by the Weierstrass equation $y^2 - 4x^3 + g_2(z)\,x + g_3(z) = 0$, one can take $\xi = y$, $\eta = \eta(x, z)$, and $\zeta = \zeta(x, z)$. That is, one needs only to bring the function $f(x, z) = -4x^3 + g_2(z)\,x + g_3(z)$ of two variables into its normal form. Suppose that $g_2(z)$, $g_3(z)$, and $\Delta(z) = g_2(z)^3 - 27\,g_3(z)^2$ have orders of zeros at $z = z_0$ as in one of the lines in Table 6.3.2. It then follows from Section 6.3.2 and Lemma 6.3.1 that there exists a minimal resolution of singularities $f : S \to W$ such that the singular fiber S_{z_0} has the Kodaira type T as indicated by Table 6.3.2. Therefore the singular point $(x, y, z) = (x_0, 0, z_0)$ of W is an A - D - E singularity of the type attached to T.

Since Proposition 6.3.20 implies that every Kodaira type is realized by a relatively minimal elliptic fibration without multiple singular fibers and a local section, this implies that every rational double point is isomorphic to a singular point of a Weierstrass model W of such an elliptic fibration. Given $g_2(z)$ and $g_3(z)$, one can also bring

the singularity of W into its normal form by means of an explicit computation. In its normal form, the minimal resolution of singularities can be performed by means of explicit substitutions of variables corresponding to the successive blowups. This leads to an explicit construction of the elliptic surface S from the Weierstrass data $g_2(z)$ and $g_3(z)$. ⊘

Remark 6.3.9. If C is compact, then C is a complex projective algebraic curve, and it follows from the G.A.G.A. principle that L is an algebraic line bundle over C, the bundle $\mathrm{P}(L^0 \oplus L^2 \oplus L^3)$ is a complex projective algebraic variety, the respective sections g_2 and g_3 are algebraic, and therefore W is a complex projective algebraic surface. Corollary 6.2.57 now implies that the elliptic surface S is projective algebraic, in the sense that there is a complex analytic diffeomorphism from S onto a nonsingular complex projective algebraic surface. Every elliptic surface with a section is obtained from a relatively minimal elliptic surface with a section by means of blowing up, and blowing up of projective algebraic surfaces leads to projective algebraic surfaces. This leads to another proof of Corollary 6.2.28, that every compact elliptic surface with a section is projective algebraic. ⊘

6.3.2 The Elliptic Surface of a Weierstrass Family

Assume, conversely to the situation in Section 6.3.1, that C is a complex analytic curve, $\widetilde{L}$ is a holomorphic complex line bundle over C, and $\widetilde{g}_2$ and $\widetilde{g}_3$ are holomorphic sections of $\widetilde{L}^4$ and $\widetilde{L}^6$, respectively. Assume furthermore that the holomorphic section

$$\widetilde{\Delta} := \widetilde{g}_2^3 - 27\,\widetilde{g}_3^2 \tag{6.3.5}$$

of $\widetilde{L}^{12}$ is not identically equal to zero. Let $\widetilde{W}$ be the analytic subset of the complex projective plane bundle $\mathrm{P}(\widetilde{L}^0 \oplus \widetilde{L}^2 \oplus \widetilde{L}^3)$ over C, which is defined as in (6.3.4), with L, g_2, and g_3 replaced by $\widetilde{L}$, $\widetilde{g}_2$, and $\widetilde{g}_3$, respectively. Let $\widetilde{p} : \widetilde{W} \to C$ be the restriction to $\widetilde{W}$ of the projection $\pi : \mathrm{P}(\widetilde{L}^0 \oplus \widetilde{L}^2 \oplus \widetilde{L}^3) \to C$. As for W, $\widetilde{W}$ has only isolated singularities, and $\widetilde{p} : \widetilde{W} \to C$ is an elliptic fibration with the proviso that $\widetilde{W}$ is allowed to have singularities. The mapping $\widetilde{\tau} : C \to \widetilde{W}$ defined by

$$\widetilde{\tau}(c) = (c,\ \{0\} \oplus \{0\} \oplus \widetilde{L}_c^3)$$

for every $c \in C$ is a holomorphic section of $\widetilde{p}$ such that $\widetilde{\tau}(C)$ is contained in the nonsingular part $\widetilde{W}^\circ = \widetilde{W} \setminus \widetilde{W}^*$.

According to Lemma 6.2.53, there exists a minimal resolution of singularities $\widetilde{f} : \widetilde{S} \to \widetilde{W}$ of $\widetilde{W}$, which according to Lemma 6.2.56 is unique up to isomorphism. The mapping $\widetilde{\varphi} := \widetilde{p} \circ \widetilde{f} : \widetilde{S} \to C$ is an elliptic fibration, and $\widetilde{\sigma} := \widetilde{f}^{-1} \circ \widetilde{\tau}$ is a holomorphic section of $\widetilde{\varphi}$. Successively blowing down -1 curves in fibers of the elliptic fibrations, when these occur, we arrive at a relatively minimal elliptic fibration $\varphi : S \to C$ such that $\widetilde{\varphi} = \varphi \circ \widetilde{b}$, if $\widetilde{b} : \widetilde{S} \to S$ denotes the mapping that blows down the -1 curves in fibers of the elliptic fibrations. The mapping $\sigma := \widetilde{b} \circ \widetilde{\sigma} : C \to S$ is holomorphic and $\varphi \circ \sigma = \varphi \circ \widetilde{b} \circ \widetilde{\sigma} = \widetilde{\varphi} \circ \widetilde{\sigma}$ is equal to the identity in C, which shows that σ is a holomorphic section of φ.

For the relatively minimal elliptic fibration $\varphi : S \to C$ with the holomorphic section $\sigma : C \to S$ we have the line bundle $L = \mathfrak{f}^*$, the holomorphic sections g_2 and g_3 of L^4 and L^6, respectively, and the Weierstrass model $S \xrightarrow{f} W \xrightarrow{p} C$, as in Theorem 6.3.6. Because $\varphi = p \circ f$, we have $\widetilde{p} \circ \widetilde{f} = \widetilde{\varphi} = \varphi \circ \widetilde{b} = p \circ f \circ \widetilde{b}$, which leads to the commutative diagram

$$\begin{array}{ccccc} \widetilde{S} & \xrightarrow{\widetilde{f}} & \widetilde{W} & \xrightarrow{\widetilde{p}} & C \\ \widetilde{b}\downarrow & & & & \| \\ S & \xrightarrow{f} & W & \xrightarrow{p} & C \end{array} \tag{6.3.6}$$

We leave it to the reader to insert the mappings $\widetilde{\tau} : C \to \widetilde{W}, \widetilde{\varphi} : \widetilde{S} \to C, \widetilde{\sigma} : C \to \widetilde{S}$, $\tau : C \to W, \varphi : S \to C$, and $\sigma : C \to S$, where for instance $f = \tau \circ \varphi$ and $\tau = f \circ \sigma$.

Let $\mathrm{ord}_c(m)$ denote the order at $c \in C$ of a meromorphic section m of a holomorphic line bundle K over C, where m has a zero of order $\mathrm{ord}_c(m)$ at c if $\mathrm{ord}_c(m) \in \mathbb{Z}_{>0}$, a pole of order $-\mathrm{ord}_c(m)$ at c if $\mathrm{ord}_c(m) \in \mathbb{Z}_{<0}$, and no zero or pole at c if $\mathrm{ord}_c(m) = 0$. We write $\mathrm{ord}_c(m) = \infty$ if m is identically equal to zero. Let $[K]$ denote the element of $\mathrm{H}^1(C, \mathcal{O}^\times)$ that corresponds to the isomorphism class of the holomorphic line bundle K over C. We also recall the homomorphism δ from the additive group of divisors on C to the multiplicative group $\mathrm{H}^1(C, \mathcal{O}^\times)$. With this notation, we have the following converse to Theorem 6.3.6.

Theorem 6.3.10 *For every $c \in Z$, let k_c be the unique $k \in \mathbb{Z}_{\geq 0}$ such that $0 \leq \mathrm{ord}_c(\widetilde{g}_2) - 4k < 4$ or $0 \leq \mathrm{ord}_c(\widetilde{g}_3) - 6k < 6$. Then $Z := \{c \in C \mid k_c > 0\}$ is contained in the discrete subset $C^{\mathrm{sing}} := \{c \in C \mid \widetilde{\Delta} = 0\}$ of C, and*

$$[\widetilde{L}] = [L]\,\delta\left(\sum_{c \in Z} k_c \{c\}\right). \tag{6.3.7}$$

Furthermore the following conditions are equivalent:

(i) *For every $c \in C$ we have $\mathrm{ord}_c(\widetilde{g}_2) < 4$ or $\mathrm{ord}_c(\widetilde{g}_3) < 6$.*
(ii) *There is an isomorphism $\iota : \widetilde{L} \to L$ of holomorphic line bundles over C such that $g_2 = \iota^4 \circ \widetilde{g}_2$ and $g_3 = \iota^6 \circ \widetilde{g}_3$.*
(iii) *The elliptic fibration $\widetilde{\varphi} = \widetilde{p} \circ \widetilde{f} : \widetilde{S} \to C$ is relatively minimal.*
(iv) *The elliptic fibration $\widetilde{\varphi} = \widetilde{p} \circ \widetilde{f} : \widetilde{S} \to C$ is relatively minimal, $\widetilde{L}$ isomorphic to the dual of the Lie algebra bundle of $\widetilde{\varphi} : \widetilde{S} \to C$, and the sequence $\widetilde{S} \xrightarrow{\widetilde{f}} \widetilde{W} \xrightarrow{\widetilde{p}} C$ exhibits $\widetilde{W}$ as the Weierstrass model of $\widetilde{\varphi} : \widetilde{S} \to C$.*

Proof. If $k_c > 0$ then $\widetilde{\mathfrak{g}}_2(c) = 0$ and $\widetilde{\mathfrak{g}}_3(c) = 0$, and hence $\widetilde{\Delta}(c) = 0$, see (6.3.5). Hence $Z \subset C^{\mathrm{sing}}$, where C^{sing} is a discrete subset of C because the holomorphic section $\widetilde{\Delta}$ does not vanish identically and C is connected.

Let $c_0 \in C$, and let C_0 be an open neighborhood of c_0 in C over which we have a trivialization of the holomorphic line bundles $\widetilde{L}$ and $L = \mathfrak{f}^*$. These trivializations induce trivializations of all powers of $\widetilde{L}$ and L over C_0, which means that the

holomorphic sections $\widetilde{g}_2$, $\widetilde{g}_3$, g_2, and g_3 of $\widetilde{L}^4$, $\widetilde{L}^6$, L^4, and L^6, respectively, all are identified with holomorphic complex-valued functions on C_0, which we denote by the same letters. By shrinking C_0 if necessary, we arrange that $C_0 \cap C^{\mathrm{sing}} = \{c_0\}$ if $C_0 \cap C^{\mathrm{sing}} \neq \emptyset$.

In order to keep the notation simple, in the following discussion we will denote the parts of $\widetilde{W}$, $\widetilde{S}$, S, and W that are mapped to C_0 by the same letters $\widetilde{W}$, $\widetilde{S}$, S, and W, respectively. In this way the surface $\widetilde{W}$ is identified with the set of all $(w,\ [x]) \in C_0 \times \mathbb{P}^2$ such that $[x]$ belongs to the Weierstrass curve W_c, defined by the equation (2.3.6), with g_2 and g_3 replaced by $\widetilde{g}_2(c)$ and $\widetilde{g}_3(c)$, respectively. We have $\widetilde{p} : \widetilde{W} \to C_0 : (c,\ [x]) \mapsto c$ and the fiber of $\widetilde{p}$ over c is equal to $\{c\} \times \widetilde{W}_c$. Note that if $\widetilde{w} = (c,\ [x])$ is a singular point of $\widetilde{W}$, then $\widetilde{W}_c$ has $[x]$ as its unique singular point, and it follows from Section 2.3.2 that $\widetilde{\Delta}(c) = 0$, that is, $\widetilde{p}(\widetilde{w}) = c \in C^{\mathrm{sing}}$, which by our assumption implies that $c = c_0$.

The vector field $v_{\widetilde{W}}$ on the nonsingular part $\widetilde{W}^\circ = \widetilde{W} \setminus \widetilde{W}^*$ of $\widetilde{W}$, defined by $\mathrm{d}c/\,\mathrm{d}t = 0$ and (2.3.7), (2.3.8) with g_2 and g_3 replaced by $\widetilde{g}_2(c)$ and $\widetilde{g}_3(c)$, respectively, is holomorphic and tangent to the fibers of $\widetilde{p}$. The diffeomorphism $\widetilde{b} \circ \widetilde{f}^{-1}$ from $\widetilde{p}^{-1}(C_0 \setminus \{c_0\})$ onto $S' := \varphi^{-1}(C_0 \setminus \{c_0\})$ intertwines $v_{\widetilde{W}}$ with a holomorphic vector field v' on S', which is tangent to the fibers of φ, and therefore defines a holomorphic section over $C \setminus \{c_0\}$ of the Lie algebra bundle $\mathfrak{f}$ of $\varphi : S \to C_0$. The trivialization of $\mathfrak{f}^*$ corresponds to a holomorphic section v of $\mathfrak{f}$ without zeros over C_0, and $v' = u\,v$ for a holomorphic function u on $C_0 \setminus \{c_0\}$.

It follows from the discussion of the Weierstrass normal form of elliptic curves in Section 2.3.2 that $\widetilde{g}_2 = u^4\, g_2$ and $\widetilde{g}_3 = u^6\, g_3$; hence $\widetilde{\Delta} = u^{12}\,\Delta$, if $\Delta := {g_2}^3 - 27\,{g_3}^2$. Because $\widetilde{\Delta}$ and Δ are holomorphic on C_0, and $\widetilde{\Delta}$ is nonzero in $C_0 \setminus C^{\mathrm{sing}}$, it follows that u and Δ are nonzero in $C_0 \setminus C^{\mathrm{sing}}$. If $c_0 \in C^{\mathrm{sing}}$, then the fact that $\widetilde{\Delta}$ and Δ have a zero of finite order at c_0, in combination with $\widetilde{\Delta} = u^{12}\,\Delta$, leads to estimates for $u(c)$ as $c \to c_0$ that imply that u defines a meromorphic function on C_0 for which $\mathrm{ord}_{c_0}(\widetilde{\Delta}) = 12\ \mathrm{ord}_{c_0}(u) + \mathrm{ord}_{c_0}(\Delta)$. Then $\widetilde{g}_2 = u^4\, g_2$ and $\widetilde{g}_3 = u^6\, g_3$ imply that $\mathrm{ord}_{c_0}(\widetilde{g}_2) = 4\ \mathrm{ord}_{c_0}(u) + \mathrm{ord}_{c_0}(g_2)$ and $\mathrm{ord}_{c_0}(\widetilde{g}_3) = 6\ \mathrm{ord}_{c_0}(u) + \mathrm{ord}_{c_0}(g_3)$, respectively. It follows from Table 6.3.2 that $0 \leq \mathrm{ord}_{c_0}(g_2) < 4$ or $0 \leq \mathrm{ord}_{c_0}(g_3) < 6$. In combination with $\mathrm{ord}_{c_0}(\widetilde{g}_2) \geq 0$ and $\mathrm{ord}_{c_0}(\widetilde{g}_3) \geq 0$ this implies in either case that $\mathrm{ord}_{c_0}(u) \geq 0$. That is, $\mathrm{ord}_{c_0}(u) = k_{c_0}$, and the holomorphic function u on $C_0 \setminus \{c_0\}$ extends to a holomorphic function on C_0, with a finite order k_{c_0} of its zero at c_0. We have $\mathrm{ord}_{c_0}(u) > 0$ if and only if $c_0 \in Z$.

Let $C_{g_2,\,g_3}$ denote the Weierstrass curve defined by (2.3.5), and let $v_{g_2,\,g_3}$ denote the Hamiltonian vector field on $C_{g_2,\,g_3}$ defined by (2.3.7). If $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $\Phi_\lambda : (x,\ y) \mapsto (\lambda^2\, x,\ \lambda^3\, y)$, then Φ_λ is a diffeomorphism from $C_{g_2,\,g_3}$ onto $C_{\lambda^4\, g_2,\,\lambda^6\, g_3}$, which intertwines $v_{g_2,\,g_3}$ with $\lambda^{-1}\, v_{\lambda^4\, g_2,\,\lambda^6\, g_3}$.

Let $c \in C$, $\widetilde{l} \in \widetilde{L}_c$, $\widetilde{l} \neq 0$. Then, for $k = 2,\ 3$, we have uniquely determined complex numbers $\widetilde{g}_{k,\,\widetilde{l}}(c)$ such that $\widetilde{g}_k(c) = \widetilde{g}_{k,\,\widetilde{l}}(c)\,\widetilde{l}^{2\,k}$, and it follows that $\widetilde{g}_{k,\,\widetilde{l}}(c) = \lambda^{2\,k}\,\widetilde{g}_{k,\,\lambda\,\widetilde{l}}$. Let $v_{\widetilde{l}}$ denote the element of $\mathfrak{f}_c$ that is assigned as above to $\widetilde{v}_{\widetilde{l}} := v_{g_2,\,g_3}$ with $g_k = \widetilde{g}_{k,\,\widetilde{l}}(c)$. Then $v_{\lambda\,\widetilde{l}}$ is the element of $\mathfrak{f}_c$ that is assigned to $\widetilde{v}_{\lambda\,\widetilde{l}} = v_{g_2,\,g_3}$ with $g_k = \widetilde{g}_{k,\,\lambda\,\widetilde{l}}(c) = \lambda^{-2k}\,\widetilde{g}_{k,\,\widetilde{l}}(c)$. Because

$$\mathrm{T}\Phi_\lambda \circ \widetilde{v}_{\lambda\widetilde{l}} = \lambda\, \widetilde{v}_{\widetilde{l}} \circ \Phi_\lambda,$$

it follows that $v_{\lambda\widetilde{l}} = \lambda^{-1}\, v_{\widetilde{l}}$. Therefore, if $\omega_{\widetilde{l}}$ denotes the linear form on $\mathfrak{f}_c$ such that $\omega_{\widetilde{l}}(v_{\widetilde{l}}) = 1$, the linear form that is dual to $v_{\widetilde{l}}$, we obtain that the mapping $\widetilde{l} \mapsto \omega_{\widetilde{l}}$ is a linear isomorphism ι_c from $\widetilde{L}_c$ onto $L_c := \mathfrak{f}_c^*$. This defines an isomorphism ι from the holomorphic line bundle $\widetilde{L}$ over $C \setminus C^{\mathrm{sing}}$ onto the line bundle L over $C \setminus C^{\mathrm{sing}}$.

A nonvanishing local holomorphic section of $\widetilde{L}$ over C_0 corresponds to a local trivialization of $\widetilde{L}$, and the corresponding section of $\mathfrak{f}$ is of the form $u\, v$, where v is a nonvanishing holomorphic section of $\mathfrak{f}$ that defines the local trivialization of $\mathfrak{f}$, and u is a holomorphic function on C_0 with divisor equal to the sum of the $k_c\, c$ such that $c \in C_0 \cap Z$. It follows that if $\widetilde{l}$ is a local holomorphic section without zeros of $\widetilde{L}$ over any open subset C_0 of C, then $\iota \circ \widetilde{l}$ is a meromorphic section of L over C_0 with divisor equal to

$$\mathrm{Div}(\iota \circ \widetilde{l}) = -\sum_{c \in C_0 \cap Z} k_c\, \{c\}.$$

This implies (6.3.7) in view of Lemma 2.1.2. Note that ι extends to an isomorphism from $\widetilde{L}|_{C\setminus Z}$ onto $L|_{C\setminus Z}$, which at each $c \in Z$ has a pole of order k_c.

(i) $\Rightarrow$ (ii) Condition (i) means that $Z = \emptyset$; hence ι is an isomorphism from $\widetilde{L}$ onto L. That $g_2 = \iota^4 \circ \widetilde{g}_2$ and $g_3 = \iota^6 \circ \widetilde{g}_3$ follows from the equations $g_2(P) = g_2$, $g_3(P) = g_3$ in Section 2.3.2, where P is the period lattice of the Hamiltonian vector field (2.3.7), (2.3.8) on the Weierstrass curve defined by (2.3.5).

(ii) $\Rightarrow$ (iii) Write $\widetilde{W}' = \widetilde{p}^{-1}(C \setminus C^{\mathrm{sing}})$ and $W' = p^{-1}(C \setminus C^{\mathrm{sing}})$, which are dense open subsets of $\widetilde{W}$ and W, contained in the nonsingular parts $\widetilde{W}^\circ$ and W° of $\widetilde{W}$ and W, respectively. We have the complex analytic diffeomorphism $\widetilde{\iota} = f \circ \widetilde{b} \circ \widetilde{f}^{-1}$ from $\widetilde{W}'$ onto W'. Condition (ii) implies that we can identify $\widetilde{L}$, $\widetilde{g}_2$, and $\widetilde{g}_3$ with L, g_2, and g_3, respectively. After this identification, $\widetilde{\tau}$ leaves the section $\widetilde{\tau}(C) \cap \widetilde{W}'$ pointwise fixed and leaves the Hamiltonian vector fields on the fibers of $\widetilde{p}$ invariant, which implies that $\widetilde{\iota}$ is the identity on $\widetilde{W}'$, and $f \circ \widetilde{b} = \widetilde{f}$ on $\widetilde{W}'$, hence on $\widetilde{W}$. Because $\widetilde{f}$ and f are minimal resolutions of singularities of $\widetilde{W} = W$, it follows from Lemma 6.2.56 that $\widetilde{b}$ is a complex analytic diffeomorphism from $\widetilde{S}$ onto S. Because $\widetilde{b}$ contracts the -1 curves in fibers of $\widetilde{\varphi}$ to points, if these occur, the conclusion is that such -1 curves do not occur, that is, the fibration $\widetilde{\varphi} : \widetilde{S} \to C$ is relatively minimal.

(iii) $\Rightarrow$ (iv) Condition (iii) means that we may take $S = \widetilde{S}$ and $\widetilde{b}$ equal to the identity. The Hamiltonian vector field $\widetilde{v}$ at the beginning of the proof of Theorem 6.3.10 is nonzero at the section $\widetilde{\sigma}(C_0)$, and therefore corresponds to a holomorphic section of $\mathfrak{f}$ without zeros. That is, the holomorphic function u such that $\widetilde{v} = u\, v$ has no zeros, which implies that $c_0 \notin Z$. This proves that $Z = \emptyset$, that is, (i) holds, hence (ii), which in turn implies (iv) with $\widetilde{S} = S$.

(iv) $\Rightarrow$ (i) Theorem 6.3.6 implies that $\mathrm{ord}_c(g_2) < 4$ or $\mathrm{ord}_c(g_3) < 6$ for every $c \in C$, whereas (iv) implies that $g_2 = \iota^4 \circ \widetilde{g}_2$ and $g_3 = \iota^6 \circ \widetilde{g}_3$ for an isomorphism $\iota : \widetilde{L} \to L$. □

Theorem 6.3.10 is due to Kas [102, Theorem 2], where we have added the equivalent condition (iii). Together, Theorems 6.3.6 and 6.3.10 lead to a classification of

elliptic surfaces. This classification is quite different from Kodaira's classification as described in Section 6.4.

Remark 6.3.11. If C is compact, then all surfaces are projective algebraic, and it follows from the commutative diagram (6.3.6), in which the mappings $\widetilde{f}$, $\widetilde{b}$, and f are birational transformations, that W is birationally equivalent to $\widetilde{W}$.

Assume that some, hence all, of the conditions (i)–(iv) in Theorem 6.3.10 do not hold. Because (iii) does not hold, there exists a -1 curve E in a fiber $\widetilde{S}_c$ of $\widetilde{\varphi} = \widetilde{p} \circ \widetilde{f} : \widetilde{S} \to C$ over a point $c \in C$. Because $\widetilde{f}$ is a minimal resolution of singularities, it does not contract E to a point, hence $\widetilde{f}(E)$ is a curve, contained in the fiber $\widetilde{W}_c$ of $\widetilde{p}$ over c. As every Weierstrass curve is irreducible, $\widetilde{f}(E) = \widetilde{W}_c$ and E is equal to the proper transform of $\widetilde{W}_c$ under $\widetilde{f}$. Because $\widetilde{b}$ contracts E to a point, there is no map $g : S \to \widetilde{S}$ such that $\widetilde{f} = g \circ \widetilde{b}$.

If $\widetilde{W}$ has no singular point on $\widetilde{W}_c$, then $\widetilde{f}$ is a diffeomorphism on an open neighborhood of E in $\widetilde{S}$, when $\widetilde{W}_c$ is a smoothly embedded $\mathbb{P}^1$, which never happens for a Weierstrass curve. Therefore $\widetilde{W}$ is singular at some point $\widetilde{w} \in \widetilde{W}_c$, which then is the unique singular point of $\widetilde{W}_c$. It follows that $D := \widetilde{f}^{-1}(\{\widetilde{w}\})$ is a curve in $\widetilde{S}$ not equal to E. As the surjective mapping $f \circ \widetilde{b} : \widetilde{S} \to W$ contracts E to a point and maps $D \cup E = \widetilde{f}^{-1}(\widetilde{W}_c) = (\widetilde{p} \circ \widetilde{f})^{-1}(\{c\}) = (p \circ f \circ \widetilde{b})^{-1}(\{c\}) = (f \circ \widetilde{b})^{-1}(W_c)$ onto the fiber W_c of p over c, it maps D onto W_c. As $\widetilde{f}$ contracts D to a point, there is also no map $\widetilde{\tau} : \widetilde{W} \to W$ such that $\widetilde{\tau} \circ \widetilde{f} = f \circ \widetilde{b}$. ⊘

Example 5. If $C = \mathbb{P}^1$, then the holomorphic line bundles over C are determined up to isomorphisms by their degrees. For any $n \in \mathbb{Z}$, let $\mathrm{O}(N)$ denote the line bundle over $\mathbb{P}^1$ such that for every $(z_0,\ z_1) \in \mathbb{C}^2 \setminus \{(0,\ 0)\}$, its fiber $\mathrm{O}(N)_{[z_0:z_1]}$ over $[z_0 :\ z_1]$ is equal to the one-dimensional vector space of all homogeneous functions of degree N on $[z_0 :\ z_1] = \mathbb{C}\,(z_0,\ z_1) \subset \mathbb{C}^2$. If $N \geq 0$, then the restriction map defines an isomorphism from the space $\mathrm{H}_N(\mathbb{C}^2)$ of all homogeneous poynomials on $\mathbb{C}^2$ of degree N onto the space of all holomorphic sections of $\mathrm{O}(N)$, and because every nonzero element of $\mathrm{H}_N(\mathbb{C}^2)$ has N zeros, when counted with multiplicities, it follows that the degree of $\mathrm{O}(N)$ is equal to N. The dual of $\mathrm{O}(N)$ is canonically isomorphic to $\mathrm{O}(-N)$, hence $\deg \mathrm{O}(-N) = -\deg \mathrm{O}(N) = -N$. In other words, any line bundle L over $\mathbb{P}^1$ is isomorphic to $\mathrm{O}(N)$, where $N = \deg L$. If $N < 0$, then L has no nonzero holomorphic sections.

It follows from Theorem 6.3.6 that if $\varphi : S \to \mathbb{P}^1$ is a relatively minimal elliptic fibration with a holomorphic section over $\mathbb{P}^1$, then the line bundle L^4 or L^6, where L is equal to the dual $\mathfrak{f}^*$ of the Lie algebra bundle $\mathfrak{f}$, admits a nonzero holomorphic section, which can happen only if $L \simeq \mathrm{O}(N)$ for some $N \in \mathbb{Z}_{\geq 0}$. This correponds to the observation in Lemma 6.2.34 that the degree $\deg(\mathfrak{f}) = -N$ of $\mathfrak{f}$, which is equal to the self-intersection number $E \cdot E$ in S of any section E of φ, is equal to the nonpositive integer $-\chi(S,\ \mathcal{O})$, where $\chi(S,\ \mathcal{O}) > 0$ if and only if φ has at least one singular fiber. The fibration $\varphi : S \to \mathbb{P}^1$ is a *rational* elliptic surface if and only if $\deg(\mathfrak{f}) = -1$, that is, $N = 1$. The fibration $\varphi : S \to \mathbb{P}^1$ is an elliptic K3 surface if and only if $N = -\deg(\mathfrak{f}) = 2$, see Section 6.3.4.

Conversely, let $N \in \mathbb{Z}_{\geq 0}$, and let g_2 and g_3 be homogeneous polynomials on $\mathbb{C}^2$ of respective degrees $4\,N$ and $6\,N$. Assume that, as holomorphic sections of

the respective line bundles $\mathrm{O}(4N)$ and $\mathrm{O}(6N)$, these satisfy the conditions that $\Delta := {g_2}^3 - 27\,{g_3}^2$ is not identically equal to zero and at any zero of Δ, g_2 has a zero of order < 4 and g_3 has a zero of order < 6. Then it follows from Theorem 6.3.10 that the Weierstrass model defined by g_2 and g_3 is a relatively minimal elliptic fibration over $\mathbb{P}^1$ with a global section and $\mathfrak{f}^* \simeq \mathrm{O}(N)$, that is, $\mathfrak{f} \simeq \mathrm{O}(-N)$. Theorem 6.3.6 implies that every relatively minimal elliptic fibration over $\mathbb{P}^1$ with a global section is isomorphic to such a Weierstrass model.

In Heckman and Looijenga [81, end of the proof of Proposition 2.1], in the case of a rational elliptic surface, when $N = 1$, the condition is introduced that ${g_2}^3$ and ${g_3}^2$ are linearly independent sections of $\mathrm{O}(12)$, which implies condition (i) in Theorem 6.3.10. The condition of Heckman and Looijenga implies also that the discriminant function Δ is not identically equal to zero, and then is equivalent to the condition that the modulus function $J = {g_2}^3/\Delta$ is not equal to a constant. Since rational elliptic surfaces with constant modulus functions do exist, the condition of Heckman and Looijenga is stronger than condition (i) in Theorem 6.3.10. ⊘

Remark 6.3.12. Schmickler–Hirzebruch [176, pp. 119–122] determined the Weierstrass model of each relatively minimal elliptic fibration over $\mathbb{P}^1$ with a section, that has at most three singular fibers. The starting point is the observation [176, Satz 1 on p. 38] that for such fibrations the singular fibers of Kodaira type I_b or I_b^* satisfy $b \leq 4$, which makes the list of configurations of singular fibers finite and manageable. In particular, the holomorphic Euler number $\chi(S, \mathcal{O})$, there called the arithmetic genus, is ≤ 2. Of the list, 24 have $\chi(S, \mathcal{O}) = 1$ and therefore are rational elliptic surfaces, whereas 11 have $\chi(S, \mathcal{O}) = 2$, and therefore are K3 elliptic surfaces; see Section 6.3.4. The explicit formulas for g_2 and g_3 then are used in [176, §8] in order to compute, in each case, the Picard–Fuchs equation, i.e., the hypergeometric differential equation that is satisfied by the periods. ⊘

Remark 6.3.13. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with a holomorphic section, where C is compact. The mapping degree j of the modulus function $J : C \to \mathbb{P}^1$ is equal to 1 if and only if J is an isomorphism if and only if for every $a \in \mathbb{P}^1$ there is exactly one $c \in C$ such that the modulus $J(c)$ of the fiber S_c over c of the elliptic fibration $\varphi : S \to C$ is equal to a. Here it is allowed that S_c is a singular fiber when $J(c)$ is defined as the limit for $C^{\mathrm{reg}} \ni c' \to c$.

If $j = 1$ then we can identify C with $\mathbb{P}^1$, where $J(z) = z$ if z is an affine coordinate on $\mathbb{P}^1$. In particular we have an elliptic surface with a Weierstrass model as in Example 5 for some $N \in \mathbb{Z}_{>0}$, where $N = \chi(S, \mathcal{O}) = -\deg \mathfrak{f}$. The polynomial solutions $g_2(z)$, $g_3(z)$ of the equation $z = J(z) = g_2(z)^3/(g_2(z)^3 - 27\,g_3(z)^2)$, subject to the condition that there are no $z \in \mathbb{C}$ such that g_2 and g_3 have a zero at z of respective orders ≥ 4 and ≥ 6, are $g_2(z) = z^{2a+1}\,(z-1)^{2b+1}\,\gamma(z)^2$, $g_3(z) = z^{3a+1}\,(z-1)^{3b+2}\,c\,\gamma(z)^3$, where $a,\ b \in \{0,\ 1\}$, $27\,c^2 = 1$, and $\gamma(z)$ is a polynomial of some degree $d \in \mathbb{Z}_{\geq 0}$, with at most simple zeros and no zeros at $z = 0$ or $z = 1$. This observation is closely related to Proposition 6 in the PhD thesis of Bas Edixhoven at Utrecht University in 1989, which states, "Up to isomorphism there are exactly four elliptic curves over $\mathrm{Spec}(\mathbb{Z}[x,\ x^{-1},\ (x - 12^3)^{-1}])$."

The singular fiber over $z = 0$ has Kodaira type II or IV* if $a = 0$ or $a = 1$, respectively. The singular fiber over $z = 1$ has Kodaira type III or III* if $b = 0$ or $b = 1$, respectively. Over each zero of γ we have a singular fiber of Kodaira type I_0^*. In homogeneous coordinates (z_0, z_1), where $z = -z_1/z_0$, and g_2 and g_3 are homogeneous polynomials in (z_0, z_1) of the respective degrees $4N$ and $6N$. In combination with the requirement that g_2 and g_3 not have zeros at $z_0 = 0$ of respective degrees ≥ 4 and ≥ 6, this leads to the conclusion that over $z_0 = 0$ we have a singular fiber of Kodaira type I_1^* or I_1 if $a + b + d = 2N - 2$ or $a + b + d = 2N - 1$, respectively.

If $N = 1$, the case of rational elliptic surfaces, the possible configurations of singular fibers therefore are I_1^* II III, I_1 II III I_0^*, I_1 II III*, and I_1 IV* III. If $N \geq 2$, where $N = 2$ is the case of K3 elliptic surfaces, we have the following eight possible configurations of singular fibers: I_1^* II III $(2N-2)$ I_0^*, I_1^* II III* $(2N-3)$ I_0^*, I_1^* IV* III $(2N-3)$ I_0^*, I_1^* IV* III* $(2N-4)$ I_0^*, I_1 II III $(2N-1)$ I_0^*, I_1 II III* $(2N-2)$ I_0^*, I_1 IV* III $(2N-2)$ I_0^*, and I_1 IV* III* $(2N-3)$ I_0^*. In all cases the singular values of φ can be positioned arbitrarily on $C \simeq \mathbb{P}^1$. With $J(z) \equiv z$, the isomorphism classes of the elliptic fibrations are in bijective correspondence with the zero-sets of γ, the sets of points in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over which we have a singular fiber of Kodaira type I_0^*. In Schmickler-Hirzebruch [176, p. 120], there appear four elliptic surfaces with $J(z) \equiv z$, because she classified the elliptic surfaces over $\mathbb{P}^1$ with at most three singular fibers. Here the possible configurations of singular fibers are I_1^* II III, I_1^* IV* III*, I_1 II III*, and I_1 IV* III, where the second one is a K3 elliptic surface, see Section 6.3.4, and the other three are rational elliptic surfaces.⊘

6.3.3 Some Moduli Spaces of Elliptic Surfaces

In this subsection we will discuss a number of quotient spaces that are constructed by means of Mumford's *geometric invariant theory*, explained very briefly in the next paragraphs.

6.3.3.1 Geometric Invariant Theory

If G is an algebraic group acting algebraically on a projective algebraic variety X, then a *categorical quotient* of X by G is an algebraic morphism φ from X to a projective algebraic variety Y that is constant on the G-orbits, and has the categorical property that if ψ is another algebraic morphism from X to a projective algebraic variety Z, then there is a unique algebraic morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \varphi$. The latter property makes a categorical quotient, if it exists, unique up to algebraic isomorphisms. The categorical quotient $\varphi : X \to Y$ is called *universal* if for every algebraic morphism $\eta : Y' \to Y$ the projection to the second component of $\{(x, y') \in X \times Y' \mid \varphi(x) = \eta(y')\}$ is a categorical quotient for the action of G on the first component. If $Y' \subset Y$ and η is the identity viewed as a morphism $Y' \to Y$, then this implies that the restriction of φ to $X' = \varphi^{-1}(Y')$ defines a categorical quotient $X' \to Y'$. See Mumford, Fogarty, and Kirwan [145, Definition 0.5 and 0.7].

Assume that $X \subset \mathbb{P}^n$ is a projective algebraic variety and G a reductive algebraic subgroup of $\mathrm{GL}(\mathbb{C}^{n+1})$ that leaves the cone $\{z \in \mathbb{C}^{n+1} \setminus \{0\} \mid \mathbb{C}z \in X\}$ over X invariant. Then the induced action of G on X is algebraic. The element $\mathbb{C}z \in X$ is called *semistable* if the origin does not belong to the closure of the G-orbit $G \cdot z$ of z in $\mathbb{C}^{n+1}$, and *stable* if $G \cdot z$ is closed and $\dim(G \cdot z) = \dim G$. Let X^{ss} and X^{s} denote the sets of all semistable and stable elements of X, respectively. These are G-invariant Zariski-open subsets of X, and $X^{\mathrm{s}} \subset X^{\mathrm{ss}}$. Then Mumford's geometric invariant theory [145, Chapter 1 and Appendix B to Chapter 1] yields a surjective universal categorical quotient $\varphi : X^{\mathrm{ss}} \to Y$ for the G-action on X^{ss}, where in X^{s} the fibers of φ coincide with the G-orbits. The remark preceding [145, Converse 1.12] implies that Y is a projective algebraic variety, where the compactness of Y also follows from the homeomorphism from the reduced phase space $\mu^{-1}(0)/K$ onto Y in [145, Theorem 8.3].

If $\lambda : \mathbb{C}^\times \to G$ is a one-parameter subgroup of G, then $\mathbb{C}^{n+1}$ is the direct sum of the subspaces E_w, $w \in \mathbb{Z}$, such that $\lambda(t)$ acts on E_w as multiplication by t^w. If $z \in \mathbb{C}^{n+1} \setminus \{0\}$ then the w such that the E_w-component of z is nonzero are called the λ-weights of z. The *numerical stability criterion* of [145, §1 in Chapter 2], see also Miranda [135, Theorem 3.5], states that $\mathbb{C}z$ is not semistable or not stable if and only if there exists a one-parameter subgroup λ of G such that the λ-weights of z are all positive or nonnegative, respectively.

6.3.3.2 Miranda's Moduli Spaces of Elliptic Fibrations over $\mathbb{P}^1$

Let $\mathrm{H}_k(\mathbb{C}^n)$ denote the vector space of all homogeneous polynomials of degree l in n complex variables. Then $(L,\ (g_2,\ g_3)) \mapsto (g_2 \circ L,\ g_3 \circ L)$ defines a (right) action of the reductive complex algebraic group $\mathrm{GL}(\mathbb{C}^2)$ on the space $U = U_N := \mathrm{H}_{4\,N}(\mathbb{C}^2) \times \mathrm{H}_{6\,N}(\mathbb{C}^2)$, to which the pairs $(g_2,\ g_3)$ belong that define the Weierstrass models of relatively minimal elliptic fibrations over $\mathbb{P}^1$ as in Example 5 after Remark 6.3.11. In U we have the $\mathrm{GL}(\mathbb{C}^2)$-invariant open subset U_{ell} of all pairs $(g_2,\ g_3)$ such that the element $\Delta(g_2,\ g_3) := {g_2}^3 - 27\,{g_3}^2$ of $\mathrm{H}_{12\,N}(\mathbb{C}^2)$ is nonzero, and for every $z \in \mathbb{C}^2 \setminus \{0\}$ the order of the zero of g_2 at z is < 4 or the order of the zero of g_3 is < 6. It follows from Theorems 6.3.6 and 6.3.10 that the set $U_{\mathrm{ell}}/\mathrm{GL}(\mathbb{C}^2)$ of all $\mathrm{GL}(\mathbb{C}^2)$-orbits in U_{ell} is in bijective correspondence with the category of isomorphism classes of relatively minimal elliptic fibrations $\varphi : S \to \mathbb{P}^1$ such that $\chi(S,\ \mathcal{O}) = -\deg \mathfrak{f} = N$.

Miranda [136] defined a projective algebraic variety that is closely related to $U_{\mathrm{ell}}/\mathrm{GL}(\mathbb{C}^2)$. He replaced the action of $\mathrm{GL}(\mathbb{C}^2)$ by the action of $\mathbb{C}^\times \times \mathrm{SL}(\mathbb{C}^2)$ via the surjective twofold covering homomorphism $(\lambda,\ A) \mapsto \lambda\,A : \mathbb{C}^\times \times \mathrm{SL}(\mathbb{C}^2) \to \mathrm{GL}(\mathbb{C}^2)$. The right action of $\lambda \in \mathbb{C}^\times$ sends $(g_2,\ g_3)$ to $(\lambda^{4\,N}\,g_2,\ \lambda^{6\,N}\,g_3)$. In order to obtain a projective embedding of the weighted projective space $(U \setminus \{0\})/\mathbb{C}^\times$, Miranda [136, Section 3] used the polynomial mapping $h : (g_2,\ g_3) \mapsto ({g_2}^3,\ {g_3}^2) : U \to T$, where $T := \mathrm{H}_3(\mathrm{H}_{4\,N}(\mathbb{C}^2)) \times \mathrm{H}_2(\mathrm{H}_{6\,N}(\mathbb{C}^2))$. For each $\lambda \in \mathbb{C}^\times$ we have $h(g_2 \circ \lambda,\ g_3 \circ \lambda) = h(\lambda^{4\,N}\,g_2,\ \lambda^{6\,N}\,g_3) = \lambda^{12}\,h(g_2,\ g_3)$, and it follows that h induces a bijective mapping from $(U \setminus \{0\})/\mathbb{C}^\times$ onto an algebraic subvariety M_N of the projec-

tive space $\mathrm{P}(T)$. The mapping h intertwines the action of $\mathrm{SL}(\mathbb{C}^2)$ on U with an action of $\mathrm{SL}(\mathbb{C}^2)$ on T by means of linear transformations, corresponding to an algebraic action of $\mathrm{SL}(\mathbb{C}^2)$ on M_N. Geometric invariant theory yields a surjective universal categorical quotient $\varphi : M_N^{\mathrm{ss}} \to \mathrm{MES}_N$ for the $\mathrm{SL}(\mathbb{C}^2)$-action, with MES_N projective algebraic. Miranda [136, Proposition 5.1] proved that $\pi(g_2,\ g_3) := \mathbb{C}\, h(g_2,\ g_3) \in M_N^{\mathrm{ss}}$ if and only if $\mathrm{ord}_z\, g_2 \le 2\,N$ or $\mathrm{ord}_z\, g_3 \le 3\,N$ for every $z \in \mathbb{C}^2 \setminus \{0\}$, and $\pi(g_2,\ g_3) \in M_N^{\mathrm{s}}$ if and only if $\mathrm{ord}_z\, g_2 < 2\,N$ or $\mathrm{ord}_z\, g_3 < 3\,N$ for every $z \in \mathbb{C}^2 \setminus \{0\}$. The name MES_N stands for "Miranda's compactification of the moduli space of isomorphism classes of stable elliptic fibrations $\varphi : S \to \mathbb{P}^1$ such that $\chi(S,\ \mathcal{O}) = N$." We have $\dim\ \mathrm{MES}_N = (4\,N + 1) + (6\,N + 1) - 1 - 3 = 10\,N - 2$.

If we write $U^{\mathrm{ss}} := \pi^{-1}(M_N^{\mathrm{ss}})$, $U^{\mathrm{s}} := \pi^{-1}(M_N^{\mathrm{s}})$, and $\mathrm{MES}_N^{\mathrm{s}} := \varphi(M_N^{\mathrm{s}})$, then $\varphi \circ \pi : U^{\mathrm{ss}} \to \mathrm{MES}_N$ is an algebraic morphism of projective algebraic varieties that is constant on the $\mathrm{GL}(\mathbb{C}^2)$-orbits inducing a surjective mapping from $U^{\mathrm{ss}}/\,\mathrm{GL}(\mathbb{C}^2)$ onto MES_N, and its restriction to U^{s} induces a bijective mapping from $U^{\mathrm{s}}/\,\mathrm{GL}(\mathbb{C}^2)$ onto a Zariski-open subset $(\mathrm{MES}_N)^{\mathrm{s}}$ of MES_N. If $N > 1$, then $U_{\mathrm{ell}} \subset U^{\mathrm{s}}$, and the set $U_{\mathrm{ell}}/\,\mathrm{GL}(\mathbb{C}^2)$ is identified with a Zariski-open subset $(\mathrm{MES}_N)_{\mathrm{ell}} \subset (\mathrm{MES}_N)^{\mathrm{s}}$ of the projective algebraic variety MES_N.

For rational elliptic surfaces, when $N = 1$, we write $\mathrm{MRES} := \mathrm{MES}_1$ = "Miranda's compactification of the moduli space of isomorphism classes of stable rational elliptic surfaces." Then $(g_2,\ g_3) \in U^{\mathrm{ss}}$ if and only if $\mathrm{ord}_z\, g_2 \le 2$ or $\mathrm{ord}_z\, g_3 \le 3$ for every $z \in \mathbb{C}^2 \setminus \{0\}$, and $(g_2,\ g_3) \in U^{\mathrm{s}}$ if and only if $\mathrm{ord}_z\, g_2 < 2$ or $\mathrm{ord}_z\, g_3 < 3$ for every $z \in \mathbb{C}^2 \setminus \{0\}$. Furthermore, $\Delta(g_2,\ g_3) = 0$ if and only there exists $q \in \mathrm{H}_2(\mathbb{C}^2)$ such that $g_2 = 3\,q^2$ and $g_3 = q^3$, when $(g_2,\ g_3) \notin U^{\mathrm{s}}$, and $(g_2,\ g_3) \in U^{\mathrm{ss}}$ if and only if q has simple zeros. It follows that $U^{\mathrm{s}} \subset U_{\mathrm{ell}}$, but not $U_{\mathrm{ell}} \subset U^{\mathrm{ss}}$. If $(g_2,\ g_3) \in U_{\mathrm{ell}}$ are the Weierstrass data of a rational elliptic surface $\kappa : S \to \mathbb{P}^1$, then Table 6.3.2 implies that $(g_2,\ g_3) \in U^{\mathrm{ss}}$ if and only if κ has no singular fibers of Kodaira type II*, III*, or IV*, whereas $(g_2,\ g_3) \in U^{\mathrm{s}}$ if and only if in addition, S has no singular fibers of type I_b^*, $b \in \mathbb{Z}_{>0}$. That is, $(g_2,\ g_3) \in U^{\mathrm{s}}$ if and only if every irreducible component of every fiber of κ has multiplicity one. We have $\dim\ \mathrm{MRES} = \dim\ \mathrm{MES}_1 = 10 \cdot 1 - 2 = 8$, in agreement with the naive count $\dim(U/\,\mathrm{GL}(\mathbb{C}^2)) = \dim U - \mathrm{GL}(\mathbb{C}^2) = 12 - 4 = 8$.

The complex projective algebraic variety MRES is equal to the compactification $\mathcal{M}^M$ in Heckman and Looijenga [81, Introduction] of the moduli space $\mathcal{M}$ of isomorphism classes of rational elliptic surfaces with 12 singular fibers of Kodaira type I_1, which implies that $\mathcal{M}$ is dense in $\mathcal{M}^M = \mathrm{MRES}$. Since $(\mathrm{MRES})^{\mathrm{s}}$ contains $\mathcal{M}$, it is dense in MRES as well. In Heckman and Looijenga there is an "altogether different" compactification $\mathcal{M}^*$ of $\mathcal{M}$, and one of the main goals of [81] is to study a compactification of $\mathcal{M}$ that dominates both $\mathcal{M}^M$ and $\mathcal{M}^*$.

6.3.3.3 The Cubic Pencil Quotient and the Aronhold Morphism

According to (a) in Theorem 9.1.3, every rational elliptic surface S is obtained from a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ by successively blowing up base points of the anticanonical pencils. Here $\mathfrak{C}$ is defined by a pair $(q^0,\ q^1) \in V := \mathrm{H}_3(\mathbb{C}^3) \times \mathrm{H}_3(\mathbb{C}^3)$

of linearly independent homogeneous polynomials of degree three in three variables. Proposition 4.4.3 implies that the Aronhold invariants define a polynomial mapping $A : (q^0, q^1) \mapsto (g_2, g_3) : V \to U$, where $g_2(z_0, z_1) = 27\,S(z_0\,q^0 + z_1\,q^1)$, $g_3(z_0, z_1) = -27\,T(z_0\,q^0 + z_1\,q^1)$, and the rational elliptic surface S is isomorphic to the Weierstrass model defined by g_2 and g_3. The pencil defined by (q^0, q^1) has at least one smooth member if and only if $A(q^0, q^1) \in U_{\text{ell}}$. We write $V_{\text{ell}} = A^{-1}(U_{\text{ell}})$ for the set of all $v \in V$ such that the pencil defined by v has at least one smooth member.

The pencils defined by (q^0, q^1) and (r^0, r^1) are isomorphic by means of a projective linear transformation in $\mathbb{P}^2$ if and only if there exist $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(\mathbb{C}^2)$ and $g \in \mathrm{GL}(\mathbb{C}^3)$ such that $r^0 = (a\,q^0 + b\,q^1) \circ g$ and $r^1 = (c\,q^0 + d\,q^1) \circ g$. The pairs $\mu^{-3}\,1 \in \mathrm{GL}(\mathbb{C}^2)$, $\mu\,1 \in \mathrm{GL}(\mathbb{C}^3)$ are the ones that act trivially; hence we have an effective linear action of the reductive algebraic group $H := (\mathrm{GL}(\mathbb{C}^2) \times \mathrm{GL}(\mathbb{C}^3))/\mathbb{C}^\times$ on the vector space V. Note that $\dim V = 2 \cdot 10 = 20$ and $\dim H = 4 + 9 - 1 = 12$, and we expect an orbit space V/H of dimension $20 - 12 = 8$, but at this stage V/H is only a set. The mapping $A : V \to U$ intertwines the action of H on V with the action of $\mathrm{GL}(\mathbb{C}^2)$ on U, and therefore induces a mapping $A/ : V/H \to U/G$. The restriction to V_{ell}/H of $A/$ is the surjective mapping $(A/)_{\text{ell}} : V_{\text{ell}}/H \to U_{\text{ell}}/G$ that assigns to each isomorphism class of cubic pencils in $\mathbb{P}^2$ with at least one smooth member the corresponding isomorphism class of rational elliptic surfaces. Remark 9.1.5 implies that the mapping $(A/)_{\text{ell}}$ is finite in the sense that each fiber of $(A/)_{\text{ell}}$ is finite.

If V' denotes the set of linearly independent (q^0, q^1), then $\Pi : (q^0, q^1) \to \mathbb{C}\,q^0 + \mathbb{C}\,q^1$ is a morphism from V' onto the projective algebraic variety $P = \mathrm{G}_2(\mathrm{H}_3(\mathbb{C}^3))$ of all two-dimensional linear subspaces of $\mathrm{H}_3(\mathbb{C}^3)$, where the fibers of Π are the orbits under the aforementioned $\mathrm{GL}(\mathbb{C}^2)$-action on V'. In the sequel, the Plücker embedding, see Section 1.3, will be used in order to view P as an algebraic subvariety of the projective space of $\Lambda^2(\mathrm{H}_3(\mathbb{C}^3))$. If $p_0 \in P$, the pair (q_0^0, q_0^1) is a basis of p_0, the vector space C a linear complement of p_0 in $\mathrm{H}_3(\mathbb{C}^3)$, and $P_C := \{p \in P \mid p \cap C = 0\}$, then the morphism that assigns to each $p \in P_C$ the unique (q^0, q^1) such that $q^i + C = q_0^i + C, i = 1, 2$, is a section for Π over the Zariski-open subset P_C of P. These sections exhibit $\Pi : V' \to P$ as an algebraic principal $\mathrm{GL}(\mathbb{C}^2)$-bundle, and it follows that $\Pi : V' \to P$ is a surjective universal categorical quotient. Since (q^0, q^1) and (r^0, r^1) define the same pencil of cubic curves in $\mathbb{P}^2$ if and only if $\Pi(q^0, q^1) = \Pi(r^0, r^1)$, the variety P can be viewed as the space of all pencils of cubic curves in $\mathbb{P}^2$. We have $p \in P_{\text{ell}} := \Pi(V_{\text{ell}})$ if and only if the pencil has at least one nonsingular member, while blowing up the base points yields a rational elliptic surface. According to Lemma 2.2.1, the automorphism group of $\mathbb{P}^2$ is the action of the reductive algebraic group $\mathrm{PGL}(\mathbb{C}^3) \simeq \mathrm{GL}(\mathbb{C}^3)/\mathbb{C}^\times$. The action of $\mathrm{PGL}(\mathbb{C}^3)$ is replaced by the action of $\mathrm{SL}(\mathbb{C}^3)$ via the threefold covering homomorphism $\mathrm{SL}(\mathbb{C}^3) \to \mathrm{PGL}(\mathbb{C}^3)$. Geometric invariant theory provides a surjective categorical quotient $\Phi : P^{\text{ss}} \to \mathrm{MCP}$ for the action of $\mathrm{SL}(\mathbb{C}^3)$, where MCP is projective algebraic. Here MCP stands for "the compactification of Miranda [135] of the moduli space of isomorphism classes of stable pencils of cubic curves in the projective plane."

The sets $V^{ss} := \Pi^{-1}(P^{ss})$ and $V^{s} := \Pi^{-1}(P^{s})$ are Zariski open H-invariant subsets of V', and the restriction of Π to V^{ss} is a surjective categorical quotient $V^{ss} \to P^{ss}$ for the $\mathrm{GL}(\mathbb{C}^2)$-action. In combination with the fact that $\Phi : P^{ss} \to \mathrm{MCP}$ is a surjective categorical quotient for the $\mathrm{SL}(\mathbb{C}^3)$-action, it follows that $\Phi \circ \Pi : V^{ss} \to \mathrm{MCP}$ is a surjective categorical quotient for the H-action in V^{ss}, whose fibers in V^{s} are the H-orbits in V^{s}. Because $\dim P = 2 \cdot (10 - 2) = 16$ and $\dim(\mathrm{SL}(\mathbb{C}^3)) = 9 - 1 = 8$, we have $\dim \mathrm{MCP} = 16 - 8 = 8 = \dim \mathrm{MRES}$, confirming the naive counts $\dim(V/H) = \dim V - \dim H = 2 \cdot 10 - (4 + 9 - 1) = 8 = \dim(U/\mathrm{GL}(2, \mathbb{C}))$.

Assume that the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ defined by $v \in V'$ has at least one smooth member, and let $\kappa : S \to \mathbb{P}^1$ be the rational elliptic surface defined by $\mathfrak{C}$. The main theorem in Miranda [135] states that $p = \Pi(v) \in P^{ss}$ if and only if κ has no singular fibers of Kodaira type II*, III*, or IV*, and $p \in P^{s}$ if and only if the pencil $\mathfrak{C}$ has at least one smooth member and every irreducible component of every fiber of κ has multiplicity one. In view of the previous descriptions of $U_{\mathrm{ell}} \cap U^{ss}$ and U^{s} it follows that that $A^{-1}(U_{\mathrm{ell}} \cap U^{ss}) = V_{\mathrm{ell}} \cap V^{ss}$, $U_{\mathrm{ell}} \cap U^{ss} = A(V_{\mathrm{ell}} \cap V^{ss})$, $A^{-1}(U^{s}) = V^{s} \subset V_{\mathrm{ell}}$, and $A(V^{s}) = U^{s}$. This corresponds to Miranda [136, remark on p. 388].

A straightforward computation by means of a formula manipulation program, applied to the description of the nonsemistable elements of P in Miranda [135, Proposition 5.1], yields that if $(q^0, q^1) \notin V^{ss}$, then $(g_2, g_3) = A(q^0, q^1) \notin U^{ss}$. Here it is allowed that all members of the pencil $\mathfrak{C}$ defined by $(q^0, q^1) \in V'$ are singular, which we will assume in the sequel. In view of Proposition 4.4.3 this happens if and only if $\Delta(g_2, g_3) = 0$ if $(g_2, g_3) = A(q^0, q^1)$, or equivalently $g_2 = 3\,Q^2$, $g_3 = Q^3$ for some $Q \in \mathrm{H}_2$. Furthermore, Q has two distinct simple zeros in $\mathbb{P}^1$ if and only if $(g_2, g_3) \in U^{ss}$, which implies that $(q^0, q^1) \in V^{ss}$. It follows from Corollary 3.2.9, and the fact that all members of $\mathfrak{C}$ have a curve in common if there are infinitely many base points, that all members have a common singular point b, or q^0 and q^1 have a homogeneous polynomial of degree one or two in three variables as a common factor. In each case we can bring (q^0, q^1) into a relatively simple normal form, while the formula for Q obtained by a formula manipulation program yields that if Q does not have two simple zeros, the normal form can be simplified further. A computation with Plücker coordinates as in the proof of Miranda [135, Proposition 5.1] yields that $(q^0, q^1) \notin V^{ss}$ in every case. It follows that $A^{-1}(U^{ss}) = V^{ss}$ and $U^{ss} = A(V^{ss})$.

Because $\varphi \circ \pi \circ A : V^{ss} \to \mathrm{MRES}$ is a morphism that is constant on the H-orbits, and $\Phi \circ \Pi : V^{ss} \to \mathrm{MCP}$ is a categorical quotient for the H-action on V^{ss}, it follows that there is a unique morphism $\mathrm{AM} : \mathrm{MCP} \to \mathrm{MRES}$ such that $\varphi \circ \pi \circ A = \mathrm{AM} \circ \Phi \circ \Pi$. The morphism AM is surjective because $\varphi \circ \pi \circ A : V^{ss} \to \mathrm{MRES}$ is surjective. The sets $(\mathrm{MCP})^{s} := \Pi(P^{s})$ and $(\mathrm{MCP})_{\mathrm{ell}} := \Pi(P_{\mathrm{ell}})$ are Zariski open in MCP, $(\mathrm{MCP})^{s} \subset (\mathrm{MCP})_{\mathrm{ell}}$, and the restriction to V^{s} of $\Phi \circ \Pi$ induces a bijective mapping from V^{s}/H onto $(\mathrm{MCP})^{s}$. Because the mapping $(A/)_{\mathrm{ell}} : V_{\mathrm{ell}}/H \to U_{\mathrm{ell}}/\mathrm{GL}(\mathbb{C}^2)$ is finite, the restriction to $(\mathrm{MCP})^{s}$ of MA is a finite morphism from $(\mathrm{MCP})^{s}$ onto $(\mathrm{MRES})^{s}$. Because the polynomial map A is

defined in terms of Aronhold's invariants, I propose to call the morphism AM between the aforementioned compactifications of moduli spaces the *Aronhold morphism*.

It is argued in Remark 9.2.28 that the generic fiber of AM has $8640 = 2^6 \cdot 3^3 \cdot 5$ elements. For all the Manin elliptic surfaces considered in Remark 9.2.28, the corresponding pencils of cubic curves in $\mathbb{P}^2$ have nine base points, with no three base points on a projective line in $\mathbb{P}^2$. The example in Remark 9.1.5 of the rational elliptic surface S with the configuration of the singular fibers $\mathrm{I}_9\ 3\ \mathrm{I}_1$ corresponds to a stable element of U and V, with two nonisomorphic pencils of cubic curves defining S. Here the interesting feature is that the first pencil has one base point, whereas the second pencil has two base points, a very good reason why the second pencil cannot be obtained from the first one by means of a projective linear transformation.

6.3.3.4 The QRT Quotient and the Frobenius Morphism

Let B denote the nine-dimensional vector space space of all bihomogeneous polynomials of bidegree 2, 2 on $\mathbb{C}^2$ as in Lemma 3.3.3 and Remark 2.5.7, $(B \times B)'$ the Zariski-open subset of $B \times B$ of all linearly independent pairs $(p^0,\ p^1)$, and $P := \mathrm{G}_2(B)$ the variety of all two-dimensional linear subspaces of B, identified with an algebraic subvariety of $\mathrm{P}(\Lambda^2 B)$ by means of the Plücker embedding. Every $(p^0,\ p^1) \in (B \times B)'$ defines a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, where $(q^0,\ q^1)$ defines the same pencil if and only if $\mathbb{C}\, q^0 + \mathbb{C}\, q^1 = \mathbb{C}\, p^0 + \mathbb{C}\, p^1$ if and only if $(q^0,\ q^1)$ belongs to the orbit of $(p^0,\ p^1)$ for the left action of $\mathrm{GL}(\mathbb{C}^2)$ on $B \times B$. The mapping $\Pi : (p^0,\ p^1) \mapsto \mathbb{C}\, p^0 + \mathbb{C}\, p^1 : (B \times B)' \to P$ is a surjective universal categorical quotient, and we will view P as the space of all pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. According to Lemma 9.2.30, the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is equal to the union of its identity component $A^\circ = \mathrm{PGL}(\mathbb{C}^2) \times \mathrm{PGL}(\mathbb{C}^2)$ acting componentwise and the coset $\sigma\, A^\circ$, where $\sigma : (x,\ y) \mapsto (y,\ x)$ is the symmetry switch. Replacing each copy of $\mathrm{PGL}(\mathbb{C}^2)$ by $\mathrm{SL}(\mathbb{C}^2)$, we obtain a reductive algebraic group G, acting on P in an algebraic fashion. Therefore geometric invariant theory yields a surjective universal categorical quotient $\Phi : P^{\mathrm{ss}} \to \mathrm{MBP}$ where MBP is projective algebraic and $\dim \mathrm{MBP} = 2 \cdot (9 - 2) - (3 + 3) = 8$. Here MBP stands for "the GIT compactification of the moduli space of isomorphism classes of stable pencils of biquadratic curves."

If the pencil $\mathbb{C}\, p^0 + \mathbb{C}\, p^1$ has at least one smooth member, then, blowing up its base points, we arrive at a rational elliptic surface S together with a Manin QRT automorphism τ of S; see Corollary 3.3.10, Corollary 3.4.2, and Theorem 4.3.2. It follows from Example 5 and Proposition 7.8.1 that such a pair $(S,\ \tau)$ has the Weierstrass data $(X,\ Y,\ g_2) \in D := \mathrm{H}_2(\mathbb{C}^2) \times \mathrm{H}_3(\mathbb{C}^2) \times \mathrm{H}_4(\mathbb{C}^2)$, where $g_3 = 4\, X^3 - g_2\, X - Y^2$. Mimicking the construction of MES_N of Miranda [136], we might use a mapping such as $(X,\ Y,\ g_2) \mapsto (X^6,\ X^4\, g_2,\ X^3\, Y^2,\ X^2\, {g_2}^2,\ Y^4,\ {g_2}^3)$ in order to obtain an $\mathrm{SL}(\mathbb{C}^2)$-equivariant embedding of $(D \setminus \{0\})/\mathbb{C}^\times$ onto a projective algebraic variety E, where $\mathbb{C}^\times$ acts on the right on D by means of scalar multiplication. Geometric invariant theory provides a surjective universal categorical quotient $\varphi : E^{\mathrm{ss}} \to \mathrm{MRES}_{\mathrm{QRT}}$ for the $\mathrm{SL}(\mathbb{C}^2)$-action on E, where $\mathrm{MRES}_{\mathrm{QRT}}$ is projective

algebraic and dim $\mathrm{MRES}_{\mathrm{QRT}} = (2+1)+(3+1)+(4+1)-1-3 = 8$. Let $\pi : D \setminus \{0\} \to E$ denote the composition of the projection $D \setminus \{0\} \to (D \setminus \{0\})/C^\times$ and the embedding $(D \setminus \{0\})/C^\times \to E$. The double acronym $\mathrm{MRES}_{\mathrm{QRT}}$ stands for "a compactification of the moduli space of isomorphism classes of rational elliptic surfaces provided with a Manin QRT automorphism."

Corollary 2.4.7 and Proposition 2.5.6 yield the Weierstrass data $(X,\ Y,\ g_2)$ of the QRT map τ that arises from the pencil $\mathbb{C}\, p^0 + \mathbb{C}\, p^1$, by means of a trihomogeneous polynomial map $F : B \times B \to D$, of tridegree $(2,\ 3,\ 4)$, induced by the Frobenius invariants of biquadratic polynomials defined in Section 2.5.1. If $(\mathrm{MBP})'$ denotes the Zariski open subset $\Phi \circ \Pi((\pi \circ F)^{-1}(E^{\mathrm{ss}}) \cap \Pi^{-1}(P^{\mathrm{ss}}))$ of MBP, then, as for the Aronhold morphism, the mapping F induces a unique algebraic morphism $\mathrm{FM} : (\mathrm{MBP})' \to \mathrm{MRES}_{\mathrm{QRT}}$, called the *Frobenius morphism*, such that $\mathrm{FM} \circ \Phi \circ \Pi = \varphi \circ \pi \circ F$. It is argued in Remark 9.2.29 that the generic fiber of FM has $288 = 2^5 \cdot 3^2$ elements.

Let $W : (X,\ Y,\ g_2) \mapsto (g_2,\ g_3)$, where $g_3 = 4\,X^3 - g_2\,X - Y^2$. If $(X,\ Y,\ g_2)$ corresponds to an element of E that is not semistable, then $(g_2,\ g_3)$ corresponds to an element of M_1 that is not semistable, and therefore the polynomial mapping $W : D \to U$ induces a surjective morphism WM from a Zariski-open subset $\mathrm{MRES}'_{\mathrm{QRT}}$ of $\mathrm{MRES}_{\mathrm{QRT}}$ onto MRES. Here WM stands for *Weierstrass morphism*, since it is defined in terms of the Weierstrass equation. The composition $\mathrm{WM} \circ \mathrm{FM}$ is a surjective morphism from a Zariski-open subset of MBP to MRES. It assigns to a generic isomorphism class of pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ the isomorphism class of rational elliptic surfaces obtained by blowing up the base points. It is argued in Remark 9.2.29 that the generic fiber of WM has 120 elements. Therefore the generic fiber of $\mathrm{WM} \circ \mathrm{FM}$ has $120 \cdot 288 = 2^8 \cdot 3^3 \cdot 5 = 34560$ elements, four times the number $8640 = 2^6 \cdot 3^3 \cdot 5$ for the Aronhold morphism.

6.3.3.5 The Partial Discriminant Morphism and the Eisenstein Morphism

In Corollary 2.5.10 the Weierstrass data $(g_2(z_0,\ z_1),\ g_3(z_0,\ z_1))$ of the elliptic curve $z_0\, p^0 + z_1\, p^1 = 0$ were obtained by applying the Eisenstein invariants D and $-E$ to the partial discriminants Δ_2 and Δ_1 of the biquadratic polynomial $z_0\, p^0 + z_1\, p^1$. Here

$$((z_0,\ z_1),\ (x_0,\ x_1),\ (y_0,\ y_1)) \mapsto (z_0\, p^0 + z_1\, p^1)((x_0,\ x_1),\ (y_0,\ y_1))$$

is an element of $\mathrm{H}_{(1,\,2,\,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$, a trihomogeneous polynomial of tridegree $(1,\ 2,\ 2)$ on $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$. Similarly (2.4.5) implies that $((z_0,\ z_1),\ (x_0,\ x_1)) \mapsto \Delta_2(z_0\, p^0 + z_1\, p^1)(x_0,\ x_1)$ is an element of $\mathrm{H}_{(2,\,4)}(\mathbb{C}^2 \times \mathbb{C}^2)$, a bihomogeneous polynomial of bidegree $(2,\ 4)$ on $\mathbb{C}^2 \times \mathbb{C}^2$. We will denote the mapping from $\mathrm{H}_{(1,\,2,\,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$ to $\mathrm{H}_{(2,\,4)}(\mathbb{C}^2 \times \mathbb{C}^2)$ defined by (2.4.5) also by Δ_2. It is a polynomial mapping, homogeneous of degree two.

On $\mathrm{H}_{(1,\,2,\,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$ and $\mathrm{H}_{(2,\,4)}(\mathbb{C}^2 \times \mathbb{C}^2)$ we have the respective actions of the Cartesian products of three and two copies of $\mathrm{GL}(\mathbb{C}^2)$, and Δ_2 induces a

mapping between the orbit spaces. Similarly the Eisenstein mapping $D \times (-E) : \mathrm{H}_{(2,4)}(\mathbb{C}^2 \times \mathbb{C}^2) \to \mathrm{H}_4(\mathbb{C}^2) \times \mathrm{H}_6(\mathbb{C}^2)$ induces a mapping between the orbit spaces, where $(D \times (-E)) \circ \Delta_2 = g_2 \times g_3$, the mapping that assigns to p the Weierstrass data of the rational elliptic surface obtained by blowing up the base points of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Here the orbit spaces are just sets. As in the previous subsections, the respective orbit spaces of $\mathrm{H}_{(1,2,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$ and $\mathrm{H}_{(2,4)}(\mathbb{C}^2 \times \mathbb{C}^2)$ can be replaced by GIT quotients $\mathrm{MBP}_{\mathrm{axes}}$ and $\mathrm{M}(2, 4)$, where the subscript "axes" stands for the fact that in the group action we did not include the symmetry switch $(x, y) \mapsto (y, x)$, which interchanges the horizontal axes with the vertical axes in $\mathbb{P}^1 \times \mathbb{P}^1$. The notation $\mathrm{M}(2, 4)$ stands for "moduli space of isomorphism classes of $(2, 4)$-curves in $\mathbb{P}^1 \times \mathbb{P}^1$." Both these GIT quotients are eight-dimensional projective algebraic varieties. The symmetry switch leads to a morphism $\mathrm{SM} : \mathrm{MBP}_{\mathrm{axes}} \to \mathrm{MBP}$, called the *symmetry morphism*. It is a twofold branched covering, branching over the isomorphism classes of pencils of biquadratic curves that are invariant under the symmetry switch. If the two-dimensional vector space p of biquadratic polynomials is invariant under the symmetry switch σ, then either σ is equal the identity on p, in which case we have a symmetric QRT map as in Chapter 10, or p contains a symmetric and an antisymmetric biquadratic polynomial when we have an antisymmetric QRT map as in Example 7, or σ is equal to minus the identity on p when all members of the pencil contain the symmetry axis $x = y$, hence are reducible, and the base locus contains the symmetry $x = y$. The partial discriminant Δ_2 induces a morphism PDM, called the *partial discriminant morphism*, from a Zariski-open subset of $\mathrm{MBP}_{\mathrm{axes}}$ to $\mathrm{M}(2, 4)$. The Eisenstein map induces a morphism EM, called the *Eisenstein morphism*, from a Zariski-open subset of $\mathrm{M}(2, 4)$ to MRES, where the factorization $(D \times (-E)) \circ \Delta_2 = g_2 \times g_3$ implies $\mathrm{EM} \circ \mathrm{PDM} = \mathrm{WM} \circ \mathrm{FM} \circ \mathrm{SM}$, on the domain where the left- and right-hand side both are defined. In the next paragraphs it is argued that the generic fiber of the partial discriminant morphism PDM has $2^7 = 128$ elements; hence the generic fiber of the Eisenstein morphism EM has $2 \cdot (2^5 \cdot 3^2) \cdot (2^3 \cdot 3 \cdot 5)/2^7 = 2^2 \cdot 3^3 \cdot 5 = 540$ elements.

Let p be a nonzero element of $\mathrm{H}_{(1,2,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$ such that the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by p has at least one smooth member. Assume that the elliptic surface S obtained by blowing up the base points of $\mathfrak{B}$ satisifies the conditions in Remark 9.2.29. Then all fibers of S are irreducible, hence the base points of $\mathfrak{B}$ are simple, while Corollary 3.1.6 implies that the zero-set S_p of p in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is smooth and the projection $\pi_{(x,y)} : (z, x, y) \mapsto (x, y) : S_p \to \mathbb{P}^1 \times \mathbb{P}^1$ exhibits S_p as a rational elliptic surface isomorphic to S; see Section 3.3.5. We write $S_p = S$ in the sequel. Lemma 5.1.6 implies that no two of the base points lie on the same horizontal or vertical axis, where the latter condition implies that if B denotes the set of the eight base points, then the $\pi_x(b)$, $b \in B$ are eight distinct points in $\mathbb{P}^1$ if $\pi_x : (x, y) \mapsto x$. The notation for the projections is bad, but mnemonic.

The projection $\pi_{(z,x)} : (z, x, y) \mapsto (z, x) : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is a twofold branched covering, branching over the zero-set C of $f = \Delta_2(p) \in \mathrm{H}_{(2,4)}(\mathbb{C}^2 \times \mathbb{C}^2)$, where $C = \pi_{(z,x)}(\operatorname{Fix} \iota_2^S)$, where $\operatorname{Fix} \iota_2^S \subset S$ denotes the set of all fixed points of the vertical involution ι_2^S of S defined in Corollary 3.4.2. If we write $p = z_0\, p^0 + z_1\, p^1$, then the discriminant $d(x_0, x_1)$ of $(z_0, z_1) \mapsto \Delta_2(p)((z_0, z_1), (x_0, x_1))$ is equal

to 16 times the resultant of the two polynomials $(y_0,\ y_1) \mapsto p^0((x_0,\ x_1),\ (y_0,\ y^1))$ and $(y_0,\ y_1) \mapsto p^1((x_0,\ x_1),\ (y_0,\ y^1))$, and therefore the curve C branches over the points $\pi_x(b), b \in B$. Because there are eight such points, each zero of the polynomial $d(x_0,\ x_1)$ of degree eight is simple; hence C is smooth, connected = irreducible, and has a second-order contact with the z-axis $A_b := \mathbb{P}^1 \times \{\pi_{(z,\,x)}(b)\}$ at the point $\pi_{(z,\,x)}(b)$. The Riemann–Hurwitz formula (2.3.18) yields that C is a hyperelliptic curve of genus $\mathrm{g}(C) = 1 - 2 + 8/2 = 3$.

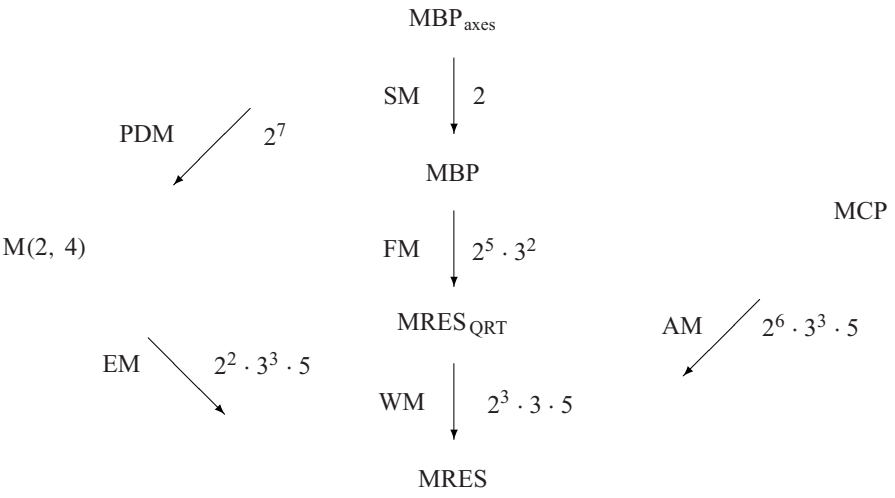
The -1 curve in S that appears as the blowup of the base point $b \in \mathbb{P}^1 \times \mathbb{P}^1$ is equal to $E_b = \mathbb{P}^1 \times \{b\} \subset S$, and $\pi_{(z,\,x)}(E_b) = A_b$. The curve ${\pi_{(z,\,x)}}^{-1}(A_b) \cap S$ is equal to the union of E_b and $E'_b := \iota_2^S(E_b)$. The -1 curves E'_b and E_b intersect only at one point over b, where the intersection is transversal. If $b \neq \beta$, then E_b and E'_b are disjoint from both E_β and E'_β. Let $I \subset B$. If instead of blowing down the E_b, $b \in B$ we blow down the E'_i, $i \in I$, and the E_j, $j \in B \setminus I$, then we have performed the zigzags at the intersection points of E_i and E'_i for all $i \in I$, to the surface $\mathbb{P}^1 \times \mathbb{P}^1$ viewed as a ruled surface with the vertical axes as the fibers. See Section 6.2.3 for zigzags of ruled surfaces. If the number of elements of I is even (respectively odd), we arrive at a surface Σ that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (respectively Σ_1). In the first case there is no -1 curve in Σ, hence there is no -1 curve in S that is disjoint from the $E'_i, i \in I$, and the E_j, $j \in B \setminus I$, whereas in the second case there is a unique such -1 curve in S. I learned the idea of blowing down the E'_i, $i \in I$, and E_j, $j \in B \setminus I$, from Vakil [198, p. 44], where it is stated that there is always a ninth -1 curve in S disjoint from these eight mutually disjoint -1 curves.

If $i \neq j$, then we can arrange by means of projective linear transformations in the two components of $\mathbb{P}^1 \times \mathbb{P}^1$ that $i = ([1 : 0],\ [1 : 0])$ and $j = ([0 : 1],\ [0 : 1])$. The biquadratic polynomial (2.4.1) vanishes at these points if and only if $A_{22} = 0$ and $A_{00} = 0$. The zigzags at these two points correspond to the birational transformation defined by replacing y_0 and y_1 by $x_0\, y_0$ and $x_1\, y_1$, respectively. The correspondence between biquadratic polynomials and holomorphic two-vector fields on $\mathbb{P}^1 \times \mathbb{P}^1$ in Lemma 3.3.3 and the description in Section 3.2.4 of the proper transform of the anticanonical line bundle under a blowup yield that the proper transform of p is the biquadratic polynomial p' obtained by replacing $(y_0,\ y_1)$ by $(x_0\, y_0,\ x_1\, y_1)$ in p, and then dividing the resulting polynomial by $x_0\, x_1$. This amounts to replacing $A_{00} = 0$, A_{10}, and A_{20} by A_{10}, A_{20}, and 0, respectively, leaving the A_{i1} unchanged, and replacing A_{02}, A_{12}, and $A_{22} = 0$ by 0, A_{02}, and A_{11}, respectively. It follows that $\Delta_2(p) = \Delta_2(p')$, and $\Delta_2(p') \in \mathbb{C}\, \Delta_2(p)$ if p' is the proper transform of p for any even number of zigzags at base points. In the notation of Lemma 5.1.1, where we identify the e_i, $1 \leq i \leq 8$, with the e_b, $b \in B$, the action ι_2 of ι_2^S on $\mathrm{H}_2(S,\ \mathbb{Z})$ is given by $\iota_2(l_1) = l_1 + 4\, l_2 - \sum_i\ e_i = f - l_1 + 2\, l_2$, $\iota_2(l_2) = l_2$, and $\iota_2(e_i) = l_2 - e_i$, see Lemma 5.1.2, where Lemma 5.1.6 implies that $d_2 = 4$. Because $Q = f^\perp/\mathbb{Z}\, f$ is generated by $f_2 - f_1 + \mathbb{Z}\, f$ and the $e_i - e_8 + \mathbb{Z}\, f$, $1 \leq i \leq 7$, it follows that ι_2 acts as minus the identity on Q. It follows from Remark 9.2.29 that the blowdowns for different choices of I with $\#(I)$ even lead to distinct elements of $\mathrm{MBP}_{\mathrm{axes}}$, where the element $q = l_2 - l_1 + \mathbb{Z}\, f$ is not identified with $-q = l_1 - l_2 + \mathbb{Z}\, f$, because the symmetry switch is not included in the group that has been divided out. The morphism $\mathrm{SM} : \mathrm{MBP}_{\mathrm{axes}} \to \mathrm{MBP}$ identifies the two elements of $\mathrm{SM} : \mathrm{MBP}_{\mathrm{axes}}$

defined by I and $B \setminus I$, because ι_2^S maps the I-choice to the $B \setminus I$-choice of the E_i, E_i'. Because the sets $I \subset B$ such that $\#(I) \in 2\,\mathbb{Z}$ are in bijective correspondence with the seven-dimensional vector space $\{x \in (\mathbb{Z}/2\,\mathbb{Z})^8 \mid \sum_i x_i = 0\}$ over the field $\mathbb{Z}/2\,\mathbb{Z}$, there are 2^7 such I's.

Assume conversely that $p' \in \mathrm{H}_{(1,\,2,\,2)}(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2)$ and $\Delta_2(p') = \Delta(p)$. Then the restrictions to S and $S' := S_{p'}$ of $\pi_{(z,\,x)}$ exhibit S and S' as twofold branched coverings over the same curve C = the zero-set in $\mathbb{P}^1 \times \mathbb{P}^1$ of $\Delta_2(p) = \Delta_2(p')$. For each $s \in {\pi_{(z,\,x)}}^{-1}(C)$ there exist holomorphic local coordinates $(u,\ v)$ and $(u,\ w)$ near s and $\pi_{(z,\,x)}(s)$ in S and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, such that $\pi_{(z,\,x)}(u,\ v) = (u,\ w)$ with $w = v^2$, where $w = 0$ is the local equation for C. A similar statement holds for S'. Because $\mathbb{P}^1 \times \mathbb{P}^1$, the Cartesian product of two Riemann spheres, is simply connected, the proof of Lemma 2.3.6, with C, C', and P replaced by respectively S, S', and $\mathbb{P}^1 \times \mathbb{P}^1$, yields a complex analytic diffeomorphism ψ from S onto S' such that $\pi_{(z,\,x)} \circ \psi = \pi_{(z,\,x)}|_S$, and any other such diffeomorphism is equal to $\psi \circ \iota_2^S$. If $E_{b'}$ denotes the blowup of a base point b' of the pencil defined by p', then there exists a unique $b \in B$ such that $\pi_x(b') = \pi_x(b)$ and $\pi_{(z,\,x)} \circ \psi^{-1}(E_{b'}) = \pi_{(z,\,x)}(E_{b'}) = A_b = \pi_{(z,\,x)}(E_b)$; hence $\psi^{-1}(E_{b'}) = E_b$ or $\psi^{-1}(E_{b'}) = E_b'$. It follows that the pencil defined by p' is isomorphic to a pencil defined by a p' as in the previous paragraph while the replacement of ψ by $\psi \circ \iota_2^S$ corresponds to the involution in the fiber of the morphism SM. This concludes the argument that the generic fiber of PDM has 2^7 elements.

The morphisms between GIT quotients are summarized in the following commutative diagram.



Each GIT quotient is an eight-dimensional complex projective algebraic variety. The Aronhold morphism AM and the symmetry morphism SM are surjective morphisms from the whole GIT quotients MCP and $\mathrm{MBP}_{\mathrm{axes}}$ onto the whole GIT quotients MRES and MBP, respectively. In the other cases the arrows are algebraic morphisms

between the GIT quotients, defined on large Zariski-open subsets of the source GIT quotients. Further determinations of the semistable and stable elements, by means of the numerical stability criterion, are needed in order to obtain more information about the domains of definition. Since the passage in Section 4.1 from a pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ to a pencil of cubic curves in $\mathbb{P}^2$ depends on the choice of a base point of $\mathfrak{B}$, I do not see an algebraic morphism from a large Zariski-open subset of MBP to MCP such that its composition with AM is equal to WM $\circ$ FM. If it existed, its generic fiber would have four elements.

6.3.3.6 Moduli of QRT Roots

If D, X_ρ, and Y_ρ are the respective homogeneous polynomials of degree 1, 2, and 3 in (z_0, z_1) defined in Proposition 10.1.6, then these define a QRT root on the rational elliptic surface defined by $g_2 \in \mathrm{H}_4$ and $g_3 \in \mathrm{H}_6$, given in (10.1.8), (10.1.9). A categorical quotient of dimension $(1+1)+(2+1)+(3+1)-4 = 5$ could be the point of departure for a moduli space of QRT roots. The map $(d,\ X_\rho,\ Y_\rho) \mapsto (X,\ Y,\ g_2)$, given by (10.1.7) and (10.1.8), (10.1.9 induces a map that assigns to the isomorphism class of a QRT root the isomorphism class of the QRT map that is its square. The image of this map corresponds to a codimension-$(8-5=3)$ subvariety of the moduli space of QRT maps.

6.3.4 K3 Surfaces

A *K3 surface* is defined as a compact connected complex two-dimensional complex analytic manifold S such that $\mathrm{q}(S) := \mathrm{H}^1(S,\ \mathcal{O}) = 0$, and the Chern class of the canonical line bundle K_S of S is equal to zero. It follows from Theorem 6.2.23(iii), (iv) that $\mathrm{q}(S) = 0$ if and only if $\mathrm{b}_1(S) = 0$, which is a homological condition. Furthermore, if this is the case, then (2.1.8) implies that the Chern homomorphism $\mathrm{c} : \mathrm{Pic}(S) \to \mathrm{H}^2(M,\ \mathbb{Z})$ is injective, and therefore $\mathrm{c}(\mathrm{K}_S) = 0$ if and only if K_S is trivial if and only if there exists a homolomorphic complex two-form ω without zeros on S. In the next paragraph we collect some immediate consequences of the definition.

Let S be a K3 surface. In view of (6.2.27) this implies that $\mathrm{p_g}(S) := \dim \mathrm{H}^2(S,\ \mathcal{O}) = \dim \mathrm{H}^0(S,\ \Omega^2) = 1$, hence $\chi(S,\ \mathcal{O}) = 2$. If L is a holomorphic line bundle over S, then the triviality of K_S implies that $\mathrm{K}_S \cdot L = 0$; hence (6.2.31) yields $\chi(S,\ \mathcal{O}(L)) = 2+(L\cdot L)/2$. Therefore $L\cdot L$ is even. If $L\cdot L \geq -2$, then (6.2.33) with K_S trivial yields that $\dim \mathrm{H}^0(S,\ \mathcal{O}(L)) + \dim \mathrm{H}^0(S,\ \mathcal{O}(L^{-1})) \geq 2+(L\cdot L)/2 \geq 1$. If L is nontrivial, Lemma 3.3.2 implies that one of the dimensions is zero, hence the other is positive, and replacing L by L^{-1} if necessary we obtain a holomorphic section λ of L, which has zeros because L is nontrivial. Therefore $D := \mathrm{Div}(\lambda)$ is a strictly positive divisor in S, and $[L] = \delta(D)$ in view of Lemma 2.1.2. If L^n is trivial for some $n \in \mathbb{Z}_{>0}$, then $n^2\,(L\cdot L) = L^n \cdot L^n = 0$; hence replacing L by

L^{-1} if necessary, L has a nonzero holomorphic section λ. Because λ^n is a nonzero holomorphic section of the trivial bundle L^n, it has no zeros; therefore λ has no zeros, and L is trivial. This shows that the group $\mathrm{Pic}(S)$, and therefore also its image $\mathrm{c}(\mathrm{Pic}(S))$ in $\mathrm{H}^2(S,\,\mathbb{Z})$ under the injective homomorphism c, has no torsion. If D is any divisor in S then $D\cdot D = \delta(D)\cdot\delta(D)$ is even; in particular, S does not contain any -1 curve. Noether's formula (6.2.32), where $\mathrm{c}(\mathrm{K}_S) = 0$ implies that $\mathrm{K}_S\cdot\mathrm{K}_S = 0$, yields that $2 - 2\,\mathrm{b}_1(S) + \mathrm{b}_2(S) = \chi_{\mathrm{top}}(S) = 12\,\chi(S,\,\mathcal{O}) = 24$, and therefore $\mathrm{b}_2(S) = 22$. Theorem 6.2.23 implies that $\mathrm{H}^0(S,\,\Omega^1) = 0$ and $\mathrm{b}_2^+(S) = 3$. It will turn out that the (co)homology group $\mathrm{H}^2(S,\,\mathbb{Z}) \simeq \mathrm{H}_2(S,\,\mathbb{Z})$ has no torsion, and with respect to the intersection form it is isomorphic to the lattice (6.3.8).

We are especially interested in *elliptic K3 surfaces*, K3 surfaces that at the same time are elliptic surfaces. For the definition of the basic member corresponding to an elliptic fibration, see Definition 6.4.5

Lemma 6.3.14 *Assume that S is a K3 surface that admits a fibration $\varphi : S \to C$. Then $C \simeq \mathbb{P}^1$ and identifying C with $\mathbb{P}^1$, there exist a branched covering $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ and an elliptic fibration $\psi : S \to \mathbb{P}^1$ such that $\varphi = \pi\circ\psi$. The elliptic fibration $\psi : S \to \mathbb{P}^1$ is relatively minimal and has no multiple singular fibers. The basic member corresponding to $\psi : S \to \mathbb{P}^1$ is a K3 surface.*

Proof. The pullback by means of the nonconstant holomorphic map $\varphi : S \to C$ induces an injective linear mapping $\mathrm{H}^0(C,\,\Omega^1) \to \mathrm{H}^0(S,\,\Omega^1) = 0$; hence $\mathrm{g}(C) := \dim \mathrm{H}^0(C,\,\Omega^1) = 0$, that is, $C \simeq \mathbb{P}^1$. The Stein factorization lemma, Lemma 6.1.3, yields a fibration $\psi : S \to \widetilde{C}$ with connected fibers and a branched covering $\pi : \widetilde{C} \to C$ such that $\varphi = \pi\circ\psi$. Therefore, $\widetilde{C} \simeq \mathbb{P}^1$. If A is a smooth fiber of ψ, then the adjunction formula (6.2.5), where Lemma 6.1.2 and the triviality of K_S imply $A\cdot A = \mathrm{K}_S\cdot A = 0$, yields $\mathrm{g}(A) = 1$; hence the fibration $\psi : S \to \mathbb{P}^1$ is elliptic. It is relatively minimal because S does not contain any -1 curve.

Let ω be a holomorphic complex two-form without zeros on S. If S_{c_0} is a fiber of ψ of type ${}_m\,\mathrm{I}_b$, and v is the vector field in Corollary 6.2.47, then $\mathrm{i}_v\,\omega$ is a holomorphic one-form on $\psi^{-1}(C_0)$ with only zeros at the intersection points of the irreducible components of S_{c_0}, and $\mathrm{i}_v\,\omega = \psi^*\alpha$ for a holomorphic one-form α on C_0 without zeros. Corollary 6.2.47(iii) implies that $\mathrm{i}_v\,\omega$ has a zero of order $m-1$ along S_{c_0}; hence $m = 1$. Therefore ψ has no multiple singular fibers.

Let $\beta : B \to C$ be the basic member of the family $\mathcal{F}(J,\,M)$ to which $\varphi : S \to C$ belongs. In the gluing construction in Section 6.4.3 of any member of $\mathcal{F}(J,\,M)$ from the basic member, we can arrange that the intersections $C_i \cap C_j$ of the open covering belong to C^{reg} when the diffeomorphisms Φ_{ij} on $\beta^{-1}(C_i\cap C_j)$ act as translations on the elliptic fibers and therefore preserve any holomorphic complex two-form on $\beta^{-1}(C_i\cap C_j)$. It follows that every member of $\mathcal{F}(J,\,M)$ carries a holomorphic complex area form without zeros if some member has this property. In particular, B carries a holomorphic complex two-form without zeros, K_B is trivial, and $\mathrm{p_g}(B) = 1$. Since the configurations of the singular fibers of $\beta : B \to C$ and $\varphi : S \to C$ are the same, it follows from (6.2.39) that $\chi_{\mathrm{top}}(B) = \chi_{\mathrm{top}}(S)$, in which case Lemma 6.2.30 yields $\chi(B,\,\mathcal{O}) = \chi(S,\,\mathcal{O}) = 2$. Therefore $\mathrm{q}(B) = 1 + \mathrm{p_g}(B) - \chi(B,\,\mathcal{O}) = 0$, and we conclude that B is a K3 surface. □

For nonalgebraic K3 surfaces, the first statement in the next corollary is a straightforward generalization of Kodaira [110, I, Theorem 15]. For projective algebraic surfaces, I learned it from Pjateckiĭ-Šapiro and Šafarevič [160, Corollary 3 on p. 560], where in [160, p. 558] they define a pencil of elliptic curves as an elliptic fibration. I learned the second statement in Corollary 6.3.15 from Oguiso and Shioda [155, Appendix 1]. Their application of the second statement is discussed in Remark 9.2.5 in this book.

Corollary 6.3.15 *Let S be a K3 surface. Then S is an elliptic K3 surface if and only if there is a nontrivial holomorphic line bundle L over S such that $L \cdot L = 0$. There exists an elliptic fibration of S with a holomorphic section if and only there exist holomorphic line bundles L, M over S such that $L \cdot L = M \cdot M = 0$ and $L \cdot M = 1$.*

Proof. If $\varphi : S \to C$ is any fibration, then Lemma 6.1.2 implies that $F \cdot F = 0$ for every fiber F of φ. Lemma 2.1.2 implies that there exists a holomorphic line bundle L over S with a nonzero holomorphic section λ such that $F = \mathrm{Div}(\lambda)$ and $[L] = \delta(F)$. Since F is nonzero, λ has zeros; hence L is not trivial, and $L \cdot L = F \cdot F = 0$.

Now assume conversely that L is a nontrivial holomorphic line bundle over S such that $L \cdot L = 0$, while the second paragraph of Section 6.3.4 yields that $\dim \mathrm{H}^0(S, \mathcal{O}(L)) \geq 2$ or $\dim \mathrm{H}^0(S, \mathcal{O}(L^{-1}) \geq 2$. If S is not projective algebraic, then Theorem 6.2.25 and the first paragraph of its proof imply that there exists an elliptic fibration $\varphi : S \to C$ such that every irreducible curve in S is contained in a fiber of φ. We therefore assume in the sequel that S is projective algebraic, and employ arguments of Pjateckiĭ-Šapiro and Šafarevič [160, p. 574 and §3] in order to prove that S admits a fibration, when Lemma 6.3.14 implies that S admits an elliptic fibration. In view of Lemma 6.1.6, it suffices to find L as in the assumption such that in addition $L \cdot E \geq 0$ for every effective divisor E in S.

The torsion-free groups $\mathrm{Pic}(S) = \delta(\mathrm{Div}(S))$, $\mathrm{c}(\mathrm{Pic}(S))$ a subgroup of $\mathrm{H}^2(S, \mathbb{Z})$, and $\mathrm{NS}(S) := \mathrm{H}(\mathrm{Div}(S))$ subgroup of $\mathrm{H}_2(S, \mathbb{Z})$ are canonically isomorphic; see the second paragraph of this subsection and the first two paragraphs of Section 7.2. Therefore the assumption is equivalent to the existence of a nonzero $x \in \mathrm{NS}(S)$ such that $x \cdot x = 0$. In the sequel we will write $\Lambda := \mathrm{NS}(S)$ and Λ^+ the set of all $x \in \Lambda$ such that $x = \mathrm{H}(X)$ for some effective divisor X in S. Let $h \in \Lambda$ denote the homology class of a hyperplane section H in S. Since the generic H is smooth and connected, it is irreducible, and because S is equal to the union of these H's and $H \cdot H > 0$, we have $h \cdot e > 0$ for every nonzero $e \in \Lambda^+$. The second paragraph of this subsection implies that if $x \in \Lambda$ is nonzero and $x \cdot x \geq -2$, then $x \in \Lambda^+$ or $-x \in \Lambda^+$ when $x \cdot h > 0$ or $-(x \cdot h) = (-x) \cdot h > 0$, respectively, hence $x \in \Lambda^+$ if and only if $x \cdot h > 0$. In particular, if $x \cdot x = 0$ and $x \cdot h = 0$ then $x = 0$, hence the intersection form is nondegenerate on Λ. Therefore $b_2^+(S) = 3$ implies in view of (6.2.28) that Λ is an even lattice of signature $(1, \rho - 1)$ if $\rho := \mathrm{rank}\, \Lambda$. This means that the intersection form is negative definite on the orthogonal complement $\{z \in \Lambda \mid z \cdot h = 0\}$ of h. It follows that if $x \cdot x \geq 0$, $x \cdot h > 0$, $y \cdot y \geq 0$, and $y \cdot h > 0$, then $x \cdot y \geq 0$, with equality if and only if $x \cdot x = y \cdot y = 0$ and x and y are proportional in Λ. Indeed, if $z := (y \cdot h)\, x - (x \cdot h)\, y$ then $z \cdot h = 0$; hence $h \cdot h > 0$ implies that

$0 \geq z\cdot z = (y\cdot h)^2\,(x\cdot x) - 2\,(y\cdot h)\,(x\cdot h)\,(x\cdot y) + (x\cdot h)^2\,(y\cdot y) \geq -2\,(y\cdot h)\,(x\cdot h)\,(x\cdot y)$, and hence $x \cdot y \geq 0$, with equality if and only if $x \cdot x = y \cdot y = 0$ and $z = 0$.

Let $\mathrm{P} := \{p \in \Lambda \mid p \cdot p = -2\}$. For any $p \in P$, the reflection $r_p : x \mapsto x + (x \cdot p)\, p$ in Λ preserves the intersection form. Note that ${r_p}^{-1} = r_p$. Let $x \cdot x = 0$ and $x \cdot h > 0$. Then $r_p(x) \cdot r_p(x) = x \cdot x = 0$ and $r_p(x) \cdot h = x \cdot r_p(h) > 0$, because $r_p(h) \cdot r_p(h) = h \cdot h > 0$ and $r_p(h) \cdot h = h \cdot h + (h \cdot p)^2 \geq h \cdot h > 0$. Suppose there exists $e \in \Lambda^+$ such that $x \cdot e < 0$. We have $e = \mathrm{H}(E)$ for a divisor E in S. There are finitely many irreducible curves E_j and $m_j \in \mathbb{Z}_{>0}$ such that $E = \sum m_j\, E_j$, where $E_j \cdot H > 0$ for every j. Then $0 > x \cdot e = \sum_j m_j\, x \cdot \mathrm{H}(E_j)$ implies that there exists j such that $x \cdot \mathrm{H}(E_j) < 0$; hence $E_j \cdot E_j < 0$ in view of the previous paragraph with $y = E_j$. It follows from (6.2.8) with $A = E_j$, $\mathrm{vg}(E_j) \in \mathbb{Z}_{\geq 0}$, $E_j \cdot E_j < 0$, and K_S trivial that $E_j \cdot E_j = -2$, when actually $\mathrm{vg}(E_j) = 0$ and E_j is an embedded complex projective line in S. Therefore $p := \mathrm{H}(E_j) \in P \cap \Lambda^+$, $p \cdot h > 0$, and $x \cdot p < 0$; hence $r_p(x) \cdot h = x \cdot h + (x \cdot p)\,(p \cdot h) \leq x \cdot h - 1$ because $(x \cdot p)\,(p \cdot h) \in \mathbb{Z}_{<0}$. By induction on i we find $x_i \in \Lambda$ such that $0 < x_i \cdot h \leq x \cdot h - i$, $x_0 = x$, and for each $i > 0$ there exists $p_i \in P$ such that $x_i = \gamma_{p_i}(x_{i-1})$. The inequalities for $x_i \cdot h$ imply that the inductive procedure has to stop, which, because always $x_i \cdot x_i = 0$ and $x_i \cdot h > 0$, happens when we reach an i such that $x_i \cdot e \geq 0$ for every $e \in \Lambda$. Therefore, if Γ denotes the group of automorphisms of the lattice Λ generated by the reflections r_p, $p \in P$, there exists a $\gamma \in \Gamma$ such that $\gamma(x) = \mathrm{H}(D)$ for a nonzero effective divisor D in S such that $D \cdot D = 0$ and $D \cdot E \geq 0$ for every effective divisor E in S. Lemma 2.1.2 implies the existence of a holomorphic line bundle L over S and a nonzero holomorphic section λ of L such that $[L] = \delta(D)$ and $\mathrm{Div}(\lambda) = D$; hence $L \cdot E = D \cdot E \geq 0$ for every effective divisor E in S.

If E and F are a holomorphic section and a fiber of a fibration of S, respectively, then $E \cdot F = 1$, $F \cdot F = 0$, and (6.2.42) implies that $E \cdot E = \chi(S, \mathcal{O}) = -2$. Then $E' = E + F$ satisfies $E' \cdot F = 1$ and $E' \cdot E' = 0$. Lemma 2.1.2 yields holomorphic line bundles L and M over S such that $[L] = \delta(F)$ and $[M] = \delta(E')$ when $L \cdot L = M \cdot M = 0$ and $L \cdot M = 1$. In order to prove the "if" part of the second statement in Corollary 6.3.15, assume that $x,\ y \in \Lambda$, $x \cdot x = y \cdot y = 0$, and $x \cdot y = 1$, where the latter equation implies that $x \neq 0$ and $y \neq 0$. It follows that $(x + y) \cdot (x + y) = 2 > 0$, and Corollary 6.2.26 implies that S is projective algebraic. It follows from $x \cdot x = y \cdot y = 0$ that $x \in \Lambda^+$ or $-x \in \Lambda^+$, and $y \in \Lambda^+$ or $-y \in \Lambda^+$. If $x' = \epsilon\, x$ and $y' = \eta\, y$ with $\epsilon,\ \eta \in \{-1,\ 1\}$ and $x',\ y' \in \Lambda^+$, then $x' \cdot x' = y' \cdot y' = 0$, $x' \cdot h > 0$, and $y' \cdot h > 0$, hence $\epsilon\,\eta = \epsilon\,\eta(x \cdot y) = x' \cdot y' \geq 0$, and therefore $\epsilon = \eta$. That is, replacing both x and y by their opposites if necessary, we can arrange that $x \cdot x = y \cdot y = 0$, $x \cdot y = 1$, and $x,\ y \in \Lambda^+$. Let γ, D, and L be as in the previous paragraph. Then L is nontrivial because $\gamma(x) \neq 0$ hence $D \neq 0$, $L \cdot L = D \cdot D = \gamma(x) \cdot \gamma(x) = 0$, and $L \cdot E = D \cdot E = \gamma(x) \cdot E \geq 0$ for every effective divisor E. The second paragraph of Section 6.3.4 yields that $\dim \mathrm{H}^0(S,\ \mathcal{O}(L)) \geq 2$. Because $\dim \mathrm{H}^0(S,\ \mathcal{O}(L_{\mathrm{f}})) = 1$, $L_{\mathrm{f}} \cdot L_{\mathrm{f}} = 0$, see Lemma 6.1.6, it also implies that the fixed part L_{f} of L is trivial, that is, $L = L_{\mathrm{m}}$. It follows that the linear system of L contains a fibration $\kappa_V : S \to \mathrm{P}(V)$ for a suitable two-dimensional linear subspace V of $\mathrm{H}^0(S,\ \mathcal{O}(L))$. Because $y \in \mathcal{E}$ we have $\gamma(y) \in \mathcal{E}$, that is, $\gamma(y) = \mathrm{H}(E)$ for an effective divisor E in S. For each fiber F of κ_V we have $\mathrm{H}(F) = \mathrm{H}(D)$, hence

$F \cdot E = D \cdot E = \gamma(x) \cdot \gamma(y) = x \cdot y = 1$, and Lemma 6.1.5 implies the fibration κ_V admits a holomorphic section and has connected fibers, while Lemma 6.1.3 and Lemma 6.3.14 imply that the fibration $\kappa_V : S \to \mathrm{P}(V)$ is elliptic. □

Corollary 6.3.16 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with a holomorphic section. Then the following conditions are equivalent:*

(i) *S is a K3 surface.*
(ii) *$C \simeq \mathbb{P}^1$ and $\mathfrak{f}^* \simeq \mathrm{O}(2)$. That is, S is as in Example 5 with $N = 2$.*
(iii) *$C \simeq \mathbb{P}^1$ and $\varphi : S \to C$ is isomorphic to a Weierstrass model $S \xrightarrow{f} W \xrightarrow{p} \mathbb{P}^1$, in the following way. The complex analytic surface W, with singularities allowed, is defined by (6.3.4), where g_2 and g_3 are holomorphic sections of $\mathrm{O}(8)$ and $\mathrm{O}(12)$ over $\mathbb{P}^1$, defined by homogeneous polynomials in two variables of degree 8 and 12, respectively, such that at every point of $\mathbb{P}^1$ the order the zero of g_2 is < 4 or the order of the zero of g_3 is < 6. Furthermore, $f : S \to W$ is a minimal resolution of singularities and $\varphi = p \circ f$.*

Proof. (i) ⇔ (ii). Every holomorphic line bundle over $\mathbb{P}^1$ of degree $N \in \mathbb{Z}$ is isomorphic to $\mathrm{O}(N)$; see Example 5. Theorem 6.2.18 implies that $\mathrm{K}_S \simeq \varphi^*(\mathrm{K}_C\ \mathfrak{f}^*)$; hence K_S is trivial if and only if $-2 + \deg \mathfrak{f}^* = \deg \mathrm{K}_C + \deg \mathfrak{f}^* = 0$ if and only if the degree $N = \chi(S, \mathcal{O})$ of $\mathfrak{f}^*$ that appears in Example 5 is equal to 2. We have observed at the beginning of Section 6.3.4 that S is a K3 surface if K_S is trivial and $\chi(S, \mathcal{O}) = 2$.

The equivalence between (ii) and (iii) follows from Theorems 6.3.6 and 6.3.10. □

Two compact complex analytic manifolds are said to belong to a complex analytic family if they are isomorphic to fibers of a proper complex analytic submersion from a complex analytic manifold onto a connected complex analytic manifold. The complex analytic manifolds M and M' are called *deformation equivalent* if there exists a sequence M_j, $0 \le j \le n$, such that $M \simeq M_0$, $M' \simeq M_n$, and M_j, M_{j-1} belong to a complex analytic family for each $1 \le j \le n$.

Theorem 6.3.17 *Every deformation of a K3 surface is a K3 surface, and any two K3 surfaces are deformation equivalent.*

Proof. We sketch the proof of Kodaira [110, I, Section 5], which interestingly goes via elliptic K3 surfaces, first nonalgebraic and then with holomorphic sections.

Deformation-equivalent surfaces are real diffeomorphic to each other, and therefore have the same Betti numbers. In a holomorphic family $T \ni t \mapsto S_t$ over a simply connected manifold T the cohomology groups $\mathrm{H}^2(S_t, \mathbb{Z})$ are canonically identified with each other, $t \mapsto \mathrm{c}(\mathrm{K}_{S_t})$ is a continuous mapping from T to a discrete space, hence constant, and it follows that if S is a K3 surface and S' is deformation equivalent to S, then $\mathrm{c}(\mathrm{K}_{S'}) = 0$ and $\mathrm{b}_1(S') = 0$, and therefore the first paragraph of this subsection yields that S' is a K3 surface.

We now describe the proof that all K3 surfaces are deformation equivalent to each other. Let S_0 be a K3 surface. Using the deformation theory of Kodaira, Nirenberg,

and Spencer, Kodaira proved that there exists an arbitrarily small deformation S of S_0 such that the right-hand side of (6.2.28) is a subgroup N of $\mathrm{H}^2(S,\ \mathbb{Z})$ of rank one on which the intersection form vanishes. It follows from the previous paragraph that S is a K3 surface, nonalgebraic in view of Corollary 6.2.26. Because there exists a nontrivial holomorphic over S, for which $L \cdot L = 0$, the second paragraph of the proof of Corollary 6.3.15 yields that S admits an elliptic fibration $\varphi : S \to C$. Lemma 6.3.14 implies that $C \simeq \mathbb{P}^1$, the fibration is relatively minimal, and there are no multiple singular fibers. The basic member $\beta : B \to \mathbb{P}^1$ of the family $\mathcal{F}(J,\ M)$ is as in Corollary 6.3.16. Because $D \cdot D = \delta(D) \cdot \delta(D) = 0$ for every divisor D on S, it follows from the description of the singular fibers in Section 6.2.6 that all the fibers of φ are irreducible, that is, every singular fiber is of type I_1 or II.

Kodaira then proved that the members of $\mathcal{F}(J,\ M)$ are deformation equivalent to each other, and therefore $\varphi : S \to \mathbb{P}^1$ is deformation equivalent to $\beta : B \to \mathbb{P}^1$. Because $\beta : B \to \mathbb{P}^1$ has the same configuration of singular fibers as $\varphi : S \to \mathbb{P}^1$, all the fibers of $\beta : B \to \mathbb{P}^1$ are irreducible, and it follows from Remark 6.3.7 that the surface W in the Weierstrass model is smooth and $B \simeq W$. If in the definition (6.3.4) we add the coefficients of g_2 and g_3 as parameters, then we obtain a complex analytic manifold. Let ψ denote the projection onto the parameter space. Let $\mathcal{P}$ be the set of all parameters p such that the fiber of ψ over p is a smooth elliptic surface without reducible singular fibers. It follows from Table 6.3.2 that the condition that all singular fibers are of type I_1 or II defines a connected open subset $\mathcal{P}$ of the parameter space. Then $\psi : \psi^{-1}(\mathcal{P}) \to \mathcal{P}$ exhibits the elliptic K3 surfaces with a holomorphic section and no reducible fibers as a single deformation family. Since every K3 surface is deformation equivalent to a member of this family, the K3 surfaces form a single deformation equivalence class. □

Corollary 6.3.18 *Every K3 surface is simply connected.*

Proof. Quartic surfaces in $\mathbb{P}^3$ are probably the first examples in the literature of K3 surfaces. Let p be a homogeneous polynomial of degree 4 in four variables such that its zero-set $p = 0$ defines a nonsingular quartic surface S in $\mathbb{P}^3$. Then the complex 4-form $\Theta = (1/p)\ \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ in $\mathbb{C}^4$ is homogeneous of degree zero. If $E = \sum_j x_j\, \partial/\partial x_j$ denotes the Euler vector field, $\mathrm{i}_E\, \Theta$ is a basic complex 3-form for the principal $\mathbb{C}^\times$ bundle $\pi : \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$, that is, there is a unique meromorphic complex 3-form θ on $\mathbb{P}^3$ without zeros and with simple zeros along S. The residue $\omega = \mathrm{Res}_S\, \theta$ of θ on S is a holomorphic complex 2-form on S without zeros; hence K_S is trivial. Every nonsingular complete intersection of hypersurfaces in a projective space is simply connected; see Shafarevich [182, pp. 401, 402]. Therefore S is simply connected, hence $\mathrm{b}_1(S) = 0$, and Theorem 6.2.23(iii) implies that $\mathrm{q}(S) = 0$. Therefore S is a K3 surface. Since any K3 surface is deformation equivalence hence homeomorphic to S, it is simply connected. □

As all K3 surfaces S are homeomorphic, and their (co)homology groups $\mathrm{H}^2(S,\ \mathbb{Z}) \simeq \mathrm{H}_2(S,\ \mathbb{Z})$ are isomorphic, where the isomorphisms preserve the intersection form. Let $a \in \mathrm{H}^2(S,\ \mathbb{Z})$, $n \in \mathbb{Z}_{>0}$, and $n\, a = 0$. If j is the homomorphism $\mathrm{H}^2(S,\ \mathbb{Z}) \to \mathrm{H}^2(S,\ \mathcal{O})$ in the exact sequence (2.1.8), then $n\, j(a) = j(n\, a) = 0$; hence $j(a) = 0$

because a vector space has no torsion. Therefore $a = \mathrm{c}(L)$ for a holomorphic line bundle L over S. We have $n\,(L \cdot L) = n\,\mathrm{c}(L) \cdot \mathrm{c}(L) = (n\,a) \cdot a = 0$, hence $L \cdot L = 0$ because $\mathbb{Z}$ has no torsion. As in the proof of Theorem 6.3.17 we conclude that $\dim \mathrm{H}^0(S,\, \mathcal{O}(L)) + \dim \mathrm{H}^0(S,\, \mathcal{O}(L^{-1}) \geq 2$. Let λ be a nonzero section of L or L^{-1}. Then λ^n is a nonzero section of L^n or L^{-n}, where $\mathrm{c}(L^{\pm n}) = \pm n\,a = 0$; hence $L^{\pm n}$ is trivial because c is injective. Therefore λ^n is a nonzero constant, which implies that λ has no zeros; hence L is trivial and $a = \mathrm{c}(L) = 0$. This argument shows that $\mathrm{H}^2(S,\, \mathbb{Z})$ is torsion-free; hence, in view of Poincaré duality, $\mathrm{H}^2(S,\, \mathbb{Z})$ is a unimodular lattice with respect to the intersection form. The isomorphism class of the $\mathrm{H}^2(S,\, \mathbb{Z})$ for K3 surfaces S is called *the K3 lattice* $\Lambda = \Lambda_{\mathrm{K3}}$.

It is a theorem of Todorov that for any $\Omega \in \Lambda \otimes \mathbb{C}$ such that $\Omega \cdot \Omega = 0$ and $\Omega \cdot \overline{\Omega} > 0$, there exist a K3 surface S, an isomorphism $i : \mathrm{H}^2(S,\, \mathbb{Z}) \to \Lambda$, and a holomorphic two-form ω on S, such that $i([\omega]) = \Omega$, where $[\omega]$ denotes the de Rham cohomology class of ω, viewed as an element of $\mathrm{H}^2(S,\, \mathbb{C}) \simeq \mathrm{H}^2(S,\, \mathbb{Z}) \otimes \mathbb{C}$. This statement is known as the *surjectivity of the period map for K3 surfaces*; see Siu [187] for a simple proof. The statement that the set of $i([\omega])$'s is open is the deformation-theoretic argument used in the proof of Theorem 6.3.17 in order to obtain the arbitrarily small deformation S of S_0 such that the right-hand side of (6.2.28) is the subgroup N with the desired properties. In addition to the surjectivity of the period map one has the *weak Torelli theorem for K3 surfaces*, which states that two K3 surfaces S and S' are isomorphic if there exists a latice isomorphism $\iota : \mathrm{H}^2(S,\, \mathbb{Z}) \to \mathrm{H}^2(S',\, \mathbb{Z})$ that sends $\mathfrak{H}^{2,\,0}(S)$ to $\mathfrak{H}^{2,\,0}(S')$. Here $\mathfrak{H}^{2,\,0}$ is defined as in (6.2.35). One also has a strong Torelli theorem for K3 surfaces; see Barth, Hulek, Peters, and van de Ven [11, Chapter VIII, Section 11]. For the history of both Torelli theorems, see [11, p. 372]

Let $\alpha \in \Lambda$. Because $\mathrm{b}_2^+(S) = 3$, there exists $\Omega \in \Lambda \otimes \mathbb{C}$ such that $\Omega \cdot \Omega = 0$, $\Omega \cdot \overline{\Omega} > 0$, and $\alpha \cdot \Omega = 0$. The surjectivity of the period map yields a K3 surface S, an isomorphism $i : \mathrm{H}^2(S,\, \mathbb{Z}) \to \Lambda$, and a holomorphic two-form ω on S such that $i([\omega]) = \Omega$. If $a := i^{-1}(\alpha)$, then (6.2.28) implies that $a = \mathrm{c}(L)$ for a holomorphic line bundle L. It follows that $\alpha \cdot \alpha = a \cdot a = L \cdot L$ is even, and we conclude that Λ is an even unimodular lattice. According to Serre [181, Theorem 5 on p. 54], every even unimodular lattice of signature $(r^+,\, r^-)$ with $r^+ \leq r^-$ is isomorphic to $m\,\mathrm{U} \oplus n\,\mathrm{E}_8^-$, where $r^+ = m$ and $r^- = m + 8\,n$. Here $\oplus$ means orthogonal direct sum, $k\,A$ denotes the orthogonal direct sum of k copies of a lattice A, $\mathrm{U} = \mathbb{Z}\,p + \mathbb{Z}\,q$ with $p \cdot p = q \cdot q = 0$ and $p \cdot q = 1$, and E_8^- is the root lattice of type E_8 with minus the inner product as the bilinear form, like the lattice Q in Lemma 9.2.3. For our $\mathrm{H}^2(S,\, \mathbb{Z})$ we have $r^+ = \mathrm{b}_2^+(S) = 3$ and $r^- = \mathrm{b}_2^-(S) = 19$, hence $m = 3$, $n = 2$, and therefore

$$\Lambda_{\mathrm{K3}} \simeq 3\,\mathrm{U} \oplus 2\,\mathrm{E}_8^-\,. \tag{6.3.8}$$

Remark 6.3.19. Kodaira [110, I, Theorem 19] states that if S is a compact connected nonsingular complex analytic surface, then S carries a holomorphic complex two-form without zeros if and only if S is a K3 surface, a complex torus as in case (a) in Remark 6.2.27, or an elliptic surface that is a nontrivial principal elliptic curve bundle over an elliptic curve. The third case is discussed in more detail in Section 6.4.4. It is a theorem of Siu [188] that every K3 surface is Kähler, and any constant Hermitian

metric on a complex torus is evidently a Kähler structure. On the other hand, for the third case in [110, I, Theorem 19] the first Betti number b_1 of the surface is equal to three. Because for a compact connected Kähler manifold the odd Betti numbers are even, see for instance Griffiths and Harris [74, p. 117], it follows that the third case in [110, I, Theorem 19] is precisely the class of all compact connected complex surfaces that carry a holomorphic complex two-form without zeros and that are not Kähler. The theorem of Siu [188] implies that for every compact connected complex analytic surface S the first Betti number is even if and only if S is Kähler. Using Kodaira's classification of surfaces, this theorem had been conjectured in Kodaira and Morrow [112, p. 85], with the K3 surfaces as the only hitherto unconfirmed case. Barth, Hulek, Peters, and van de Ven [11, Chapter IV, Theorem 3.1] gave a proof that does not use Kodaira's classification. ⊘

For some more basic facts about K3 surfaces, see Griffiths and Harris [74, pp. 590–594] and Barth, Hulek, Peters, and van de Ven [11, p. 245 and Chapter VIII, Section 2–14]. The research literature on K3 surfaces is extensive, and still growing.

For an example in this book of a K3 elliptic surface with an element of its Mordell–Weil group, see Section 11.9.2.

6.3.5 Existence of Local Models

We have the following converse of Lemma 6.2.38.

Proposition 6.3.20 *Let $p_1(z)$, $p_2(z)$ be multivalued holomorphic functions on a punctured open neighborhood of the origin in $\mathbb{C}$ as in Table 6.2.39, where $a(z)$ and $b(z)$ are holomorphic functions of z on an open neighborhood of $z = 0$, and $a(0) \neq 0$. Then there exists a relative minimal elliptic fibration $\varphi : S \to D$ without multiple singular fibers over an open neighborhood D of the origin in $\mathbb{C}$ and a holomorphic section v without zeros of the Lie algebra bundle $\mathfrak{f}$ over D such that the period lattice of the vector field v on the fiber over $z \in D \setminus \{0\}$ is generated by $p_1(z)$ and $p_2(z)$. Every germ at $z = 0$ of a meromorphic function $J(z)$ and matrix M that satisfy the conditions in Table 6.2.40 are equal to the respective modulus function and monodromy matrix of an elliptic fibration as described above.*

Proof. Despite the fact that we don't have the elliptic fibration S with the singular fiber over 0 of the given type yet, we will indicate the case of the period functions $p_1(z)$, $p_2(z)$ in Table 6.2.39 by the symbol of the Kodaira type in Table 6.2.39.

Although the period functions $p_1(z)$, $p_2(z)$ are multivalued holomorphic functions on $D \setminus \{0\}$, it follows from the fact that the period matrix has integral coefficients that the lattice $P(z) = \mathbb{Z}\, p_1(z) + \mathbb{Z}\, p_2(z)$ is a single-valued holomorphic function of $z \in D \setminus \{0\}$. If we insert the period functions of Table 6.2.39 in (2.3.4) with $P = P(z)$, then an investigation of the asymptotic behavior for $z \to 0$ shows that the single-valued holomorphic functions $g_2 : z \mapsto g_2(P(z))$ and $g_3 : z \mapsto g_3(P(z))$ on $D \setminus \{0\}$ are bounded when $z \to 0$. It therefore follows from Riemann's theorem on

removable singularities that g_2 and g_3 extend to holomorphic functions on D, which we denote by the same letters. The asymptotic analysis actually leads to the orders of the zeros of g_2 and g_3 at $z = 0$ as in Table 6.3.2. This implies that $\operatorname{ord}_0 g_2 < 4$ or $\operatorname{ord}_0 g_3 < 6$.

Let W be as in (6.3.4) with C and L replaced by D and the trivial line bundle over D, respectively, and let $f : S \to W$ be a minimal resolution of singularities of W. Then it follows from Theorem 6.3.10 that $\varphi := p \circ f : S \to D$ is a relatively minimal elliptic fibration with $\sigma := f^{-1} \circ \tau : D \to S$ as a holomorphic section. If v_W denotes the holomorphic vector field on $W \setminus W^*$ induced by the Hamiltonian vector field (2.3.7) on each Weierstrass curve (2.3.5) with $g_2 = g_2(z)$, $g_3 = g_3(z)$, then f^{-1} intertwines v_W with a holomorphic vector field v on $f^{-1}(W \setminus W^*)$ that is tangent to the fibers of φ. Because $\tau(D) \subset W \setminus W^*$, we have $\sigma(D) = f^{-1} \circ \tau(D) \subset f^{-1}(W \setminus W^*)$, and it follows from Theorem 6.2.18 that v corresponds to a holomorphic section over D of the Lie algebra bundle $\mathfrak{f}$ of $\varphi : S \to D$, where v has no zeros in D.

The period lattice of v_z is equal to the period lattice of $(v_W)_z$, which is equal to $P(z)$ according to the arguments following "Now assume conversely ..." in Section 2.3.2. This shows that $p_1(z)$ and $p_2(z)$ form a $\mathbb{Z}$-basis of the period lattice of v_z.

Let $J(z)$ be a germ at $z = 0$ of a meromorphic function. Below we describe holomorphic solutions g_2 and g_3 of the equation $J = {g_2}^3/({g_2}^3 - 27\,{g_3}^2)$ such that g_2, g_3, and $\Delta = {g_2}^3 - 27\,{g_3}^2$ satisfy the conditions in Table 6.3.2. These holomorphic functions $g_2(z)$ and $g_3(z)$ lead to a Weierstrass model of an elliptic surface over an open neighborhood of $z = 0$ with a singular fiber over $z = 0$ of the Kodaira type given in Table 6.3.2. In the description, $u(z)$ denotes a unit, a germ of a holomorphic function such that $u(0) \neq 0$.

Suppose that $J(z)$ has a pole at $z = 0$ of order $b \in \mathbb{Z}_{>0}$ when we rewrite the equation $J = {g_2}^3/({g_2}^3 - 27\,{g_3}^2)$ as ${g_3}^2 = (1 - 1/J)\,{g_2}^3/27$. Then $g_2(z) \equiv 1$, and $g_3(z) = ((1 - 1/J(z))/27)^{1/2}$ are holomorphic solutions leading to type I_b, whereas $g_2(z) = z^2$ and $g_3(z) = z^3\,((1 - 1/J(z))/27)^{1/2}$ lead to type I_b^*. If $J(z)$ has no pole at $z = 0$ and $J(0) \notin \{0, 1\}$, then the above choices lead to types I_0 and I_0^*, respectively.

Next consider the case that $J(0) = 0$ when we rewrite the equation as ${g_2}^3 = 27\,{g_3}^2\,J/(J - 1)$. Let $J(z) = z^{3k}\,u(z)$, where $k \in \mathbb{Z}_{>0}$ and $u(z)$ is a unit. Then $g_3(z) \equiv 1$ and $g_2(z) = 3\,z^k\,(u(z)/(J(z) - 1))^{1/3}$ are holomorphic solutions leading to type I_0, whereas $g_3(z) = z^3$ and $g_2(z) = 3\,z^{2+k}\,(u(z)/(J(z) - 1))^{1/3}$ lead to type I_0^*. Let $J(z) = z^{1+3k}\,u(z)$, where $k \in \mathbb{Z}_{\geq 0}$ and $u(z)$ is a unit. Then $g_3(z) = z$ and $g_2(z) = 3\,z^{1+k}\,(u(z)/(J(z) - 1))^{1/3}$ lead to type II, whereas $g_3(z) = z^4$ and $g_2(z) = 3\,z^{3+k}\,(u(z)/(J(z) - 1))^{1/3}$ lead to type IV*. Let $J(z) = z^{2+3k}\,u(z)$, where $k \in \mathbb{Z}_{\geq 0}$ and $u(z)$ is a unit. Then $g_3(z) = z^2$ and $g_2(z) = 3\,z^{2+k}\,(u(z)/(J(z) - 1))^{1/3}$ lead to type IV, whereas $g_3(z) = z^5$ and $g_2(z) = 3\,z^{4+k}\,(u(z)/(J(z) - 1))^{1/3}$ lead to type II*.

Finally, consider the case that $J(0) = 1$ when we rewrite the equation as ${g_2}^2 = (J - 1)\,{g_2}^3/27\,J$. Let $J(z) = 1 + z^{2k}\,u(z)$, where $k \in \mathbb{Z}_{>0}$ and $u(z)$ is a unit. Then $g_2(z) \equiv 1$ and $g_3(z) = z^k\,(u(z)/27\,J(z))^{1/2}$ are holomorphic solutions leading to

type I_0, whereas $g_2(z) = z^2$ and $g_3(z) = z^{3+k}\,(u(z)/27\,J(z))^{1/2}$ lead to type I_0^*. Let $J(z) = 1 + z^{1+2\,k}\,u(z)$, where $k \in \mathbb{Z}_{\geq 0}$ and $u(z)$ is a unit. Then $g_2(z) = z$ and $g_3(z) = z^{2+k}\,(u(z)/27\,J(z))^{1/2}$ lead to type III, whereas $g_2(z) = z^3$ and $g_3(z) = z^{5+k}\,(u(z)/27\,J(z))^{1/2}$ lead to type III*.

If J is a constant, when $\deg_0 J = \infty$, we write $z^\infty \equiv 0$ in the above formulas.

□

Remark 6.3.21. The existence of the local models in Proposition 6.3.20 was proved by Kodaira in [109, II, pp. 581–602]. In [109, II, pp. 598, 599] the proof for type I_1 is given, as in our proof of Proposition 6.3.20, by reference to a family of Weierstrass curves. The type I_b with $b > 1$ is constructed in [109, II, pp. 599, 600] from type I_1 by means of a suitable b-fold covering, whereas type I_b^* with $b \geq 0$ is constructed in [109, II, pp. 600, 601] from type $\mathrm{I}_{2\,b}$ by dividing out a symmetry group of order two, the inverse of the construction of the twofold covering in the proof of Lemma 6.2.38 for type I_b^*.

For all the other types, the construction in [109, II, pp. 582–596] starts with finding $q(z)$ with $J(z) = \mathcal{J}(q(z))$ in the form $q(z) = p_2(z)/p_1(z)$, with $p_1(z)$, $p_2(z)$ as in Table 6.2.39, with monodromy matrix M as in Table 6.2.40. If m is the order of M, then the substution $z = \zeta^m$ leads to elliptic curves $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\,q(\zeta))$ that define an elliptic fibration $\varphi^\sim : S^\sim \to D^\sim$ without singular fibers over an open neighborhood E of the origin in the complex ζ-plane. On it there is an action of $\mathrm{U}(m) = \{\epsilon \in \mathbb{C} \mid \epsilon^m = 1\} \simeq \mathbb{Z}/m\,\mathbb{Z}$ of automorphisms acting as multiplication by ϵ on ζ, which implies that the fibers over $\zeta \neq 0$ are permuted, but the fiber over $\zeta = 0$ is mapped to itself. If we write $\mathrm{pow}(m) : \zeta \mapsto \zeta^m$ and $D = \mathrm{pow}(m)(D^\sim)$, then there is a unique mapping $\psi : S^\sim/\,\mathrm{U}(m) \to D$ such that $\psi \circ \pi = \mathrm{pow}(m) \circ \varphi^\sim$, where $\pi : S^\sim \to S^\sim/\,\mathrm{U}(m)$ denotes the canonical projection that assigns to every point the $\mathrm{U}(m)$-orbit to which it belongs. Over the complement of the origin in D, ψ is an elliptic fibration with modulus function J and monodromy matrix M.

The action of any finite group of holomorphic transformations is proper. A theorem of H. Cartan [27, Theorem 4 on p. 97] states that if G is a discrete group that acts properly on a complex analytic space W by means of biholomorphic transformations, then the orbit space $X = W/G$ is a complex analytic space, and X is normal if W is normal. It follows that the $\mathrm{U}(m)$ orbit space $X = S^\sim/\,\mathrm{U}(m)$ is a normal complex analytic surface, and it follows from Lemma 6.2.53 that there exists a minimal resolution of singularities $f : S' \to X$ of X. Then $\varphi' := \psi \circ f : S' \to D$ is an elliptic fibration, and successively blowing down -1 curves in fibers of the elliptic fibrations, if these are present, we obtain a modification $g : S' \to S$ and unique mapping $\varphi : S \to D$ such that $\varphi' = \varphi \circ g$, where φ is a relatively minimal elliptic fibration. The mapping $\zeta \mapsto (\zeta, 0)$ induces a holomorphic section $\sigma : D \to S$ of φ, which concludes Kodaira's construction of the local model. In Kodaira [109, II, pp. 582–598], all the resolutions of singularities are constructed explicitly, and no general theory of resolutions of singularities of surfaces is used. It follows from Kodaira loc. cit., that there exists a -1 curve in a singular fiber of φ' in the cases II, III, IV, whereas in all other cases the elliptic fibration $\psi : Y \to D$ is already relatively minimal.

In the cases other than II, III, IV, the resolution of singularities $f : S' \to S^{\sim}/\mathrm{U}(m)$ contracts every irreducible component of the singular fiber, except the one with the highest multiplicity, to a point. On the other hand, in the Weierstrass model every irreducible component is contracted to a point except the one that meets the section, this irreducible component having multiplicity one. Therefore, except for type I_1, the intermediate singular surface $S^{\sim}/\mathrm{U}(m)$ in Kodaira's construction is quite different from the Weierstrass surface W model in our proof of Proposition 6.3.20. ⊘

Remark 6.3.22. If we call two local models equivalent if they are isomorphic after a biholomorphic change of coordinates in the base curve, then prescribing the value and the degree of the modulus function at c_0 in Table 6.2.40 leaves in each case two possibilities for the equivalence class. In these two cases the monodromy matrices have opposite signs and the Euler numbers of the singular fiber differ by 6. For compact elliptic surfaces, see Remark 6.4.6. ⊘

6.3.6 A Fiber System of Lie Groups

The first basic result in this subsection is that if two relatively minimal elliptic fibrations over the same curve have the same modulus function and monodromy representation, and both fibrations have a holomorphic section, then these fibrations are isomorphic. See Theorem 6.3.23 below. It is not assumed that the base curve, and therefore the elliptic surface, is compact. This generalization is essential in some applications.

Theorem 6.3.23 *Let $\varphi : S \to C$ and $\varphi' : S' \to C$ be relatively minimal fibrations over C, with the same modulus function and the same monodromy representation. Let $\sigma : C \to S$ and $\sigma' : C \to S'$ be a holomorphic sections of φ and φ', respectively. Then there exists a complex analytic diffeomorphism α from S onto S' such that $\varphi' \circ \alpha = \varphi$ and $\alpha \circ \sigma = \sigma'$.*

Proof. It follows from Table 6.2.40 that the monodromy matrix around a singular value of the elliptic fibration determines the type of the singular fiber. It follows that $c \in C$ is a singular value of $\varphi : S \to C$ if and only if c is a singular value of φ', and if this is the case, then the singular fiber S_c has the same Kodaira type as the singular fiber S'_c. This allows us to write C^{sing} and C^{reg}, without reference to $\varphi : S \to C$ or $\varphi' : S' \to C$.

Let $\mathfrak{f}$ and $\mathfrak{f}'$ denote the line bundles over C of Lie algebras defined by $\varphi : S \to C$ and $\varphi' : S' \to C$, respectively, as in Section 6.2.7. For each $c \in C^{\mathrm{reg}}$, let P_c and P'_c denote the period lattices in $\mathfrak{f}_c$ and $\mathfrak{f}'_c$, respectively. Choose a base point c_* in C^{reg}. Then the condition that $\varphi : S \to C$ and $\varphi' : S' \to C$ have the same monodromy representation means that there is an isomorphism $\iota_{c_*} : P_{c_*} \to P'_{c_*}$ such that for each loop γ in C^{reg} based at c_*, we have $\iota_{c_*} \circ p_\gamma = p'_\gamma \circ \iota_{c_*}$, where p_γ and p'_γ denote the parallel transports in P and P' along γ, respectively. Using the parallel

transport along any curve in C^{reg} from c_* to $c \in C^{\mathrm{reg}}$, we obtain an isomorphism ι of real two-dimensional vector bundles from $\mathfrak{f}|_{C^{\mathrm{reg}}}$ onto $\mathfrak{f}'|_{C^{\mathrm{reg}}}$ such that $\iota(P_c) = P'_c$ for every $c \in C^{\mathrm{reg}}$.

Let C_0 be an open subset of C^{reg} over which P is trivial, and let v_1 and v_2 be the holomorphic sections of $\mathfrak{f}$ over C_0 such that for each $c \in C_0$, $v_1(c)$ and $v_2(c)$ form the $\mathbb{Z}$-basis of P_c corresponding to the trivialization. Then $v'_1 = \iota(v_1)$ and $v'_2 = \iota(v_2)$ are holomorphic sections of $\mathfrak{f}'$ such that for each $c \in C_0$, $v'_1(c)$ and $v'_2(c)$ form the $\mathbb{Z}$-basis of P'_c. The assumption that the modulus function J of $\varphi : S \to C$ is equal to the modulus function J' of $\varphi' : S' \to C$ implies that

$$\mathcal{J}(v_2(c)/v_1(c)) = J(c) = J'(c) = \mathcal{J}(v'_2(c)/v'_1(c));$$

see Section 2.3.3. Applying an element of $\mathrm{PSL}(2, \mathbb{Z})$ if necessary, we can arrange that $v_2(c)/v_1(c) = v'_2(c)/v'_1(c)$ for every $c \in C_0$, which means that $\iota : \mathfrak{f}_c \to \mathfrak{f}'_c$ is a complex linear mapping for every $c \in C_0$, which moreover depends holomorphically on c because v_1 and v'_1 are holomorphic sections of $\mathfrak{f}$ and $\mathfrak{f}'$, respectively. This shows that $\iota : \mathfrak{f}|_{C^{\mathrm{reg}}} \to \mathfrak{f}'|_{C^{\mathrm{reg}}}$ is an isomorphism of holomorphic complex line bundles over C^{reg}. If v is any local holomorphic section of $\mathfrak{f}$ without zeros, then $p_i = v_i/v$ are the period functions of v, and it follows that the period functions of $\iota \circ v$ are equal to $v'_i/\iota \circ v = \iota \circ v_i/\iota \circ v = v_i/v = p_i$. That is, on each open subset of C^{reg} over which P is trivial, $\iota \circ v$ and v have the same period functions.

Now let $c_0 \in C$, where it is allowed that $c_0 \in C^{\mathrm{sing}}$, that is, S_{c_0} is a singular fiber of φ. Let v and v' be holomorphic sections of $\mathfrak{f}$ and $\mathfrak{f}'$ without zeros over an open neighborhood of C_0 of c_0 in C. By shrinking C_0 if necessary, we can arrange that $C_0 \cap C^{\mathrm{sing}} \subset \{c_0\}$, and that we have a holomorphic local coordinate function $z : C_0 \to \mathbb{C}$ such that $z(c_0) = 0$. Then the period functions of $\iota \circ v$ are equal to the period functions p_i of v, and we obtain from Lemma 6.2.38 that, again shrinking C_0 if necessary, there is a holomorphic function u without zeros on C_0 such that the functions $u\, p_i$ form a $\mathbb{Z}$-basis of period functions of v'. This implies that $u\, v' = \iota \circ v$, which proves that the isomorphism ι from $\mathfrak{f}|_{C^{\mathrm{reg}}}$ onto $\mathfrak{f}'|_{C^{\mathrm{reg}}}$ extends to an isomorphism of holomorphic line bundles from $\mathfrak{f}$ onto $\mathfrak{f}'$, which we again denote by ι.

With the notation $L := \mathfrak{f}^* = \mathfrak{f}^{-1}$ and $L' := (\mathfrak{f}')^* = (\mathfrak{f}')^{-1}$, the isomorphism $\iota : \mathfrak{f} \to \mathfrak{f}'$ induces, for each $k \in \mathbb{Z}$, the isomorphism $\iota^{-k} : L^k \to (L')^k$. Let g_2 and g_3 be the holomorphic sections over C of the respective line bundles L^4 and L^6 defined in in Section 6.3.1, and g'_2 and g'_3 the analogous ones with L and P replaced by L' and P', respectively. Then $\iota(P) = P'$ implies that $g'_2 = \iota^{-4} \circ g_2$ and $g'_3 = \iota^{-6} \circ g_3$. Let $p : W \to C$ and $f : S \to W$ be the Weierstrass model of Theorem 6.3.6. Let $p' : W' \to C$ and $f' : S' \to W'$ be the analogous Weierstrass model with $\varphi : S \to C$ and $\sigma : C \to S$ replaced by $\varphi' : S \to C$ and $\sigma' : C \to S'$, respectively. It follows from (6.3.4) and $g'_2 = \iota^{-4} \circ g_2$, $g'_3 = \iota^{-6} \circ g_3$, that the mapping $(c,\ (x_0,\ x_1,\ x_2)) \mapsto (c,\ (x_0,\ \iota^{-2}(x_1),\ \iota^{-3}(x_2))$ induces a complex analytic diffeomorphism $\beta : W \to W'$ such that $p = p' \circ \beta$ and $\beta \circ \infty = \infty'$, where we recall that $p = f \circ \sigma$, $\varphi = p \circ f$, $\infty = f \circ \sigma$, and the same identities with primes. Because both $f : S \to W$ and $\beta^{-1} \circ f' : S' \to W$ are minimal

resolutions of singularities of W, Lemma 6.2.56 implies that there is a complex analytic diffeomorphism $\alpha : S \to S'$ such that $f = (\beta^{-1} \circ f') \circ \alpha$, or equivalently, $\beta \circ f = f' \circ \alpha$. We have $\varphi' \circ \alpha = p' \circ f' \circ \alpha = p' \circ \beta \circ f = p \circ f = \varphi$. Finally, $f' \circ \sigma' = \infty' = \beta \circ \infty = \beta \circ f \circ \sigma = f' \circ \alpha \circ \sigma$ shows that for each $c \in C$ the elements $\sigma'(c)$ and $\alpha \circ \sigma(c)$ belong to the same fiber of f', and therefore are equal if this fiber lies over a nonsingular point of W', which in turn is the case if the fiber S'_c of φ' is irreducible. Because the set of such c is dense in C and σ' and $\alpha \circ \sigma$ are continuous, it follows that $\sigma' = \alpha \circ \sigma$. □

Remark 6.3.24. The closest statement to Theorem 6.3.23 in Kodaira [109, II] that I could find is the sentence "Λ'_ρ can be extended to a biholomorphic fibre map of $B|_{E_\rho}$ onto $V|_{E_\rho}$" on p. 623. Throughout [109, II], Kodaira assumed that the elliptic surface is embedded in a compact one, whereas in Theorem 6.3.23 no such assumption has been made. ⊘

Definition 6.3.25. Let $\mathrm{Aut}(S)$ denote the group of all complex analytic diffeomorphisms of S, the *automorphisms* of S if S is a complex projective algbraic surface. For any fibration $\varphi : S \to C$, let $\mathrm{Aut}(S)_\varphi$ denote the set of all $\alpha \in \mathrm{Aut}(S)$ that preserve the fibers of φ, that is, satisfy $\varphi \circ \alpha = \varphi$. If $\varphi : S \to C$ is an elliptic fibration, then we denote by $\mathrm{Aut}(S)^+_\varphi$ the set of all $\alpha \in \mathrm{Aut}(S)_\varphi$ that act on each smooth fiber of φ, which is an elliptic curve, as a translation. Here the plus sign refers to the convention to think of the composition in a translation group as an addition. $\mathrm{Aut}(S)_\varphi$ is a normal subgroup of $\mathrm{Aut}(S)$, and $\mathrm{Aut}(S)^+_\varphi$ is a normal subgroup of $\mathrm{Aut}(S)_\varphi$. The group $\mathrm{Aut}(S)^+_\varphi$ will be called the *Mordell–Weil group* of the elliptic surface $\varphi : S \to C$. See Remark 7.1.2 for some comments on the origin of this name.

⊘

The following lemma grew from the analysis of the freedom in the diffeomorphism $\beta : S \to S'$ in Theorem 6.3.23. Note that the equation $\alpha \circ \sigma = \sigma$ means that α leaves the curve $E := \sigma(C)$ pointwise fixed, where E intersects each fiber of φ in exactly one point, at which the fiber is smooth and the intersection is transversal.

Lemma 6.3.26 *Let $\varphi : S \to C$ be an elliptic fibration with modulus function J. For any $m \in \mathbb{Z}_{>0}$, let $\mathrm{U}(m) = \{\lambda \in \mathbb{C} \mid \lambda^m = 1\} \simeq \mathbb{Z}/m\,\mathbb{Z}$ denote the group of all mth roots of unity. In the sequel we respectively take $m = 6$, $m = 4$, or $m = 2$ if $J \equiv 0$, $J \equiv 1$, or otherwise. Then each $\alpha \in \mathrm{Aut}(S)_\varphi$ acts on the Lie algebra bundle $\mathfrak{f}$ of φ as multiplication by a uniquely determined complex number $\lambda(\alpha) \in \mathrm{U}(m)$, and $\lambda : \mathrm{Aut}(S)_\varphi \to \mathrm{U}(m)$ is a homomorphism of groups with kernel equal to the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$.*

Assume that σ is a holomorphic section of φ. Then for each $c \in \mathrm{U}(m)$, there exists a unique $\alpha \in \mathrm{Aut}(S)_\varphi$ such that $\alpha \circ \sigma = \sigma$ and $\lambda(\alpha) = c$. It follows that the homomorphism $\lambda : \mathrm{Aut}(S)_\varphi \to \mathrm{U}(m)$ is surjective, and that

$$\mathrm{Aut}(S)_{\varphi,\sigma} := \{\alpha \in \mathrm{Aut} \mid \varphi \circ \alpha = \varphi \text{ and } \alpha \circ \sigma = \sigma\} \simeq \mathrm{U}(m)$$

is a complementary subgroup to $\ker \lambda = \mathrm{Aut}(S)^+_\varphi$ in $\mathrm{Aut}(S)_\varphi$.

Finally, $\beta : S \to S'$ is another mapping as in Theorem 6.3.23 if and only if $\beta \in \alpha\ \mathrm{Aut}(S)_{\varphi,\sigma}$.

Proof. For each $c \in C$, the element $\alpha \in \mathrm{Aut}(S)_\varphi$ acts on $\mathfrak{f}_c$ as multiplication by λ_c, where $\lambda_c P_c = P_c$ if $c \in C^{\mathrm{reg}}$. It follows from Section 2.3.1 and the paragraph preceding Remark 2.3.3 that $\lambda_c \in \mathrm{U}(m)$. Because $\mathrm{U}(m)$ is finite and λ_c is a holomorphic function of c, it follows that α acts on $\mathfrak{f}$ as multiplication by a constant $\lambda = \lambda(\alpha) \in \mathrm{U}(m)$. The mapping $\lambda : \mathrm{Aut}(S)_\varphi \to \mathrm{U}(m)$ is a homomorphism with kernel equal to $\mathrm{Aut}(S)^+_\varphi$.

Conversely, let σ be a holomorphic section of φ and $\lambda \in \mathrm{U}(m)$. Then multiplication by λ defines an automorphism ι of $\mathfrak{f}$ such that $\iota(P_c) = P_c$ for every $c \in C^{\mathrm{reg}}$. Table 6.2.40 yields that in the cases $J \equiv 0$ and $J \equiv 1$ the monodromy matrices act on the $\mathbb{Z}$-bases 1, q with $q = \mathrm{e}^{2\pi \mathrm{i}/6}$ and $q = \mathrm{e}^{2\pi \mathrm{i}/4}$ as multiplication by a sixth and fourth root of unity, respectively. Therefore ι commutes with the monodromy, which is trivially the case if $\lambda = \pm 1$. Now the proof of Theorem 6.3.23 yields that ι is induced by a unique $\alpha \in \mathrm{Aut}(S)_{\varphi,\sigma}$.

Finally, let $\alpha : S \to S'$ be as in Theorem 6.3.23. Then $\beta : S \to S'$ is a complex analytic diffeomorphism if and only if $\gamma := \alpha^{-1} \circ \beta$ is a complex analytic diffeomorphism of S, $\varphi' \circ \beta = \varphi$ if and only if $\varphi \circ \gamma = \varphi \circ \alpha^{-1} \circ \beta = \varphi' \circ \beta = \varphi$, and $\beta \circ \sigma = \sigma'$ if and only if $\gamma \circ \sigma = \alpha^{-1} \circ \beta \circ \sigma = \alpha^{-1} \circ \sigma' = \sigma$. That is, $\beta : S \to S'$ is another mapping as in Theorem 6.3.23 if and only if $\gamma \in \mathrm{Aut}(S)_{\varphi,\sigma}$. □

If, in the notation of Lemma 6.3.26, $\alpha \in \mathrm{Aut}(S)_{\varphi,\sigma}$ and $\lambda(\alpha) = -1$, that is, α acts on each smooth fiber of φ as an inversion, then α is called the *inversion about the section* $E = \sigma(C)$.

Lemma 6.3.27 *Let σ, $\sigma' : C \to S$ be holomorphic sections of the elliptic fibration $\varphi : S \to C$. Then there is a unique $\alpha \in \mathrm{Aut}(S)^+_\varphi$ such that $\sigma' = \alpha \circ \sigma$.*

Let $c_0 \in C$ and assume that $\sigma(c_0)$ belongs to the same irreducible component Θ of S_{c_0} as $\sigma'(c_0)$. Let v be a nowhere vanishing holomorphic section of $\mathfrak{f}$ over an open neighborhood C_0 of c_0 in C, viewed as a vector field on $\varphi^{-1}(C_0)$ with zeros in $\varphi^{-1}(C_0) \cap S^{\mathrm{sing}}$ as in Theorem 6.2.18. Then, by shrinking C_0 if necessary, there is a holomorphic function $T : C_0 \to \mathbb{C}$ such that

$$\alpha(s) = \mathrm{e}^{T(\varphi(s))\, v}(s) \quad \text{for every} \quad s \in \varphi^{-1}(C_0). \tag{6.3.9}$$

Proof. The first statement follows from Theorem 6.3.23 with $S = S'$, while the uniqueness follows from Lemma 6.3.26.

Because $\Theta \cap S^{\mathrm{reg}}$ is connected, the vector field v has no zeros on $\Theta \cap S^{\mathrm{reg}}$, and the flow of v for complex times leaves $\Theta \cap S^{\mathrm{reg}}$ invariant, it follows that the flow of v defines a transitive action of the additive group $\mathbb{C}$ on $\Theta \cap S^{\mathrm{reg}}$. Because $\sigma(C)$ and $\sigma'(C)$ are contained in S^{reg}, we conclude that there exists a $T_0 \in \mathbb{C}$ such that $\sigma'(c_0) = \mathrm{e}^{T_0\, v}(\sigma(c_0))$. Using the implicit function theorem it follows that, shrinking the open neighborhood C_0 of c if necessary, there is a holomorphic function $T : C_0 \to \mathbb{C}$, such that $T(c_0) = T_0$ and $\sigma'(c) = \mathrm{e}^{T(c)\, v}(\sigma(c))$ for every $c \in C_0$. This in turn implies that for every $c \in C_0$ and $t \in \mathbb{C}$,

$$\alpha(\mathrm{e}^{t\, v}(\sigma(c)) = \mathrm{e}^{t\, v}(\sigma'(c)) = \mathrm{e}^{t\, v}(\mathrm{e}^{T(c)\, v}(\sigma(c))) = \mathrm{e}^{T(c)\, v}(\mathrm{e}^{t\, v}(\sigma(c))),$$

where the first identity follows from the fact that α leaves the vector field v invariant and $\alpha \circ \sigma = \sigma'$. For every $c \in C_0 \cap C^{\mathrm{reg}}$ the flow of v on S_c is transitive, which implies that for every $s \in S_c$ such that $\varphi(s) = c$ we have $s = \mathrm{e}^{t\,v}(\sigma(c))$ for some $t \in \mathbb{C}$. Therefore (6.3.9) holds with C_0 replaced by $C_0 \cap C^{\mathrm{reg}}$. Because $\varphi^{-1}(C_0 \cap C^{\mathrm{reg}})$ is dense in $\varphi^{-1}(C_0)$, a continuity argument yields (6.3.9). □

The following concept was introduced by Kodaira [109, II, Definition 9.1].

Definition 6.3.28. Let Y be a complex analytic manifold. A *complex analytic fiber system of complex Lie groups over* Y is a complex analytic manifold G, together with a complex analytic submersion $\psi : G \to Y$ such that

(i) For each $y \in Y$ the fiber $G_y := \psi^{-1}(\{y\})$ of ψ over y is provided with a group structure, and
(ii) The mapping $(g_1,\ g_2) \mapsto g_1\, {g_2}^{-1}$ is a complex analytic mapping from

$$G \times_\psi G := \{(g_1,\ g_2) \in G \times G \mid \psi(g_1) = \psi(g_2)\}$$

to G.

In other words, a complex analytic fiber system of complex Lie groups is a complex analytic Lie groupoid with source map equal to the target map. See for instance Libermann [121, Section 6] for the definition of differentiable groupoids, where the case that source map = target map is mentioned in point 3) on p. 72.

Because ψ is a submersion, we have for each $y \in Y$ that G_y is a closed complex analytic submanifold of G, and (ii) implies that with this complex structure G_y is a complex analytic Lie group. ⊘

For every $c \in C$ we denote by $\mathfrak{F}_c$ the group of all restrictions to S_c of $\alpha \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$, where C_0 is an open neighborhood of c in C.

Theorem 6.3.29 *The $\mathfrak{F}_c$, $c \in C$, have a unique structure of a complex analytic fiber system $\psi : \mathfrak{F} \to C$ of commutative one-dimensional complex Lie groups such that for each open subset C_0 of C and $\alpha \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ the mapping $\iota_{C_0}(\alpha) : c \mapsto \alpha|_{S_c} \in \mathfrak{F}_c$ is a holomorphic section of ψ. For each open subset C_0 of C, the mapping ι_{C_0} is an isomorphism from $\mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ onto the group $\mathrm{H}^0(C_0,\ \mathcal{O}(\mathfrak{F}))$ of all holomorphic sections of ψ over C_0. For each $c \in C$, the Lie algebra of $\mathfrak{F}_c$ is canonically isomorphic to $\mathfrak{f}_c$.*

The set F of all $(g,\ s) \in \mathfrak{F} \times S$ such that $\psi(g) = \varphi(s)$ is a smooth 3-dimensional complex analytic submanifold of $\mathfrak{F} \times S$, and the mapping $A : (g,\ s) \mapsto g(s)$ is a complex analytic map from F to S. For each $c \in C$, the restriction of A to $\mathfrak{F}_c \times (S_c \cap S^{\mathrm{reg}})$ defines a free and transitive action of $\mathfrak{F}_c$ on $S_c \cap S^{\mathrm{reg}}$, which is equal to the translational action of $\mathfrak{F}_c$ on S_c when $c \in C^{\mathrm{reg}}$. If $g \in \mathfrak{F}_c$ leaves one of the connected components of $S_c \cap S^{\mathrm{reg}}$ invariant, then g belongs to the identity component of $\mathfrak{F}_c$, g leaves every connected component of $S_c \cap S^{\mathrm{reg}}$ invariant, and g acts as the identity on $S_c \cap S^{\mathrm{sing}}$.

Proof. Let $c_0 \in C$. For every $s_0,\ s_0' \in S_{c_0} \cap S^{\mathrm{reg}}$ there exist an open neighborhood C_0 of c_0 in C and holomorphic sections $\sigma,\ \sigma'$ of φ over C_0 such that $\sigma(c_0) = s_0$ and

$\sigma'(c_0) = s_0'$. We therefore obtain from Lemma 6.3.27 with S replaced by $\varphi^{-1}(C_0)$ that there exists and $\alpha \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ such that $\sigma' = \alpha \circ \sigma$, hence $s_0' = g(s_0)$ for the element $g := \alpha|_{S_{c_0}}$ of the group $\mathfrak{F}_{c_0}$. Note that every element g of $\mathfrak{F}_{c_0}$ arises in this way.

Furthermore, if s_0 and s_0' belong to the same irreducible component of S_{c_0}, then it follows from (6.3.9) that g acts on S_{c_0} as $\mathrm{e}^{T_0\, v}$ with $T_0 = T(c_0)$. Because the flow of v leaves every connected component of $S_{c_0} \cap S^{\mathrm{reg}}$ invariant, g leaves every connected component of $S_{c_0} \cap S^{\mathrm{reg}}$ invariant. Also, because the flow of v on S_{c_0} belongs to the identity component $\mathfrak{F}_{c_0}{}^{\mathrm{o}}$ of $\mathfrak{F}_{c_0}$, we have $g \in \mathfrak{F}_{c_0}{}^{\mathrm{o}}$, and because $v = 0$ on S^{sing}, see (6.2.16), it follows that $g = \mathrm{e}^{T\, v}|_{S_{c_0}}$ is equal to the identity on $S_{c_0} \cap S^{\mathrm{sing}}$. If $g(s_0) = s_0$, that is, $s_0' = s_0$, then we can arrange that $T = 0$, hence $g = 1$. We have proved all the statements in Theorem 6.3.29 about the action of $\mathfrak{F}_{c_0}$ on the fiber S_{c_0} over c_0.

For any given $\beta_0 \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$, the elements $\beta \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ near β_0 are of the form $\beta = \beta_0 \circ \alpha = \alpha \circ \beta_0$, where $\alpha = \beta \circ {\beta_0}^{-1}$ leaves $\Theta \cap S^{\mathrm{reg}}$ invariant, and therefore has a representation of the form (6.3.9), where we shrink the open neighborhood C_0 of c in C if necessary. This proves the first two statements in Theorem 6.3.29, where the third statement follows from the fact that v is a holomorphic section of $\mathfrak{f}$. Because ψ is a submersion, F is a smooth codimension-one complex analytic submanifold of $\mathfrak{F} \times S$, and the analyticity of the action $A : F \to S$ follows from (6.3.9). □

Remark 6.3.30. If $\sigma : C \to S$ is a global holomorphic section of φ, then the mapping $\Psi : g \mapsto A(g, \sigma(\psi(g)))$ is a complex analytic diffeomorphism from $\mathfrak{F}$ onto S^{reg} such that $\psi = \varphi \circ \Psi$. On the other hand, if $\varphi : S \to C$ admits a global holomorphic section, then $\varphi : S \to C$ is isomorphic to the basic member $\beta : B \to C$ in Section 6.4.2. Therefore, if φ admits a global holomorphic section, then Theorem 6.3.29 corresponds to Kodaira [109, II, Theorem 9.1], which says that $B^{\#} := B^{\mathrm{reg}}$ has a unique structure of a complex analytic fiber system of abelian groups over C. The theorem of Kodaira [109, II, Theorem 9.1] concludes with a description of the structure of the complex Lie groups $\mathfrak{F}_{c_0}$ when $c_0 \in C^{\mathrm{sing}}$, which we will discuss at the end of this subsection. His table [109, II, Table I after Theorem 9.1] also contains the monodromy matrices and values of the modulus function at c_0, which we have collected in Table 6.2.40. ⊘

Corollary 6.3.31 *Let $c \in C$. Let S_c^{irr} denote the finite set of all irreducible components of S_c, and $S_c^{\mathrm{irr},\,1}$ the subset of all $\Theta \in \mathrm{S}^{\mathrm{irr}}$ such that $\mu_\Theta = 1$. Then the component group $\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}}$ of $\mathfrak{F}_c$ acts on S_c^{irr}, preserving the multiplicities and the intersection diagram. Its action on the subset $S_c^{\mathrm{irr},\,1}$ of all irreducible components of multiplicity one is free and transitive, and therefore the number of connected components of $\mathfrak{F}_c$ is finite, equal to the number $n_c^{(1)}$ of multiplicity-one irreducible components of S_c.*

For each open neighborhood C_0 of c in C, the mapping that assigns to each $\alpha \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ its restriction α_c to S_c is a homomorphism of groups from $\mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ to $\mathfrak{F}_c$. For each $\alpha \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ the action of α on S_c^{irr} is equal to the action of $\alpha_c\, \mathfrak{F}_c{}^{\mathrm{o}} \in \mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}}$ on S_c^{irr}.

Proof. The first statement follows from the fact that each element of $\mathfrak{F}_c$ is equal to the restriction to S_c of a complex analytic diffeomorphism from an open neighborhood of S_c in S to itself.

The other statements follow from Theorem 6.3.29 and the fact that for each $\Theta \in S_c^{\mathrm{irr}}$, we have $\Theta \subset S^{\mathrm{sing}}$ if $\mu_\Theta > 1$, whereas in the other case that $\mu_\Theta = 1$ we have that $\Theta \cap S^{\mathrm{sing}}$ is finite and $\Theta \setminus S^{\mathrm{sing}}$ is a connected component of $S_c \cap S^{\mathrm{reg}}$. Therefore the mapping $\Theta \mapsto \Theta \setminus S^{\mathrm{sing}}$ is bijective from $S_c^{\mathrm{sing},\,1}$ onto the set of all connected components of $S_c \cap S^{\mathrm{reg}}$. □

In the remainder of this subsection we determine, for each Kodaira type of a fiber S_c, the structure of the complex Lie group $\mathfrak{F}_c$, and the way it acts on the set S_c^{irr} of the irreducible components of S_c. It follows from the structure theory of commutative Lie groups that $\mathfrak{F}_c \simeq \mathfrak{F}_c{}^{\mathrm{o}} \times (\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}})$.

Type I_b

If $b = 0$ then S_c is an elliptic curve, and $\mathfrak{F}_c$ is the group of all translations on S_c, isomorphic to $\mathbb{C}/P$, where P is a period lattice in $\mathbb{C}$. As a real Lie group, $\mathfrak{F}_c$ is isomorphic to the real two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

If $b = 1$ then S_c is irreducible with one ordinary double point p. It follows from Corollary 6.3.31 that $\mathfrak{F}_c$ is connected. The normalization $\widetilde{S_c}$ of S_c is a complex projective line with two points $p_1,\ p_2$ lying over p. The connected group $\mathfrak{F}_c$ acts on this complex projective line by automorphisms that leave p_1 and p_2 fixed, and the action on the complement of p_1 and p_2, which is isomorphic to $S_c \cap S^{\mathrm{reg}}$, is free. In an affine coordinate where p_1 and p_2 correspond to ∞ and 0, respectively, these automorphisms are multiplications by nonzero complex numbers. Therefore $\mathfrak{F}_c \simeq \mathbb{C}^\times$, the multiplicative group of all nonzero complex numbers.

If $b > 1$, then S_c is a cycle of b rational curves, each with multiplicity one, intersecting each other successively and transversally. Because the action of $\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}}$ on S_c^{irr} is transitive, there exists $g \in \mathfrak{F}_c$ that maps one of the rational curves to the next one in the cycle. Because g preserves the cyclic intersection diagram, g acts cyclically, and the powers of g define a transitive action of $\mathbb{Z}/b\,\mathbb{Z}$ on S_c^{irr}. Because the action of $\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}}$ on S_c^{irr} is free, it follows that $\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}} \simeq \mathbb{Z}/b\,\mathbb{Z}$. Each $\Theta \in S_c^{\mathrm{irr}}$ is a complex projective line with two intersection points p_1 and p_2 with the adjacent irreducible components. The connected onedimensional Lie group $\mathfrak{F}_c{}^{\mathrm{o}}$ acts by automorphisms on Θ, leaving p_1 and p_2 fixed, and the action on $\Theta \setminus \{p_1,\ p_2\} = \Theta \cap S^{\mathrm{reg}}$ is free. As in the case $b = 1$ we conclude that $\mathfrak{F}_c{}^{\mathrm{o}} \simeq \mathbb{C}^\times$, and therefore $\mathfrak{F}_c \simeq \mathbb{C}^\times \times (\mathbb{Z}/b\,\mathbb{Z})$ when $b \in \mathbb{Z}_{>0}$.

Type I_b^*

S_c consists of $b+5$ rational curves Θ_i, $0 \le i \le b+4$, such that the $b+1$ curves Θ_i, $2 \le i \le b+2$, form a chain of rational curves which have multiplicity 2, and the end curves Θ_2 and Θ_{b+2} are intersected once by the multiplicity-one curves Θ_0, Θ_1 and Θ_{b+3}, Θ_{b+4}, respectively, and there are no other intersections. $\mathfrak{F}_c{}^{\mathrm{o}}$ acts on each multiplicity-one curve Θ by automorphisms, leaving its single intersection point p fixed, and acting freely and transitively on $\Theta \cap S^{\mathrm{reg}} = \Theta \setminus \{p\}$. In an affine coordinate t on Θ such that $t(p) = \infty$, any automorphism g of Θ that leaves p fixed are of the form $t \mapsto u\,t + v$, where $u \in \mathbb{C}^\times$ and $v \in \mathbb{C}$. If $u \ne 1$, then g has the fixed point

$t = v/(1-u)$, and because the action of ${\mathfrak{F}_c}^{\circ}$ on $\Theta \setminus \{p\}$ is free, it follows that ${\mathfrak{F}_c}^{\circ}$ acts on Θ by means of translations $t \mapsto t + v$, $v \in \mathbb{C}$. It follows that ${\mathfrak{F}_c}^{\circ} \simeq \mathbb{C}$, the additive group of all complex numbers.

In order to investigate the action of $\mathfrak{F}_c/{\mathfrak{F}_c}^{\circ}$ on the set of the multiplicity-one components Θ_0, Θ_1, Θ_{b+3}, Θ_{b+4}, we use the beginning of the part of the proof of Theorem 6.3.23 that deals with the Kodaira type I_b^*. There the set of multiplicity-one components corresponds to the subgroup

$$\Gamma = \{(-b\,k_2/4 + k_1/2 + \mathbb{Z},\ k_2/2 + \mathbb{Z}) \mid k_1,\ k_2 \in \mathbb{Z}\}$$

of $(\mathbb{R}/\mathbb{Z})^2$. If b is even, $\Gamma = ((1/2)\,\mathbb{Z}/\mathbb{Z})^2 \simeq (\mathbb{Z}/2\,\mathbb{Z})^2$. If b is odd, then Γ is equal to the additive subgroup of $(\mathbb{R}/\mathbb{Z})^2$ generated by the element $(-b/4 + \mathbb{Z},\ 1/2 + \mathbb{Z})$, and therefore it is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$. This leads to the following conclusions.

If b is even, then $\mathfrak{F}_c \simeq \mathbb{C} \times ((\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z}))$. Each nontrivial element of $\mathfrak{F}_c/{\mathfrak{F}_c}^{\circ} \simeq (\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$, which has order two, permutes the multiplicity-one curves as $(\Theta_0\,\Theta_1)\,(\Theta_{b+3}\,\Theta_{b+4})$, $(\Theta_0\,\Theta_{b+3})\,(\Theta_1\,\Theta_{b+4})$, or $(\Theta_0\,\Theta_{b+4})\,(\Theta_1\,\Theta_{b+3})$. The last two of these are those for which the chain is inverted, whereas for the other two elements of $\mathfrak{F}_c/{\mathfrak{F}_c}^{\circ}$ each multiplicity-two curve is fixed.

If b is odd, then $\mathfrak{F}_c \simeq \mathbb{C} \times (\mathbb{Z}/4\,\mathbb{Z})$. The component group $\mathfrak{F}_c/{\mathfrak{F}_c}^{\circ}$ is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$. A generator of it acts on $S_c^{\mathrm{irr},\,1}$ as $(\Theta_0\,\Theta_{b+3}\,\Theta_1\,\Theta_{b+4})$ or its inverse $(\Theta_0\,\Theta_{b+4}\,\Theta_1\,\Theta_{b+3})$ when the chain of multiplicity-two components is inverted. The other two permutations of $S_c^{\mathrm{irr},\,1}$ are the identity and $(\Theta_0\,\Theta_1)\,(\Theta_{b+3}\,\Theta_{b+4})$ when each of the multiplicity-two components is fixed.

Type II
S_c is an irreducible curve; hence $\mathfrak{F}_c$ is connected. S_c has one singular point p, where S_c has an ordinary cusp, and the desingularization $\widetilde{S_c}$ of S_c is a complex projective line with one point $\widetilde{p}$ over p. The group $\mathfrak{F}_c$ acts on $\widetilde{S_c}$ by means of automorphisms that leave the point $\widetilde{p}$ fixed. Moreover, $\mathfrak{F}_c$ acts freely on $\widetilde{S_c} \setminus \{\widetilde{p}\}$, the part of $\widetilde{S_c}$ that is projected isomorphically onto $S_c \setminus \{p\} = S_c \cap S^{\mathrm{reg}}$. As for type I_b^*, we conclude that $\mathfrak{F}_c \simeq \mathbb{C}$, acting as translations on $\widetilde{S_c} \setminus \{\widetilde{p}\}$.

Type II*
In this case there is only one irreducible component Θ with multiplicity one. The curve Θ is a complex projective line with only one intersection point $\{p\}$, where $\Theta \cap S^{\mathrm{sing}} = \{p\}$. As for type II, we conclude that $\mathfrak{F}_c \simeq \mathbb{C}$, acting by means of translations on $\Theta \setminus \{p\}$.

Type III
S_c is the union of two rational curves Θ_i, $i \in \mathbb{Z}/2\mathbb{Z}$, of multiplicity 1 that intersect each other at one point s, with a second-order contact. We have ${\mathfrak{F}_c}^{\circ} \simeq \mathbb{C}$ acting by means of translations on $\Theta_i \setminus \{s\}$. Because every group of order two is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$, we have $\mathfrak{F}_c \simeq \mathbb{C} \times (\mathbb{Z}/2\,\mathbb{Z})$, and the nontrivial element of $\mathfrak{F}_c/{\mathfrak{F}_c}^{\circ}$ switches the two irreducible components of S_c.

Type III*
Again S_c has two irreducible components of multiplicity one, each with one intersection point. As for type III, we conclude that $\mathfrak{F}_c \simeq \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})$, where ${\mathfrak{F}_c}^{\circ} \simeq \mathbb{C}$

acts on the complement of the intersection point of each multiplicity-one component by means of translations, and the nontrivial element of $\mathfrak{F}_c/\mathfrak{F}_c{}^{\circ}$ switches the two multiplicity-one components of S_c.

Type IV
S_c is the union of three rational curves Θ_i, $i \in \mathbb{Z}/3\mathbb{Z}$, each of multiplicity one, intersecting each other at a single point s and with their tangent spaces in general position. Because $\Theta_i \cap S^{\mathrm{reg}} = \Theta_i \setminus \{s\}$, $\mathfrak{F}_c{}^{\circ}$ acts on it by means of translations, and therefore $\mathfrak{F}_c{}^{\circ} \simeq \mathbb{C}$. Because every group of order three is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$, it follows that $\mathfrak{F}_c \simeq \mathbb{C} \times (\mathbb{Z}/3\,\mathbb{Z})$, and $\mathfrak{F}_c/\mathfrak{F}_c{}^{\circ} \simeq \mathbb{Z}/3\,\mathbb{Z}$ permutes the curves Θ_i cyclically.

Type IV*
Again S_c has three irreducible components of multiplicity one, each with only one intersection point. As for the type IV, we conclude that $\mathfrak{F}_c \simeq \mathbb{C} \times (\mathbb{Z}/3\,\mathbb{Z})$, where $\mathfrak{F}_c{}^{\circ} \simeq \mathbb{C}$ acts on the complement of the intersection point of each multiplicity-one component by means of translations, and $\mathfrak{F}_c/\mathfrak{F}_c{}^{\circ} \simeq \mathbb{Z}/3\,\mathbb{Z}$ cyclically permutes the three multiplicity-one components of S_c.

Remark 6.3.32. Let S_r be a reducible fiber of $\varphi : S \to C$. In each case, the component group $\mathfrak{F}_r/\mathfrak{F}_r{}^{\circ}$ is isomorphic to the weight lattice modulo the root lattice of the root system R of type A_l, D_l, E_l, if the intersection diagram of the irreducible components of S_c is of type $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(1)}$, $\mathrm{E}_l^{(1)}$, respectively. See Section 6.2 for the intersection diagrams. In the next paragraph we describe the isomorphism.

The *root lattice* $Q(R)$ is defined as the additive group generated by R, the group generated by the simple roots α_i, $1 \le i \le l$. The vertices α_i, $0 \le i \le l$, of the extended Dynkin diagram correspond to the irreducible components of S_r, where $\mu_0 = 1$ and $\sum_{i=0}^{l} \mu_i \alpha_i = 0$ if the μ_i denote the multiplicities. For the inner product $(\cdot|\cdot)$ corresponding to minus the intersection form we have $(\alpha_i|\alpha_i) = 2$ for every i, and it follows from Bourbaki [23, Lemma 2 on p. 143] that the coroot lattice $Q(R^{\vee})$ is equal to the root lattice $Q(R)$. Therefore the *weight lattice* $P(R)$ as defined in Bourbaki [23, p. 167] is equal to the set of vectors ϖ such that $(\varpi|\alpha_i) \in \mathbb{Z}$ for every $1 \le i \le n$. The *fundamental weights*, the ϖ_j such that $(\varpi_j|\alpha_i) = \delta_{i,j}$, $1 \le i, j \le l$, form a $\mathbb{Z}$-basis of $P(R)$. The *affine root hyperplanes* are the $\{x \in Q \otimes \mathbb{R} \mid (x|\alpha) = n\}$, where $\alpha \in R$ and $n \in \mathbb{Z}$. The connected components of the complement in $Q \otimes \mathbb{R}$ of all the affine root hyperplanes are called the *alcoves*. Let A be the alcove such that the origin is a vertex of A and A is contained in the Weyl chamber $\{x \in Q \otimes \mathbb{R} \mid (x|\alpha_i) > 0 \ \forall i\}$. The walls of A are those contained in the walls $\{x \in Q \otimes \mathbb{R} \mid (x|\alpha_i) = 0\}$ of the Weyl chamber and one additional wall contained in $\{x \in Q \otimes \mathbb{R} \mid (x|\alpha_0) = 1\}$. In this way the set of all walls of A corresponds bijectively to the set Σ of all vertices of the extended Dynkin diagram, where in our application $\Sigma = S_r^{\mathrm{irr}}$, the set of all irreducible components of S_r. The fundamental weights ϖ_i together with 0 are the vertices of A, where ϖ_i and 0 are the unique vertices of A not contained in the walls $(x|\alpha_i) = 0$ and $(x|\alpha_0) = 1$, respectively, This defines a bijection from Σ onto the set of vertices of A. The *affine Weyl group* W_a, the group generated by the orthogonal reflections in the affine root hyperplanes, acts freely and transitively on the set of all alcoves; see Bourbaki [23,

p. 174]. The mapping that assigns to each element s of W_a its linear part ds is a homomorphism from W_a onto the Weyl group W with kernel equal to the group of translations over elements of $Q(R)$, where the latter group is also denoted by $Q(R)$. See Bourbaki [23, Proposition 1 on p. 73]. A translation over ϖ maps each affine root hyperplane to an affine root hyperplane if and only if $\varpi \in P(R)$. For each $\varpi \in P(R)$ there is a unique $s \in W_a$ such that $s(A + \varpi) = A$, and the mapping $x \mapsto s(x + \varpi)$ permutes the walls and the vertices of A. The linear part d$s \in W$ of s is an automorphism of the root system R that leaves the extended Dynkin diagram invariant and therefore also leaves the set $\Sigma^1 = S_r^{\mathrm{irr},\,1}$ of all α_i with $0 \le i \le l$ and $\mu_i = 1$ invariant. It follows from Bourbaki [23, Corollary on p. 177] that for each $\varpi \in P(R)$ such that $\varpi \notin Q(R)$ there is a unique $1 \le i \le n$ such that $\mu_i = 1$ and $\varpi + Q(R) = \varpi_i + Q(R)$ in $P(R)/Q(R)$, and therefore the above action of $P(R)/Q(R)$ on Σ^1 is free and transitive. It follows from Bourbaki [23, (XII) on pp. 251, 257, 262, 266, and (VIII) on p. 269] that for each of the types A_l, D_l, E_l the action of $P(R)/Q(R)$ on Σ^1 is isomorphic to the action of $\mathfrak{F}_r/\mathfrak{F}_r^\circ$ on the set $S_r^{\mathrm{irr},\,1}$ of all multiplicity-one irreducible components of S_r, as described in the above list. This implies that $P(R)/Q(R) \simeq \mathfrak{F}_r/\mathfrak{F}_r^\circ$, where the statement about the actions on $\Sigma^1 = S_r^{\mathrm{irr},\,1}$ is more precise.

The weight lattice modulo the root lattice is also isomorphic to the center of the simply connected complex Lie group with simple Lie algebra of the corresponding type, which in turn is isomorphic to the fundamental group of the adjoint group. While these Lie groups don't seem to play any role in the theory of elliptic surfaces, the root lattice appears as Q_r^{irr} in Section 7.5. ⊘

6.4 Kodaira's Classification of Elliptic Surfaces

In this section we assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration without multiple singular fibers. Kodaira's classification of such elliptic surfaces in Section 6.4.2 is formulated in terms of the monodromy representation M introduced in Section 6.4.1, and another monodromy representation M_J, which is defined by an arbitrary *nonconstant* meromorphic function J on S, that is, a nonconstant holomorphic mapping $J : C \to \mathbb{P}^1$, which becomes the modulus function of the elliptic fibration. In Section 6.4.1 we will define the monodromy representation M_J, which, in contrast to M, takes its values in $\mathrm{PSL}(2,\,\mathbb{Z}) := \mathrm{SL}(2,\,\mathbb{Z})/\{\pm 1\}$ instead of $\mathrm{SL}(2,\,\mathbb{Z})$.

6.4.1 Monodromy of the Modulus Function

Throughout this subsection, we will use the notations of Section 2.3.3.

Let C_J denote the complement in C of the discrete set $J^{-1}(\{0,\,1,\,\infty\})$. Choose a base point $c_* \in C_J$, and a point $q_* \in H$ such that $J(c_*) = \mathcal{J}(q_*)$; see Proposition 2.3.5. If $\gamma : [0,\,1] \to C_J$ is a continuous real curve in C_J such that

$\gamma(0) = \gamma(1) = c_*$, then, because the modular function defines a complex analytic covering map

$$\mathcal{J} : H \setminus \mathcal{J}^{-1}(\{0,\ 1\}) \to \mathbb{C} \setminus \{0,\ 1\},$$

there exists a unique continuous curve $t \mapsto q(t) : [0,\ 1] \to H$ such that $q(0) = q_*$ and $J(\gamma(t)) = \mathcal{J}(q(t))$ for every $0 \leq 1 \leq t$. Then

$$\mathcal{J}(q(1)) = J(\gamma(1)) = J(\gamma(0)) = \mathcal{J}(q(0)) = \mathcal{J}(q_*),$$

and it follows from Proposition 2.3.5 that there is a unique $M = M_J(\gamma) \in \mathrm{PSL}(2,\ \mathbb{Z})$ such that $q(1) = \Phi_M(q_*)$.

If δ is another loop based at c_*, and $t \mapsto r(t)$ is the unique continuous curve in H such that $r(0) = q_*$ and $J(\delta(t)) \equiv \mathcal{J}(r(t))$, then the curve $\Phi_M \circ r$ satisfies $\Phi_M(r(0)) = \Phi_M(q_*) = q(1)$, and $\mathcal{J}(\Phi_M(r(t))) = \mathcal{J}(r(t)) = J(\delta(t))$ for every $0 \leq t \leq 1$. Therefore the concatenation of the curve q followed by the curve $\Phi \circ r$ is a continuous curve $t \mapsto s(t)$ starting at q_* such that $\mathcal{J}(s(t))$ is equal to the value of J on the concatenation $\gamma\,\delta$ of γ followed by δ. We have

$$\Phi_{M_J(\gamma\,\delta)}(q_*) = s(1) = \Phi_{M_J(\gamma)}(r(1)) = \Phi_{M_J(\gamma)}(\Phi_{M_J(\delta)}(q_*)) = \Phi_{M_J(\gamma)\,M_J(\delta)}(q_*),$$

and therefore $M_J(\gamma\,\delta) = M_J(\gamma)\,M_J(\delta)$.

Because the modular group action is proper and free on $H \setminus \mathcal{J}^{-1}(\{0,\ 1\})$, the element $M_J(\gamma) \in \mathrm{PSL}(2,\ \mathbb{Z})$ depends continuously on γ, and because $\mathrm{PSL}(2,\ \mathbb{Z})$ is discrete it follows that $M_J(\gamma) = M_J([\gamma])$ depends only on the homotopy class $[\gamma] \in \pi_1(C_J,\ c_*)$. The mapping $M_J : [\gamma] \mapsto M_J([\gamma])$ is a homomorphism from the fundamental group $\pi_1(C_J,\ c_*)$ to $\mathrm{PSL}(2,\ \mathbb{Z})$, which is called the *monodronomy representation defined by the meromorphic function J on C*.

Replacing q_* by $\Phi_N(q_*)$ forces the homomorphism M_J has to be composed with the conjugation by N in $\mathrm{PSL}(2,\ \mathbb{Z})$. Therefore, although this is not expressed in the notation, the noncommutativity of $\mathrm{PSL}(2,\ \mathbb{Z})$ makes that in general, the homomorphism M_J will depend on the choice of the point q_* in H.

Lemma 6.4.1 *Let $\varphi : S \to C$ be an elliptic fibration with a nonconstant modulus function $J : C \to \mathbb{P}^1$. With the notation $C_J = C \setminus J^{-1}(\{0,\ 1,\ \infty\})$, choose a base point $c_* \in C' := C_J \cap C^{\mathrm{reg}}$ and a positively oriented $\mathbb{Z}$-basis $v_1,\ v_2$ in the period group P_{c_*}. Let $q_* \in H$ be such that $v_2 = q_*\,v_1$, which implies that $\mathcal{J}(q_*) = J(c_*)$, and let ι denote the homomorphism from $\pi_1(C',\ c_*)$ to $\pi_1(C_J,\ c_*)$, defined by the inclusion of C' in C_J.*

Then the monodromy representation M of the elliptic surface $\varphi : S \to C$ is a lift of the monodromy representation M_J defined by the meromorphic function J on C, in the sense that $\pm(M(\gamma)) = M_J(\iota([\gamma]))$ for every $[\gamma] \in \pi_1(C',\ c_*)$. Here $\pm$ is the projection $M \mapsto \{M,\ -M\}$ from $\mathrm{SL}(2,\ \mathbb{Z})$ onto $\mathrm{PSL}(2,\ \mathbb{Z}) := \mathrm{SL}(2,\ \mathbb{Z})/\{\pm 1\}$.

Proof. Let γ be a loop in C' based at c_*. Let $t \mapsto v_i(t)$, $i = 1,\ 2$, be the unique continuous real curves in the bundle of period groups such that $v_i(0) = v_i$ and $v_i(t) \in P_{\gamma(t)}$ for every $0 \leq t \leq 1$. Then the periods $v_i' := v_i(t)$ are given by (2.3.11), with $M = M([\gamma])$.

Because for every $0 \le t \le 1$, the periods $v_1(t)$ and $v_2(t)$ form a positively oriented $\mathbb{R}$-basis of the complex one-dimensional space $\mathfrak{f}_{\gamma(t)}$, there is a unique $q(t) \in H$ such that $v_2(t) = q(t)\, v_1(t)$. Because $v_1(t)$, $v_2(t)$ depend continuously on t, $t \mapsto q(t)$ is a continuous real curve in H. We have $q(0) = q_*$ and $\mathcal{J}(q(t)) = J(\gamma(t))$ for every $0 \le t \le 1$. Therefore $q(1) = \Phi_{M_J}(q_*)$, where $M_J = M_J(\iota([\gamma]))$.

On the other hand, we have

$$v_2' = v_2(1) = q(1)\, v_1(1) = q(1)\, v_1',$$

which in view of (2.3.11) and (2.3.12) with $M = M([\gamma])$ and $q = q_*$ implies that $q(1) = \Phi_{\pm M([\gamma])}(q_*)$. Because $q_* \in H \setminus J^{-1}(\{0,\ 1\})$ and the modular group action is free on $H \setminus J^{-1}(\{0,\ 1\})$, it now follows from

$$\Phi_{M_J}(q_*) = q(1) = \Phi_{\pm M([\gamma])}(q_*)$$

that $M_J = \pm M([\gamma])$. This completes the proof of the lemma. □

Kodaira [109, II, p. 579] gave the following formula for the number of lifts of the monodromy representation of a nonconstant meromorphic function on a compact Riemann surface.

Lemma 6.4.2 *Let C be any compact connected complex analytic curve, and J any nonconstant meromorphic function J on C. Furthermore, let F be any finite subset of C that contains $J^{-1}(\{0,\ 1,\ \infty\})$. Then $F \neq \emptyset$, the monodromy representation*

$$M_J : \pi_1(C \setminus J^{-1}(\{0,\ 1,\ \infty\}),\ c_*) \to \mathrm{PSL}(2,\ \mathbb{Z})$$

of J has at least one lift to a homomorphism

$$M : \pi_1(C \setminus F,\ c_*) \to \mathrm{SL}(2,\ \mathbb{Z}),$$

and the number of possible lifts is equal to 2^{2g+r-1}, where g is the genus of C and r is equal to the number of elements of F.

Proof. If $F = \emptyset$, then $J^{-1}(\{\infty\}) \subset J^{-1}(\{0,\ 1,\ \infty\}) \subset F = \emptyset$ implies that $J : C \to \mathbb{C}$ is a holomorphic function, hence constant in view of the maximum principle. Because we assumed that J is not constant, it follows that $r = \#(F) > 0$. Let c_i, $1 \le i \le r$, be an enumeration of F. Choose a base point c_* in $C \setminus F \subset C_J := C \setminus \{0,\ 1,\ \infty\}$. If $g = \mathrm{g}(C)$ denotes the genus of C, then the fundamental group $\pi_1(C_J,\ c_*)$ has $2g + r$ generators γ_i, $1 \le i \le 2g$, δ_j, $1 \le j \le r$, with a single relation

$$\gamma_1\, \gamma_2\, \gamma_1^{-1}\, \gamma_2^{-1} \cdots \gamma_{2g-1}\, \gamma_{2g}\, \gamma_{2g-1}^{-1}\, \gamma_{2g}^{-1}\, \delta_1 \cdots \delta_r = 1. \tag{6.4.1}$$

Here the γ_i correspond to a $\mathbb{Z}$-basis of $\mathrm{H}_1(C,\ \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, and $\delta_j = \tau_j\, \lambda_j\, \tau_j^{-1}$, where τ_j is a path from c_* to close to c_j, and λ_j is a small loop going around c_j in the positive direction. See Seifert and Threlfall [178, Kap. 6] for the model of a compact oriented surface that leads to this classical result, where [178, Satz on p. 170] is the statement for $r = 0$. It follows that for any group G a mapping h from this set of

generators to G extends to a homomorphism from $\pi_1(C_J,\, c_*)$ to G if and only if (6.4.1) holds with γ_i and δ_j replaced by $h(\gamma_i)$ and $h(\delta_j)$, respectively.

Because M_J is a homomorphism, (6.4.1) holds with γ_i and δ_j replaced by the elements $M_J(\gamma_i)$ and $M_J(\delta_j)$ of $\mathrm{PSL}(2,\, \mathbb{Z}) = \mathrm{SL}(2,\, \mathbb{Z})/\{\pm 1\}$, respectively. Therefore, if M is any lift of M_J to $\mathrm{SL}(2,\, \mathbb{Z})$ on the generators, that is, $M_J(\gamma_i) = \pm(M(\gamma_i))$, $M_J(\delta_j) = \pm(M(\delta_j))$, then (6.4.1) holds with γ_i and δ_j on the left-hand side replaced by the elements $M(\gamma_i)$ and $M(\delta_j)$ of $\mathrm{SL}(2,\, \mathbb{Z})$, respectively, and the right hand side replaced by $+1$ or -1. The lift M on the generators extends to a homomorphism $\pi_1(C_J,\, c_*) \to \mathrm{SL}(2,\, \mathbb{Z})$ if and only if we have the plus sign on the right-hand side of (6.4.1). Because $r \geq 1$, a change of sign in $M(\delta_r)$ suffices to obtain the plus sign, which proves the existence of a lift. A different choice of the lift on the generators amounts to changing the signs of the elements of $\mathrm{SL}(2,\, \mathbb{Z})$ assigned to the generators. In this operation the left hand side of (6.4.1), with γ_i and δ_j replaced by the respective elements $M(\gamma_i)$ and $M(\delta_j)$, does not change sign if and only if the sign change of $M(\delta_r)$ compensates the product of the sign changes of the $M(\gamma_i)$ and $M(\delta_j)$, $1 \leq j \leq r-1$. We therefore have 2^{2g+r-1} possible lifts. □

6.4.2 The Basic Member

Theorem 6.4.3 below generalizes the existence theorem of Kodaira [109, II, Section 8] to the situation that the base curve C is allowed to be noncompact. If C is compact, then the combination of Lemma 6.4.2 with Theorem 6.4.3 implies that for every nonconstant meromorphic function J on C there are 2^{2g+r-1} nonisomorphic relatively minimal elliptic fibrations $\beta : B \to C$ admitting a holomorphic section such that the modulus function of $\beta : B \to C$ is equal to J. Because every relatively minimal elliptic surface with a holomorphic section is projective algebraic, see Corollary 6.2.28, this shows that these elliptic fibrations constitute a vast province in algebraic geometry.

Theorem 6.4.3 *Let C be a connected complex analytic curve, and J a nonconstant meromorphic function J on C. Furthermore, let D be a discrete subset of C that contains $J^{-1}(\{0,\, 1,\, \infty\})$, and assume that $M : \pi_1(C \setminus D,\, c_*) \to \mathrm{SL}(2,\, \mathbb{Z})$ is a lift of the monodromy representation M_J of the meromorphic function J. Then there exists a relatively minimal elliptic fibration $\beta : B \to C$ with at least one holomorphic section $\gamma : C \to B$ such that J is equal to the modulus function of $\beta : B \to C$, and M is equal to the monodromy representation of $\beta : B \to C$. Moreover, such an elliptic fibration over C is unique up to isomorphisms of elliptic fibrations.*

Proof. The uniqueness in Theorem 6.4.3 follows from Theorem 6.3.23.

We begin with the construction in Kodaira [109, II, pp. 581, 582] of the elliptic fibration over $C \setminus D$ with the desired properties. Let $\pi : U \to C \setminus D$ denote the universal covering of $C \setminus D$, where the fundamental group $\Gamma := \pi_1(C \setminus D,\, c_*)$ acts on U by means of deck transformations $u \mapsto \gamma \cdot u$, $\gamma \in \Gamma$. In this way the curve

$C \setminus D = \pi(U)$ is identified with the orbit space U/Γ of the action of Γ on U by deck transformations.

Because the modular function $\mathcal{J} : H \setminus \mathcal{J}^{-1}(\{0, 1, \infty\}) \to \mathbb{C} \setminus \{0, 1\}$ is a covering, see Section 2.3.3, and U is simply connected, there is a holomorphic mapping $q : U \to H \setminus \mathcal{J}^{-1}(\{0, 1\})$ such that $J \circ \pi = \mathcal{J} \circ q$.

The discrete additive group $\mathbb{Z}^2$ acts properly and freely on $U \times \mathbb{C}$ if $(k_1, k_2) \in \mathbb{Z}^2$ sends $(u, t) \in U \times \mathbb{C}$ to $(u, t + k_1 + k_2\, q(u))$. The projection $(u, t) \mapsto u$ induces a holomorphic mapping $p_1 : (U \times \mathbb{C})/\mathbb{Z}^2 \to U$, whose fiber over $u \in U$ is equal to the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\, q(u))$, which has modulus equal to $\mathcal{J}(q(u)) = J(\pi(u))$.

Let (k_1, k_2), $(l_1, l_2) \in \mathbb{Z}^2$, $\gamma \in \Gamma$, and $k_i = \sum_j M(\gamma)^i_j\, l_j$, that is, $k = M(\gamma)^{\mathrm{t}}\, l$. Then $q(\gamma \cdot u) = (M(\gamma)^1_2 + M(\gamma)^2_2\, q)/(M(\gamma)^1_1 + M(\gamma)^2_1\, q)$ implies that

$$l_1 + l_2\, q(\gamma \cdot u) = f_\gamma(u)\, (k_1 + k_2\, q(u)), \tag{6.4.2}$$

where

$$f_\gamma(u) := 1/(M(\gamma)^1_1 + M(\gamma)^2_1\, q(u)). \tag{6.4.3}$$

If $j = M(\delta)^{\mathrm{t}}\, k = M(\delta)^{\mathrm{t}}\, M(\gamma)^{\mathrm{t}}\, l = (M(\gamma) \circ M(\delta)^{\mathrm{t}}\, l = M(\gamma\, \delta)^{\mathrm{t}}\, l$, then $f_{\gamma\, \delta}(u)(j_1 + j_2\, q(u)) = l_1 + l_2\, q((\gamma\, \delta) \cdot u) = l_1 + l_2\, q(\gamma \cdot (\delta \cdot u)) = f_\gamma(\delta \cdot u)(k_1 + k_2\, q(\delta \cdot u)) = f_\gamma(\delta \cdot u)\, f_\delta(u)\, (j_1 + j_2\, q(u))$ shows that

$$f_{\gamma\, \delta}(u) = f_\gamma(\delta \cdot u)\, f_\delta(u), \quad \gamma,\, \delta \in \Gamma, \quad u \in U. \tag{6.4.4}$$

The formula $(\delta, l) \cdot (\gamma, k) = (\delta \cdot \gamma, k + M(\gamma)^{\mathrm{t}}(l)))$ defines a group structure on $\Gamma \times \mathbb{Z}^2$, with subgroup $\Gamma \times \{0\}$ and normal subgroup $\{1\} \times \mathbb{Z}^2$. That is, the group is a semidirect product $\Gamma \ltimes \mathbb{Z}^2$ of Γ and $\mathbb{Z}^2$. It follows from (6.4.4) that the formula

$$(\gamma, k) \cdot (u, t) = (\gamma \cdot u,\ f_\gamma(u)\, (t + k_1 + k_2\, q(u)), \tag{6.4.5}$$

for $(\gamma, k) \in \Gamma \ltimes \mathbb{Z}^2$ and $(u, t) \in U \times \mathbb{C}$, defines an action of the bigger group $\Gamma \ltimes \mathbb{Z}^2$ on $U \times \mathbb{C}$. This action is proper and free, and because $\mathbb{Z}^2$ is a normal subgroup of $\Gamma \ltimes \mathbb{Z}^2$, it induces a proper and free action of Γ on $(U \times \mathbb{C})/\mathbb{Z}^2$ by complex analytic diffeomorphisms when the orbit space $R := (U \times \mathbb{C})/(\Gamma \ltimes \mathbb{Z}^2)$ is canonically isomorphic to $((U \times \mathbb{C})/\mathbb{Z}^2)/\Gamma$. The projection $(u, t) \to u$ induces a holomorphic mapping $\psi : R \to U/\Gamma \simeq C \setminus F$, where for each $u \in U$ the fiber of ψ over $c = \pi(u) = \Gamma \cdot u$ is isomorphic to the fiber of p_2 over u, which is an elliptic curve with modulus equal to $J(\pi(u)) = J(c)$. It follows that $\psi : R \to C \setminus D$ is an elliptic fibration without singular fibers and with modulus function equal to J. The mapping $u \mapsto (u, 0) : U \to U \times \mathbb{C}$ induces a holomorphic section $o : C \setminus D \to R$.

If $v'_i = \sum_j M^j_i\, v_j$, then $\sum_i l_i\, v'_i = \sum_j k_j\, v_j$, where $k_j = \sum_i l_i\, M^j_i$. It therefore follows from (6.4.2) that the monodromy matrix in $\mathrm{SL}(2, \mathbb{Z})$, defined in Section 6.2.11 as the action of $\gamma \in \Gamma$ on the period lattice, is equal to the matrix $M(\gamma)$. That is, the elliptic fibration $\psi : R \to C \setminus F$ has the prescribed monodromy representation $M : \Gamma \to \mathrm{SL}(2, \mathbb{Z})$.

It follows from Proposition 6.3.20 that there exist an open neighborhood C_0 of D in C, equal to the union of disjoint open disks around the points of D, and a

relatively minimal fibration $\varphi : S \to C_0$ without multiple singular fibers such that the modulus function and the monodromy of $\varphi : S \to C_0$ are equal to $J|_{C_0}$ and given by the $M(\gamma)$, respectively, where the γ are small loops in C_0 around the points in D. The fact that the singular fibers of φ over the points of D are not multiple, hence contain points of S^{reg}, implies that, shrinking C_0 if necessary, there exists a holomorphic section $\sigma : C_0 \to S$ of φ.

Write $C_0' := (C \setminus D) \cap C_0$, $R' := \psi^{-1}(C_0')$, and $S' := \varphi^{-1}(C_0')$. Because $\psi : R' \to C_0'$ and $\varphi : S' \to C_0'$ are relatively minimal elliptic fibrations that have the same modulus function and the same monodromy representation, it follows from Theorem 6.3.23 that there exists a complex analytic diffeomorphism α from R' onto S' such that $\varphi \circ \alpha = \psi$ on R' and $\alpha \circ o = \sigma$ on C_0'. On the disjoint union $R \sqcup S$ of R and S we identify the point $r \in R'$ with the point $\alpha(r) \in S'$. Gluing R and S together in this way, with $\alpha : R' \to S'$ as the gluing map, we obtain a complex analytic surface B, where the inclusions $R \hookrightarrow R \sqcup S$ and $S \hookrightarrow R \sqcup S$ induce complex analytic diffeomorphisms ι_R and ι_S from R and S onto an open subset of B, respectively. Because $\varphi \circ \alpha = \psi$ on R', there is a unique holomorphic mapping $\beta : B \to C$ such that $\beta \circ \iota_R = \psi$ and $\beta \circ \iota_S = \varphi$, and it follows that $\beta : B \to C$ is a relatively minimal elliptic fibration with modulus function equal to J and monodromy representation equal to M.

On the other hand $\alpha \circ o = \sigma$ implies that there is a unique holomorphic mapping $\tau : C \to B$ such that $\tau|_{S \setminus F} = \iota_R \circ o$ and $\tau|_{C_0} = \iota_S \circ \sigma$. It follows that $\beta \circ \tau|_{S \setminus D} = \beta \circ \iota_R \circ o = \psi \circ o$ is equal to the identity on $C \setminus D$ and $\beta \circ \tau|_{C_0} = \beta \circ \iota_S \circ \sigma = \varphi \circ \sigma$ is equal to the identity on C_0. Because C is equal to the union of $C \setminus D$ and C_0, we conclude that $\beta \circ \tau$ is equal to the identity on C, that is, $\tau : C \to B$ is a holomorphic section of $\beta : B \to C$. □

Remark 6.4.4. According to Theorem 6.3.6, the elliptic fibration $\beta : B \to C$ is isomorphic to a Weierstrass model defined by holomorphic sections g_2 and g_3 of L^4 and L^6, respectively, where L is a holomorphic line bundle L. Let $f_\gamma(u)$ be defined by (6.4.3). Then (6.4.4) implies that $(\gamma,\ (u,\ t)) \mapsto (\gamma \cdot u,\ f_\gamma(u)\, t)$ defines an action of γ on the trivial line bundle $U \times \mathbb{C}$ over U, and the proof of Theorem 6.4.3 exhibits L over $C \setminus D$ as the orbit space $U \times_\Gamma \mathbb{C}$. Furthermore, $g_2(u) := g_2(\mathbb{Z} + \mathbb{Z}q(u))$ and $g_3(u) := g_3(\mathbb{Z} + \mathbb{Z}q(u))$ define the holomorphic sections g_2 and g_3 of the respective line bundles L^4 and L^6 over $C \setminus D$. Theorem 6.4.3 can also be proved by showing directly that L, g_2, and g_3 have a holomorphic extension to C, whence Theorem 6.3.10 yields the existence of the desired elliptic fibration. ⊘

Definition 6.4.5. Let C be a connected complex analytic curve, J a nonconstant meromorphic function on C, and M a lift of the monodromy representation of J. Write $\mathcal{F}(J,\ M)$ for the family of all isomorphism classes of relatively minimal elliptic fibrations without multiple singular fibers, with modulus function equal to J and monodromy representation equal to M. Then the isomorphism class of the elliptic fibration $\beta : B \to C$ in Theorem 6.4.3 is called the *basic member* of the family $\mathcal{F}(J,\ M)$. Sometimes the elliptic fibration $\beta : B \to C$ itself is also called the basic member. ⊘

Remark 6.4.6. In every pair of cases in which the modulus functions have the same behavior near c_0, and the lifts of M_J differ by their sign, we read off from Table 6.2.40 that the Euler numbers of the singular fibers differ by ± 6. Because the lifts of a given M_J differ by an even number of sign changes in the monodromies around the point $c_0 \in D$, it follows that for compact elliptic surfaces the sum of the Euler numbers of all the singular fibers always differ by an integral multiple of 12. This agrees with the fact that the sum of the Euler numbers of the singular fibers, which is equal to the Euler number of the elliptic surface S, is equal to $12\,\chi(S,\,\mathcal{O})$, where the integer $\chi(S,\,\mathcal{O})$ is the holomorphic Euler number of S; see Lemma 6.2.30. ⊘

Remark 6.4.7. The singular fibers of Kodaira type I_0^* play a very special role, since these are the only ones that occur over points in C around which the monodromy of J is trivial, but the monodromy itself is nontrivial, hence equal to -1. Moreover, the modulus function J is regular at c_0, and can take any finite complex value at c_0. See the discussion of type I_0^* in the proof of Lemma 6.2.38, and Proposition 6.3.20 for the existence of an elliptic fibration with a singular fiber of type I_0^* and arbitrarily prescribed modulus function on a neighborhood of it.

A singular fiber of Kodaira type I_0^* occurs in the billiard map, as the limit of the billiard trajectories that become tangential to the boundary. See Section 11.2.2. ⊘

6.4.3 The Family $\mathcal{F}(J,\,M)$

Let $\varphi : S \to C$ belong to the family $\mathcal{F}(J,\,M)$. According to the beginning of the proof of Theorem 6.3.23, this implies that there is an isomorphism ι from $\mathfrak{f}|_{C^{\mathrm{reg}}}$ onto $\mathfrak{f}'_{C^{\mathrm{reg}}}$, where $\mathfrak{f}$ and $\mathfrak{f}'$ denote the Lie algebra bundle over C of $\varphi : S \to C$ and $\beta : B \to C$, respectively. For every $c_0 \in C$ there are an open neighborhood C_0 of c_0 in C and a holomorphic section $\sigma_0 : C_0 \to \varphi^{-1}(C_0)$ of φ over C_0. According to Theorem 6.3.23, there is a complex analytic diffeomorphism α_0 from $\varphi^{-1}(C_0)$ onto $\beta^{-1}(C_0)$ such that $\beta \circ \alpha_0 = \varphi$ on $\varphi^{-1}(C_0)$ and $\alpha_0 \circ \sigma_0 = \gamma$ on C_0. The diffeomorphism α_0 induces and isomorphism ι_0 from $\mathfrak{f}|_{C_0}$ onto $\mathfrak{f}'|_{C_0}$ that, because the modulus function J is not constant, is equal to $\pm\,\iota$ over $C_0 \cap C^{\mathrm{reg}}$. According to Lemma 6.3.26 we can choose α_0 such that $\iota_0 = \iota$ over $C_0 \cap C^{\mathrm{reg}}$. It follows that ι has a holomorphic extension to an isomorphism from $\mathfrak{f}$ onto $\mathfrak{f}'$, which we also denote by ι.

Let C_i denote a covering of C with sufficiently small open disks, with sections σ_i of φ over C_i and diffeomorphisms $\alpha_i : \varphi^{-1}(C_i) \to \beta^{-1}(C_i)$ such that $\beta \circ \alpha_i = \varphi$ on $\varphi^{-1}(C_i)$, $\alpha_i \circ \sigma_i = \gamma$ on C_i, and α_i induces the isomorphism ι over C_i. Then $\alpha_{i\,j} := \alpha_i \circ {\alpha_j}^{-1}$ is a complex analytic diffeomorphism of $\beta^{-1}(C_i \cap C_j)$ such that $\beta \circ \alpha_{i\,j} = \beta$ and $\alpha_{i\,j}$ induces the identity on $\mathfrak{f}'$. That is, $\alpha_{i\,j} \in \mathrm{H}^0(C_i \cap C_j,\,\mathcal{O}(\mathfrak{F}))$. The $\alpha_{i\,j}$ obviously satisfy the cocycle condition

$$\alpha_{i\,j} \circ \alpha_{j\,k} = \alpha_{i\,k} \quad \text{on} \quad \beta^{-1}(C_i \cap C_j \cap C_k). \tag{6.4.6}$$

If, conversely, for any open covering $\{C_i\}$ of C we have elements $\alpha_{i\,j} \in \mathcal{O}(\mathfrak{F})_{C_i \cap C_j}$ that satisfy the cocycle condition (6.4.6), then one has the equivalence relation $\sim$ in the disjoint union U of the elliptic surfaces $\beta^{-1}(C_i)$ by writing $b_i \sim b_j$ for $b_i \in \beta^{-1}(C_i)$ and $b_j \in \beta^{-1}(C_j)$ if both points belong to $\beta^{-1}(C_i \cap C_j)$ and $b_i = \alpha_{i\,j}(b_j)$. Then $S := U/\sim$ is a complex analytic surface and the projections $\beta : \beta^{-1}(C_i) \to C_i$ range together to a complex analytic mapping $\varphi : S \to C$ that is a relatively minimal elliptic fibration without multiple singular fibers, with modulus function equal to J and monodromy representation equal to M. This proves the following generalization of Kodaira [109, II, Theorem 10.1] to the case in which noncompact elliptic surfaces are allowed. It follows from Theorem 6.4.11 that Theorem 6.4.8 also holds when the modulus function J is constant.

Theorem 6.4.8 *Let $\mathfrak{F}$ denote the common complex analytic fiber system of Lie groups of the members of $\mathcal{F}(J,\ M)$, where $\mathcal{F} \simeq B^{\mathrm{reg}}$ as in Remark 6.3.30. Let $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}))$ denote the cohomology group of 1-cocycles on C with values in the sheaf $\mathcal{O}(\mathfrak{F})$ of germs of holomorphic sections of $\psi : \mathfrak{F} \to C$. Then the above description defines a free and transitive action of $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}))$ on the family $\mathcal{F}(J,\ M)$.*

Lemma 6.3.27 leads to an identification between the space Σ_{C_0} of all holomorphic sections of β over C_0 and the group $\mathrm{H}^0(C_0,\ \mathcal{O}(\mathfrak{F}))$. In this way the sheaf over C of germs of holomorphic sections of β, which in Kodaira [109, II, Theorem 10.1] is denoted by $\Omega(B^{\#})$, is identified with the sheaf of commutative groups $\mathcal{O}(\mathfrak{F})$. Here $B^{\#} = B^{\mathrm{reg}} := \{b \in B \mid \mathrm{T}_b\beta \neq 0\}$.

Remark 6.4.9. Assume that C hence S is compact. Let $\mathfrak{F}^{\mathrm{o}}$ denote the union over all $c \in C$ of the connected components $\mathfrak{F}_c^{\mathrm{o}}$ of the groups $\mathfrak{F}_c$ containing their identity elements, as in Theorem 6.3.31. Then $\mathfrak{F}^{\mathrm{o}}$ is an open subset of $\mathfrak{F}$, and the restriction to $\mathfrak{F}^{\mathrm{o}}$ of $\psi : \mathfrak{F} \to C$ exhibits $\mathfrak{F}^{\mathrm{o}}$ as a complex analytic fiber system of complex Lie groups over C as in Definition 6.3.28. The sheaf $\mathcal{O}(\mathfrak{F}^{\mathrm{o}})$ of germs of holomorphic sections of $\psi : \mathfrak{F}^{\mathrm{o}} \to C$ is a subsheaf of the sheaf $\mathcal{O}(\mathfrak{F})$. It follows from Theorem 6.3.31 that the quotient sheaf $\mathfrak{Q} := \mathcal{O}(\mathfrak{F})/\mathcal{O}(\mathfrak{F}^{\mathrm{o}})$ is trivial over $C \setminus C^{\mathrm{red}}$, whereas for each $r \in C^{\mathrm{red}}$ the stalk of $\mathfrak{Q}$ over r is equal to the component group $\mathfrak{F}_r/\mathfrak{F}_r^{\mathrm{o}}$, which acts freely and transitively on the set $S_r^{\mathrm{irr},\,1}$ of all multiplicity-one components of the fiber S_r of φ over r. Here C^{red} denotes the finite set of all $r \in C$ such that the fiber S_r is reducible. Because $\mathfrak{Q}$ is a skyscraper sheaf, all cohomology groups $\mathrm{H}^k(C,\ \mathfrak{Q})$ for $k \in \mathbb{Z}_{>0}$ vanish, and the short exact sequence $0 \to \mathcal{O}(\mathfrak{F}^{\mathrm{o}}) \to \mathcal{O}(\mathfrak{F}) \to \mathfrak{Q} \to 0$ induces the long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}})) \to \mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F})) \to \mathrm{H}^0(C,\ \mathfrak{Q}) \\ \to \mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}})) \to \mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F})) \to 0 \end{aligned} \tag{6.4.7}$$

in cohomology. Lemma 7.1.1 implies that $\mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F}))$ is canonically isomorphic to the Mordell–Weil group $\mathrm{Aut}(S)_\varphi^+$, under which isomorphism $\mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$ corresponds to the subgroup $\mathrm{Aut}(S)_\varphi^{+,\,0}$ of all $\alpha \in \mathrm{Aut}(S)_\varphi^+$ that leave every irreducible component of every reducible fiber of φ invariant. The group $\mathrm{H}^0(C,\ \mathfrak{Q})$ is canonically isomorphic to the group $\Gamma := \prod_{r \in C^{\mathrm{red}}} \mathfrak{F}_r/\mathfrak{F}_r^{\mathrm{o}}$, and the third arrow in (6.4.7)

is the homomorphism ϖ in (7.3.2) and Corollary 7.3.3. The sixth arrow in (6.4.7) is surjective, and its kernel is equal to the image of $\mathrm{H}^0(C,\ \Omega)$ in $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$, which in turn is isomorphic to $\Gamma/\varpi(\mathrm{Aut}(S)^+_\varphi)$. In this way the cohomology group $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}))$, which acts freely and transitively on the family $\mathcal{F}(J,\ M)$, is canonically isomorphic to the quotient of $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$ by a subgroup that is canonically isomorphic to $\Gamma/\varpi(\mathrm{Aut}(S)^+_\varphi)$. This is a more detailed version of Kodaira's [109, III, Theorem 11.1], which just states the surjectivity of the sixth arrow in (6.4.7).

The point of the passage to the other cohomology group $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$ is that the exponential mapping $\mathcal{O}(\mathfrak{f}) \to \mathcal{O}(\mathfrak{F})$, which assigns to each local holomorphic section $v : C_0 \to \mathfrak{f}$ of the Lie algebra bundle $\mathfrak{f}$, viewed as a holomorphic vector field on $\varphi^{-1}(C_0)$, the time one flow e^v of v, defines a short exact sequence $0 \to P \to \mathcal{O}(\mathfrak{f}) \to \mathcal{O}(\mathfrak{F}^{\mathrm{o}}) \to 0$, where P denotes the sheaf of period groups P_c, $c \in C$ as discussed in Section 6.2.11. This short exact sequence induces the long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(C,\ P)) \to \mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{f})) \to \mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}})) \\ \to \mathrm{H}^1(C,\ P)) \to \mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{f})) \to \mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}})) \to \mathrm{H}^2(C,\ P) \to 0, \end{aligned} \tag{6.4.8}$$

where (6.2.24) for $(n,\ p,\ q) = (1,\ 2,\ 0)$ implies that $\mathrm{H}^2(C,\ \mathcal{O}(\mathfrak{f})) = 0$. In order to obtain more information about $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$, hence about $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}))$, Kodaira observed in the proof of [109, III, Theorem 11.7] that the cohomology group $\mathrm{H}^2(C,\ P)$ is isomorphic to the homology group $\mathrm{H}_0(C,\ P)$, presumably based on an argument similar to the proof of the Poincaré duality (2.1.17). Kodaira also stated in [107] that $\mathrm{H}_0(C,\ P) \simeq (\mathbb{Z} \times \mathbb{Z})/\mathcal{M}$. Here $c_* \in C^{\mathrm{reg}}$, hence $P_{c_*} \simeq \mathbb{Z}^2$, and $\mathcal{M}$ is the subgroup of $\mathbb{Z}^2$ generated by the elements $M(\gamma)(p) - p$ such that $p \in \mathbb{Z}^2$ and $\gamma \in \pi^1(C^{\mathrm{reg}},\ c_*)$, where $M(\gamma)$ denotes the monodromy matrix along γ as defined in Section 6.2.11. Kodaira [109, III, Theorem 11.7] observed that if the elliptic fibration $\varphi : S \to C$ has singular fibers, then $\mathrm{H}^2(C,\ P) \simeq \mathbb{Z}^2/\mathcal{M}$ is finite, and used this in order to prove his theorem [109, III, Theorem 11.8] that in this case every member of the family $\mathcal{F}(J,\ M)$ is a deformation of a projective algebraic surface.

For what happens with rational elliptic surfaces, see Proposition 9.2.10 and its proof. ⊘

6.4.4 Constant Modulus, Trivial Monodromy, No Singular Fibers

In Lemma 6.4.1, Lemma 6.4.2, and Theorem 6.4.3, the assumption that the modulus function is not constant is essential, because if J is constant, then the monodromy representation of J is not defined. Also in the constructions of Kodaira [109, II, Section 8] it was assumed throughout that the meromorphic function J on C is not constant.

In the following lemma, we choose a base point $c_* \in C^{\mathrm{reg}}$ and a positively oriented $\mathbb{Z}$-basis $(v_1^*,\ v_2^*)$ of the period group P_{c_*} in $\mathfrak{f}_{c_*}$, when $q := v_2^*/v_1^* \in \mathbb{C}$ and $\mathrm{Im}(q) > 0$. Note that $J(c_*) = \mathcal{J}(q)$. Let $M : \pi_1(C^{\mathrm{reg}},\ c_*) \to \mathrm{SL}(2,\ \mathbb{Z})$ be the

monodromy representation of φ defined by $(v_1^*,\ v_2^*)$ as in Section 6.2.11. We use the modular function $\mathcal{J}$ and the action of $\mathrm{SL}(2,\ \mathbb{Z})$ on the complex upper half plane, where the stabilizer subgroup $\mathrm{SL}(2,\ \mathbb{Z})_q$ of q in $\mathrm{SL}(2,\ \mathbb{Z})$ is isomorphic to $\mathbb{Z}/6\,\mathbb{Z}$ if $\mathcal{J}(q) = 0$, to $\mathbb{Z}/4\,\mathbb{Z}$ if $\mathcal{J}(q) = 1$, and to $\mathbb{Z}/2\,\mathbb{Z}$ if $\mathcal{J}(q) \notin \{0,\ 1\}$. See Section 2.3.3 for the basic facts about the modular function and the modular group action, and in particular (2.3.15) and (2.3.16) for the description of $\mathrm{SL}(2,\ \mathbb{Z})_q$ when $\mathcal{J}(q) = 0$ and $\mathcal{J}(q) = 1$, repectively.

Lemma 6.4.10 *Assume that the modulus function $J : C \to \mathbb{P}^1$ of the elliptic fibration $\varphi : S \to C$ is constant. Then $\mathfrak{M} := M(\pi_1(C^{\mathrm{reg}},\ c_*)) \subset \mathrm{SL}(2,\ \mathbb{Z})_q$. For every $c_0 \in C^{\mathrm{reg}}$ there exists an open neighborhood C_0 of c_0 in C^{reg} such that $\varphi : \varphi^{-1}(C_0) \to C_0$ is a complex analytic principal $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\,q)$-bundle. The monodromy group $\mathfrak{M}$ is the obstruction to $\varphi : \varphi^{-1}(C^{\mathrm{reg}}) \to C^{\mathrm{reg}}$ being a principal $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\,q)$-bundle.*

Proof. For any $c \in C^{\mathrm{reg}}$ and any path γ in C^{reg} from c_* to c, the analytic continuation along γ in the bundle P of the oriented basis $(v_*^1,\ v_*^2)$ of P_{c_*} leads to an oriented basis $(v_c^1,\ v_c^2)$ of P_c, where $\mathcal{J}(q(c)) = \mathrm{J}(S_c) = J(c) = J$, if we write $q(c) = v_c^2/v_c^2$. Because locally $q(c)$ depends holomorphically on c and $\mathcal{J}(q(c)) = J$ is constant and $\mathcal{J}$ acts like a branched covering map, it follows that $q(c)$ is constant along γ; hence $q(c) = q$ for all analytic continuations. This means in the notation of (2.3.12) that $\Phi_{M([\gamma])}(q) = q$; hence $M([\gamma]) \in \mathrm{S}(2,\ \mathbb{Z})_q$ for every loop γ based at c_*.

For every $c_0 \in C^{\mathrm{reg}}$ there exists an open neighborhood C_0 in C^{reg} together with a holomorphic section $(v^1,\ v^2)$ of $\mathrm{SF}(P)$ over C_0 that is obtained by analytic continuation from $(v_*^1,\ v_*^2)$. Then v^1 is a holomorphic section of $\mathfrak{f}$ without zeros, which we view as a fiber-tangent holomorphic vector field on $\varphi^{-1}(C_0)$. Because v_c^1 and $q\,v_c^2$ form a $\mathbb{Z}$-basis of P_c, $c \in C_0$, we have, for every $s \in \varphi^{-1}(C_0)$, that $\mathrm{e}^{t\,v^1}(s) = s$ if and only if $t \in \mathbb{Z} + \mathbb{Z}\,p$. Shrinking C_0 if necessary, we also obtain a holomorphic section σ of φ over C_0, when the mapping $(c,\ t) \mapsto \mathrm{e}^{t\,v^1}(\sigma(c))$ induces a complex analytic diffeomorphism from $C_0 \times \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\,q)$ onto $\varphi^{-1}(C_0)$, the inverse of which is a trivialization, exhibiting $\varphi : \varphi^{-1}(C_0) \to C_0$ as a principal $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\,q)$-bundle. □

The following theorem is the analogue of Theorems 6.4.3 and 6.4.8 when the modulus function is a constant, that is, any two nonsingular fibers are complex analytic diffeomorphic.

Theorem 6.4.11 *Let C be a complex analytic curve and $J \in \mathbb{C}$. Choose $q \in H$ such that $\mathcal{J}(q) = J$. Let D be a discrete subset of C, $c_* \in C \setminus D$, and M a homomorphism from $\pi_1(C \setminus D,\ c_*)$ to $\mathrm{SL}(2,\ \mathbb{Z})_q$. Then there exists a relatively minimal elliptic fibration $\beta : B \to C$ with at least one holomorphic section, modulus function equal to the constant J, $C^{\mathrm{sing}} \subset D$, and the monodromy representation equal to M. Such an elliptic fibration over C is unique up to isomorphisms of elliptic fibrations.*

Let $\mathcal{F}(J,\ M)$ denote the family of isomorphism classes of relatively minimal elliptic fibrations over C with constant modulus J, $C^{\mathrm{sing}} \subset F$, and the monodromy representation equal to M. Let $\mathfrak{F}$ denote the common complex analytic fiber system of Lie groups of the members of $\mathcal{F}(J,\ M)$. Then the description in Section 6.4.8

leads to a bijective correspondence between $\mathcal{F}(J,\ M)$ *and the cohomology group* $\mathrm{H}^1(C,\ \mathcal{O}(\mathfrak{F}))$ *of one-cocycles on* C *with values in the sheaf* $\mathcal{O}(\mathfrak{F})$ *of commutative groups over* C*, defined in Theorem* 6.3.29.

Proof. The proofs of Theorems 6.4.3 and 6.4.8 apply also in the situation that J is a constant and $M(\pi_1(C \setminus D,\ c_*)) \subset \mathrm{SL}(2,\ \mathbb{Z})_q$, when the functions $q(u)$ and $f_\gamma(u)$ are equal to the constants q and $f_\gamma = 1/(M(\gamma)_1^1 + M(\gamma)_1^2\, q)$, respectively. □

The following lemma is a supplement to Theorem 6.4.11 in a more precise way than Lemma 6.4.2 supplements Theorem 6.4.3. In it, we discuss relatively minimal elliptic fibrations over a compact Riemann surface C of genus $\mathrm{g}(C)$ and with constant modulus function J. We denote by $\#(T)$ the number of singular fibers of Kodaira type T in the elliptic fibration. By "a combination of Kodaira types can occur" we mean that there exists a relatively minimal elliptic fibration over C with constant modulus function J such that its configuration of singular fibers is the given combination of Kodaira types.

Lemma 6.4.12 *If* $J = 0$ *then there are singular fibers only of Kodaira type* I_0^*, II, II^*, IV, *or* IV^*, *there exists* $k \in \mathbb{Z}_{\geq 0}$ *such that*

$$\#(\mathrm{II}) + 2\,\#(\mathrm{IV}) + 3\,\#(\mathrm{I}_0^*) + 4\,\#(\mathrm{IV}^*) + 5\,\#(\mathrm{II}^*) = 6\,k, \tag{6.4.9}$$

and every combination of types of singular fibers that satisfies these conditions can occur.

If $J = 1$ *then there are singular fibers only of Kodaira type* I_0^*, III, III^*, IV, *or* IV^*, *there exists* $k \in \mathbb{Z}_{\geq 0}$ *such that*

$$\#(\mathrm{III}) + 2\,\#(\mathrm{I}_0^*) + 3\,\#(\mathrm{III}^*) = 4\,k, \tag{6.4.10}$$

and every configuration of singular fibers that satisfies these conditions can occur.

If $J \neq \{0,\ 1\}$ *then there are singular fibers only of Kodaira type* I_0^*, *there exists* $k \in \mathbb{Z}_{\geq 0}$ *such that*

$$\#(\mathrm{I}_0^*) = 2\,k, \tag{6.4.11}$$

and every configuration of singular fibers that satisfies these conditions can occur.

In each of the cases (6.4.9), (6.4.10), *and* (6.4.11), *we have* $k = \chi(S,\ \mathcal{O})$, *the holomorphic Euler number of the elliptic surface* S *with the given configuration of singular fibers. There are* $m^{2\ \mathrm{g}(C)}$ *isomorphism classes of relatively minimal elliptic fibrations with a section and the given configuration of singular fibers, where* $m = 6$, $m = 4$, *and* $m = 2$ *if* $J = 0$, $J = 1$, *and* $J \neq \{0,\ 1\}$, *respectively.*

Proof. We recall from the introduction to this subsection that $\mathrm{SL}(2,\ \mathbb{Z})_q$ is isomorphic to $\mathbb{Z}/6\,\mathbb{Z}$, $\mathbb{Z}/\,4\mathbb{Z}$, or $\mathbb{Z}/2\,\mathbb{Z}$ if $J = 0$, $J = 1$, or $J \neq \{0,\ 1\}$, respectively. The fact that $\mathrm{SL}(2,\ \mathbb{Z})_q$ is commutative implies that in the proof of Lemma 6.4.2 we have no restrictions on the choice of the $M(\gamma_i) \in \mathrm{SL}(2,\ \mathbb{Z})_q$, which proves the last statement in the lemma, with $m = \#(\mathrm{SL}(2,\ \mathbb{Z})_q)$. Because the $M(\delta_j)$ commute, their product is equal to one if and only if the sum of their orders is an integral multiple of the order of $\mathrm{SL}(2,\ \mathbb{Z})_q$, which leads to (6.4.9), (6.4.10), and (6.4.11), where the existence of the

elliptic fibrations follows from Theorem 6.4.11. It follows from (6.2.39) and Lemma 6.2.30 that the number k in (6.4.9), (6.4.10), and (6.4.11) is equal to $\chi(S, \mathcal{O})$. □

The following lemma can be viewed as an addendum to Corollary 6.2.33.

Lemma 6.4.13 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration without multiple singular fibers. If the monodromy is trivial, then φ has no singular fibers, there are nonvanishing holomorphic sections v_1 and v_2 of $\mathfrak{f}$ that at each point form a $\mathbb{Z}$-basis of P, $q := v_2/v_1$ is a holomorphic function on C such that $\operatorname{Im} q > 0$ everywhere, and the modulus function $J = \mathcal{J} \circ q$ is a holomorphic function on C. If S is compact, then the following four statements are equivalent:*

(i) *The monodromy is trivial.*
(ii) *The Lie algebra bundle $\mathfrak{f}$ is trivial.*
(iii) *The basic member is isomorphic to $S = C \times (\mathbb{C}/P)$, where $P = \mathbb{Z} + \mathbb{Z}q$, $q \in \mathbb{C}$, $\operatorname{Im}(q) > 0$, and the fibration is the projection onto the first component.*
(iv) *There exist P as in (vi) and $\eta \in \mathrm{H}^1(C, \mathcal{O}(\mathbb{C}/P))$ such that S is isomorphic to the holomorphic principal $\mathbb{C}/P$-bundle over C defined by η.*

Proof. The first statement follows from inspection of Table 6.2.40, and (ii) follows. If C is compact, then the maximum principle implies that q is constant. Let $\sigma : C \to S$ be a holomorphic section of φ. If we view the holomorphic section v_1 of $\mathfrak{f}$ as a holomorphic vector field on S, then the mapping $(c, t) \mapsto \mathrm{e}^{t\,v_1}(\sigma(c)) : C \times \mathbb{C} \to S$ defines a complex analytic diffeomorphism from $C \times (\mathbb{C}/P)$ onto S, which, moreover, intertwines the projection onto the first factor with φ. This proves (iii), which in turn implies (iv), as follows. The complex analytic fiber system $\mathfrak{F}$ of Lie groups over C is isomorphic to the trivial one $C \times (\mathbb{C}/P)$, with $\mathfrak{F}_c = \{c\} \times (\mathbb{C}/P) \simeq \mathbb{C}/P$ for every $c \in C$. Therefore the description in Section 6.4.8 of the family $\mathcal{F}(J, M)$ with $\mathfrak{F} = C \times (\mathbb{C}/P)$ yields that S is obtained by gluing together the trivial fibrations $C_\alpha \times (\mathbb{C}/P)$ by means of gluing maps $(c, u) \mapsto (c, u + \eta_{\alpha\beta}(c))$, $c \in C_\alpha \cap C_\beta$, where the C_α form an open covering of C and the holomorphic maps $\eta_{\alpha\beta} : C_\alpha \cap C_\beta \to \mathbb{C}/P$ define the cohomology class $\eta \in \mathrm{H}^1(C, \mathcal{O}(\mathbb{C}/P))$. This implies (iv). Since every principal torus bundle has trivial monondromy, (iv) implies (i). Finally, assume that (ii) holds and $\varphi : S \to C$ has a holomorphic section. Then in the Weierstrass model of Theorem 6.3.6 the holomorphic sections g_2 and g_3 are constants such that ${g_2}^3 - 27\,{g_3}^2 \neq 0$, and S is isomorphic to $C \times W_{g_2, g_3}$, where W_{g_2, g_3} is the Weierstrass curve defined by (2.3.6) and φ is the projection onto the first component. This proves (iii). □

Lemma 6.4.13 shows that $\mathrm{H}^1(C, \mathcal{O}(\mathbb{C}/P))$ is the classifying space of the isomorphism classes of elliptic fibrations $\varphi : S \to C$ with trivial monodromy and given constant modulus function J. In this case P is a fixed lattice in $\mathbb{C}$, $\mathfrak{f} \simeq \mathbb{C}$, $\mathfrak{F} = \mathfrak{F}^{\mathrm{o}} \simeq \mathbb{C}/P$, and (6.4.8) yields the exact sequence

$$\mathbb{C} \to \mathrm{H}^0(C, \mathcal{O}(\mathbb{C}/P)) \to \mathrm{H}^1(C, P) \xrightarrow{i} \mathrm{H}^1(C, \mathcal{O}) \xrightarrow{j} \mathrm{H}^1(C, \mathcal{O}(\mathbb{C}/P)) \xrightarrow{\delta} \mathrm{H}^2(C, P) \to 0.$$

Here $\mathrm{H}^0(C,\,\mathcal{O}(\mathbb{C}/P))$ is the space of all holomorphic mappings $\mu : C \to \mathbb{C}/P$. Since all tangent spaces of $\mathbb{C}/P$ are canonically isomorphic to $\mathbb{C}$, the derivative of μ is a holomorphic function on the compact and connected C, hence constant. If $\mathrm{d}\mu = 0$ then μ maps to a point, whereas μ is an unbranched covering map if $\mathrm{d}\mu \neq 0$. The unbranched coverings of $\mathbb{C}/P$ are the elliptic curves $\mathbb{C}/Q$ such that Q is a cofinite subgroup of P. We assume from now that C is not isomorphic to such a $\mathbb{C}/Q$, when the homomorphism $\mathbb{C} \to \mathrm{H}^0(C,\,\mathcal{O}(\mathbb{C}/P)) \simeq \mathbb{C}/P$ is surjective, and therefore the homomorphism i is injective. It follows from (2.1.13) with $L = \mathbb{C}$ and the definition in Section 2.1.5 of the genus g of the compact Riemann surface C that $\dim \mathrm{H}^1(C,\,\mathcal{O}) = g$, whereas the universal coefficient theorem yields $\mathrm{H}^k(C,\,P) \simeq \mathrm{H}^k(C,\mathbb{Z}) \otimes P$. Because $\mathrm{H}^1(C,\,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, $\mathrm{H}^2(C,\,\mathbb{Z}) \simeq \mathbb{Z}$, and $P \simeq \mathbb{Z}^2$, we have $\mathrm{H}^1(C,\,P) \simeq \mathbb{Z}^{4g}$ and $\mathrm{H}^2(C,\,P) \simeq \mathbb{Z}^2$. The coboundary operator δ is surjective from $\mathrm{H}^1(C,\,\mathcal{O}(\mathbb{C}/P))$ onto $\mathrm{H}^2(C,\,P) \simeq P$, with kernel equal to the image of $\mathrm{H}^1(C,\,\mathcal{O}) \simeq \mathbb{C}^g$ under the homomorphism j. Because the kernel of j is equal to the image under i of $\mathrm{H}^1(C,\,P) \simeq \mathbb{Z}^{4g}$, and therefore is countable, the classifying space $\mathrm{H}^1(C,\,\mathcal{O}(\mathbb{C}/P))$ has complex dimension g. The subgroup $i(\mathrm{H}^1(C,\,P)) \simeq \mathbb{Z}^{4g}$ of the real $2g$-dimensional vector space $\mathrm{H}^1(C,\,\mathcal{O})$ cannot be discrete, and therefore $\ker\delta = j(\mathrm{H}^1(C,\,\mathcal{O})) \simeq \mathrm{H}^1(C,\,\mathcal{O})/i(\mathrm{H}^1(C,\,P)))$ is not Hausdorff.

Since $\mathfrak{f}$ is trivial, it follows from Theorem 6.2.18 that $\mathrm{K}_S \simeq \varphi^*(\mathrm{K}_C)$. Therefore the canonical bundle K_S of S is trivial if and only if K_C is trivial if and only if $\mathrm{g}(C) = 1$, that is, C is an elliptic curve as well. Note that the triviality of K_S means that S carries a holomorphic complex area form ω without zeros, which is unique up to a nonzero factor. The principal $\mathbb{C}/P$-bundles over elliptic curves are the second and third cases in Kodaira [110, I, Theorem 19]. The third case, which does not admit a Kähler structure, has been discussed in detail in Example 3.

Viewed as a real four-dimensional manifold with the real or imaginary part of ω as the symplectic form, see Remark 6.2.22, the real two-dimensional torus $\mathbb{C}/P$ acts freely, by means of symplectomorphisms, and its orbits are Lagrange submanifolds of S. For the classification of the somewhat wider class of symplectic torus actions with coisotropic principal orbits, see [49].

Remark 6.4.14. Let $\deg(\mathfrak{f}) = 0$, when Corollary 6.2.33 implies that $\varphi : S \to C$ is a locally trivial holomorphic fiber bundle. Lemma 6.4.13 implies that $\varphi : S \to C$ is a holomorphic principal $\mathbb{C}/P$-bundle if and only if the monodromy is trivial if and only if $\mathfrak{f}$ is trivial. If $g := \mathrm{g}(C) = 0$, then every degree-zero holomorphic line bundle over C is trivial.

Assume that $g > 0$. There are no singular fibers and the modulus function J, a holomorphic function on the compact curve C, is constant. Lemma 6.4.12 therefore implies that there are m^{2g} isomorphism classes of relatively minimal elliptic fibrations with a section, no singular fibers, and modulus function equal to the constant J, where $m = 6$, $m = 4$, and $m = 2$ if $J = 0$, $J = 1$, and $J \notin \{0,\,1\}$, respectively. Since only one of these is (iii) in Lemma 6.4.13, for each of the $m^{2g} - 1$ other cases the monodromy is not trivial, the Lie algebra bundle $\mathfrak{f}$ is not trivial, and the locally trivial holomorphic $\mathbb{C}/P$-bundle is not a holomorphic principal $\mathbb{C}/P$-bundle.

In the Weierstrass model of Theorem 6.3.6 we have $L = \mathfrak{f}^*$, and holomorphic sections g_2 and g_3 of L^4 and L^6, respectively, not both identically zero. If g_2 is

nonzero, then $\deg(\mathrm{Div}(g_2)) = \deg(L^4) = -4\deg(\mathfrak{f}) = 0$ implies that g_2 has no zeros, and therefore L^4 is trivial. Similiarly L^6 is trivial if g_3 is nonzero. If $J \neq 0$ and $J \neq 1$, then both g_2 and g_3 are nonzero; hence $L^2 = L^6\,(L^4)^{-1}$ is trivial. As discussed at the end of Section 2.1.5, the space of isomorphism classes of degree-zero holomorphic line bundles over C is isomorphic to the complex g-dimensional complex torus $T = \mathrm{H}^1(C,\,\mathcal{O})/\,\mathrm{H}^1(S,\,\mathbb{Z})$, where $\mathrm{H}^1(S,\,\mathbb{Z})$ is a rank-$2g$ lattice in $\mathrm{H}^1(C,\,\mathcal{O})$. The elements in T of order m form a subgroup isomorphic to $(\mathbb{Z}/m\,\mathbb{Z})^{2g}$, which has m^{2g} elements. Since this number is equal to the number of isomorphism classes of relatively minimal elliptic fibrations with a section, no singular fibers, and constant modulus J, the conclusion is that the degree-zero Lie algebra bundles $\mathfrak{f}$ of such elliptic fibrations are precisely the holomorphic line bundles F over C such that F^m is trivial, with $m = 6$, $m = 4$, or $m = 2$ if $J = 0$, $J = 1$, or $J \notin \{0,\,1\}$, respectively. ⊘

Chapter 7
Automorphisms of Elliptic Surfaces

In this chapter we will assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration over a *compact* Riemann surface C, that is, a compact connected complex analytic manifold of complex dimension one. Because C is compact and the mapping φ is proper, it follows that S is compact. We also assume that φ has no multiple singular fibers. After Lemma 7.1.1 we will make the stronger assumption that $\varphi : S \to C$ admits at least one section, which then implies that the surface S is projective algebraic; see Corollary 6.2.28. After Lemma 7.2.2 we also assume that $\chi(S, \mathcal{O}) > 0$, which according to Lemmas 6.2.30 and 6.2.29 happens if and only if *the topological Euler number of S is strictly positive*, if and only if $\varphi : S \to C$ *has at least one singular fiber*. If the fibration $\varphi : S \to C$ has no singular fibers, then Corollary 6.2.33 implies that $\varphi : S \to C$ is a locally trivial complex analytic fiber bundle. See also Remark 6.4.14. In this case much can be said, but the conclusions are very different from the quite uniform description when $\chi(S, \mathcal{O}) > 0$. Recall that the natural domain of definition of any QRT transformation is a rational elliptic surface, whence $\chi(S, \mathcal{O}) = 1$, see Lemma 9.1.2(iii).

In Definition 6.3.25 we introduced the group $\mathrm{Aut}(S)$ of all complex analytic diffeomorphisms of S, the group $\mathrm{Aut}(S)_\varphi$ of all $\alpha \in \mathrm{Aut}(S)$ that preserve the fibers of φ, and the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$, of all $\alpha \in \mathrm{Aut}(S)_\varphi$ that act as a translation on each smooth fiber. In Sections 7.1–7.6 we collect a number of facts about the Mordell–Weil group that are known in the literature. The analysis in Section 7.7 of the asymptotic behavior, for $k \to \infty$, of the set of k-periodic fibers, viewed as a subset of C, seems to be new. Chapter 7 is concluded by a description, in Section 7.8, of the Mordell–Weil group in the Weierstrass model and Manin's homomorphism.

7.1 The Mordell–Weil Group and the Set of Sections

Let $\psi : \mathfrak{F} \to C$ denote the complex analytic fiber system of complex Lie groups that extends the system of the groups $\mathfrak{F}_c$ of all translations on the smooth fibers S_c, $c \in C^{\mathrm{reg}}$; see Theorem 6.3.29.

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_7,
© Springer Science+Business Media, LLC 2010

Note that for every $c \in C^{\mathrm{sing}}$ the Lie group $\mathfrak{F}_c$ is not compact, and therefore the mapping ψ is not proper as soon as the fibration $\varphi : S \to C$ has singular points. Let J and M denote the modulus function and the monodromy representation of $\varphi : S \to C$, respectively, and let $\beta : B \to C$ be the basic member of $\mathcal{F}(J, M)$. It follows from Theorem 6.4.8 that the complex analytic fiber system $\psi : \mathfrak{F} \to C$ is isomorphic to the complex analytic fiber system of complex Lie groups of the basic member, which in turn is isomorphic to the restriction of β to the open dense subset B^{reg} of B. It follows that there exists a complex analytic extension of $\psi : \mathfrak{F} \to C$ to an elliptic fibration $\widehat{\psi} : \widehat{\mathfrak{F}} \to C$, where $\widehat{\mathfrak{F}}$ is a compactification of $\mathfrak{F}$, and the isomorphism class of $\widehat{\psi} : \widehat{\mathfrak{F}} \to C$ is equal to the basic member of $\mathcal{F}(J, M)$.

The following lemma follows immediately from Theorem 6.3.29 and Lemma 6.3.27.

Lemma 7.1.1 *The mapping ι_C defined in Theorem* 6.3.29 *is an isomorphism from $\mathrm{Aut}(S)^+_\varphi$ onto the group $\mathrm{H}^0(C, \mathcal{O}(\mathfrak{F}))$ of all holomorphic sections of $\psi : \mathfrak{F} \to C$.*

If the elliptic fibration $\varphi : S \to C$ has a holomorphic section o, then the mapping $g \mapsto A(g, o(\psi(g))) : \mathfrak{F} \to S^{\mathrm{reg}}$ has an extension to an isomorphism ι_o from the elliptic surface $\widehat{\mathfrak{F}}$ onto S, in the sense that ι_o is a complex analytic diffeomorphism from $\widehat{\mathfrak{F}}$ onto S such that $\widehat{\psi} = \varphi \circ \iota_o$. Furthermore, the mapping $\alpha \mapsto \alpha \circ o$, or alternatively $\alpha \mapsto \alpha(o(C))$, is a bijective mapping from the commutative group $\mathrm{Aut}(S)^+_\varphi$ onto the set Σ_φ of all holomorphic sections of φ. The group $\mathrm{Aut}(S)^+_\varphi$ acts freely and transitively on the set Σ_φ of all holomorphic sections of φ.

It follows that if $\varphi : S \to C$ and $\varphi' : S' \to C$ have the same modulus function and monodromy representation, then their Mordell–Weil groups are isomorphic. If $\varphi' : S' \to C$ is an element of the basic member of $\mathcal{F}(J, M)$, then it has a holomorphic section. For this reason it is no restriction of the generality if in the study of the Mordell–Weil groups, we restrict ourselves to relatively minimal elliptic fibrations with at least one section.

Remark 7.1.2. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration over a complex projective curve C, with at least one holomorphic section $o : C \to S$. Let K denote the field of rational functions on C. Using a rational trivialization of the line bundle $L = \mathfrak{f}^*$, the set Σ_φ of all holomorphic sections of φ corresponds bijectively to the set $E(K)$ of all K-points of the Weierstrass curve E, defined by equation (2.3.6) with the constants $g_2, g_3 \in \mathbb{C}$ replaced by elements of K, denoted by the same letters. Under this bijection, the section o corresponds to the point $[0 : 0 : 1]$ at infinity on $E(K)$. See Corollary 7.8.3.

In this way the situation is analogous to the situation in arithmetic algebraic geometry, in which one has an elliptic curve E over a number field K, a finite extension of $\mathbb{Q}$. If $E(K) \neq \emptyset$, then $E(K)$ is a commutative group with one of the elements of $E(K)$ chosen as the zero element, and the group $E(K)$ is called the *Mordell–Weil group of E*.

In view of this analogy, Shioda [184] and Persson [156, p. 7] call the set of all sections Σ_φ of the elliptic fibration $\varphi : S \to C$, which becomes a commutative group after choosing one section o as the zero element, the Mordell–Weil group of S and denote it by $E(K)$. Because the group structure of $\mathrm{Aut}(S)^+_\varphi$ does not depend

on the choice of a section, I prefer to call $\mathrm{Aut}(S)^+_\varphi$ the Mordell–Weil group, instead of calling Σ_φ a group. ⊘

Remark 7.1.3. The Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ is equal to the group $\mathrm{H}^0(C,\ \mathcal{O}(\mathfrak{F}))$ in Theorem 6.3.29. Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and $c \in C$. It then follows from the definition of $\mathfrak{F}_c$ that the restriction α_c of α to the fiber S_c in S over c is an element of the group $\mathfrak{F}_c$. If $c \in C^{\mathrm{reg}}$, then α_c is a translation on the elliptic curve S_c. If $c \in C^{\mathrm{sing}}$, we refer to Section 6.3.6 for a description of the action of an element $g \in \mathfrak{F}_c$ on the singular fiber S_c, which separates into the action on the connected components of $S_c \cap S^{\mathrm{reg}}$ and on the action on the set of singular points of S_c. If $c = r \in C^{\mathrm{red}}$, that is, the fiber S_r is reducible, then the action of the component group $\mathfrak{F}_r/{\mathfrak{F}_r}^{\mathrm{o}}$ of $\mathfrak{F}_r$ on the set S_r^{irr} of irreducible components of S_r is described in Corollary 6.3.31 with $c = r$. ⊘

Remark 7.1.4. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with at least one holomorphic section E. Then there exists an automorphism ι of S such that ι is an involution, that is, ι^2 is equal to the identity, ι preserves the fibers of φ, that is, $\varphi \circ \iota = \varphi$, and ι acts as an inversion on some, hence every, smooth = elliptic fiber. We can, for instance, take ι equal to the inversion about the section E as defined after Lemma 6.3.26.

Because the composition of a translation and an inversion on an elliptic curve is an inversion, we have for every $\alpha \in \mathrm{Aut}(S)^+_\varphi$ that $\iota' := \alpha \circ \iota$ is an inversion on every smooth fiber. Because an inversion on any elliptic curve is an involution, and because the smooth fibers are dense in S, it follows that ι' is an involution on S. Because $\alpha = \iota' \circ \iota$, the conclusion is that *every element of the Mordell–Weil group is reversible*, in the sense that it can be written as the composition of two involutory automorphisms of S that act as inversions on each smooth = elliptic fiber. ⊘

7.2 The Néron–Severi Group

Our elliptic surface S is projective algebraic; see Corollary 6.2.28. The *Néron–Severi group* $\mathrm{NS}(S)$ of S is defined as the group of divisors on S modulo algebraic equivalence; see Griffiths and Harris [74, p. 461]. On a projective algebraic surface two divisors are algebraically equivalent if and only if they are homologous; see Griffiths and Harris [74, p. 462]. In other words, the homomorphism $\mathrm{H} : \mathrm{Div}(S) \to \mathrm{H}_2(S,\ \mathbb{Z})$ that assigns to any divisor D its homology class induces an injective homomorphism from $\mathrm{NS}(S)$ to $\mathrm{H}_2(S,\ \mathbb{Z})$, which allows us to identify $\mathrm{NS}(S)$ with a subgroup of $\mathrm{H}_2(S,\ \mathbb{Z})$. Because $\mathrm{H}_2(S,\ \mathbb{Z})$ is a finitely generated commutative group, $\mathrm{NS}(S)$ is finitely generated, with $\mathrm{rank}\,\mathrm{NS}(S) \leq \mathrm{rank}\,\mathrm{H}_2(S,\ \mathbb{Z})$. In this section we give a detailed description of the Néron–Severi group of S, where the proofs at the same time yield properties of the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ and its action on $\mathrm{NS}(S)$.

It follows from the GAGA principle of Serre [180, Proposition 18] that each holomorphic line bundle L over S admits a nonzero rational section s, when Lemma 2.1.2 implies that $L \simeq \delta(\mathrm{Div}(s))$. Therefore the homomorphism δ in (2.1.6) is surjective. The formula $\mathrm{c}(\delta(D)) = \mathrm{pd}(\mathrm{H}(D))$, see (2.1.24), where $\mathrm{pd} : \mathrm{H}_2(S,\ \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(S,\ \mathbb{Z})$

is the Poincaré duality of (2.1.17), shows that pd induces an isomorphism from $\mathrm{NS}(S)$ onto the image $\mathrm{c}(\mathrm{Pic}(S))$ in $\mathrm{H}^2(S, \mathbb{Z})$ of the Picard group of S under the Chern homomorphism c, and that $\mathrm{c} \circ \delta$ induces an isomorphism from $\mathrm{NS}(S)$ onto $\mathrm{c}(\mathrm{Pic}(S))$.

Two divisors D and D' are called *linearly equivalent*, notation $D \sim D'$, if $\delta(D) = \delta(D')$, which in view of the exact sequence (2.1.4) means that there is a meromorphic function f such that $D' = D + \mathrm{Div}(f)$. That is, the homomorphism δ in (2.1.6) induces an isomorphism from the group $\mathrm{Div}(S)/\sim$ of all divisors modulo linear equivalence onto the Picard group $\mathrm{Pic}(S)$ of S. It follows from $\mathrm{c}(\delta(D)) = \mathrm{pd}(\mathrm{H}(D))$ that $D \sim D'$ implies that $\mathrm{H}(D) = \mathrm{H}(D')$, where $\mathrm{H}(D) = \mathrm{H}(D')$ if and only if $\delta(D - D')$ belongs to the kernel $\mathrm{Pic}(S)^\circ := \ker c$ in $\mathrm{Pic}(S)$ of the Chern homomorphism. The exact sequence (2.1.8) implies that $\mathrm{Pic}(S)^\circ \simeq \mathrm{H}^1(S, \mathcal{O})/\mathrm{H}^1(S, \mathbb{Z})$, and therefore $\mathrm{Pic}(S)^\circ$ is equal to the connected component of the identity element in the Picard group, which explains the notation. It follows that the homomorphism δ induces an isomorphism from $\mathrm{NS}(S)$ onto the component group $\mathrm{Pic}(S)/\mathrm{Pic}(S)^\circ$ of the Picard group of S.

Lemma 7.2.1 *Let D be a divisor on S. Let E_0 be a holomorphic section of φ and F_0 a fiber of φ. Then there exist $n \in \mathbb{Z}$, a holomorphic section E of φ, finitely many distinct irreducible components C_j of fibers of φ, and corresponding $m_j \in \mathbb{Z}_{>0}$ such that*

$$D \sim (D \cdot F_0)\, E_0 + (E - E_0) + \sum_j m_j\, C_j - n\, F_0.$$

Proof. Let $n \in \mathbb{Z}$ and $[L_n] = \delta(D_n)$, where $D_n := D + E_0 - (D \cdot F_0)\, E_0 + n\, F_0$. Let F be any nonsingular fiber of φ. Lemma 6.1.2 implies that $D \cdot F = D \cdot F_0$ and $F_0 \cdot F = F \cdot F = 0$. Lemma 6.2.2 with $A = F$, $\mathrm{g}(F) = 1$, and $F \cdot F = 0$ implies that $\mathrm{K}_S \cdot F = 0$. Since $E_0 \cdot F = 1$ because E_0 is a holomorphic section, we have $L_n \cdot F = 1$, $\deg(\iota_F^*(\mathrm{K}_S \otimes L_n^*)) = \mathrm{K}_S \cdot F - L_n \cdot F = -1$, and the restriction of any holomorphic section of $\mathrm{K}_S \otimes L_n^*$ to F is equal to zero. Because the union of the nonsingular fibers of φ is dense in S, it follows that $\mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S \otimes L_n^*)) = 0$. Therefore (6.2.33) and (6.2.31) imply that $\dim \mathrm{H}^0(S, \mathcal{O}(L_n)) \geq \chi(S, \mathcal{O}(L_n)) = \chi(S, \mathcal{O}) + (L_n \cdot L_n - \mathrm{K}_S \cdot L_n)/2$. Because $D_0 \cdot F_0 = D \cdot F = 1$, $F_0 \cdot F_0 = 0$, and $\mathrm{K}_S \cdot F_0 = \mathrm{K}_S \cdot F = 0$, we have $L_n \cdot L_n = D_n \cdot D_n = (D_0 + n\, F_0) \cdot (D_0 + n\, F_0) = D_0 \cdot D_0 + 2n$ and $\mathrm{K}_S \cdot L_n = \mathrm{K}_S \cdot D_n = \mathrm{K}_S \cdot D_0$. Hence there exists $n \in \mathbb{Z}$ such that $\dim \mathrm{H}^0(S, \mathcal{O}(L_n)) \geq \chi(S, \mathcal{O}(L_n)) > 0$, when there is a nonzero holomorphic section λ of L_n. Then Lemma 2.1.2 implies that $D_n \sim \mathrm{Div}(\lambda) = \sum_j m_j\, C_j$, where the C_j are finitely many distinct irreducible curves in S and $m_j \in \mathbb{Z}_{>0}$, and $1 = D_n \cdot F = \sum_j m_j\, C_j \cdot F$. Let J and K be the set of indices j such that C_j is and is not contained in a fiber of φ, respectively. Then $C_j \cdot F = 0$ if $j \in J$ and $C_k \cdot F > 0$ if $k \in K$; hence $1 = \sum_{k \in K} m_k\, C_k \cdot F$, which in turn implies that $K = \{k\}$ and $m_k = C_k \cdot F = 1$. Since any irreducible complex analytic curve E in S such that $E \cdot F = 1$ is a holomorphic section of φ, this completes the proof of the lemma. □

Let $\mathcal{N}$ be the set of all $D \in \mathcal{D} := \mathrm{Div}(S)/\sim$ such that $D \cdot F = 0$ for some, hence every, fiber F of φ, and let $\mathcal{F}$ denote the subgroup of $D \in \mathcal{D}$ generated by all irreducible curves that are contained in some fiber of φ. Then $\mathcal{N}$ is a subgroup of $\mathcal{D} \simeq \mathrm{Pic}(S)$, and $\mathcal{F} \subset \mathcal{N}$.

Lemma 7.2.2 *Let E_0 be a holomorphic section of φ. Then the mapping*

$$\mu = \mu_{E_0} : \alpha \mapsto \alpha(E_0) - E_0 + \mathcal{F}$$

is an isomorphism of groups from $\mathrm{Aut}(S)^+_\varphi$ *onto* $\mathcal{N}/\mathcal{F}$*, where* μ *does not depend on the choice of* E_0*. If* $\alpha^{\mathcal{D}}$ *denotes the action of* α *on* $\mathcal{D}$*, then* $\alpha^{\mathcal{D}}(D) - D \in \mathcal{N}$ *if* $D \in \mathcal{D}$*, and* $\alpha^{\mathcal{D}}(D) - D \in \mathcal{F}$ *if* $D \in \mathcal{N}$*.*

Proof. If $D \in \mathcal{N}$ then Lemma 7.2.1 with $D \cdot F_0 = 0$ implies that there exists a holomorphic section E of φ such that $D \in E - E_0 + \mathcal{F}$. Lemma 7.1.1 yields an $\alpha \in \mathrm{Aut}(S)^+_\varphi$ such that $E = \alpha(E_0)$. This proves that the mapping μ_{E_0} is surjective.

Let F be a nonsingular fiber of φ not equal to one of the finitely many fibers of φ that contain the irreducible components of $D - (E - E_0)$, and let $\iota_F : F \to S$ denote the embedding of F into S. Let L_0 and L be holomorphic line bundles over S such that $[L_0] = \delta(E_0)$ and $[L] = \delta(E)$. According to Lemma 2.1.2 there are nonzero holomorphic sections s_0 and s of L_0 and L such that $\mathrm{Div}(s_0) = E_0$ and $\mathrm{Div}(s) = E$. Let $\iota_F^* s_0$ and $\iota_F^* s$ be the respective holomorphic sections of $\iota_F^* L_0$ and $\iota_F^* L$ as in Lemma 2.1.3. Because E_0 and E intersect F in unique respective points f_0 and f, where the intersection is transversal, $\mathrm{Div}(\iota_F^* s_0) = \{f_0\}$ and $\mathrm{Div}(\iota_F^* s) = \{f\}$, and hence $\iota_F^* \circ \delta(D) = \iota_F^* \circ \delta(E - E_0) = \delta(\{f\} - \{f_0\})$ as an identity in the group $\mathrm{Pic}(F)^\circ$ of isomorphism classes of holomorphic line bundles over F with Chern class equal to zero. Because F is an elliptic curve, with translation group $\mathrm{Aut}(F)^\circ$, the mapping $\epsilon_F : \tau \mapsto \delta(\{\tau(f_0)\} - \{f_0\})$ is an isomorphism from $\mathrm{Aut}(F)^\circ$ onto $\mathrm{Pic}(F)^\circ$, which does not depend on the choice of $f_0 \in F$. This is a rephrasing of *Abel's theorem for elliptic curves*; see for instance Farkas and Kra [61, Corollary 1 on p. 95]. Let G denote the Cartesian product over all nonsingular fibers F of φ of the groups $\mathrm{Aut}(F)^\circ$, and H the subgroup of all $g \in G$ such that $g_F = 1$ for all but finitely many F's. Then the homomorphisms $\epsilon_F^{-1} \circ \iota_F^* \circ \delta$ define a homomorphism $\nu : \mathcal{N}/\mathcal{F} \to G/H$. If $D \in \mathcal{N}$ and $\nu(D + \mathcal{F}) = 1$, then in the above notation the respective intersection points of E and E_0 with F are equal for all but finitely many F's, which implies that the irreducible curves E and E_0 have infinitely many points in common, and therefore $E = E_0$, and hence $D \in \mathcal{F}$. This proves that the homomorphism ν is injective.

If $\alpha \in \mathrm{Aut}(S)^+_\varphi$, then $\alpha(E_0) \cap F = \alpha(E_0) \cap \alpha(F) = \alpha(E_0 \cap F)$, and therefore in the above notation $f = \alpha(f_0)$; hence $\epsilon_F^{-1} \circ \iota_F^* \circ \delta(\alpha(E_0) - E_0) = \rho_F(\alpha)$, the restriction of α to F. We conclude that $\nu \circ \mu_{E_0} = \rho$, where $\rho : \mathrm{Aut}(S)^+_\varphi \to G/H$ denotes the homomorphism defined by the ρ_F's. Because ν and ρ are homomorphisms that do not depend on E_0 and ν is injective, μ_{E_0} is a homomorphism that does not depend on E_0. If $\rho(\alpha) = 1$, then $\rho_F(\alpha) = 1$ for all but finitely many F's, and because the union of these F's is dense in S and α is continuous, it follows that $\alpha = 1$, that is, the homomorphism ρ is injective, whence $\nu \circ \mu_{E_0} = \rho$ implies that μ_{E_0} is injective. We have proved that $\mu_{E_0} : \mathrm{Aut}(S)^+_\varphi \to \mathcal{N}/\mathcal{F}$ is an isomorphism of groups that does not depend on the choice of E_0.

If $D \in \mathcal{D}$, F a fiber of φ, and $\alpha \in \mathrm{Aut}(S)^+_\varphi$, then $\alpha(D) \cdot F = \alpha(D) \cdot \alpha(F) = D \cdot F$; hence $(\alpha(D) - D) \cdot F = 0$, and therefore $\alpha(D) - D \in \mathcal{N}$. If $D \cdot F = 0$, then there exists a $\beta \in \mathrm{Aut}(S)^+_\varphi$ such that $D \in \beta(E_0) - E_0 + \mathcal{F}$. Since α leaves $\mathcal{F}$ invariant and

$\mathrm{Aut}(S)^+_\varphi$ is commutative, it follows that $\alpha(D)-D \in (\alpha\circ\beta(E_0)-\alpha(E_0))-(\beta(E_0)-E_0)+\mathcal{F} = (\beta\circ\alpha(E_0)-\alpha(E_0))-(\beta(E_0)-E_0)+\mathcal{F} = \mu_{\alpha(E_0)}(\beta)-\mu_{E_0}(\beta) = \mathcal{F}$, because $\mu_{\alpha(E_0)} = \mu_{E_0}$. □

Lemma 7.2.3 below will be used in the proof of Lemma 7.2.4 and Corollary 7.3.3, and in Section 7.5. Its verification is a straightforward calculation. I learned it from Cox and Zucker [41, Table 1.14], where in the case III* the coefficient 3/2 of C_3 or of C_7 has to be replaced by 2. Recall that Lemma 6.2.11 implies that the kernel of the intersection matrix consists of the multiples of the vector with jth coefficient equal to the multiplicity μ_j of Θ_j.

Lemma 7.2.3 *Let F be a reducible fiber of φ and Θ_j a numbering of the set F^{irr} of all irreducible components of F as in Section* 6.2.6*, where $\mu_j = \mu_{\Theta_j}$ is the multiplicity of Θ_j in the divisor F. Note that $\mu_0 = 1$. Let $k \neq 0$ and $\mu_k = 1$. Then the solutions m_j of the equations $\sum_j m_j\,\Theta_j\cdot\Theta_0 = 1$, $\sum_j m_j\,\Theta_j\cdot\Theta_k = -1$, and $\sum_j m_j\,\Theta_j\cdot\Theta_l = 0$ for every $l \notin \{0,\ k\}$ are given by $m_j = m_0\,\mu_j + c_j$, where the numbers $c_j = c_{r,\,k,\,j}$ are as follows. In each case, not all c_j are integers.*

Type A *If the intersection diagram is of type $\mathrm{A}^{(1)}_{b-1}$, then $c_i = j\,(b-k)/b$ for $1 \le j \le k$ and $c_j = k\,(b-j)/b$ for $k \le j \le b-1$. That is, $c_0 = c_b = 0$, $c_k = k\,(b-k)/b$, and the function $j \mapsto c_j$ is linear on the interval $[0,\ k]$ and on the interval $[k,\ b]$.*

Type D *Let the intersection diagram be of type $\mathrm{D}^{(1)}_{b+4}$, with Θ_0, Θ_1 the multiplicity-1 components at one end of the chain of multiplicity-2 components and Θ_{b+3}, Θ_{b+4} the multiplicity-1 components at the other end. If $k = 1$, then $c_j = 1$ for $1 \le j \le b+2$, $c_{b+3} = c_{b+4} = 1/2$.*

If $k = b+3$, then $c_j = j/2$ for $1 \le j \le b+2$, $c_{b+3} = (b+4)/4$, and $c_{b+4} = (b+2)/4$. The case $k = b+4$ is obtained from the case $k = b+3$ by interchanging $b+3$ and $b+4$.

Type E *If the intersection diagram is of type $\mathrm{E}^{(1)}_6$, the multiplicity-one components are Θ_0, Θ_1, and Θ_6. Let $k = 1$. At the branch Θ_0, Θ_2, Θ_4 we have $c_0 = 0$, $c_2 = 1$, $c_4 = 2$; at the branch Θ_1, Θ_3, Θ_4 we have $c_1 = 4/3$, $c_3 = 5/3$, $c_4 = 6/3$; and at the branch Θ_6, Θ_5, Θ_4 we have $c_6 = 2/3$, $c_5 = 4/3$, $c_4 = 6/3$. The case $k = 6$ is obtained by interchanging 1 and 3 by 6 and 5, respectively.*

If the intersection diagram is of type $\mathrm{E}^{(1)}_7$, the multiplicity-one components are Θ_0 and Θ_7. At the branch Θ_0, Θ_1, Θ_3, Θ_4 we have $c_0 = 0$, $c_1 = 1$, $c_3 = 2$, $c_4 = 3$; at the branch Θ_7, Θ_6, Θ_5, Θ_4 we have $c_7 = 3/2$, $c_6 = 4/2$, $c_5 = 5/2$, $c_4 = 6/2$; and at the branch Θ_2, Θ_4 we have $c_2 = 3/2$, $c_4 = 6/2$.

If the intersection diagram is of type $\mathrm{E}^{(1)}_8$, then Θ_0 is the only multiplicity-one component, and there is no $k \neq 0$ such that $\mu_k = 1$.

From now on in Chapter 7 it will be assumed that $\chi(S,\ \mathcal{O}) > 0$.

Lemma 7.2.4 *Let $D \in \mathrm{Div}(S)$, $D\cdot F_0 = 0$ for some, hence every, fiber F_0 of φ, and $D\cdot D \ge 0$. Then there are finitely many fibers F_i of φ and corresponding integers n_i such that $D \sim \sum_i n_i\,F_i$, which in turn implies that $D\cdot D = 0$.*

Proof. Let E_0 be a holomorphic section of φ. According to Lemma 7.2.2 there is a unique $\alpha \in \mathrm{Aut}(S)^+_\varphi$ such that $D \sim \alpha(E_0) - E_0 + U$, and $U = \sum_i U_i$, and $U_i = \sum_\Theta m_\Theta \Theta$, where the Θ are the irreducible components of a fiber F_i of φ, $m_\Theta \in \mathbb{Z}$, and all sums are finite. For each i there is a unique irreducible component Θ_i of F_i such that $E_0 \cdot \Theta_i = 1$, $\mu_{\Theta_i} = 1$, and $E_0 \cdot \Theta = 0$ for every other irreducible component Θ of F_i. Because $\alpha \in \mathrm{Aut}(S)_\varphi$, $E := \alpha(E_0)$ is a holomorphic section of φ and $E \cdot \alpha(\Theta_i) = \alpha(E_0) \cdot \alpha(\Theta_i) = E_0 \cdot \Theta_i = 1$. In other words, if F is a reducible fiber of φ and Θ_{F,E_0} is the irreducible component of F_i that is intersected by E_0, then

$$\Theta_{F,\alpha(E_0)} = \alpha(\Theta_{F,E_0}). \tag{7.2.1}$$

Assume first that α does not permute the irreducible components of any reducible fiber. Then $(E-E_0)\cdot U = 0$; hence $0 \le D\cdot D = (E-E_0)\cdot(E-E_0)+\sum_i U_i\cdot U_i$. We have $(E - E_0) \cdot (E - E_0) = E \cdot E - 2\, E \cdot E_0 + E_0 \cdot E_0 = -2\,\chi(S,\, \mathcal{O}) - 2\, E \cdot E_0$; where we have used that the self-intersection number of every section is equal to $-\chi(S,\, \mathcal{O})$; see Lemma 6.2.34. Furthermore, if $E \ne E_0$ then $E \cdot E_0 \ge 0$ because both E and E_0 are irreducible, hence $(E - E_0) \cdot (E - E_0) < 0$. On the other hand, Lemma 6.2.11 implies that $U_i \cdot U_i \le 0$, with equality if and only if there exists an $n_i \in \mathbb{Z}$ such that $m_\Theta = n_i\, \mu_\Theta$ for every irreducible component Θ of F_i, that is, if and only if $U_i = n_i\, F_i$. It follows that we can have $D \cdot D \ge 0$ only if $E = E_0$ and $D \sim \sum_i n_i\, F_i$.

In the general case we observe that $\varphi : S \to C$ has only finitely many reducible fibers, each of which has only finitely many irreducible components, which implies that the action of α on the set of all these irreducible components has a finite order $m \in \mathbb{Z}_{>0}$, when $(\alpha^m(E_0)-E_0)\cdot U = 0$ for any $\mathbb{Z}$-linear combination U of irreducible curves contained in fibers of φ. It follows from Lemma 7.2.2 that $m\, D \in m\,(\alpha(E_0) - E_0)+\mathcal{F} = \alpha^m(E_0) - E_0 + \mathcal{F}$. Because $(m\, D)\cdot(m\, D) = m^2\, D\cdot D \ge 0$, the previous paragraph with α replaced by α^m implies that $m\, D$ is linearly equivalent to a $\mathbb{Z}$-linear combination of fibers. Therefore $m\, D\cdot U = 0$; hence $D\cdot U = 0$ for every irreducible component U of any fiber of φ.

Let Θ_j be a numbering of the irreducible components of the reducible fiber S_r and suppose that E_0 and E intersect the reducible fiber F in the respective irreducible components Θ_0 and $\Theta_k \ne \Theta_0$. Then $0 = D\cdot\Theta_l = E\cdot\Theta_l - E_0\cdot\Theta_l + \sum_j m_{\Theta_j}\,\Theta_l$ for all l implies in view of (7.2.1) and Lemma 7.2.3 that $m_{\Theta_j} = m_{\Theta_0}\,\mu_j + c_j$, where not all c_j are integers. Because all m_{Θ_j} and μ_j are integers, this leads to a contradiction, and the conclusion is that α does not permute any irreducible components of reducible fibers. □

Corollary 7.2.5 *Pullback by means of φ defines an isomorphism from* $\mathrm{Pic}(C)^\circ$ *onto* $\mathrm{Pic}(S)^\circ$, *from* $\mathrm{H}^1(C,\, \mathcal{O})$ *onto* $\mathrm{H}^1(S,\, \mathcal{O})$, *and from* $\mathrm{H}^0(C,\, \Omega^1)$ *onto* $\mathrm{H}^0(S,\, \Omega^1)$.

Proof. The existence of a holomorphic section σ of φ for which $\sigma^* \circ \varphi^* = (\varphi \circ \sigma)^*$ is the identity shows that φ^* is injective in each case.

If $D \in \mathrm{Div}(S)$ is homologically trivial, then $D \cdot F = 0$ and $D \cdot D = 0$, and Lemma 7.2.4 implies that $D \sim \sum_i n_i\, S_{c_i}$, where the c_i are finitely distinct points in C and $n_i \in \mathbb{Z}$. For any holomorphic section E of φ we have $0 = D \cdot E =$

$\sum_i n_i S_{c_i} \cdot E = \sum_i n_i$. That is, $\delta(D) = \varphi^*(\delta(\Gamma))$, where $\Gamma = \sum_i n_i \{c_i\}$ and $\deg \delta(\Gamma) = \sum_i n_i = 0$. This proves that $\mathrm{Pic}(S)^\circ = \varphi^*(\mathrm{Pic}(C)^\circ)$.

In view of the exact sequence (2.1.8) we have $\mathrm{H}^1(S, \mathcal{O})/\mathrm{H}^1(S, \mathbb{Z}) \simeq \mathrm{Pic}(S)^\circ \simeq \mathrm{Pic}(C)^\circ \simeq \mathrm{H}^1(C, \mathcal{O})/\mathrm{H}^1(C, \mathbb{Z})$, hence $\dim \mathrm{H}^0(S, \Omega^1) = \dim \mathrm{H}^1(S, \mathcal{O}) = \dim \mathrm{H}^1(C, \mathcal{O}) = \dim H^0(C, \Omega^1)$, where in the first and third identities we have used that S and C are Kähler; see (6.2.36). Since the complex linear mappings $\varphi^* : \mathrm{H}^1(C, \mathcal{O}) \to \mathrm{H}^1(S, \mathcal{O})$ and $\varphi^* : \mathrm{H}^0(C, \Omega^1) \to \mathrm{H}^1(S, \Omega^1)$ are injective and all vector spaces involved have the same dimension $\mathrm{g}(C)$, both pullbacks by φ are linear isomorphisms. □

It follows from Corollary 6.2.32 that $\varphi^* : \mathrm{H}^0(C, \Omega^1) \to \mathrm{H}^0(S, \Omega^1)$ is an isomorphism if S is a relatively minimal compact Kähler elliptic surface without multiple singular fibers and ϱ is nontrivial. If S had a holomorphic section, then the last condition would imply that $\chi(S, \mathcal{O}) > 0$, see Lemma 6.2.29, and we would be in the situation of Corollary 7.2.5. I have no example in which the conditions of Corollary 6.2.32 hold, ϱ is nontrivial, and there is no holomorphic section.

Definition 7.2.6. If A is a commutative group, written additively, then A_{tor} is defined as the set of all $a \in A$ such that $m\,a = 0$ for some $m \in \mathbb{Z}_{>0}$. A_{tor} is a subgroup of A, called the *torsion subgroup* of A. If the group A is finitely generated, then A_{tor} is finite, and there is a unique $r = \mathrm{rank}\, A \in \mathbb{Z}_{\geq 0}$, called the *rank* of A, such that A is isomorphic to the Cartesian product of $\mathbb{Z}^r$ and finite cyclic groups $\mathbb{Z}/{p_i}^{n_i}\mathbb{Z}$, where the primes p_i need not be distinct. The orders ${p_i}^{n_i}$ are unique up to permutation, and $\#(A_{\mathrm{tor}}) = \prod_i {p_i}^{n_i}$. See for instance Fuchs [65, Chapter 2, §10].

A *lattice* is a finitely generated torsion-free commutative group Λ provided with a nondegenerate symmetric $\mathbb{Q}$-valued bilinear form $(a, b) \mapsto a \cdot b : \Lambda \times \Lambda \to \mathbb{Q}$. The lattice is called *integral* if $a \cdot b \in \mathbb{Z}$ for all $a, b \in \Lambda$. The lattice is called *even* if $a \cdot a \in 2\mathbb{Z}$ for every $a \in \Lambda$. In view of the formula $(a+b)\cdot(a+b) = a\cdot a + 2\,(a\cdot b) + b\cdot b$ this implies that the lattice is integral.

Write $\Lambda = \mathrm{NS}(S)$ and let $f \in \Lambda$ denote the common homology class of the fibers of φ, see Lemma 6.1.2, where $f \cdot f = 0$. Let

$$f^\perp := \{c \in \Lambda \mid c \cdot f = 0\} \tag{7.2.2}$$

denote the orthogonal complement of f in Λ with respect to the intersection form. Because $f \cdot f = 0$, $\mathbb{Z}\, f$ is a subgroup of $f^\perp$, and we have the quotient group

$$Q := f^\perp/\mathbb{Z}\, f. \tag{7.2.3}$$

For each reducible fiber S_r of φ, let Λ_r denote the subgroup of Λ generated by the homology classes of the irreducible components of S_r, and let Q_r denote the image in Q of Λ_r under the projection $c \mapsto c + \mathbb{Z}\, f$ from $f^\perp$ onto Q. Let Λ^{irr} denote the subgroup of Λ generated by f and the Λ_r, $r \in C^{\mathrm{red}}$, and Q^{irr} the image of Λ^{irr} in Q, the subgroup of Q generated by the Q_r, $r \in C^{\mathrm{red}}$. Here C^{red} is the finite set of all $c \in C$ such that the fiber S_c of φ over c is reducible. Finally, if $\alpha \in \mathrm{Aut}(S)^+_\varphi$, then α has an induced action α_* on $\mathrm{H}_2(S, \mathbb{Z})$ that leaves $\Lambda = \mathrm{NS}(S)$ invariant, and

$\alpha^\Lambda := (\alpha_*)|_\Lambda$ is an automorphism of the group Λ that preserves the intersection form. Because α leaves each fiber of φ invariant, we have $\alpha^\Lambda(f) = f$, and the action α^Λ of α on Λ passes to an action α^Q of α on Q. With this notation, we have the following conclusions.

Theorem 7.2.7 *The Néron–Severi group $\Lambda := \mathrm{NS}(S)$ is an integral lattice with respect to the intersection form. If $e_0 = [E_0]$ is the homology class of a holomorphic section E_0 of φ, then Λ is the direct sum of $\mathbb{Z}\,e_0 + \mathbb{Z}\,f$ and the orthogonal complement $(\mathbb{Z}\,e_0 + \mathbb{Z}\,f)^\perp$ of $\mathbb{Z}\,e_0 + \mathbb{Z}\,f$ in Λ. If $c \in f^\perp$ and $c \cdot c \geq 0$ then $c \in \mathbb{Z}\,f$. The intersection form passes to a symmetric bilinear form on $Q := f^\perp/\mathbb{Z}\,f$, with respect to which Q is a negative definite even lattice, with* $\operatorname{rank} Q = \operatorname{rank} \mathrm{NS}(S) - 2$. *The projection $c \mapsto c + \mathbb{Z}\,f$ defines an isomorphism of lattices from $(\mathbb{Z}\,e_0 + \mathbb{Z}\,f)^\perp$ onto Q. The intersection matrix with respect to any $\mathbb{Z}$-basis of Λ has one positive eigenvalue, where all the other eigenvalues are negative.*

Let $r \in C^{\mathrm{red}}$, when the intersection diagram of S_r is of type $\mathrm{A}_n^{(1)}$, $\mathrm{D}_n^{(1)}$, or $\mathrm{E}_n^{(1)}$, see Remark 6.2.12*, which is a function of the Kodaira type of S_r; see Section* 6.2.6*. Then Q_r, provided with minus the intersection form, is a root lattice of type A_n, D_n, or E_n, respectively; hence Q_r is a negative definite even lattice with respect to the intersection form. We have* $\operatorname{rank} Q_r = n_r - 1$ *if n_r is the number of irreducible components of S_r. The Q_r, $r \in C^{\mathrm{red}}$, are mutually orthogonal with respect to the intersection form, and Q^{irr} is the direct sum of the Q_r, $r \in C^{\mathrm{red}}$.*

The mapping $\mu : \alpha \mapsto \alpha^\Lambda(e_0) - e_0 + Q^{\mathrm{irr}}$ is an isomorphism from the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ onto the quotient group Q/Q^{irr}, which does not depend on the choice of the section E_0. It follows that $\mathrm{Aut}(S)^+_\varphi$ has $\leq \operatorname{rank} \mathrm{NS}(S) - 2$ generators, and

$$\operatorname{rank}\ \mathrm{Aut}(S)^+_\varphi = \operatorname{rank} \mathrm{NS}(S) - 2 - \sum_{r \in C^{\mathrm{red}}} (n_r - 1). \tag{7.2.4}$$

For each $\alpha \in \mathrm{Aut}(S)^+_\varphi$, $\alpha^\Lambda - 1$ maps Λ to $f^\perp$, and $f^\perp$ to Λ^{irr}; hence $\alpha^Q - 1$ maps Q to Q^{irr}.

The homomorphism $\alpha \mapsto \alpha^\Lambda$ from the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ to the automorphism group $\mathrm{Aut}(\Lambda)$ of the lattice $\Lambda = \mathrm{NS}(S)$ is injective.

Proof. We provide the arguments for the statements that do not immediately follow from Lemmas 7.2.1, 7.2.2, and 7.2.4. For any $c \in \Lambda$, we have $c - (\epsilon\, e_0 + \phi\, f) \in (\mathbb{Z}\,e_0 + \mathbb{Z}\,f)^\perp$ if and only if $\epsilon = c \cdot f \in \mathbb{Z}$ and $\phi = c \cdot e_0 - (c \cdot f)\,(e_0 \cdot e_0) \in \mathbb{Z}$, which proves the direct sum decomposition $\Lambda = (\mathbb{Z}\,e_0 + \mathbb{Z}\,f) \oplus (\mathbb{Z}\,e_0 + \mathbb{Z}\,f)^\perp$. On the $\mathbb{Z}$-basis e_0 and f of $\mathbb{Z}\,e_0 + \mathbb{Z}\,f$, the intersection matrix is equal to $\left(\begin{smallmatrix} -\chi(S,\mathcal{O}) & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which has one positive and one negative eigenvalue. Because the kernel of the intersection form on $f^\perp$ is equal to $\mathbb{Z}\,f$, which has zero intersection with the orthogonal complement of $\mathbb{Z}\,e_0 + \mathbb{Z}\,f$ in $\Lambda \otimes \mathbb{Q}$, and the intersection form on Q is negative definite, the other eigenvalues of the intersection matrix on Λ are negative.

Let $r \in C^{\mathrm{red}}$. Then it follows from Lemma 6.2.11 that Q_r is a negative definite lattice, and according to Remark 6.2.12 it is a root lattice of type A_n, D_n, or E_n. If $\theta = [\Theta]$ denotes the homology classes of the irreducible components Θ of S_r, and $m_\theta \in \mathbb{Z}$, then

$$\Big(\sum_\theta m_\theta\,\theta\Big)\cdot\Big(\sum_\theta m_\theta\,\theta\Big)=\sum_\theta {m_\theta}^2\,\theta\cdot\theta+2\sum_{\theta\neq\theta'} m_\theta\,m_{\theta'}\,\theta\cdot\theta',$$

in combination with $\theta\cdot\theta=-2$, see Lemma 6.2.10, shows that the lattice Q_r is even; hence Q^{irr} is a negative definite and even lattice. Lemma 7.2.1 implies that every $c\in Q$ can be written as $e-e_0+u$, where e is the homology class of a holomorphic section and $u\in Q^{\mathrm{irr}}$. It follows that $c\cdot c=e\cdot e+e_0\cdot e_0-2\,e\cdot e_0+2\,(e-e_0)\cdot u+u\cdot u$, when $e\cdot e+e_0\cdot e_0=-2\,\chi(S,\,\mathcal{O})$ and $u\cdot u\in 2\,\mathbb{Z}$ imply that $c\cdot c\in 2\,\mathbb{Z}$. This shows that the lattice Q is even, whereas Lemma 7.2.4 implies that it is negative definite.

If $\alpha^\Lambda(e_0)=e_0$ then $\alpha(E_0)$ is homologous to E_0. Because $E_0\cdot E_0=-\chi(S,\,\mathcal{O})<0$, see Lemma 6.2.34, and both E_0 and $\alpha(E_0)$ are irreducible curves in S, it follows that $\alpha(E_0)=E_0$. According to Lemma 7.1.1, the action of $\mathrm{Aut}(S)^+_\varphi$ on the set of sections is free, and therefore α is equal to the identity on S. This proves the last statement in the theorem. □

Remark 7.2.8. For rational elliptic surfaces, the homomorphism $\alpha\mapsto\alpha^\Lambda$ from the larger group $\mathrm{Aut}(S)$ to $\mathrm{Aut}(\Lambda)$ is injective; see Lemma 9.2.15.

For any compact Kähler surface S, the intersection form on $\mathrm{NS}(S)\otimes\mathbb{Q}$ is non-degenerate and on any $\mathbb{Q}$-basis its matrix has one positive eigenvalue, where all the other eigenvalues are negative. This is known as the *index theorem for projective algebraic surfaces*; see Griffiths and Harris [74, pp. 126, 164, 471, 472]. Note that the description in Theorem 7.2.7 for our elliptic surface S is much more detailed.

Lemma 7.2.1, 7.2.2, 7.2.4, and Theorem 7.2.7 follow from Shioda [184, Section 1–5], see also Shioda [183]. The formula (7.2.4) is due to Tate [194, p. 15] and Shioda [183, Corollary 1.5]. The statement about the isomorphism μ also follows from Morrison and Persson [143, Theorem 3.1,(2) and Proposition 3.4(2)]. I have followed Heckman and Looijenga [81, Proposition 1.3] in the use of the lattices Q and Q^{irr}, the definition of which is independent of the choice of a holomorphic section. ⊘

7.3 Eichler–Siegel transformations

In this section we discuss $\alpha\in\mathrm{Aut}(S)^+_\varphi$ that leave each irreducible component of each reducible fiber of φ invariant, when the action of α on the Néron–Severi group is given by a simple explicit formula.

Let Λ be an integral lattice with an element $f\in\Lambda$ such that $e\cdot f=1$, $f\cdot f=0$, and $v\cdot v\in 2\,\mathbb{Z}$ for every $v\in f^\perp:=\{c\in\Lambda\mid c\cdot f=0\}$. An *automorphism of the lattice* Λ is defined as a group automorphism $A:\Lambda\to\Lambda$ such that $A(c)\cdot A(c')=c\cdot c'$ for every $c,\,c'\in\Lambda$. Let $\mathrm{Aut}(\Lambda)^Q_f$ denote the group of all automorphisms A of Λ such that $A(f)=f$ and the induced action A^Q of A on $Q:=f^\perp/\mathbb{Z}\,f$ is trivial, that is, $A(v)-v\in\mathbb{Z}\,f$ for every $v\in f^\perp$.

For every $q\in Q=f^\perp/\mathbb{Z}\,f$, the mapping

$$\mathrm{E}_q : c \mapsto c + (c \cdot f)\, v - \Big((c \cdot v) + \frac{v \cdot v}{2}\,(c \cdot f)\Big)\, f \tag{7.3.1}$$

from Λ to Λ does not depend on the choice of $v \in q$, an element $v \in f^\perp$ such that $q = v + \mathbb{Z}\, f$. The mapping $\mathrm{E}_q : \Lambda \to \Lambda$ was introduced by Eichler [57, I, §3, (3.2)], who mentioned that it appeared in a different context in Siegel [185]. For this reason it is called an *Eichler–Siegel transformation*, although the two names also appear in the other order in the literature.

Lemma 7.3.1 *Assume that there exists $e \in \Lambda$ such that $e \cdot f = 1$. Then the mapping $q \mapsto \mathrm{E}_q$ is an isomorphism from the group Q onto the group $\mathrm{Aut}(\Lambda)_f^Q$. The inverse is equal to the mapping $M : \mathrm{Aut}(\Lambda)_f^Q \to Q : A \mapsto A(e) - e + \mathbb{Z}\, f$, which does not depend on the choice of the element $e \in \Lambda$ such that $e \cdot f = 1$.*

Proof. The Eichler–Siegel transformation is a group homomorphism. A direct calculation, simplified by the equations $f \cdot f = v \cdot f = 0$, shows that $\mathrm{E}_q(c) \cdot \mathrm{E}_q(c') = c \cdot c'$ for every $c,\ c' \in \Lambda$. $f \cdot f = v \cdot f = 0$ implies that $\mathrm{E}_q(f) = f$, hence $\mathrm{E}_q(f^\perp) \subset f^\perp$, where $\mathrm{E}_q(c) - c \in \mathbb{Z}\, f$ if $c \in f^\perp$ shows that the induced action of E_q on Q is trivial. $f \cdot f = v \cdot f = 0$ also implies that $\mathrm{E}_q(c) - c \in f^\perp$, and therefore the induced action of E_q on $\Lambda / f^\perp$ is also trivial. Because finally the action of E_q on $\mathbb{Z}\, f$ is trivial, it follows that $\mathrm{E}_q : \Lambda \to \Lambda$ is bijective, and we have proved that $\mathrm{E}_q \in \mathrm{Aut}(\Lambda)_f^Q$.

If $A \in \mathrm{Aut}(\Lambda)_f^Q$ and $e,\ e' \in \Lambda$, such that $e \cdot f = e' \cdot f = 1$, then $A(e) - e - (A(e') - e') = A(e - e') - (e - e') \in \mathbb{Z}\, f$, because $e - e' \in f^\perp$ and the action of A on $f^\perp / \mathbb{Z}\, f$ is trivial. This shows that the mapping M does not depend on the choice of the element e such that $e \cdot f = 1$. We have $M(A \circ B^{-1}) = A(B^{-1}(e)) - e + \mathbb{Z}\, f = A(B^{-1}(e)) - B^{-1}(e) - (B(B^{-1}(e)) - B^{-1}(e) + \mathbb{Z}\, f = M(A) - M(B)$, and therefore $M : \mathrm{Aut}(\Lambda)_f^Q \to Q$ is a homomorphism of groups.

Let $A \in \mathrm{Aut}(\Lambda_f^Q)$ and $M(A) = 0$, then $A(e) - e = m\, f$ for some $m \in \mathbb{Z}$. For any $c \in \Lambda$ we have $c - (c \cdot f)\, e \in f^\perp$, and because A acts trivially on $f^\perp / \mathbb{Z}\, f$ there exists $n(c) \in \mathbb{Z}$ such that $A(c - (c \cdot f)\, e) = c - (c \cdot f)\, e + n(c)\, f$, where $c \mapsto n(c) = e \cdot (n(c)\, f) = e \cdot (A(c - (c \cdot f)\, e) - (c - (c \cdot f)\, e))$ is a homomorphism from Λ to $\mathbb{Z}$. It follows that $A(c) = c + l(c)\, f$, where $l(c) := (c \cdot f)\, m + n(c)$, and $c \cdot c = A(c) \cdot A(c) = c \cdot c + 2\, l(c)\, c \cdot f$ shows that $l(c) = 0$ when $c \cdot f \neq 0$. Writing $c = e + (c - e)$, where $e \cdot f = 1 \neq 0$ and $(c - e) \cdot f = -1 \neq 0$, it follows that $l(c) = l(e) + l(c - e) = 0 + 0 = 0$ for every $c \in \Lambda$; hence $A = 1$. This proves that the homomorphism M is injective.

Because $\mathrm{E}_q(e) - e \in v + \mathbb{Z}\, f$, we have $M(\mathrm{E}_q) = q$, which implies that M is bijective, hence an isomorphism of groups, with $\mathrm{E} : q \mapsto \mathrm{E}_q$ as its inverse. It follows that $\mathrm{E} : Q \to \mathrm{Aut}_f^Q$ is an isomorphism of groups with M as its inverse. □

In the following lemma $\Lambda = \mathrm{NS}(S)$ is the Néron–Severi lattice of the elliptic surface S.

Lemma 7.3.2 *For an element $\alpha \in \mathrm{Aut}(S)_\varphi^+$ of the Mordell–Weil group of S the following conditions are equivalent:*

(i) *The action of α on Q^{irr} is trivial.*

(ii) *α leaves every irreducible component of every reducible fiber of φ invariant.*
(iii) *For every reducible fiber S_r of φ, α leaves at least one multiplicity-one component of S_r invariant.*
(iv) *The action of α on Λ^{irr} is trivial.*
(v) $(\alpha^{\Lambda} - 1)^3 = 0$.
(vi) *The action of α on Λ is unipotent, that is, there exists $m \in \mathbb{Z}_{>0}$ such that* $(\alpha^{\Lambda} - 1)^m = 0$.
(vii) *The action α^{Q} of α on Q is trivial.*
(viii) *For some holomorphic section E of φ, $\alpha(E)$ intersects every reducible fiber in the same irreducible component as E does.*
(ix) *If $e = \mathrm{H}(E)$ for a holomorphic section E of φ, and $u \in \Lambda^{\mathrm{irr}}$, then* $(\alpha^{\Lambda}(e) - e) \cdot u = 0$.
(x) *For every $e \in \Lambda$ such that $e \cdot f = 1$ we have $\alpha^{\Lambda} = \mathrm{E}_q$, where $q = \alpha^{\Lambda}(e) - e + \mathbb{Z} f$.*

Proof. If (i) holds and Θ is an irreducible component of a reducible fiber then $\alpha^{\Lambda}([\Theta]) = [\Theta] + n f$ for some $n \in \mathbb{Z}$, hence $\alpha(\Theta) \cdot \Theta = \Theta \cdot \Theta = -2 < 0$; see Lemma 6.2.10. Because $\alpha(\Theta)$ and Θ are irreducible curves, it follows that $\alpha(\Theta) = \Theta$, which proves (ii). The implication (ii) $\Rightarrow$ (iii) is obvious. If (iii) holds, then Corollary 6.3.31 implies that for each reducible fiber S_r the action α_r on the set S_r^{irr} of all irreducible components of S_r is trivial, which implies (ii). Because α leaves each fiber of φ invariant, the implication (ii) $\Rightarrow$ (iv) is obvious. If (iv) holds then Theorem 7.2.7 implies that $(\alpha^{\Lambda} - 1)^3(\Lambda) \subset (\alpha^{\Lambda} - 1)^2(f^{\perp}) \subset (\alpha^{\Lambda} - 1)(\Lambda^{\mathrm{irr}}) = 0$, hence (v), where the implication (v) $\Rightarrow$ (vi) is obvious. Because α^{Q} preserves the negative definite intersection form on Q and any unipotent orthogonal transformation is equal to the identity, we have (vi) $\Rightarrow$ (vii). Because (vii) $\Rightarrow$ (i) is obvious, we have proved the equivalence of the conditions (i)–(vii).

If E is a holomorphic section of φ, then we have for each reducible fiber F a unique irreducible component $\Theta = \Theta_{F,E}$ such that $E \cdot \Theta = 1$, when $E \cdot \Theta = 0$ for every other irreducible component Θ of F. Furthermore, $\Theta_{F,\alpha(E)} = \alpha(\Theta_{F,E})$; see (7.2.1). Therefore, if (viii) holds, we have $\alpha(\Theta_{F,E}) = \Theta_{F,\alpha(E)} = \Theta_{F,E}$, hence (iii), where conversely (ii) implies that $\alpha(E) \cdot \Theta_{F,E} = \alpha(E) \cdot \alpha(\Theta_{F,E}) = E \cdot \Theta_{F,E} = 1$, hence (viii). Obviously (ix) with $u = [\Theta_{F,E}]$ leads to (viii), whereas (iv) implies that $\alpha^{\Lambda}(e) \cdot u = \alpha^{\Lambda}(e) \cdot \alpha^{\Lambda}(u) = e \cdot u$ for every $u \in \Lambda^{\mathrm{irr}}$, which implies (ix). This adds (viii) and (ix) to the list of equivalent conditions. The equivalence between (x) and (viii) follows from Lemma 7.3.1, which completes the proof of the equivalence between (i)–(x). □

Recall that for each $\alpha \in \mathrm{Aut}(S)^{+}_{\varphi}$ and $c \in C$ the action $\iota_c(\alpha)$ of α on S_c is an element of $\mathfrak{F}_c$, the fiber over over c of the fiber system of Lie groups $\mathfrak{F}$ defined in Section 6.3.6. If the fiber S_c of φ is reducible, then the elements of $\mathfrak{F}_c$ permute the irreducible components of S_c, and the component group $\mathfrak{F}_c/\mathfrak{F}_c^{\circ}$ acts freely and transitively on the finite set $S_c^{\mathrm{irr},1}$ of all multiplicity-one irreducible components of S_c. It follows that the group $\mathfrak{F}_c/\mathfrak{F}_c^{\circ}$ is finite and its number of elements is equal to the number $n_c^{(1)}$ of multiplicity-one irreducible components of S_c. See Corollary 6.3.31.

For each $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and $r \in C^{\mathrm{red}}$, write $\varpi(\alpha)_r = \iota_r(\alpha)\,\mathfrak{F}^\circ_c \in \mathfrak{F}_r/\mathfrak{F}^\circ_r$ for the action of α on the set $S^{\mathrm{irr},\,1}_r$ of all multiplicity-one irreducible components of S_r. This defines a homomorphism

$$\varpi : \mathrm{Aut}(S)^+_\varphi \to \Gamma := \prod_{r\in C^{\mathrm{red}}} \mathfrak{F}_r/\mathfrak{F}^\circ_r. \tag{7.3.2}$$

Corollary 7.3.3 *Let* $\mathrm{Aut}(S)^{+,\,0}_\varphi$ *denote the set of all* $\alpha \in \mathrm{Aut}(S)^+_\varphi$ *that satisfy any of the equivalent conditions (i)–(x) in Lemma* 7.3.2. *Then* $\mathrm{Aut}(S)^{+,\,0}_\varphi$ *is equal to the kernel of the homomorphism* ϖ *in* (7.3.2), *and therefore* $\mathrm{Aut}(S)^{+,\,0}_\varphi$ *is a subgroup of finite index in* $\mathrm{Aut}(S)^+_\varphi$, *where*

$$\#\left(\mathrm{Aut}(S)^+_\varphi/\,\mathrm{Aut}(S)^{+,\,0}_\varphi\right) \le \#(\Gamma) = \prod_{r\in C^{\mathrm{red}}} n^{(1)}_r.$$

Let Q^0 *denote the orthogonal complement in* Q *of* Q^{irr}, *that is,* $q \in Q^0$ *if and only if* $q \in Q$ *and* $q \cdot u = 0$ *for all* $u \in Q^{\mathrm{irr}}$. *Then, for every homology class* e *of a holomorphic section* E *of* φ, *the mapping* $M : \alpha \mapsto \alpha^\Lambda(e) - e + \mathbb{Z}\,f$ *defines an isomorphism from* $\mathrm{Aut}(S)^{+,\,0}_\varphi$ *onto* Q^0, *which does not depend on the choice of the section* E.

Finally, if α *is an element of* $\mathrm{Aut}(S)^+_\varphi$ *of finite order, and* $\alpha \in \mathrm{Aut}(S)^{+,\,0}_\varphi$, *then* $\alpha = 1$.

Proof. That $\mathrm{Aut}(S)^{+,\,0}_\varphi$ is equal to the kernel of ϖ follows from the characterizations (ii), (iii) in Lemma 7.3.2 of the elements of $\mathrm{Aut}(S)^{+,\,0}_\varphi$.

Let $q \in Q^0$. It follows from Theorem 7.2.7 that there exists a unique $\alpha \in \mathrm{Aut}(S)^+_\varphi$ such that $q = \alpha^\Lambda(e) - e + u$ for some $u \in Q^{\mathrm{irr}}$. That is, u is equal to a sum over all irreducible components Θ of reducible fibers of terms $m_\Theta\,\Theta$, where the coefficients m_Θ are integers. For every homology class θ modulo $\mathbb{Z}\,f$ of an irreducible component Θ of a reducible fiber we have $0 = q \cdot \theta = (\alpha^\Lambda(e) - e)\cdot\theta + u\cdot\theta$.

Let S_r be a reducible fiber of φ such that α permutes the multiplicity-one irreducible components of S_r in a nontrivial way. Because the action of $\mathrm{Aut}(S)^+_\varphi$ on $S^{\mathrm{irr},\,1}_r$ is free, see Corollary 6.3.31, it follows that $\Theta_k := \alpha(\Theta_0) \ne \Theta_0$, where Θ_0 denotes the unique irreducible component Θ of S_r intersected by E. Note that Θ_0 and Θ_k have multiplicity one in S_r. We have $E\cdot\Theta_0 = 1$, $E\cdot\Theta_k = 0$, $\alpha(E)\cdot\Theta_k = \alpha(E)\cdot\alpha(\Theta_0) = E\cdot\Theta_0 = 1$, and therefore also $\alpha(E)\cdot\Theta_0 = 0$. It follows that $(\alpha(E)-E)\cdot\Theta$ is equal to -1, 1, or 0 if $\Theta = \Theta_0$, $\Theta = \Theta_k$, or $\Theta \in S^{\mathrm{irr}}_r \setminus \{\Theta_0,\ \Theta_k\}$, respectively. In view of Lemma 7.2.3 the equation $0 \equiv (\alpha^\Lambda(e)-e)\cdot\theta + u\cdot\theta$ therefore implies that $m_{\Theta_j} = n\,\mu_j + c_j$, where $m_{\Theta_0} = n\,\mu_0 + c_0 = n$ implies that $n \in \mathbb{Z}$, which in turn implies that $c_j = m_{\Theta_j} - n\,\mu_j \in \mathbb{Z}$ for every j. Since for every type, not all the c_j in Lemma 7.2.3 are integers, we have a contradiction to the assumption that α permutes the multiplicity-one irreducible components of S_r in a nontrivial way. The conclusion is that $\alpha \in \mathrm{Aut}(S)^{+,\,0}_\varphi$, which in turn implies that $(\alpha^\Lambda(e)-e)\cdot\theta = 0$; hence $u\cdot\theta = 0$ for every homology class θ modulo $\mathbb{Z}\,f$ of any irreducible component

of any reducible fiber. Therefore $u \in Q^{\mathrm{irr}} \cap Q^0$, and because the intersection form is nondegenerate on Q, it follows that $u = 0$ and $q = \alpha^\Lambda(e) - e$. This proves that the mapping $M : \mathrm{Aut}(S)^{+,0}_\varphi \to Q^0$ is surjective.

Finally, Q is torsion-free, hence Q^0 is torsion-free, and because $M : \mathrm{Aut}(S)^{+,0}_\varphi \to Q^0$ is an isomorphism of groups it follows that $\mathrm{Aut}(S)^{+,0}_\varphi$ is torsion-free. □

If $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and E is a holomorphic section of φ, then $\alpha(E)\cdot\alpha(E) = E \cdot E = -\chi(S, \mathcal{O})$, see Lemma 6.2.34. Because Corollary 7.3.3 implies that every $q \in Q^0$ is equal to the homomology class modulo $\mathbb{Z}$ of $\alpha(E) - E$, where $\alpha \in \mathrm{Aut}(S)^{+,0}_\varphi$ is unique and the holomorphic section E of φ is arbitrary, we have

$$q \cdot q = -2(\chi(S, \mathcal{O}) + \alpha(E) \cdot E) \leq -2\,\chi(S, \mathcal{O}) \tag{7.3.3}$$

if $q \neq 0$, with equality if and only if the sections $\alpha(E)$ and E are disjoint.

Corollary 7.3.3 (x) in Lemma 7.3.2 and Lemma 7.3.1 imply that the homomorphism $\alpha \mapsto \alpha^\Lambda : \mathrm{Aut}(S)^{+,0}_\varphi \to \mathrm{Aut}(\Lambda)$ is equal to $\mathrm{E} \circ H$, the isomorphism $H : \mathrm{Aut}(S)^{+,0}_\varphi \to Q^0$ followed by the restriction to Q^0 of the Eichler–Siegel isomorphism $q \mapsto \mathrm{E}_q : Q \to \mathrm{Aut}(\Lambda)^Q_f$. Because $\mathrm{E} : q \mapsto \mathrm{E}_q$ is a polynomial of degree two on Q with values in the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda)$, see (7.3.1), *the action of* $\mathrm{Aut}(S)^{+,0}_\varphi$ *on the Néron–Severi group* $\Lambda = \mathrm{NS}(S)$ *is polynomial of degree two*. Note that if $\varphi : S \to C$ has no reducible fibers, then $\mathrm{Aut}(S)^{+,0}_\varphi = \mathrm{Aut}(S)^+_\varphi$, $Q^0 = Q$, and $M = \mu$ is an isomorphism from $\mathrm{Aut}(S)^+_\varphi$ onto Q.

Manin [129, §4], in the case of a rational elliptic surface without reducible fibers, related the quadratic nature of the action of the Mordell–Weil group on the Néron–Severi group with what he called the *Tate height*. I don't have enough knowledge of the subject to give a more detailed explanation of the name.

I learned Corollary 7.3.3 from Shioda [184, Theorem 8.7 and 8.9]. The subgroup $\mathrm{Aut}(S)^{+,0}_\varphi$ of the Mordell–Weil group can be provided with the lattice structure of Q^0 via the isomorphism $M : \mathrm{Aut}(S)^{+,0}_\varphi \xrightarrow{\sim} Q^0$. Shioda [184, Definition 8.8] called $\mathrm{Aut}(S)^{+,0}_\varphi$, provided with this lattice structure, the *narrow Mordell–Weil lattice* of the elliptic surface S.

7.4 The Number of Periodic Fibers

We begin this subsection with a local description of an element of $\mathrm{Aut}(S)^+_\varphi$ as a translation in $\mathbb{C}$ depending holomorphically on the image point in C under the mapping $\varphi : S \to C$. For each $c \in C$, write $S_c := \varphi^{-1}(\{c\})$ for the fiber of φ over the point $c \in C$. The set $S_c \cap S^{\mathrm{reg}}$ is the *fiber of* φ *in* S^{reg} *over* c. It is equal to the whole fiber S_c if and only if $c \in C^{\mathrm{reg}}$ if and only if S_c is smooth, hence an elliptic curve.

If $c \notin C^{\mathrm{reg}}$ but S_c is irreducible, then $S_c \cap S^{\mathrm{reg}}$ is connected and equal to S_c minus its unique singular point. If S_c is reducible, then the mapping that assigns to each connected component A of $S_c \cap S^{\mathrm{reg}}$ its closure B is bijective from the set of all

connected components of $S_c \cap S^{\mathrm{reg}}$ onto the finite set $S_c^{\mathrm{irr},\,1}$ of all irreducible components of S_c of multiplicity one, and $B \setminus A$ is equal to the finite set of all intersection points of B with the other irreducible components of S_c. See the description of the singular fibers in Section 6.2.6. If $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and E is a holomorphic section of φ, then $\alpha(E) \cdot \alpha(E) = E \cdot E = -\chi(S, \mathcal{O})$, see Lemma 6.2.34. Since $\mathrm{Aut}(S)^+_\varphi$ acts on $S_c^{\mathrm{irr},\,1}$ via the action of $\mathfrak{F}_c/\mathfrak{F}_c{}^{\mathrm{o}}$ on $S_c^{\mathrm{irr},\,1}$, and the latter action is free and transitive, see Corollary 6.3.31, some iterate of α will leave some, and therefore every, connected component of $S_c \cap S^{\mathrm{reg}}$ invariant.

Lemma 7.4.1 *Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$, $c_0 \in C$, and let l be the smallest positive integer such that α^l leaves the connected components of $S_{c_0} \cap S^{\mathrm{reg}}$ invariant. In the following C_0 is a sufficiently small open neighborhood of c_0 in C. Let v be a homolomorphic section of $\mathfrak{f}$ on C_0, viewed as a holomorphic vector field on $\varphi^{-1}(C_0)$, such that $v_{c_0} \neq 0$. Let p_1 and p_2 be the possibly multivalued holomorphic functions on $C_0 \setminus \{c_0\}$ such that for each $c \in C_0 \setminus \{c_0\}$, $p_1(c)$ and $p_2(c)$ form a $\mathbb{Z}$-basis of the period lattice P_c of the vector field v_c on S_c.*

Then there exist a multivalued holomorphic function $T : C_0 \setminus \{c_0\} \to \mathbb{C}$ and integers l_1, l_2, such that $\alpha(s) = \mathrm{e}^{T(\varphi(s))\,v}(s)$ whenever $\varphi(s) \in C_0 \setminus \{c_0\}$. If $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and E is a holomorphic section of φ, then $\alpha(E) \cdot \alpha(E) = E \cdot E = -\chi(S, \mathcal{O})$; see Lemma 6.2.34. Moreover, $l\,T - l_1\,p_1 - l_2\,p_2$ extends to a holomorphic function on C_0.

Proof. Lemma 6.3.27 with c_0 replaced by any point in $C_0 \setminus \{c_0\}$ implies the existence of T, whereas Lemma 6.3.27 with α replaced by α^l implies the existence of a holomorphic function $T_l : C_0 \to \mathbb{C}$ such that

$$\alpha^l(s) = \mathrm{e}^{T_l(\varphi(s))\,v}(s), \quad \varphi(s) \in C_0.$$

It follows that, for $c \in C_0 \setminus \{c_0\}$,

$$\mathrm{e}^{T_l(c)\,v_c} = \alpha^l = (\,\mathrm{e}^{T(c)\,v_c})^l = \mathrm{e}^{l\,T(c)\,v_c}$$

on S_c hence $l\,T(c) - T_l(c) \in P_c$, or equivalently $l\,T(c) - T_l(c) = l_1(c)\,p_1(c) + l_2(c)\,p_2(c)$ for uniquely determined $l_1(c)$, $l_2(c)$, which locally depend continuously on $c \in C_0 \setminus \{c_0\}$. Because any continuous $\mathbb{Z}$-valued function on a connected set is constant, this proves the lemma. □

Remark 7.4.2. If $c_0 \in C^{\mathrm{reg}}$, then p_1 and p_2, and therefore also T, extend to holomorphic functions on C_0, and the integers l_1 and l_2 can be chosen arbitrarily.

If $c_0 \in C^{\mathrm{sing}} = C \setminus C^{\mathrm{reg}}$, then we can choose p_1, p_2 as in Lemma 6.2.38. If S_{c_0} is a singular fiber of Kodaira type I_b, $b \in \mathbb{Z}_{\geq 1}$, then the period function p_1 can be chosen to be holomorphic in a neighborhood of c_0 in C, whereas p_2 has logarithmic behavior, and it follows that l_2 is uniquely determined, but l_1 can be chosen arbitrarily.

If the Kodaira type of the singular fiber S_{c_0} is not equal to I_b, $b \in \mathbb{Z}_{\geq 1}$, then no nonzero period function is single-valued or bounded in any neighborhood of c_0, and it follows that the integers l_1 and l_2 in Lemma 7.4.1 are uniquely determined. ⊘

Definition 7.4.3. Let $\varphi : S \to C$ be a relatively minimal elliptic fibration over a compact connected curve C, with at least one section E, and let $\alpha \in \mathrm{Aut}(S)^+_\varphi$. For any $c \in C$, write $S_{c_0} = \varphi^{-1}(\{c\})$ for the fiber of φ over the point c. It is called a *fixed-point fiber for* α if α acts as the identity on S_c, that is, every point of S_c is a fixed-point for α.

According to the first statement in Lemma 7.1.1, there is a unique $g \in \mathfrak{F}_c$ such that the action of α on S_c is equal to the action of g on S_c. Because the action of $\mathfrak{F}_c$ on $S_c \cap S^{\mathrm{reg}}$ is free, see Theorem 6.3.29, it follows that S_c is a fixed point fiber for α if and only if $\alpha(s) = s$ for some $s \in S_c \cap S^{\mathrm{reg}}$. Because both sections E and $\alpha(E)$ intersect S_c in unique points s and s', respectively, and $s,\ s' \in S_c \cap S^{\mathrm{reg}}$, we have that S_c is a fixed-point fiber for α if and only if $s = s'$, that is, if and only if E and $\alpha(E)$ intersect each other at some point of S_c, which necessarily belongs to $S_c \cap S^{\mathrm{reg}}$ because $E \subset S^{\mathrm{reg}}$.

If this is the case, then we can choose $l = 1$ and $l_1 = l_2 = 0$ in Lemma 7.4.1 with c_0 replaced by c, and the multiplicity of the intersection of $\alpha(E)$ and E at s is equal to the multiplicity of the zero of the function T at c. Therefore this multiplicity is both independent of the choice of the section E and of the function T. When finite, this number will be called the *multiplicity of the fixed-point fiber* S_c *for* α, and denoted by $\nu(\alpha,\ c)$.

If α is not equal to the identity, then $\alpha(E) \neq E$, and because E and $\alpha(E)$ are irreducible, it follows that all intersection points of $\alpha(E)$ and E have a finite positive intersection multiplicity, which is equal to the multiplicity of the fixed-point fiber for α through the intersection points. It follows that the topological intersection number $\alpha(E) \cdot E$ is equal to the number of fixed-point fibers of α, counted with multiplicities.

If α is equal to the identity, then every fiber is a fixed point fiber, and the multiplicity is infinite. On the other hand, it follows from Lemma 6.2.34 that $\alpha(E) \cdot E \cdot E = -\chi(S,\ \mathcal{O})$, where $\chi(S,\ \mathcal{O})$ is the holomorphic Euler number of S as defined in (6.2.30). That is, $\nu(\alpha) = -\chi(S,\ \mathcal{O}) < 0$ if α is equal to the identity. If S is a rational elliptic surface, then $\chi(S,\ \mathcal{O}) = 1$; see Lemma 9.1.2(iii).

In order to obtain uniform formulas, we will, for any $\alpha \in \mathrm{Aut}(S)^+_\varphi$, call the topological intersection number $\nu(\alpha) := \alpha(E) \cdot E$ the *number of fixed-point fibers of* α*, counted with multiplicities*. Note that this number is independent of the choice of the section E. The only oddity in this definition is that this number is equal to $-\chi(S,\ \mathcal{O})$ when all the fibers are fixed-point fibers, in which case one would expect $\nu(\alpha) = \infty$.

Remark 7.4.4. If S_c is a singular fiber and α leaves at least one connected component of $S_c \cap S^{\mathrm{reg}}$ invariant, then it follows from the last statement in Theorem 6.3.29 that α acts as the identity on $S_c \cap S^{\mathrm{sing}}$, that is, every element of $S_c \cap S^{\mathrm{sing}}$ is a fixed point for α. If S_c has an irreducible component Θ of multiplicity $\mu_\Theta > 1$, then $\Theta \subset S^{\mathrm{sing}}$, and S_c contains a whole curve Θ of fixed points for α. Nevertheless, S_c will not be counted as a fixed-point fiber for α, unless α has at least one fixed point in $S_c \cap S^{\mathrm{reg}}$, in which case every element of S_c is a fixed point for α. In other words, "S_c is a fixed point fiber for α" does not mean "S_c contains at least one fixed point for α,"

but "every element of S_c is a fixed point for α." These two interpretations agree if and only if S_c is a smooth fiber. ⊘

Definition 7.4.5. Let $k \in \mathbb{Z}$. If $c \in C$, then S_c is called a *k-periodic fiber for $\alpha \in \mathrm{Aut}(S)^+_\varphi$ with multiplicity $\nu(\alpha^k, c)$* if S_c is a fixed-point fiber for α^k with multiplicity $\nu(\alpha^k, c)$. The number $\nu(\alpha^k) := \alpha^k(E) \cdot E$ is called the *number of k-periodic fibers for α, counted with multiplicities*. This number does not depend on the choice of the section.

The only oddity in this definition is that this number is equal to $-\chi(S, \mathcal{O})$ when α^k is equal to the identity, when every fiber is a k-periodic fiber. Also note that if S_c is a k-periodic fiber for α, then for every nonzero integer m, S_c is an $m\,k$-periodic fiber for α with the same multiplicity. That is, in this definition the period need not be the minimal positive period. This convention is essential for obtaining, for a given k, the formula (7.5.2) for the number of k-periodic fibers for α, counted with multiplicities, since this formula is based on the definition of this number as the intersection number of $\alpha^k(E)$ with E.

If C_{α^k} denotes the set of all $c \in C$ such that S_c is a k-periodic fiber for α, then the properties of the topological intersection number imply the formula

$$\nu(\alpha^k) = \sum_{c \in C_{\alpha^k}} \nu(\alpha^k, c) \tag{7.4.1}$$

for $\nu(\alpha^k)$.

Remark 7.4.6. When α^k is not equal to the identity, then the number $\nu(\alpha^k)$ is an upper bound for the set-theoretic number of k-periodic fibers.

If γ is a complex conjugation on S and the automorphism α is real in the sense that α commutes with γ, then $\nu(\alpha^k)$ is also an upper bound for the number of real k-periodic fibers, the k-periodic fibers that contain fixed points of γ. ⊘

The fact that a finite power of $\alpha \in \mathrm{Aut}(S)^+_\varphi$ acts as an Eichler–Siegel transformation on the Néron–Severi group $\Lambda = \mathrm{NS}(S)$, see Corollary 7.3.3, leads to the following conclusion. Note that $\chi(S, \mathcal{O}) = 1$ if S is a rational elliptic surface, see Lemma 9.1.2(iii).

Corollary 7.4.7 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration over a compact connected curve C with at least one section. Let α be an automorphism of S that acts as translations on the fibers of φ. Recall Definition* 7.4.5 *of the number $\nu(\alpha^k) = \alpha^k(E) \cdot E$ of k-periodic fibers for α, counted with multiplicities, where E is any section.*

Then the action of α on $Q = f^\perp/\mathbb{Z}\,f$ has finite order $m \in \mathbb{Z}_{>0}$. For any $k \in \mathbb{Z}$, we have $k = n\,m + l$ for a unique $n \in \mathbb{Z}$ and $l \in \mathbb{Z}$, $0 \le l \le m-1$, and

$$\nu(\alpha^k) = \nu(\alpha^l) + \Big[\nu(\alpha^m) + \chi(S, \mathcal{O}) - \nu(\alpha^{m-l}) + \nu(\alpha^l)\Big]\, n + \big[\chi(S, \mathcal{O}) + \nu(\alpha^m)\big]\, n^2. \tag{7.4.2}$$

If the elliptic fibration $\varphi : S \to C$ has at least one singular fiber, then either α^m is equal to the identity on S, or $\nu(\alpha^k)/k^2 \to c/m^2$ as $k \to \infty$, where $c = \chi(S, \mathcal{O}) + \nu(\alpha^m) \geq \chi(S, \mathcal{O}) > 0$.

If $m = 1$, then $\nu(\alpha^k) = -\chi(S, \mathcal{O}) + (\chi(S, \mathcal{O}) + \nu(\alpha))\, k^2$ for every $k \in \mathbb{Z}$.

Proof. Write $A = \alpha^\Lambda$, when $A^k = (\alpha^k)^\Lambda$ for any $k \in \mathbb{Z}$. It follows from (7.3.3) that $\nu(\alpha^k) = A^k(e) \cdot e = \mathrm{E}_{n\,q}(A^l(e)) \cdot e$, where $q := A^m(e) - e + \mathbb{Z}\, f$. Taking the intersection number with e of (7.3.1), with q and c replaced by $n\,q$ and $c = A^l(e)$, respectively, we obtain the formula (7.4.2), using

$$A^l(e) \cdot f = A^l(e) \cdot A^l(f) = e \cdot f = 1$$

and (7.3.3) with α replaced by α^m.

If $\alpha^m \neq 1$ then it follows from Lemma 7.1.1 that $\alpha^m(E) \neq E$. Because E and $\alpha^m(E)$ are irreducible, this implies that $\nu(\alpha^m) = \alpha^m(E) \cdot E \geq 0$. Because Lemma 6.2.34 implies that $\chi(S, \mathcal{O}) > 0$, this proves the last statement in the corollary. □

In Corollary 7.5.4 we will give a formula for the number of k-periodic fibers of $\alpha \in \mathrm{Aut}(S)^+_\varphi$, counted with multiplicities, in terms of the number of fixed point fibers for α and the way α permutes the irreducible components of the reducible fibers.

In Section 7.7 we give an asymptotic analysis, for $k \to \infty$, of the set of k-periodic fibers, viewed as a subset C_{α^k} of the base curve C, all under the assumption that the element α of the Mordell–Weil group is not of finite order. This analysis yields an alternative proof of the fact that the total number of k-periodic fibers is of order k^2 as $k \to \infty$. However, the asymptotic structure, for $k \to \infty$, of the subset C_{α^k} of C contains much more detailed information, actually sufficient to reconstruct the elliptic surface S up to isomorphism; see for instance Corollary 7.7.8.

7.5 The Contributions of the Reducible Fibers

Let Λ be a lattice, which as a group is isomorphic to $\mathbb{Z}^r$ for some $r \in \mathbb{Z}_{\geq 0}$. For every homomomorphism h from the group $\mathbb{Z}^r$ to a vector space V over $\mathbb{Q}$, there is a unique $\mathbb{Q}$-linear mapping $h_\mathbb{Q} : \mathbb{Q}^r \to V$ such that $h = h_\mathbb{Q} \circ i$, where i denotes the embedding of $\mathbb{Z}^r$ into $\mathbb{Q}^r$. It follows that there are a vector space $\Lambda \otimes \mathbb{Q}$ over $\mathbb{Q}$ and an injective homomorphism of groups $i : \Lambda \to \Lambda \otimes \mathbb{Q}$ such that for every homomorphism h from Λ to a vector space V over $\mathbb{Q}$ there is a unique $\mathbb{Q}$-linear mapping $h_\mathbb{Q} : \Lambda \otimes \mathbb{Q} \to V$ such that $h = h_\mathbb{Q} \circ i$. The vector space $\Lambda \otimes \mathbb{Q}$ over $\mathbb{Q}$ with these properties is uniquely determined up to isomorphism, and is called the *tensor product of Λ with $\mathbb{Q}$*. Every $\mathbb{Z}$-basis e_i of Λ is a $\mathbb{Q}$-basis of $\Lambda \otimes \mathbb{Q}$; hence $\dim_\mathbb{Q} \Lambda \otimes \mathbb{Q} = \mathrm{rank}\, \Lambda$. If $a \in \Lambda \otimes \mathbb{Q}$ then $a = \sum_i r_i\, e_i$ for $r_i \in \mathbb{Q}$, and if $m \in \mathbb{Z}_{>0}$ is the least common multiple of the r_i, we have $m\, a \in \Lambda$. The $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form on Λ has a unique extension to a $\mathbb{Q}$-valued $\mathbb{Q}$-bilinear form on $\Lambda \otimes \mathbb{Q}$, which we again denote by $(a, b) \mapsto a \cdot b$, and which again is nondegenerate.

If A is a sublattice of Λ, then $A \otimes \mathbb{Q}$ is a $\mathbb{Q}$-linear subspace of $\Lambda \otimes \mathbb{Q}$, and its orthogonal complement $B := \{b \in \Lambda \otimes \mathbb{Q} \mid a \cdot b = 0 \ \forall a \in A \otimes \mathbb{Q}\} = \{b \in \Lambda \otimes \mathbb{Q} \mid a \cdot b = 0 \ \forall a \in A\}$ in $\Lambda \otimes \mathbb{Q}$ is a $\mathbb{Q}$-linear subspace of $\Lambda \otimes \mathbb{Q}$, where $\dim_{\mathbb{Q}}(A \otimes \mathbb{Q})^{\perp} = \dim_{\mathbb{Q}}(\Lambda \otimes \mathbb{Q}) - \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}) = \operatorname{rank} \Lambda - \operatorname{rank} A$. Because for every $b \in B$ there exists $m \in \mathbb{Z}_{>0}$ such that $m\, b \in \Lambda$, we have $B = A^{\perp} \otimes \mathbb{Q}$, where $A^{\perp} := \{b \in \Lambda \mid a \cdot b = 0 \ \forall a \in A\}$ denotes the orthogonal complement of A in Λ. It follows that $\operatorname{rank} A^{\perp} = \dim_{\mathbb{Q}} B = \operatorname{rank} \Lambda - \operatorname{rank} A$.

If the restriction of the bilinear form to A is nondegenerate, then $\Lambda \otimes \mathbb{Q}$ is the direct sum of $A \otimes \mathbb{Q}$ and $A^{\perp} \otimes \mathbb{Q}$, and we denote the $\mathbb{Q}$-linear projection from $\Lambda \otimes \mathbb{Q}$ onto $A^{\perp} \otimes \mathbb{Q}$ along $A \otimes \mathbb{Q}$ by $\pi_{A^{\perp}}$, where the $\mathbb{Q}$-linear projection π_A from $\Lambda \otimes \mathbb{Q}$ onto $A \otimes \mathbb{Q}$ along $A^{\perp} \otimes \mathbb{Q}$ is equal to $\pi_A = 1 - \pi_{A^{\perp}}$. Note that $p \cdot q = \pi_A(p) \cdot \pi_A(q) + \pi_{A^{\perp}}(p) \cdot \pi_{A^{\perp}}(q)$ for every $p,\ q \in \Lambda \otimes \mathbb{Q}$.

We apply this to $\Lambda = Q$, $A = Q^{\mathrm{irr}}$, where $A^{\perp}$ is equal to the sublattice Q^0 of Q in Corollary 7.3.3, and we write $\pi^0 := \pi_{Q^0}$ for the orthogonal projection from $Q \otimes \mathbb{Q}$ onto $Q^0 \otimes \mathbb{Q}$, where $\ker \pi^0 = Q^{\mathrm{irr}} \otimes \mathbb{Q}$.

Lemma 7.5.1 *Let $e \in \mathrm{NS}(S)$ be the homology class of any holomorphic section E of φ. The mapping $\mu^0 : \alpha \mapsto \pi^0(\alpha^{\Lambda}(e) - e + \mathbb{Z}\, f)$ is a homomorphism of groups from $\mathrm{Aut}(S)^+_{\varphi}$ to $Q^0 \otimes \mathbb{Q}$, which does not depend on the choice of the section E. The kernel of π^0 is equal to the torsion subgroup $(\mathrm{Aut}(S)^+_{\varphi})_{\mathrm{tor}}$ of the Mordell–Weil group.*

Proof. Because $Q^{\mathrm{irr}} \subset Q^{\mathrm{irr}} \otimes \mathbb{Q} = \ker \pi^0$, the projection π^0 induces a homomorphism from Q/Q^{irr} to $Q^0 \otimes \mathbb{Q}$, which we denote by the same letter π^0. We have $\mu^0 = \pi^0 \circ \mu$, where $\mu : \mathrm{Aut}(S)^+_{\varphi} \xrightarrow{\sim} Q/Q^{\mathrm{irr}}$ is the isomorphism defined in Theorem 7.2.7. Because compositions of homomorphisms are homomorphisms, it follows that μ^0 is a homomorphism.

If $\alpha \in \mathrm{Aut}(S)^+_{\varphi}$, $m \in \mathbb{Z}_{>0}$, and $\alpha^m = 1$, then $0 = \mu^0(\alpha^m) = m\, \mu^0(\alpha)$; hence $\mu^0(\alpha) = 0$ because $Q^0 \otimes \mathbb{Q}$ is torsion-free. Conversely, if $\alpha \in \mathrm{Aut}(S)^+_{\varphi}$ and $0 = \mu^0(\alpha) = \pi^0(\mu(\alpha))$, then $\alpha^{\Lambda}(e) - e + \mathbb{Q}\, f \in Q^{\mathrm{irr}} \otimes \mathbb{Q}$; hence there exists $m \in \mathbb{Z}_{>0}$ such that $m\,(\alpha^{\Lambda}(e) - e) + \mathbb{Z}\, f) \in Q^{\mathrm{irr}}$, or equivalently $\mu(\alpha^m) = m\, \mu(\alpha)$ is equal to zero in Q/Q^{irr}. Because μ is injective, it follows that $\alpha^m = 1$. □

It follows that μ^0 defines an isomorphism of groups from $\mathrm{Aut}(S)^+_{\varphi}/(\mathrm{Aut}(S)^+_{\varphi})_{\mathrm{tor}}$ onto its image $\mu^0(\mathrm{Aut}(S)^+_{\varphi})$, which is a sublattice of Q of rank equal to the rank of the Mordell–Weil group $\mathrm{Aut}(S)^+_{\varphi}$. The pullback by μ^0 of the bilinear form on Q is a $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form on $\mathrm{Aut}(S)^+_{\varphi}/(\mathrm{Aut}(S)^+_{\varphi})_{\mathrm{tor}}$, with respect to which $\mathrm{Aut}(S)^+_{\varphi}/(\mathrm{Aut}(S)^+_{\varphi})_{\mathrm{tor}}$ is a lattice, of rank equal to the rank of the Mordell–Weil group. Shioda [184, Definition 8.5] called $\mathrm{Aut}(S)^+_{\varphi}/(\mathrm{Aut}(S)^+_{\varphi})_{\mathrm{tor}}$ provided with this bilinear form the *Mordell–Weil lattice* of the elliptic surface $\varphi : S \to C$. Actually, Shioda took minus the intersection form in order to make the bilinear form positive definite, but I don't mind working all the time with the negative definite intersection form.

If $\varphi : S \to C$ has no reducible fibers, then $\mathrm{Aut}(S)^{+,0}_{\varphi} = \mathrm{Aut}(S)^+_{\varphi}$, $Q^0 = Q$, $M = \mu = \mu^0$ is an isomorphism from $\mathrm{Aut}(S)^+_{\varphi}$ onto Q, the Mordell–Weil group

is torsion-free, and the Mordell–Weil lattice is equal to the narrow Mordell–Weil lattice.

See Theorem 7.6.6 below for the image of the homomorphism μ^0 in Lemma 7.5.1 in the case that the Néron–Severi lattice is unimodular.

Lemma 7.5.2 *Let $e \in \mathrm{NS}(S)$ be the homology class of any holomorphic section E of φ and let $\alpha \in \mathrm{Aut}(S)^+_\varphi$. Let S_r be a reducible fiber of φ such that α permutes the irreducible components of S_r in a nontrivial way, and let Θ_0 and Θ_k be the irreducible components of S_r intersected by E and $\varphi(E)$, respectively. Then the Q_r-component of $u := \pi_{Q^{\mathrm{irr}}}(\alpha^\Lambda(e) - e + \mathbb{Z}\, f)$ is equal to $u_r = -\sum_j c_j\,[\Theta_j] + \mathbb{Z}\, f$, where the numbers c_j are as in Lemma 7.2.3. We have $u_r \cdot u_r = -c_k$.*

Proof. The first statement follows from Lemma 7.2.3 because $(\alpha(E) - E) \cdot \Theta_j$ is equal to -1, 1, and 0 if $j = 0$, $j = k$, and $j \notin \{0,\ k\}$, respectively.

For the second statement we use the $e_j = [\Theta_j] + \mathbb{Z}\, f$, $1 \le j \le n_r - 1$, as a $\mathbb{Z}$-basis of Q_r, when the equations for the c_j are $\sum_j \mathcal{I}_{i\,j}\, c_j = \delta_{i\,k}$ for all i and $\mathcal{I}_{i\,j} = \Theta_i \cdot \Theta_j$ denotes the intersection matrix. That is, if δ_k denotes the kth standard basis vector in $\mathbb{Q}^{n_r-1}$, then $c = (c_1,\ \ldots,\ c_{n_r-1}) \in \mathbb{Q}^{n_r-1}$ is the solution of $\mathcal{I}\, c = -\delta_k$. If the standard inner product in $\mathbb{Q}^{n_r-1}$ is denoted by $\langle \cdot,\ \cdot \rangle$, then $u_r \cdot u_r = \langle \mathcal{I}\, c,\ c \rangle = -\langle \delta_k,\ c \rangle = -c_k$. □

Following Shioda [184, p. 228], the number $c_k = -u \cdot u$ in Lemma 7.5.2 is called the *contribution* $\mathrm{contr}_r(\alpha)$ *to α of the reducible fiber* S_r, where $\mathrm{contr}_r(\alpha) = 0$ if α does not permute the irreducible components of S_r. According to Lemma 7.2.3, the contributions are as in the following lemma, which I learned from Cox and Zucker [41, (1.19)] and Shioda [184, (8.16)].

Lemma 7.5.3 *Let $r \in C^{\mathrm{red}}$, $\alpha \in \mathrm{Aut}(S)^+_\varphi$, and suppose that the action of α on the set S_r^{irr} of irreducible components of S_r is not equal to the identity. Then*

(A) *If the intersection diagram of S_r is of type $\mathrm{A}^{(1)}_{b-1}$, a cycle with b elements, and α rotates the cycle k steps, then $\mathrm{contr}_r(\alpha) = k\,(b-k)/b$. For $b = 2$ the Kodaira type is I_2 or III, for $b = 3$ we have I_2 or IV, and for $b \ge 4$ we have I_b.*

(D) *Let the intersection diagram of S_r be of type $\mathrm{D}^{(1)}_{b+4}$, that is, S_r is of Kodaira type I^*_b. If α switches the multiplicity-one components at either end of the chain of multiplicity-two components, thereby leaving each multiplicity-two component invariant, then $\mathrm{contr}_r(\alpha) = 1$. If α maps each multiplicity-one component at one end of the chain of multiplicity-two components to a multiplicity-one component at the other end of the chain, thereby inverting the chain, then $\mathrm{contr}_r(\alpha) = (b+4)/4$.*

(E) *If the intersection diagram of S_r is of type $\mathrm{E}^{(1)}_6$, that is, we have Kodaira type IV*, then $\mathrm{contr}_r(\alpha) = 4/3$.*

If the intersection diagram of S_r is of type $\mathrm{E}^{(1)}_7$ = Kodaira type III, then $\mathrm{contr}_r(\alpha) = 3/2$.*

In the next corollary we use Definition 7.4.5 of $\nu(\beta)$ for any $\beta \in \mathrm{Aut}(S)^+_\varphi$, and μ^0 is the homomorphism in Lemma 7.5.1.

Corollary 7.5.4 *Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$. Then*

$$\nu(\alpha) = -\chi(S, \mathcal{O}) - \frac{1}{2}\,\mu^0(\alpha)\cdot\mu^0(\alpha) + \frac{1}{2}\sum_{r\in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha). \tag{7.5.1}$$

For every $k \in \mathbb{Z}$, we have

$$\begin{aligned}\nu(\alpha^k) = {} & -\chi(S, \mathcal{O}) + \frac{1}{2}\sum_{r\in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha^k) \\ & + \left[\chi(S, \mathcal{O}) + \nu(\alpha) - \frac{1}{2}\sum_{r\in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha)\right] k^2,\end{aligned} \tag{7.5.2}$$

where the sum over all $r \in C^{\mathrm{red}}$ of the numbers $\mathrm{contr}_r(\alpha^k)$ is periodic as a function of k.

Proof. We have the orthogonal sum decomposition of

$$\alpha^\Lambda(e) - e + \mathbb{Z}\,f = \mu^0(\alpha) + \sum_{r\in C^{\mathrm{red}}} u_r,$$

where the $u_r \in Q_r$ are as in Lemma 7.5.2. In view of the orthogonality, the self-intersection number of the right-hand side is equal to $\mu^0(\alpha)\cdot\mu^0(\alpha)$ plus the sum over all r of $u_r \cdot u_r = -c_k = -\mathrm{contr}_r(\alpha)$. On the other hand, $\alpha(E)\cdot\alpha(E) = E\cdot E = -\chi(S, \mathcal{O})$, see Lemma 6.2.34, and $\alpha(E)\cdot E = \nu(\alpha)$, see Definition 7.4.3, yield that the self-intersection number of the left-hand side is equal to $(\alpha(E)-E)\cdot(\alpha(E)-E) = \alpha(E)\cdot\alpha(E) - 2\,\alpha(E)\cdot E + E\cdot E = -2\,(\chi(S, \mathcal{O}) + \nu(\alpha)$. A comparison of the self-intersection numbers of the left- and right-hand side yields (7.5.1).

The fact that μ^0 is a homomorphism implies that $\mu^0(\alpha^k) = k\,\mu^0(\alpha)$, hence $\mu^0(\alpha^k)\cdot\mu(\alpha^k) = k^2\,\mu^0(\alpha)\cdot\mu^0(\alpha)$. If we substitute this in (7.5.1) with α replaced by α^k, and then use (7.5.1) in order to express $\mu^0(\alpha)\cdot\mu^0(\alpha)$ in terms of $\nu(\alpha)$ and the contributions $\mathrm{contr}_r(\alpha)$, we arrive at (7.5.2).

The sum s_k over all $r \in C^{\mathrm{red}}$ of the contributions of S_r to α^k is defined in terms of the way α^k permutes the finitely many irreducible components of reducible fibers. Because any permutation of a finite set is periodic, it follows that s_k is periodic as a function of k. □

Because $a\cdot b = ((a+b)\cdot(a+b) - a\cdot a - b\cdot b)/2$, the formula (7.5.1) leads to a determination of the intersection form, that is, the lattice structure, on $\mathrm{Aut}(S)^+_\varphi/(\mathrm{Aut}(S)^+_\varphi)_{\mathrm{tor}}$, in terms of the numbers of fixed fibers and the actions on the sets of irreducible components of reducible fibers, of elements of the group $\mathrm{Aut}(S)^+_\varphi$.

Another remarkable consequence of (7.5.1) is the following characterization of periodic elements of the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\varphi$, where an element α of a group A is called *periodic*, or *of finite order*, or *a torsion element of A*, if $\alpha^m = 1$ for some $m \in \mathbb{Z}_{>0}$. We recall Definition 7.4.5 of the number $\nu(\alpha^k)$ of k-periodic

fibers for α. Corollary 7.5.5 follows from Cox and Zucker [41, (1.15), (1.18), (1.19)] and Shioda [184, (8.10), (8.12)], and more directly from Oguiso and Shioda [155, Proposition 3.5, (ii)–(iv)].

Corollary 7.5.5 *Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$ be not equal to the identity in S. Then*

$$\sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha) \le 2(\chi(S,\ \mathcal{O}) + \nu(\alpha)). \tag{7.5.3}$$

The following statements are equivalent:

(i) *α is a periodic transformation of S.*
(ii) *We have equality in (7.5.3).*
(iii) *α has no fixed points in S^{reg}, that is, $\nu(\alpha) = 0$, and*

$$\sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha) = 2\chi(S,\ \mathcal{O}). \tag{7.5.4}$$

If α is not periodic, then

$$\lim_{k\to\infty} \nu(\alpha^k)/k^2 = \chi(S,\ \mathcal{O}) + \nu(\alpha) - \frac{1}{2} \sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha) > 0. \tag{7.5.5}$$

Proof. The inequality (7.5.3) follows from (7.5.1) and $\mu^0(\alpha) \cdot \mu^0(\alpha) \le 0$. Because the intersection form is negative definite on Q, we have equality if and only if $\mu^0(\alpha) = 0$ if and only if $\alpha \in \ker \mu^0 = (\mathrm{Aut}(S)^+_\varphi)_{\mathrm{tor}}$; see Lemma 7.5.1. This proves (i) $\Leftrightarrow$ (ii).

Let $s \in S^{\mathrm{reg}}$ and $\alpha(s) = s$. Then the whole fiber through s consists of fixed points for α, and therefore we can arrange in Lemma 7.4.1 that $s_0 = s_1 = s$. We have $\alpha^k(\Phi(c,\ t)) = \Phi(c,\ t + k\ f(c))$ and $p(c) \equiv 0$. If f has a zero of finite order m at c_0, then for any $k \in \mathbb{Z}_{>0}$, $k\ f$ has a zero of the same order m at c_0, which in particular implies that α^k is not equal to the identity. On the other hand, if f has a zero of infinite order at c_0, then $f = 0$ on a neighborhood of c_0. Therefore, if α has finite order, then the set F of fixed points of α in S^{reg} is an open subset of S^{reg}, and because the set of fixed points is also closed, it is a connected component of S^{reg}. Because S is connected and $S^{\mathrm{sing}} = S \setminus S^{\mathrm{reg}}$ is a proper complex analytic subvariety that has real codimension two, S^{reg} is connected, and $F = S^{\mathrm{reg}}$ or $F = \emptyset$. Because S^{reg} is dense in S, $F = S^{\mathrm{reg}}$ implies that α is equal to the identity on S, in contradiction to the assumption. Therefore $F = \emptyset$, that is, $\nu(\alpha) = 0$. In view of (i) $\Leftrightarrow$ (ii), this proves (i) $\Leftrightarrow$ (iii).

The asymptotic formula (7.5.5) for the number of k-periodic fibers follows from (7.5.2) and the fact that the sum of the contributions to α^k is periodic, hence bounded, as a function of k. □

7.6 More about the Mordell–Weil lattices

In this section we will discuss the lattice Q, its sublattices Q^{irr} and the narrow Mordell–Weil lattice Q^0, and the Mordell–Weil lattice $(Q/Q^{\mathrm{irr}})/(Q/Q^{\mathrm{irr}})_{\mathrm{tor}}$ in more detail. For this discussion we need a number of basic facts about lattices.

Let Λ be a lattice. Because Λ is torsion-free, it is isomorphic as a group to $\mathbb{Z}^r$, where $r = \operatorname{rank}\Lambda$. That is, there are $e_1, \ldots, e_r \in \Lambda$, called a *$\mathbb{Z}$-basis* of Λ, such that the mapping

$$(m^1, \ldots, m^r) \mapsto \sum_{i=1}^{r} m^i\, e_i$$

is an isomorphism from $\mathbb{Z}^r$ to Λ. $f_1, \ldots f_r$ is a $\mathbb{Z}$-basis of Λ if and only if $f_j = \sum_i m^i_j\, e_i$ for an integral $r \times r$ matrix $M = (m^i_j)$ such that the inverse M^{-1} also is integral, which in view of Cramer's formula and $1 = \det M\ \det M^{-1}$ is equivalent to the condition that M is integral and $\det M = \pm 1$. For any $\mathbb{Z}$-basis e_i, the symmetric bilinear form is uniquely determined by the $r \times r$ matrix $\mathcal{I}_e : e_i \cdot e_j$, $1 \le i,\ j \le r$, called the *matrix of the lattice Λ with respect to the $\mathbb{Z}$-basis $e = (e_i)$*. If f is another $\mathbb{Z}$-basis of Λ, then $\mathcal{I}_f = M\,\mathcal{I}_e\,{}^{\mathrm{t}}M$, where ${}^{\mathrm{t}}M$ denotes the transpose of M; hence $\det \mathcal{I}_f = \det \mathcal{I}_e\ (\det M)^2 = \det \mathcal{I}_e$. That is, the determinant $\det \mathcal{I}_e$ is independent of the choice of the $\mathbb{Z}$-basis e of Λ, and will be called the *determinant* $\det\Lambda$ *of the lattice* Λ. Note that $\det\Lambda > 0$ if and only if the number of negative eigenvalues of $\mathcal{I}_e$ is even, in particular if the bilinear form on Λ is positive definite. Originally lattices were defined as discrete rank-r additive subgroups Λ of $\mathbb{R}^r$, with the restriction to $\Lambda \times \Lambda$ of the standard innner product of $\mathbb{R}^r$ as the bilinear form; such lattices are positive definite. The lattice Λ is called *unimodular* if Λ is integral and $|\det(\Lambda)| = 1$.

Lemma 7.6.1 *Let M be a subgroup of finite index of the lattice Λ. Then*

$$\det M / \det \Lambda = \#(\Lambda/M)^2. \tag{7.6.1}$$

Proof. Let $n := \operatorname{rank}\Lambda = \operatorname{rank} M$ and let e_i, $1 \le i \le n$, and f_i, $1 \le i \le n$, be $\mathbb{Z}$-bases of Λ and M, respectively. Then

$$f_i = \sum_{j=1}^{n} m^j_i\, e_j, \quad 1 \le i \le n,$$

for a unique $\mathbb{Z}$-valued $n \times n$ matrix $m = (m^j_i)$, and a comparison of the volumes of the parallelepipeds spanned by the $\mathbb{Z}$-bases yields that $\#(\Lambda/M) = |\det m|$. On the other hand, if $(\mathcal{I}_e)_{i\,j} := e_i \cdot e_j$ and $(\mathcal{I}_f)_{i,\,j} := f_i \cdot f_j$, then $\mathcal{I}_f = {}^{\mathrm{t}}m\,\mathcal{I}_e\,m$, and therefore

$$\det \mathcal{I}_e = (\det {}^{\mathrm{t}}m)\,(\det \mathcal{I}_f)\,(\det m) = (\det \mathcal{I}_f)\,(\det m)^2,$$

which in turn implies (7.6.1). □

The *dual, reciprocal, or polar lattice* of Λ is defined as the set Λ^* of all $\mu \in \Lambda \otimes \mathbb{Q}$ such that $\mu \cdot \lambda \in \mathbb{Z}$ for every $\lambda \in \Lambda$.

Lemma 7.6.2 *We have*

$$\det \Lambda^* = 1/\det \Lambda. \tag{7.6.2}$$

The lattice Λ is integral if and only if $\Lambda \subset \Lambda^*$, when

$$\#(\Lambda^*/\Lambda) = |\det \Lambda| = 1/|\det \Lambda^*|. \tag{7.6.3}$$

Proof. If the e_j form a $\mathbb{Z}$-basis of Λ then the $f_i \in \Lambda \otimes \mathbb{Q}$ such that $f_i \cdot e_j = \delta_{i\,j}$ form a $\mathbb{Z}$-basis of Λ^*. There are unique $m^i_j \in \mathbb{Q}$ such that $e_j = \sum_i m^i_j f_i$, when $e_h \cdot e_j = \sum_i m^i_j e_h \cdot f_i = \sum_i m^i_j \delta_{i\,h} = m^h_j$ and $\delta_{h\,j} = f_h \cdot e_j = \sum_i m^i_j f_h \cdot f_i$ shows that the matrix $M = (m^i_j)$ is equal to $\mathcal{I}_e$ and to the inverse of the matrix $\mathcal{I}_f$. It follows that $\mathcal{I}_e \mathcal{I}_f = 1$, which implies (7.6.2).

If $\Lambda \subset \Lambda^*$, the equation (7.6.3) follows from (7.6.2) and (7.6.1) with Λ and M replaced by Λ^* and Λ, respectively. □

Note that $\Lambda^* = \Lambda$ if and only if Λ is a unimodular integral lattice.

Example 6. For any reducible fiber S_r of φ, Let Q_r be the sublattice of Q generated by the homology classes modulo $\mathbb{Z}\,f$ of the irreducible components of S_r; see the paragraph preceding Theorem 7.2.7. According to Theorem 7.2.7, Q_r provided with minus the intersection form is a root lattice of type A_n, D_n, or E_n. In particular, Q_r is an integral lattice, and actually an even lattice. According to Remark 6.3.32, the weight lattice modulo the root lattice is isomorphic to the group $\mathfrak{F}_r/\mathfrak{F}_r^\circ$, which according to Corollary 6.3.31 acts freely and transitively on the set $S_r^{\mathrm{irr},\,1}$ of all multiplicity one irreducible components of S_r. According to Bourbaki [23, Proposition 26 on p. 166] the weight lattice is equal to the dual lattice of the root lattice, and therefore

$$Q_r{}^*/Q_r \simeq \mathfrak{F}_r/\mathfrak{F}_r^\circ, \tag{7.6.4}$$

whence (7.6.3) yields

$$|\det Q_r| = \#(Q_r{}^*/Q_r) = n_r^{(1)} := \#(S_r^{\mathrm{irr},\,1}). \tag{7.6.5}$$

It follows from the description of the singular fibers in Section 6.2.6 that the number $n_r^{(1)}$ is equal to $n+1$, 4, 3, 2, and 1 for the respective types A_n, D_n, E_6, E_7, and E_8. It follows that Q_r is unimodular if and only if the root lattice is of type E_8 if and only if S_r is of Kodaira type II*. In Bourbaki [23, p. 167] the number of elements of the weight lattice modulo the root lattice is called the "indice de connexion," which according to Bourbaki [23, Planche I, IV, V, VI, VII] agrees with the aforementioned number $n_r^{(1)}$. Because the Cartan matrix is equal to minus the intersection matrix, the absolute value of the determinant of which is equal to $n_r^{(1)}$, see (7.6.5), this also confirms that the "indice de connexion" is equal to the determinant of the Cartan matrix. ⊘

Lemma 7.6.3 *Let M be a subgroup of a lattice Λ. Write*

$$M' := (M \otimes \mathbb{Q}) \cap \Lambda = \{\lambda \in \Lambda \mid \exists m \in \mathbb{Z}_{>0}\, m\,\lambda \in M\}, \tag{7.6.6}$$

which is a subgroup of Λ containing M. Then $M'/M = (\Lambda/M)_{\mathrm{tor}}$, the torsion subgroup of Λ/M.

Proof. If $\lambda \in \Lambda$ then $\lambda + M \in (\Lambda/M)_{\mathrm{tor}}$ if and only if there exists $m \in \mathbb{Z}_{>0}$ such that $m\,\lambda + M = m\,(\lambda + M) = M$, the zero element of Λ/M, and this condition is equivalent to $m\,\lambda \in M$. □

Corollary 7.6.4 *We have* $\det \mathrm{NS}(S) = -\det Q$,

$$\prod_{r \in C^{\mathrm{red}}} n_r^{(1)} = |\det((Q^{\mathrm{irr}})')|\ \#((\mathrm{Aut}(S)_\varphi^+)_{\mathrm{tor}})^2, \tag{7.6.7}$$

and, with the notation of (7.3.2) *and Corollary* 7.3.3,

$$|\det(\mathrm{NS}(S))| \prod_{r \in C^{\mathrm{red}}} n_r^{(1)} = |\det(Q^0)|\ \#(\Gamma/\varpi(\mathrm{Aut}(S)_\varphi^+))^2. \tag{7.6.8}$$

If the left-hand side in (7.6.7) *is square-free, then the Mordell–Weil group is torsion-free. If the left-hand side of* (7.6.8) *is square-free, then ϖ is a surjective homomorphism from the Mordell–Weil group onto Γ.*

Proof. Theorem 7.2.7 and Lemma 7.6.3 with $\Lambda = Q$ and $M = Q^{\mathrm{irr}}$ imply $(\mathrm{Aut}(S)_\varphi^+)_{\mathrm{tor}} \simeq (Q/Q^{\mathrm{irr}})_{\mathrm{tor}} = (Q^{\mathrm{irr}})'/Q^{\mathrm{irr}}$, when Lemma 7.6.1 with $\Lambda = (Q^{\mathrm{irr}})'$ and $M = Q^{\mathrm{irr}}$ yields

$$\#((\mathrm{Aut}(S)_\varphi^+)_{\mathrm{tor}})^2 = \#((Q^{\mathrm{irr}})'/Q^{\mathrm{irr}})^2 = \det Q^{\mathrm{irr}}/\det(Q^{\mathrm{irr}})'.$$

Equation (7.6.7) follows because its left-hand side is equal to $|\det Q^{\mathrm{irr}}|$ in view of (7.6.5).

The equation $\det \mathrm{NS}(S) = -\det Q$ follows from the proof of Theorem 7.2.7. Lemma 7.6.1 with $\Lambda = Q$ and $M = Q^{\mathrm{irr}} \oplus Q^0$ yields

$$\det Q^{\mathrm{irr}}\ \det Q^0 = \det(Q^{\mathrm{irr}} \oplus Q^0) = \det Q\ \#(Q/(Q^{\mathrm{irr}} \oplus Q^0))^2. \tag{7.6.9}$$

On the other hand, Corollary 7.3.3 implies that the isomorphism μ from $\mathrm{Aut}(S)_\varphi^+$ onto Q/Q^{irr} of Theorem 7.2.7 induces an isomorphism from $\mathrm{Aut}(S)_\varphi^+/\mathrm{Aut}(S)_\varphi^{+,0}$ onto

$$(Q/Q^{\mathrm{irr}})/((Q^{\mathrm{irr}} \oplus Q^0)/Q^{\mathrm{irr}}) \simeq Q/(Q^{\mathrm{irr}} \oplus Q^0).$$

Because $\mathrm{Aut}(S)_\varphi^+/\mathrm{Aut}(S)_\varphi^{+,0} \simeq \varpi(\mathrm{Aut}(S)_\varphi^+)$ in view of Corollary 7.3.3, this leads to $\#(\varpi(\mathrm{Aut}(S)_\varphi^+)) = \#(Q/(Q^0 + Q^{\mathrm{irr}}))$. Using (7.6.5) and multiplying both sides of (7.6.9) by $\#(\Gamma)$, we arrive at (7.6.8). □

The following lemma is a rephrasing of Conway and Sloane [38, Theorem 1 on p. 100].

Lemma 7.6.5 *Let Λ be a unimodular integral lattice and M a subgroup of Λ such that the $\mathbb{Z}$-bilinear form restricted to $M \times M$ is nondegenerate. Assume, moreover, that $M' = M$, see* (7.6.6)*, and write $N := M^\perp = \{\lambda \in \Lambda \mid \lambda \cdot \mu = 0 \ \forall \mu \in M\}$. Then the orthogonal projections from $\Lambda \otimes \mathbb{Q}$ onto $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$ define group isomorphisms from $\Lambda/(M+N)$ onto M^*/M and from $\Lambda/(M+N)$ onto N^*/N.*

Proof. The condition $M' = M$ means that $M = N^\perp$, and conversely $N = M^\perp$ implies that $N' = N$. That is, we have $M' = M = N^\perp$ and $N' = N = M^\perp$.

For every $\lambda \in \Lambda$ there are unique $\overline{\mu} \in M \otimes \mathbb{Q}$ and $\overline{\nu} \in N \otimes \mathbb{Q}$ such that $\lambda = \overline{\mu} + \overline{\nu}$. For every $\mu \in M$ we have $\mathbb{Z} \ni \lambda \cdot \mu = \overline{\mu} \cdot \mu$, hence $\overline{\mu} \in M^*$. If $\overline{\mu} \in M$, then $\overline{\nu} = \lambda - \overline{\mu} \in \Lambda \cap (N \otimes \mathbb{Q}) = M^\perp = N$, hence $\lambda \in M + N$, and conversely $\lambda \in M + N$ implies that $\overline{\mu} \in M$. Therefore $\lambda \mapsto \overline{\mu}$ defines an injective homomorphism from $\Lambda/(M+N)$ to M^*/M.

Lemma 7.6.3 implies that Λ/M is torsion-free, hence it has a $\mathbb{Z}$-basis $\lambda_j + M$, $1 \le j \le n = \operatorname{rank}\Lambda - \operatorname{rank} M = \operatorname{rank} N$, where $\lambda_j \in \Lambda$. For any $\mathbb{Z}$-basis μ_i, $1 \le i \le \operatorname{rank} M$, of M, the μ_i and λ_j together form a $\mathbb{Z}$-basis of Λ, which implies that every homomorphism $M \to \mathbb{Z}$ has an extension to a homomorphism $\Lambda \to \mathbb{Z}$, which can be taken to be equal to zero on the λ_j. Let $\overline{\mu} \in M^*$. Then $\mu \mapsto \overline{\mu} \cdot \mu$ is a homomorphism from M to $\mathbb{Z}$. It has an extension to a homomorphism from Λ to $\mathbb{Z}$, which is of the form $\lambda \mapsto \overline{\lambda} \cdot \lambda$ for a unique $\overline{\lambda} \in \Lambda^* = \Lambda$. For every $\mu \in M$ we have $\overline{\lambda} \cdot \mu = \overline{\mu} \cdot \mu$, hence $\overline{\nu} := \overline{\lambda} - \overline{\mu}$ belongs to the orthogonal complement $N \otimes \mathbb{Q}$ of M in $\Lambda \otimes \mathbb{Q}$. This proves that the homomorphism from $\Lambda/(M+N)$ to M^*/M is surjective. The proof that $\lambda \mapsto \overline{\nu}$ defines an isomorphism of groups from $\Lambda/(M+N)$ onto N^*/N is analogous. □

The following theorem is Shioda [184, Theorem 9.2], where I have added (7.6.11).

Theorem 7.6.6 *Assume that the Néron–Severi lattice of S is unimodular. Then the Mordell–Weil lattice is equal to the dual of the narrow Mordell–Weil lattice, in the sense that the homomorphism μ^0 in Lemma* 7.5.1 *induces an isomorphism of groups from $\operatorname{Aut}(S)^+_\varphi/(\operatorname{Aut}(S)^+_\varphi)_{\mathrm{tor}}$ onto $(Q^0)^*$, which we again denote by μ^0. We have the identities*

$$\prod_{r \in C^{\mathrm{red}}} n_r^{(1)} = |\det Q^0| \, \#((\operatorname{Aut}(S)^+_\varphi)_{\mathrm{tor}})^2 \tag{7.6.10}$$

and, with the notation (7.3.2),

$$\#(\Gamma/\varpi(\operatorname{Aut}(S)^+_\varphi)) = \#((\operatorname{Aut}(S)^+_\varphi)_{\mathrm{tor}}). \tag{7.6.11}$$

Proof. The homomorphism μ^0 in Lemma 7.5.1 corresponds to the orthogonal projection $\pi^0 : Q/Q^{\mathrm{irr}} \to (Q^0) \otimes \mathbb{Q}$, with kernel equal to $(Q/Q^{\mathrm{irr}})_{\mathrm{tor}}$. Lemma 7.6.3 with $\Lambda = Q$ and $M = Q^{\mathrm{irr}}$ implies that $\ker \pi^0 = (Q^{\mathrm{irr}})'/Q^{\mathrm{irr}}$, and therefore μ^0 induces an injective homomorphism $Q/(Q^{\mathrm{irr}})' \to (Q^0) \otimes \mathbb{Q}$, which we again denote by μ^0. For every $q \in Q$ we have $q - \pi^0(q) \in Q^{\mathrm{irr}} \otimes \mathbb{Q}$ hence $\pi^0(q) \cdot \nu = q \cdot \nu \in \mathbb{Z}$ for every $\nu \in Q^0$, and therefore $\pi^0(q) \in (Q^0)^*$. If $\pi^0(Q) \in Q^0 \subset Q$, then

$q - \pi^0(q) \in (Q^{\mathrm{irr}} \otimes \mathbb{Q}) \cap Q = (Q^{\mathrm{irr}})'$, and therefore π^0 induces an injective homomorphism from $Q/(Q^0 + (Q^{\mathrm{irr}})')$ to $(Q^0)^*/Q^0$. The assumption that $\mathrm{NS}(S)$ is unimodular is equivalent to the assumption that Q is unimodular; see Corollary 7.6.4, whence Lemma 7.6.5 with $\Lambda = Q$, $M = (Q^{\mathrm{irr}})'$ and $N = Q^0$ implies that the homomorphism from $Q/(Q^0 + (Q^{\mathrm{irr}})')$ to $(Q^0)^*/Q^0$ is an isomorphism, and we conclude that the image of μ^0 is equal to $(Q^0)^*$.

The identities (7.6.10) and (7.6.11) follow from (7.6.7), (7.6.8), $|\det(\mathrm{NS}(S))| = 1$, and $|\det((Q^{\mathrm{irr}})')| = \#(M^*/M) = \#(N^*/N) = |\det(Q^0)|$, where we have used (7.6.3) and $M^*/M \simeq N^*/N$; see Lemma 7.6.5. □

7.7 Asymptotics of the k-Periodic Fibers

Let denote the set of all $c \in C$ such that S_c is a k-periodic fiber for α. In this section we discuss the asymptotic behavior for $k \to \infty$ of the subset C_{α^k} of the complex analytic curve C. We will assume throughout this subsection that $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is not of finite order. We begin with a discussion of the fact that the elliptic fibration $\varphi : S \to C$ can, over simply connected open subsets of C^{reg}, be viewed as a real analytic principal T-bundle, where $T = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ is the standard real two-dimensional torus group.

For each $c \in C^{\mathrm{reg}}$, the group $\mathfrak{F}_c$ of translations in the fiber S_c is a compact, connected, complex one-dimensional complex Lie group, depending in a complex analytic way on c, see Section 6.4.2. The Lie algebra $\mathfrak{f}_c$ of $\mathfrak{F}_c$ is the complex one-dimensional complex vector space of all holomorphic vector fields on S_c. Note that the complex structure of $\mathfrak{f}_c$ defines an orientation of $\mathfrak{f}_c$. The kernel of the exponential mapping $\mathfrak{f}_c \to \mathfrak{F}_c$ is the period lattice P_c in $\mathfrak{f}_c$, and the exponential mapping induces an isomorphism of complex Lie groups from $\mathfrak{f}_c/P_c$ onto $\mathfrak{F}_c$.

Let $v_1(c)$, $v_2(c)$ be a positively oriented $\mathbb{Z}$-basis of the kernel of the exponential mapping from $\mathfrak{f}_c$ onto $\mathfrak{F}_c$. Then the mapping

$$\Phi_c : (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \ni (t_1,\ t_2) \mapsto \mathrm{e}^{t_1\, v_1(c) + t_2\, v_2(c)} \tag{7.7.1}$$

is an orientation-preserving isomorphism of real Lie groups from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$, and every orientation-preserving isomorphism of real Lie groups from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$ arises in this way. Every other positively oriented $\mathbb{Z}$-basis $v_1^\sim$, $v_2^\sim$ is of the form

$$v_1^\sim = M_1^1\, v_1(c) + M_1^2\, v_2(c), \quad v_2^\sim = M_2^1\, v_1(c) + M_2^2\, v_2(c)$$

for a uniquely determined $M \in \mathrm{SL}(2,\ \mathbb{Z})$, and the isomorphism from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$ defined by $v_1^\sim$, $v_2^\sim$ is equal to the orientation-preserving automorphism

$$(t_1^\sim,\ t_2^\sim) \mapsto (t_1^\sim\, M_1^1 + t_2^\sim\, M_2^1,\ t_1^\sim\, M_1^2 + t_2^\sim\, M_2^2) \tag{7.7.2}$$

of $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, defined by the matrix $M \in \mathrm{SL}(2,\ \mathbb{Z})$, followed by the isomorphism from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$ defined by $v_1(c)$, $v_2(c)$.

If $v_1(c)$, $v_2(c)$ locally depend continuously on c, then these vector fields locally depend holomorphically on c, and therefore the isomorphism (7.7.2) locally depends in a real analytic way on c. Note that if v is a holomorphic local section of the holomorphic complex line bundle $\mathfrak{f}$ as in Lemma 6.2.38, with corresponding local holomorphic positively oriented $\mathbb{Z}$-basis $p_1(c),\ p_2(c) \in \mathbb{C}$ of the period group in $\mathbb{C}$, then we can take $v_1(c) = p_1(c)\, v(c)$ and $v_2(c) = p_2(c)\, v(c)$. The homology classes $[\gamma_i]$ of the closed real curves $\mathbb{R}/\mathbb{Z} \ni t \mapsto \mathrm{e}^{t\, v_i(c)}(s_c)$ in S_c, where s_c is any choice of a base point in S_c, form a $\mathbb{Z}$-basis of $\mathrm{H}_1(S_c,\ \mathbb{Z})$, and every $\mathbb{Z}$-basis of $\mathrm{H}_1(S_c,\ \mathbb{Z})$, locally depending continuously on c, arises in this way for a $\mathbb{Z}$-basis $v_1(c)$, $v_2(c)$ of the kernel of the exponential mapping, locally depending holomorphically on c. In this way the freedom in the choice of the orientation-preserving isomorphism from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$, or the choice of the oriented $\mathbb{Z}$-basis of the exponential mapping from $\mathfrak{f}_c$ onto $\mathfrak{F}_c$, corresponds to the freedom of the choice of an oriented $\mathbb{Z}$-basis in $\mathrm{H}_1(S_c,\ \mathbb{Z})$.

For any open subset C_0 of C^{reg}, the restriction to $\varphi^{-1}(C_0)$ of $\varphi : S \to C$ is a real analytic principal $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$-bundle over C_0 if and only if there exists an orientation-preserving isomorphism from $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ onto $\mathfrak{F}_c$, that depends continuously on $c \in C^{\mathrm{reg}}$ if and only if the monodromy representation of Section 6.2.11 is trivial over C_0, which for instance is the case if C_0 is simply connected. If the monodromy is trivial over C_0, then C_0 does not contain any small loop around any point of C^{sing}, because the monodromy around any singular fiber is nontrivial; see Table 6.2.40. If the monodromy is trivial over C^{reg} and C is compact, then the elliptic fibration $\varphi : S \to C$ is extremely special; see Section 6.4.4.

Definition 7.7.1. Let $v_1(c)$, $v_2(c)$ be a positively oriented $\mathbb{Z}$-basis of the kernel of the exponential mapping from $\mathfrak{f}_c$ to $\mathfrak{F}_c$, that depends in a continuous, hence holomorphic, multivalued way on $c \in C^{\mathrm{reg}}$, where the multivaluedness is described by the monodromy representation in Section 6.2.11. Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$, an automorphism of S that acts as translations on the fibers of φ over C^{reg}. Then the unique multivalued mapping

$$R = (R_1,\ R_2) : C^{\mathrm{reg}} \to (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}),$$

defined by

$$\alpha(s) = \mathrm{e}^{R_1(c)\, v_1(c) + R_2(c)\, v_2(c)}(s), \quad \varphi(s) = c, \quad c \in C^{\mathrm{reg}},$$

is called the *rotation map* of the automorphism α, with respect to the continuous choice of the $\mathbb{Z}$-basis of the kernel of the exponential mapping = the continuous choice of the $\mathbb{Z}$-basis of the homology groups $\mathrm{H}_1(S_c,\ \mathbb{Z})$, $c \in C^{\mathrm{reg}}$, of the smooth fibers.

The rotation map is real analytic. The real analytic real-valued area form

$$\delta_\alpha := \mathrm{d}R_1 \wedge \mathrm{d}R_2$$

on C^{reg} does not depend on the choice of the aforementioned continuous $\mathbb{Z}$-basis, and is called the *density of* α on C^{reg}. ⊘

Remark 7.7.2. As discussed in Remark 6.2.22, the elliptic fibration near a given fiber can be viewed as a Lagrangian fibration in a real four-dimensional manifold with respect to the symplectic form equal to the real or the imaginary part of the holomorphic complex two-form ω, and $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is a symplectic diffeomorphism with respect to each of these. Near a smooth fiber, or near a singular fiber of whose irreducible components are not permuted by α, the automorphism α can be written as the flow after time 1 of a Hamiltonian vector field v, which is completely integrable because it preserves the Lagrangian fibration. If we write $v = v_1\, p_1 + v_2\, p_2$, where $p_1,\ p_2$ is the $\mathbb{Z}$-basis of the period lattice, then the mapping $(v_1,\ v_2) : C^{\mathrm{reg}} \to \mathbb{R}^2$ is called the *frequency map* of the Hamiltonian vector field v. The rotation map of α is equal to the frequency map of v followed by the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, that is, the rotation map of α is equal to the frequency map of v modulo $\mathbb{Z}^2$. The points $c \in C^{\mathrm{reg}}$ where the density δ_α is nonzero are precisely the points where the frequency map of v is a local diffeomorphism, and this is the set of regular values of φ where Kolmogorov's condition for the KAM theorem holds. A theorem of Arnol'd and Avez [5, Chapter 4, Theorem 21.11 and Appendix 34] implies that, fixing the symplectic form and assuming that $\delta_\alpha(c) \neq 0$, real diffeomorphisms $\alpha^\sim$ that are close to α and are what is called global canonical transformation, have many $\alpha^\sim$-invariant tori near S_c on each of which $\alpha^\sim$ acts a as a translation. ⊘

In the next lemma we collect a number of analytic properties of the rotation map. For the existence of points in D, points in C^{reg} where the density of α is equal to zero, see Remark 7.7.14.

Lemma 7.7.3 *Let $c_0 \in C^{\mathrm{reg}}$ and let z be a local holomorphic coordinate near c_0 such that $z = 0$ at c_0. Write $q(z) = p_2(z)/p_1(z)$, where $p_1(z),\ p_2(z)$ is a positively oriented $\mathbb{Z}$-basis of the period group of S_c when $z(c) = z$, defined by a local nonzero holomorphic section v of $\mathfrak{f}$. Note that $\mathrm{Im}\, q(0) > 0$. Suppose that the rotation map of $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is not constant in a neighborhood of c_0. Let $L : \mathbb{C} \to \mathbb{R}^2$ be the bijective real linear mapping such that $L(p_1(0)) = (1,\ 0)$ and $L(p_2(0)) = (0,\ 1)$, where we note that $\det L > 0$ because $\mathrm{Im}(p_2(0)/p_1(0)) > 0$. Furthermore, define $a_k \in \mathbb{C}$ by*

$$k!\, a_k := T^{(k)}(0) - R_1(0)\, p_1^{(k)}(0) - R_2(0)\, p_2^{(k)}(0) \tag{7.7.3}$$

for every $k \in \mathbb{Z}_{\geq 0}$. Then there exist a unique $d \in \mathbb{Z}_{>0}$ such that $u_k = 0$ for $0 \leq k < d$, $u_d \neq 0$, and, asymptotically for $z \to 0$,

$$(R_1(z),\ R_2(z)) = (R_1(0),\ R_2(0)) + L(a_d\, z^d) + \mathrm{O}(|z|^{d+1}) \tag{7.7.4}$$

and, with the notation $z = x + \mathrm{i}\, y,\ x,\ y \in \mathbb{R}$,

$$\mathrm{d}R_1 \wedge \mathrm{d}R_2/\,\mathrm{d}x \wedge \mathrm{d}y = \det L\, (|a_d|\, d\, |z|^{d-1})^2 + \mathrm{O}(|z|^{2d-1}). \tag{7.7.5}$$

We have $d = 1$ if and only if the derivative $R'(c_0)$ at c_0 of the rotation map is nonzero, and then $R'(c_0)$ is the bijective linear mapping from $\mathrm{T}_{c_0}C$ onto $\mathbb{R}^2$ such

that $R'(c_0)(p_1(0)/a_1) = (1,\ 0)$ *and* $R'(c_0)(p_2(0)/a_1) = (0,\ 1)$. *The rotation map is orientation-preserving at the regular points.*

The singular points of the rotation map, which are the points where the degree of the rotation map is > 1, *and also the points where the density is equal to zero, form a discrete subset* D *of* C^{reg}. *Near a singular point, the rotation map is a* d*-fold branched covering, with* d *branches meeting at the singular point.*

The rotation map is locally constant if and only if it is globally constant if and only if the density of α *is identically equal to zero. If in addition the fibration* $\varphi : S \to C$ *has at least one section and at least one singular fiber, then* α *is periodic.*

Proof. In the notation of Lemma 7.4.1, where we replace the point $c \in C$ near c_0 by its coordinate $z = z(c)$, the rotation map $(R_1,\ R_2)$ is determined by $T(z) = R_1(z)\, p_1(z) + R_2(z)\, p_2(z)$. Because $c_0 \in C^{\mathrm{reg}}$, we have $l = 1$ and can take $l_1 = l_2 = 0$ in Lemma 7.4.1.

We use the notation $\partial^\alpha = {\partial_1}^{\alpha_1}\, {\partial_2}^{\alpha_2}$, where $|\alpha| = \alpha_1 + \alpha_2$ is the total order of differentiation and $\partial_1 = \partial/\partial x, \partial_2 = \partial/\partial y$. Here we identify functions of the complex variable z with functions of two real variables x, y, where $z = x + \mathrm{i}\, y$.

If the rotation map is not constant, then there is a unique $d \in \mathbb{Z}_{>0}$ such that $\partial^\alpha R_1(0) = \partial^\alpha R_2(0) = 0$ when $1 \le |\alpha| < d$, whereas $\partial^\alpha R_1(0) \ne 0$ or $\partial^\alpha R_2(0) \ne 0$ for at least one α with $|\alpha| = d$. For a complex analytic function h we have $\partial^\alpha h = \mathrm{i}^{\alpha_2}\ h^{(k)}$, where $h^{(k)}$ denotes the complex derivative of h of order $k = |\alpha|$.

Then $a_k = 0$ for every $0 \le k < d$, whereas for $|\alpha| = d$ we have

$$\mathrm{i}^{\alpha_2}\ d!\, a_d = \partial^\alpha R_1(0)\, p_1(0) + \partial^\alpha R_2(0)\, p_2(0).$$

If we multiply this by $x^{\alpha_1}\, y^{\alpha_2}/\alpha_1!\,\alpha_2!$ and sum over all α such that $|\alpha| = d$, we obtain

$$z^d\, a_d = R_1(z)\, p_1(0) + R_2(z)\, p_2(0) + \mathrm{O}(|z|^{d+1}),$$

which in turn implies that

$$\begin{aligned} R_1(z) &= \mathrm{Im}(z^d\, a_d/p_2(0))/\,\mathrm{Im}(p_1(0)/p_2(0)) + \mathrm{O}(|z|^{d+1}), \quad \text{and} \\ R_2(z) &= \mathrm{Im}(z^d\, a_d/p_1(0))/\,\mathrm{Im}(p_2(0)/p_1(0)) + \mathrm{O}(|z|^{d+1}), \quad \text{as} \quad z \to 0. \end{aligned}$$

Note that with the notation $q = p_2/p_1$, we have $\mathrm{Im}(p_2(0)/p_1(0)) = \mathrm{Im}(q(0)) > 0$ and $\mathrm{Im}(p_1(0)/p_2(0)) = \mathrm{Im}(1/q(0)) = \mathrm{Im}(\overline{q(0)})/|q(0)|^2 < 0$. It follows that $a_d \ne 0$. Because the real linear mapping

$$\begin{aligned} L : \mathbb{C} &\to \mathbb{R}^2 \\ : v &\mapsto (\mathrm{Im}(v/p_2(0))/\,\mathrm{Im}(p_1(0)/p_2(0)),\ \mathrm{Im}(v/p_2(0))/\,\mathrm{Im}(p_1(0)/p_2(0))) \end{aligned}$$

satisfies $L(p_1(0)) = (1,\ 0)$ and $L(p_2(0)) = (0,\ 1)$, this proves (7.7.4) and (7.7.5). It follows in particular that the density is strictly positive at every regular point of the rotation map, where $d = 1$, which implies that the rotation map is orientation-preserving.

Because the rotation map is real analytic and C^{reg} is connected, it is globally constant as soon as it is constant on any nonempty open subset of C^{reg}. If the rotation

map is constant, equal to $(k_1/k,\ k_2/k)$ for some $k_1,\ k_2 \in \mathbb{Z}$, then α^k is equal to the identity. If the rotation map is constant equal to an element $\notin (\mathbb{Q}/\mathbb{Z}) \times (\mathbb{Q}/\mathbb{Z})$, then it has no k-periodic smooth fibers. Because there are only finitely many singular fibers, it follows that the number $\nu(\alpha^k)$ of k-periodic fibers for α is a bounded function of k. Corollary 7.4.7 now implies that α has finite order if $\varphi : S \to C$ has at least one section and one singular fiber. In other words, in the latter case we cannot have a rotation map that is constant, equal to an element $\notin (\mathbb{Q}/\mathbb{Z}) \times (\mathbb{Q}/\mathbb{Z})$. □

Remark 7.7.4. The points where the density δ_α of α is equal to zero are isolated, where δ_α is strictly positive in the complement. If $\varphi : S \to C$ and α belong to a continuous family, then one might think that a zero of δ_α could be perturbed away because the density could be perturbed to an everywhere positive one. However, this is not the case, because the proof of Lemma 7.7.3 showed that the points of D correspond to the zeros of the complex-valued function $u_1(z) = T'(z) - D_1(z)\, p_1'(z) - D_2(z)\, p_1'(z)$. If $z = 0$ corresponds to a point of D where the rotation map has degree $d > 1$, then it follows from (7.7.5) that the topological multiplicity of the zero $z = 0$ of the function $u_1(z)$ is equal to $d-1$. Therefore any small perturbation of $u_1(z)$ has $d-1$ zeros near $z = 0$, where the zeros are counted with their topological multiplicities. Question: Are there situations in which, by means of global topological considerations, one could determine the number of points in D?

Viewing the elliptic fibration of a Lagrangian torus fibration as in Remarks 6.2.22 and 7.7.2, we have action-angle coordinates $(x,\ \xi)$ as in Arnol'd and Avez [5, Appendix 26] or Duistermaat [48, Section 1], in which the transformation α takes the form $(x,\ \xi) \mapsto (x,\ \xi + \Xi(x))$, where $a \mapsto \Xi(x)$ is the rotation map in action-angle coordinates. The condition that α is a symplectic diffeomorphism is equivalent to the condition that the derivative matrix $\mathrm{D}\Xi(x)$ is symmetric, which, locally in x, is equivalent to the condition that $\Xi(x) = \mathrm{d}f(x)$ for a smooth real-valued function $f(x)$ of the action variables x. The rigidity of action-angle coordinates means that there is little freedom in putting the function $f(x)$ into a normal form. The example $f(x) = {x_1}^3/3 + {x_2}^2/2$ yields a rotation map $\Xi : (x_1,\ x_2) \mapsto ({x_1}^2,\ x_2)$ equal to the *fold*, which is orientation-preserving or orientation-reversing as $x_1 > 0$ or $x_1 < 0$, and degenerate along the curve $x_1 = 0$. This example shows that within the framework of integrable Hamiltonian systems, the frequency maps of Hamiltonian vector fields, whose time-one flow is an element of $\mathrm{Aut}(S)^+_\varphi$ for an elliptic fibrations $\varphi : S \to C$ are quite special. ⊘

Definition 7.7.5. For any $c \in C^{\mathrm{reg}} \setminus D$, let P_c^α denote the preimage in T_cC of the standard lattice $\mathbb{Z}^2$ under the derivative $R'(c)$ of the rotation mapping R at the point c, where $R'(c)$ is viewed as a bijective real linear mapping from T_cC onto $\mathbb{R}^2$. Because in Lemma 7.7.3 we have $c_0 \in C^{\mathrm{reg}} \setminus D$ if and only if $d = 1$, we see that $P_c^\alpha = (1/u(c))\, P_c$, where $P_c = \mathbb{Z}\, p_1(c) + \mathbb{Z}\, p_2(c)$ is the period lattice generated by $p_1(c)$ and $p_2(c)$, and $u := T' - R_1\, p_1' - R_2\, p_2'$. In particular, T_cC/P_c^α, $\mathbb{C}/P_c$, and the fibers S_c of $\varphi : S \to C$ over c are isomorphic compact complex analytic curves, and $J(P_c^\alpha) = J(P_c) = J(c)$.

Any local positively oriented $\mathbb{Z}$-basis of the period lattice P in $\mathfrak{h}$ defines a real analytic trivialization $\Psi : \mathfrak{F} \to C^{\mathrm{reg}} \times (\mathbb{R}/\mathbb{Z})^2$ of the fiber system $\mathfrak{F}$ of Lie groups $\mathfrak{F}_c$,

$c \in C^{\mathrm{reg}}$, defined by the inverse of (7.7.1). If we view $\alpha \in \mathrm{Aut}(S)^+_\varphi$ as a holomorphic section $\alpha : C \to \mathfrak{F}$ of $\mathfrak{F}$, then the corresponding rotation map R is equal to $\pi_2 \circ \Psi \circ \alpha$, where $\pi_2 : C^{\mathrm{reg}} \times (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ denotes the projection onto the second factor. Therefore $R' = (\pi_2 \circ \Psi)' \circ \mathrm{T}\alpha$. A change of the local positively oriented $\mathbb{Z}$-basis of P corresponds to replacing Ψ by $(1,\ M)$, where $M \in \mathrm{SL}(2,\ \mathbb{Z})$ when R' gets replaced by $M \circ R'$. It follows that the P^α_c, $c \in C^{\mathrm{reg}} \setminus D$, form a real analytic lattice bundle P^α in $\mathrm{T}C$ over $C^{\mathrm{reg}} \setminus D$, which is independent of the choice of the trivialisation $\Psi : \mathfrak{F} \to C^{\mathrm{reg}} \times (\mathbb{R}/\mathbb{Z})^2$. The lattice bundle P^α will be called the *tangent lattice bundle in* $C^{\mathrm{reg}} \setminus D$ *defined by* α.

Any continuous local section V of P^α is a real analytic vector field, and the lattice bundle is *integrable* in the sense that any two such vector fields commute. Actually, if we use $(R_1,\ R_2)$ as a real analytic local coordinate system in $C^{\mathrm{reg}} \setminus D$, then the commuting vector fields $\partial/\partial R_1$ and $\partial/\partial R_2$ form a $\mathbb{Z}$-basis of the lattice of these locally defined vector fields. The density δ_α of α on $C^{\mathrm{reg}} \setminus D$ is determined in terms of P^α by the condition that it is equal to 1 on every positively oriented local $\mathbb{Z}$-basis of P^α.

As a real analytic principal $\mathrm{SL}(2,\ \mathbb{Z})$-bundle over $C^{\mathrm{reg}} \setminus D$, the positively oriented $\mathbb{Z}$-basis bundle $\mathrm{SF}(P^\alpha)$ of P^α is isomorphic to the restriction to $C^{\mathrm{reg}} \setminus D$ of $\mathrm{SF}(P)$, where P is the period lattice bundle in $\mathfrak{f}$. This is because both bundles have the same transition matrices $M \in \mathrm{SL}(2,\ \mathbb{Z})$. In particular, the monodromy of P^α is equal to the monodromy of P over $C^{\mathrm{reg}} \setminus D$. See Section 6.2.11 for the monodromy of lattice bundles. Due to nontrivial monodromy of P, hence of P^α, the aforementioned commuting vector fields in general will not extend to globally defined single-valued real analytic vector fields on $C^{\mathrm{reg}} \setminus D$. We have such vector fields only on the universal covering space of $C^{\mathrm{reg}} \setminus D$. ⊘

Remark 7.7.6. The real analytic lattice bundle P^α is complex analytic if and only if the locally defined function u is complex analytic, which, under the assumption that R is not constant, is the case if and only if the modulus function J is constant. Elliptic surfaces with constant modulus functions are very special; see Section 6.4.4.

For the proof, we observe that $T = R_1\, p_1 + R_2\, p_2$ implies, because $\overline{\partial} T = \overline{\partial} P_j = 0$, that $p_1\, \overline{\partial} R_1 + p_2\, \overline{\partial} R_2 = 0$, hence

$$0 = \overline{\partial} u = -p_1'\, \overline{\partial} R_1 - p_2'\, \overline{\partial} R_2 = \overline{\partial} R_2\, (p_2\, p_1' - p_1\, p_2')/p_1$$

if and only if $p_2\, p_1' - p_2\, p_2' = 0$ if and only if $(p_2/p_1)' = 0$ if and only if $J' = 0$. Here we have also used that $\overline{\partial} R_2 \neq 0$ because R_2 is a nonconstant real-valued function.

Figure 7.7.1 illustrates the prototypical behavior of C_{α^k} near a point of D. This picture is the preimage under the mapping $z \mapsto z^2$ of the square lattice $(-2 + 4\,k_1/50) + \mathrm{i}(-2 + 4\,k_2/50)$, where $k_1,\ k_2 \in \mathbb{Z}$ and $0 \le k_1,\ k_2 \le 50$.

Lemma 7.7.7 below states, among other things, that the tangent lattice $P^\alpha_{c_0}$ is equal to the limit, for $k \to \infty$, of the set of $c \in C^{\mathrm{reg}} \setminus D$ such that S_c is a k-periodic fiber for α, after magnifying out this set from points near c_0 by means of the factor k. Therefore the lattice bundle P^α is uniquely determined by the asymptotic behavior,

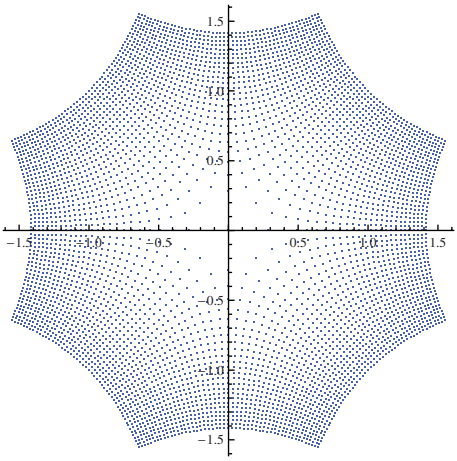


Fig. 7.7.1 Around a point of D.

for $k \to \infty$, of this set of points in $C^{\mathrm{reg}} \setminus D$. Because $P^{\alpha}_{c_0}$ changes as c_0 moves, Lemma 7.7.7 exhibits C_{α^k}, asymptotically for $k \to \infty$, as a sort of curvilinear lattice with the lattice points at distances to each other of order $1/k$.

Figures 11.2.5, 11.2.6, 11.2.7, 11.2.8 for the billiard map, and 11.4.16, 11.4.17, 11.4.18 for the Lyness map, illustrate the asymptotic behavior of the set C_{α^k} for $k \to \infty$ quite convincingly, even if k is not extremely large yet. The gray horizontal segment in the middle of Figure 11.2.5 is the set of real points in $\mathbb{C} \subset \mathbb{P}^1 = C$ over which we have a nonempty real fiber.

Lemma 7.7.7 *We use the notation of Definition* 7.4.5. *If* $c_0 \in C^{\mathrm{reg}} \cap C_{\alpha^k}$, *then* $\nu(\alpha^k,\ c_0)$ *is equal to the degree of the rotation map at the point* c_0. *In particular,* $\nu(\alpha^k,\ c_0) > 1$ *if and only if* $c_0 \in D$.

Let $c_0 \in C^{\mathrm{reg}} \setminus D$, *and let* z *be a local holomorphic coordinate such that* $z(c_0) = 0$. *Then there exists a sequence* $z_k \in C_{\alpha^k}$, $k \gg 0$, $k \gg 0$, *such that* $z_k = \mathrm{O}(1/k)$ *as* $k \to \infty$. *Moreover, for any sequence* $z_k \in C_{\alpha^k}$, $k \gg 0$, *such that* $z_k \to 0$ *as* $k \to \infty$, *and any sequence* r_k *of positive real numbers such that* $r_k \to \infty$ *and* $r_k/k \to 0$ *as* $k \to \infty$, *the set*

$$k\,(C_{\alpha^k} - z_k) \cap \{z \in \mathbb{C} \mid |z| < r_k\}, \tag{7.7.6}$$

that is, the set C_{α^k} *shifted over* z_k, *magnified by the factor* k, *and viewed in the growing disks of radius* r_k, *converges uniformly to the tangent lattice* $P^{\alpha}_{c_0}$ *as* $k \to \infty$.

If $c_0 \in D$, *then there exist for every* $\delta > 0$ *a neighborhood* C_0 *of* c_0 *in* C^{reg} *and an integer* $k_0 \in \mathbb{Z}_{>0}$ *such that for every* $k \geq k_0$ *we have* $\#(C_0 \cap C_{\alpha^k}) \leq \delta\, k^2$.

Proof. We have $c_0 \in C_{\alpha^k}$ if and only if the element $R(0) \in (\mathbb{R}/\mathbb{Z})^2$ belongs to the k-torsion subgroup $((\mathbb{Z}\,(1/k))/\mathbb{Z})^2$. In the sequel we will simplify the notation and view R as an $\mathbb{R}^2$-valued real analytic function on a neighborhood of the origin in the complex plane. According to Definition 7.4.3 and Lemma 7.4.1 with $l = 1$, $l_1 = l_2 = 0$, the multiplicity $\nu(\alpha^k,\ c_0)$ is equal to the multiplicity of the zero of the

function $T-(k_1/k)\,p_1-(k_2/k)\,p_2$ at $z=0$, which in the notation of Lemma 7.7.3 is equal to the degree d. This proves the first statement.

For the second statement, we observe that $z\in C_{\alpha^k}$ if and only if $R(z)\in(\mathbb{Z}\,(1/k))^2$. On the other hand, it follows from (7.7.4) with $d=1$ that the derivative $R'(0)$ of R at 0 is equal to the bijective linear mapping $v\mapsto L(a_1\,v)$, which implies that there is an open neighborhood U of 0 in $\mathbb{C}$ such that $R|_U$ is a real analytic local diffeomorphism from U onto an open neighborhood V of $R(0)$ in $\mathbb{R}^2$.

Let $\{x\}$ denote the fractional part of x, that is, $0\le\{x\}<1$ and $x-\{x\}\in\mathbb{Z}$. If $\varepsilon_{k,\,j}:=\{k\,R_j(0)\}/k$ and $\lambda_{k,\,j}:=R_j(0)-\varepsilon_{k,\,j}$, then for $j=1,\,2$, we have $0\le\varepsilon_{k,\,j}<1/k$ and $\lambda_{k,\,j}\in\mathbb{Z}\,(1/k)$. Therefore, if $\varepsilon_k=(\varepsilon_{k,1},\,\varepsilon_{k,2})$ and $\lambda_k=(\lambda_{k,1},\,\lambda_{k,2})$, we have $R(0)=\lambda_k+\varepsilon_k$, where $\varepsilon_k=\mathrm{O}(1/k)$ and $\lambda_k\in(\mathbb{Z}\,(1/k))^2$. For $k\gg 0$ we therefore have $\lambda_k\in V$, $z_k:=R^{-1}(\lambda_k)\in C_{\alpha^k}$, and $z_k=\mathrm{O}(1/k)$ as $k\to\infty$.

Now assume that we have any sequence $z_k\in C_{\alpha^k}$, $k\gg 0$, such that $z_k\to 0$ as $k\to\infty$. Let $|z|<r_k/k$, when $R(z)-R(z_k)=R'(0)(z-z_k)+\mathrm{o}(|z-z_k|)$. Because $z_k\in C_{\alpha^k}$ we have $R(z_k)\in(\mathbb{Z}\,(1/k))^2$, hence $z\in C_{\alpha^k}$ if and only if $R(z)-R(z_k)=(k_1/k,\,k_2/k)$ for $k_1,\,k_2\in\mathbb{Z}$, where the fact that both z and z_k converge to 0 as $k\to\infty$ implies that $k_1/k\to 0$ and $k_2/k\to 0$. Because $R'(0)(p_1(0)/a_1)=(1,\,0)$ and $R'(0)(p_2(0)/a_1)=(0,\,1)$, we obtain that

$$z-z_k=R'(0)^{-1}(k_1/k,\,k_2/k)+\mathrm{o}(1/k)=((k_1/k)\,p_1(0)+(k_2/k)\,p_2(0))/u_1+\mathrm{o}(1/k),$$

and therefore $k\,(z-z_k)\to(k_1\,p_1(0)+k_2\,p_2(0))/u_1$ as $k\to\infty$, where the convergence is uniform on the growing disks. This proves the second statement. The last statement follows from the fact that if the rotation map has degree >1 at c_0, then it decreases distances. □

Lemma 7.7.7 has the following implications, where the last statement follows from the observation that the open subset $C^{\mathrm{reg}}\setminus D$ of C is dense in C. In the sequel of this section we assume that the rotation map of α is not constant.

Corollary 7.7.8 *For every $c_0\in C^{\mathrm{reg}}\setminus D$ and every neighborhood C_0 of c_0 in C^{reg}, the number of elements of $C_0\cap C_{\alpha^k}$ is of order k^2 as $k\to\infty$. The modulus function J and the monodromy of the elliptic fibration $\varphi:S\to C$, and hence the isomorphism class of the relatively minimal elliptic fibration $\varphi:S\to C$, can be read off from the asymptotic behavior for $k\to\infty$ of the set C_{α^k}. As $k\to\infty$, the set C_{α^k} becomes everywhere dense in C.*

Proof. That the isomorphism class of the relatively minimal elliptic fibration $\varphi:S\to C$ is uniquely determined by J and the monodromy representation is the last statement in Theorem 6.4.3. □

7.7.0.1 Asymptotics Near a Singular Fiber

We next turn our attention to the asymptotic behavior, for $k \to \infty$, of the set C_{α^k} near a point $c_0 \in C^{\mathrm{sing}} := C \setminus C^{\mathrm{reg}}$, where S_{c_0} is a singular fiber of the elliptic fibration $\varphi : S \to C$.

It follows from Lemma 7.4.1 that the function

$$f := T - (l_1/l)\, p_1 - (l_2/l)\, p_2 = \lambda_1\, p_1 + \lambda_2\, p_2, \quad \lambda_j := R_j - l_j/l,$$

is holomorphic in a neighborhood of c_0. It follows from (7.7.5) for $d = 1$ and c_0 replaced by $c \in C^{\mathrm{reg}}$ near c_0 that the $\mathbb{Z}$-basis $v_1(c)$, $v_2(c)$ of the α-lattice P_c^α in $\mathrm{T}_c C$ and the α-density δ_α are given by

$$\begin{aligned}
v_1(c) &= p_1(c)/u(c), && (7.7.7)\\
v_2(c) &= p_2(c)/u(c), && (7.7.8)\\
\delta_\alpha(c) &= \det R'(c) = |u(c)|^2/\operatorname{Im}(\overline{p_1(c)}\, p_2(c)), \quad \text{where} && (7.7.9)\\
u &:= f' - \lambda_1\, p_1' - \lambda_2\, p_2'\\
&= f' - f\, p_1'/p_1 - (\operatorname{Im}(f/p_1)/\operatorname{Im}(p_2/p_1))\,(p_2/p_1)'\, p_1. && (7.7.10)
\end{aligned}$$

Note that $R'(c)\, v_1(c) = (1,\, 0)$ and $R'(c)\, v_2(c) = (0,\, 1)$.

For the proof of the second identity in (7.7.10), we have used that the real-valued functions λ_1 and λ_2 satisfy $f = \lambda_1\, p_1 + \lambda_2\, p_2$, and therefore can be expressed in terms of f, p_1, and p_2 by means of the formulas $\lambda_2 = \operatorname{Im}(f/p_1)/\operatorname{Im}(p_2/p_1)$ and

$$\lambda_1 = \operatorname{Re}(f/p_1) - (\operatorname{Im}(f/p_1)/\operatorname{Im}(p_2/p_1))\, \operatorname{Re}(p_2/p_1).$$

We have subsequently used that $p_2' = p_2\, p_1'/p_1 + (p_2/p_1)'\, p_1$, $-\operatorname{Re}(p_2/p_1) + p_2/p_1 = \mathrm{i}\, \operatorname{Im}(p_2/p_1)$, and $\operatorname{Re}(p_2/p_1) + \mathrm{i}\, \operatorname{Im}(p_2/p_1) = p_2/p_1$.

We will use Lemma 6.2.38 in order to determine the leading term in the asymptotic behavior of $u(z)$ as $z \to 0$. Note that in Table 6.2.39 we can arrange that $a(z) \equiv 1$, which leads to some simplifications in the notation. The function $b(z)$ is holomorphic in a neighborhood of $z = 0$.

For type I_b, $b \in \mathbb{Z}_{>0}$, we have $p_1(z) \equiv 1$, and $p_2(z) = (b/2\pi\, \mathrm{i})(\log z) + b(z)$, where $b(z)$ is holomorphic in a neighborhood of $z = 0$. Therefore (7.7.10) yields $u = f' - (\operatorname{Im} f/\operatorname{Im} p_2)\, p_2'$, with $\operatorname{Im} p_2(z) = (b/2\pi)\, \log(1/|z|) + \operatorname{Im} b(z) \sim (b/2\pi)\, \log(1/|z|)$ and $p_2'(z) = (b/2\pi\, \mathrm{i}\, z) + b'(z) \sim b/2\pi\, \mathrm{i}\, z$.

If $\operatorname{Im} f(0) \neq 0$, then we conclude that $u(z) \sim -\operatorname{Im} f(0)/\mathrm{i}\, z\, \log(1/|z|)$, and therefore we have, asymptotically for $z \to 0$,

$$\begin{aligned}
v_1(z) &\sim -\big[\log(1/|z|)/\operatorname{Im} f(0)\big]\, \mathrm{i}\, z, && (7.7.11)\\
v_2(z) &\sim -\big[b\, \log(1/|z|)/2\pi\, \operatorname{Im} f(0)\big]\, z\, \log z, && (7.7.12)\\
\delta_\alpha(z) &\sim (2\pi/b)\, (\operatorname{Im}(f(0))^2/|z|^2\, (\log(1/|z|))^3, && (7.7.13)
\end{aligned}$$

where (7.7.13) follows from (7.7.11) and (7.7.12) because $\delta_\alpha = |u|^2/\operatorname{Im}(\overline{p_1}\, p_2)$.

On the other hand, if $f(0) \in \mathbb{R}$, then there exist $m \in \mathbb{Z}_{>0}$ and $f_m \neq 0$ such that $f(z) \sim f(0) + f_m\, z^m$, because otherwise $f = \lambda_1 + \lambda_2\, p_2$ would be a real constant, hence $R_2 - l_2/l = \lambda_2 = 0$ and $R_1 - l_1/l = \lambda_1$ a real constant, in contradiction to our assumption that the rotation map is not a constant. It follows in this case that $u(z) \sim m\, f_m\, z^{m-1}$, and therefore, asymptotically for $z \to 0$,

$$v_1(z) \sim 1/(m\, f_m\, z^{m-1}), \tag{7.7.14}$$

$$v_2(z) \sim (b/2\pi\,\mathrm{i})\, \log z/(m\, f_m\, z^{m-1}), \tag{7.7.15}$$

$$\delta_\alpha(z) \sim (2\pi/b)\, (m\,|f_m|)^2\, |z|^{2\,m-2}\, |f_m|^2/\log(1/|z|). \tag{7.7.16}$$

For the remaining types it will be sufficient to observe that if f is identically zero, then $\lambda_1 \equiv 0$, $\lambda_2 \equiv 0$, and hence $R_1 \equiv l_1/l$, $R_2 \equiv l_2/l$, in contradiction to our assumption that the rotation map is not a constant. Therefore there exist $m \in Z_{\geq 0}$ and $f_m \neq 0$ such that $f(z) \sim f_m\, z^m$ and $f'(z) \sim m\, z^{m-1}$.

It follows from Table 6.2.39 that $p_1'(z)/p(z) \sim \alpha/z$, where α is as given in Table 7.7.9.

Table 7.7.9

Type	I_b^*	II	II*	III	III*	IV	IV*
α	$-1/2$	$-1/6$	$-5/6$	$-1/4$	$-3/4$	$-1/3$	$-2/3$

It follows that $f' - f\, p_1'/p_1 \sim f_m\, (m-\alpha)\, z^{m-1}$.

On the other hand, $\mathrm{Im}(f/p_1)\, p_1 = \mathrm{O}(|z|^m)$. For type I_b^*, $b \in \mathbb{Z}_{>0}$, we have $\mathrm{Im}(p_2/p_1) \sim (b/2\pi)\, \log(1/|z|)$, $(p_2/p_1)' \sim b/2\pi\,\mathrm{i}\, z$, and $\mathrm{Im}(f/p_1)\, p_1 = \mathrm{O}(|z|^m)$, and it follows that the last term on the right-hand side of (7.7.10) is $\mathrm{O}(|z|^{m-1}/\log(1/|z|))$ as $z \to 0$. For all the other types $p_2(z)/p_1(z)$ converges to a complex number c with $\mathrm{Im}(c) > 0$, whereas $(p_2/p_1)'$ is bounded. It follows that $u(z) \sim f_m\, (m-\alpha)\, z^{m-1}$ for all types other than I_b, $b > 0$, leading to the following leading asymptotics for $z \to 0$ of (7.7.7), (7.7.8), and (7.7.9):

$$v_1(z) \sim (1/f_m\, (m+1/2))\, z^{1/2-m}, \tag{7.7.17}$$

$$v_2(z) \sim (b/2\pi\,\mathrm{i}\; f_m\, (m+1/2))\, z^{1/2-m}\, \log z, \tag{7.7.18}$$

$$\delta_\alpha(z) \sim |f_m\, (m+1/2)|^2\, |z|^{2\,m-1}/(b/2\pi)\, \log(1/|z|), \tag{7.7.19}$$

for the Kodaira type I_b^*, $b > 0$. On the other hand,

$$v_1(z) \sim (1/f_m\, (m-\alpha))\, z^{\alpha+1-m}, \tag{7.7.20}$$

$$v_2(z) \sim (c/f_m\, (m-\alpha))\, z^{\alpha+1-m}, \tag{7.7.21}$$

$$\delta_\alpha(z) \sim |f_m\, (m-\alpha)|^2\, |z|^{2\,(m-1-\alpha)}/\mathrm{Im}(c), \quad z \to 0, \tag{7.7.22}$$

if the Kodaira type is not I_b, $b > 0$, or I_b^*, $b > 0$. Here $f(z) \sim f_m\, z^m$ as $z \to 0$, and α is as in Table 7.7.9. The complex number c is the limit of $p_2(z)/p_1(z)$ as $z \to 0$, and $\mathrm{Im}(c) > 0$. For type I_0^*, c can be any number in the complex upper half-plane. We have $c = \mathrm{e}^{2\pi\,\mathrm{i}/6}$ for types II, II*, IV, and IV*, whereas $c = \mathrm{e}^{2\pi\,\mathrm{i}/4}$ for types III and III*.

The proofs of Lemmas 7.7.3 and 7.7.7 lead to Corollary 7.7.10 below. For the notation, see Definition 7.4.5 and the description of the group $\mathfrak{F}_{c_0}$ at the end of Section 6.3.6.

When the singular fiber S_{c_0} is of Kodaira type I_b for some $b \in \mathbb{Z}_{>0}$, then the group $\mathfrak{F}_{c_0} \simeq S_{c_0} \cap S^{\mathrm{reg}}$ is isomorphic to $\mathbb{C}^\times \times (\mathbb{Z}/b\,\mathbb{Z})$, where $\mathbb{C}^\times$ is the multiplicative group of all nonzero complex numbers. The maximal compact subgroup of $\mathfrak{F}_{c_0}$ is $U \times (\mathbb{Z}/b\,\mathbb{Z})$, where $U := \{z \in \mathbb{C} \mid |z| = 1\}$ is the circle subgroup of $\mathbb{C}^\times$. For all other types of the singular fiber S_{c_0} the group $\mathfrak{F}_{c_0}$ is isomorphic to $\mathbb{C} \times (\mathbb{Z}/k\,\mathbb{Z})$ for some $k \in \mathbb{Z}_{>0}$, where $\mathbb{C}$ is the additive group of all complex numbers. In this case the maximal compact subgroup of $\mathfrak{F}_{c_0}$ is $\{0\} \times (\mathbb{Z}/k\,\mathbb{Z})$, and the restriction of α to $S_{c_0} \cap S^{\mathrm{reg}}$ belongs to the maximal compact subgroup of $\mathfrak{F}_{c_0}$ if and only if S_{c_0} is a k-periodic fiber for α.

Corollary 7.7.10 *Assume that the rotation map of $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is not constant. Then the subset D of C^{reg} mentioned in Lemma 7.7.7 is finite, and therefore $D' := D \cup C^{\mathrm{sing}}$ is a finite subset of C.*

For every $k \in \mathbb{Z}_{>0}$, let

$$\mu_k := \frac{1}{k^2} \sum_{c \in C_{\alpha^k}} \nu(\alpha^k,\, c)\, \delta_c,$$

where δ_c denotes the Dirac measure at the point c, which is the unit mass at the point c. As $k \to \infty$, the measures μ_k converge to a finite nonnegative measure μ on C. On C^{reg} the measure μ has a real analytic density, equal to $\mathrm{d}R_1 \wedge \mathrm{d}R_2$, where R_1 and R_2 are the multivalued real analytic functions on C^{reg} mentioned in Lemma 7.7.7. This density is strictly positive on $C \setminus D'$, and equal to zero on D'. When approaching a singular value $c_0 \in C^{\mathrm{sing}}$ of $\varphi : S \to C$, the density converges to infinity, unless the restriction of α to $S_{c_0} \cap S^{\mathrm{reg}}$ belongs to the maximal compact subgroup of $\mathfrak{F}_{c_0}$, when the density converges to zero. Here $\mathfrak{F}_{c_0}$ is the commutative complex one-dimensional Lie group introduced in Section 6.4.2.

If $c \in C_{\alpha^k}$ and $\nu(\alpha^k,\, c) > 1$, then $c \in D'$. There are only finitely many periodic fibers for α of multiplicity > 1, and the multiplicities of the periodic fibers for α have an upper bound.

If $\varphi : S \to C$ has at least one section and at least one singular fiber, and $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is not periodic, then the rotation map of α is not constant, and

$$\begin{aligned}\lim_{k\to\infty} \nu(\alpha^k)/k^2 = \mu(C) &= \int_{C^{\mathrm{reg}}} \mathrm{d}R_1 \wedge \mathrm{d}R_2 \\ &= (\chi(S,\, \mathcal{O}) + \nu(\alpha^m))/m^2 = \chi(S,\, \mathcal{O}) + \nu(\alpha) \\ &\quad - \frac{1}{2} \sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha) > 0;\end{aligned}$$

see Corollary 7.4.7 *and* (7.5.5).

Proof. For the statements in the third paragraph, we observe that if $c \in C$ and S_c is a periodic fiber for α, then there exists a $k_0 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}$, then S_c is k-periodic for α, if and only if $k \in k_0\,\mathbb{Z}$. Furthermore, it follows from Lemma 7.4.1 that if S_c is k_0-periodic for α of multiplicity m, then we have for every $l \in \mathbb{Z}_{\neq 0}$ that S_c is $l\,k_0$-periodic for α of the same multiplicity m. In other words, the multiplicity m depends only on the fiber, not on the period.

Let $c_0 \in C^{\mathrm{sing}} = C \setminus C^{\mathrm{reg}}$. Then it follows from (7.7.13), (7.7.16), (7.7.19), and (7.7.22) that c_0 is not an accumulation point of the discrete subset D of C^{reg} mentioned in Lemma 7.7.7. In combination with the fact that D is a discrete subset of C^{reg}, it follows that D is a discrete subset of C, and therefore finite because C was assumed to be compact.

The density δ_α is integrable, because all the functions appearing on the right-hand sides of (7.7.13), (7.7.16), (7.7.19), and (7.7.22) are integrable near $z = 0$. For type I_b, $b > 0$, we have that $p_1(0) = 1$ is a basic period of the vector field v on the connected component $S_{c_0} \cap S^{\mathrm{reg}}$ that contains s_0, and $f(0)$ represents the action of α^l on this connected component, where l is the positive integer such that α^l preserves this connected component. Therefore $\mathrm{Im}(f(0)) = 0$ if and only if the action of α^l on $S_{c_0} \cap S^{\mathrm{reg}}$ belongs to the circle subgroup U of $\mathfrak{F}_{c_0}$. In that case (7.7.16) implies that $\delta_\alpha(z) \to 0$ as $z \to 0$.

If S_{c_0} is not of Kodaira type I_b, $b \in \mathbb{Z}_{>0}$, then it follows from (7.7.19) and (7.7.22) that the density converges to infinity and zero when $m = 0$ and $m \in \mathbb{Z}_{>0}$, respectively. We have $m = 0$ if and only if $f(0) \neq 0$ if and only if S_{c_0} is not a k-periodic fiber for α. □

Remark 7.7.11. Assume that S_{c_0} is of type I_b, $b \in \mathbb{Z}_{>0}$, and the restriction of α to $S_{c_0} \cap S^{\mathrm{reg}}$ does not belong to the maximal compact subgroup of $\mathfrak{F}_{c_0}$. Then we have (7.7.13) with $\mathrm{Im}\, f(0) \neq 0$, and the integral of δ_α over the disk $|z| < r$ is, asymptotically for $r \downarrow 0$, equal to $c/(\log(1/r))^2$, where $c = (2\pi\ \mathrm{Im}\, f(0))^2/b)/2 > 0$. This function of r goes down to zero as $r \downarrow 0$, but in an infinitely steep way. For large k and small r, the fraction $\delta \sim c/\mu(C)\,(\log(1/r))^2$ of the points of C_{α^k} lies in the disk $|z| < r$. It follows that the radius

$$r \sim \mathrm{e}^{-d\,\delta^{-1/2}}, \quad d = (c/\mu(C))^{1/2},$$

of the disk centered at $z = 0$, which contains a small but not excessively small fraction δ of the points of C_{α^k}, is very small indeed.

More detailed information about the asymptotic behavior for $k \to \infty$ of the set C_{α^k} near c_0 can be obtained from the proof of (7.7.13). Because $p_2(z) = (b/2\pi\ \mathrm{i})\,(\log z) + b(z)$, the function $\mathrm{Im}\, p_2(z) = (b/2\pi)\,\log(1/|z|) + \mathrm{Im}\, b(z)$ is single-valued, and therefore $\lambda_2 = (\mathrm{Im}\, f)/(\mathrm{Im}\, p_2)$ is single-valued. Because

$$\lambda_2(z) \sim \mathrm{Im}\, f(0)/((b/2\pi)\,\log(1/|z|) + \mathrm{Im}\, b(0)), \quad z \to 0,$$

the level curves of λ_2 near $z = 0$ are approximate small circles centered at the origin. Assume that $\mathrm{Im}\, f(0) > 0$. The points of C_{α^k} near $z = 0$ lie on the level curves $\lambda_2(z) = k_2/k$, $k_2 \in \mathbb{Z}_{\geq 1}$, k_2/k small, when

$$|z| \sim C\ \mathrm{e}^{-(2\pi/b)\ \mathrm{Im}\, f(0)\, k/k_2}, \quad C = \mathrm{e}^{(2\pi/b)\ \mathrm{Im}\, b(0)}. \tag{7.7.23}$$

If $\mathrm{Im}\, f(0) < 0$, then the right-hand side of (7.7.23) would be exponentially large for small positive k_2/k, meaning that z is not in the domain where our asymptotic expansions hold. In this case we obtain the approximate small circles $\lambda_2 = -k_2/k$ for $k_2 \in \mathbb{Z}_{\geq 1}$, k_2/k small, and (7.7.23) holds with k_2 replaced by $-k_2$, or equivalently $\mathrm{Im}\, f(0)$ replaced by $|\mathrm{Im}\, f(0)|$.

Because $p_2'(z) = b/2\pi\ \mathrm{i}\ z + b'(z)$ is single-valued, the function u and the vector field v_1 defined by (7.7.10) and (7.7.7), respectively, where $p_1 = 1$, are also single-valued. The flow of v_1 leaves λ_2 invariant, and it follows from (7.7.11) that the angular velocity on the approximate circle $\lambda_2 = k_2/k$ is $\sim (1/\mathrm{Im}\, f(0))\ \log(1/|z|) \sim (2\pi/b)\, k/k_2$. Because the points of C_{α^k} on this v_1 solution curve are spaced at time intervals $1/k$, it follows that we have $b\, k_2$ points of C_{α^k} on the curve $\lambda_2 = k_2/k$, forming an approximate regular $b\, k_2$-gon on the approximate circle (7.7.23).

On the other hand. we have the multivalued vector field v_2, asymptotically given by (7.7.12), where the solution curves of the right-hand side of (7.7.12) are spirals. The points on the subsequent curves $\lambda_2 = k_2/k$, when k_2 is increased by one, are connected by solution curves of v_2, again at time intervals $1/k$. The shift by b of the number of points on the subsequent curves $\lambda_2 = k_2/k$ agrees with the fact that the continuation of the vector field v_2 along the v_1-loop returns as the vector field $v_2 + b\, v_1$, which in turn corresponds to the monodromy matrix in Table 6.2.40 for type I_b, $b \in \mathbb{Z}_{>0}$. It follows that the pattern of having $b\, k_2$ points of C_{α^k} on the k_2th loop $\lambda_2 = k_2/k$ around c_0 will persist on any open neighborhood C_0 of c_0 such that λ_2 is a proper single-valued real analytic function without stationary points on $C_0 \setminus \{c_0\}$.

It follows from (7.7.23) that for small $k_2 \in \mathbb{Z}_{\geq 1}$ the curve $\lambda_2 = k_2/k$ is exponentially small, and in numerical examples too small to be visible. However, for k_2/k small but not too small, the pattern usually is clearly visible. Figure 11.2.5, for α equal to the billiard map and period $k = 134$, shows the behavior described in this remark near $\lambda = 1$, the singular point in the middle of the picture, over which we have a singular fiber of type I_2, and the action of the billiard map on its regular part does not belong to the maximal compact subgroup.

If S_{c_0} is of type I_b, $b \in \mathbb{Z}_{>0}$, and the restriction of α to $S_{c_0} \cap S^{\mathrm{reg}}$ belongs to the maximal compact subgroup of $\mathfrak{F}_{c_0}$, then (7.7.16) implies that the density $\delta_\alpha(c)$ converges to zero as $c \to c_0$, and the asymptotic behavior for $k \to \infty$ of the set C_{α^k} is very different from the case that $\mathrm{Im}\, f(0) \neq 0$. Such zero density occurs if α is real and the singular fiber S_{c_0} over c_0 is elliptic with respect to a real structure; see Remark 8.4.6. This occurs for the elliptic billiard at $\lambda = 0$, the left boundary point of the real interval $[0,\ \lambda_0]$, and is clearly visible in Figure 11.2.5. ⊘

Remark 7.7.12. A period lattice bundle in the cotangent bundle of the base manifold was used in [48] in the analysis of global action-angle coordinates of integrable Hamiltonian systems. The dual lattice bundle in the tangent bundle is similar to the α-lattice in TC introduced in Definition 7.7.5. One of the points in [48] is that for the spherical pendulum, the lattice bundle has nontrivial monodromy around the singular fiber equal to the union of the unstable equilibrium together with its stable

= unstable manifold. This fiber is similar to a singular fiber of Kodaira type I_1, and the monodromy around it is the same as the one in Table 6.2.40 for such a singular fiber. See Remarks 6.2.22 and 7.7.2 for the interpretation of the elliptic surface as an integrable classical mechanical system with two degrees of freedom. For the monodromy of the α-lattice in $\mathrm{T}C$, see Definition 7.7.5.

If the action of α on $S_{c_0} \cap S^{\mathrm{reg}}$ does not belong to the maximal compact subgroup of $\mathfrak{F}_{c_0}$, then the asymptotic description in Remark 7.7.11 of the set $C^{\mathrm{reg}} \cap C_{\alpha^k}$ near c_0 is similar to the spectrum of the quantum-mechanical spherical pendulum near the unstable equilibrium in Cushman and Duistermaat [43], if we take h of order $1/k$. One of the points of [43] was that the monodromy of the toral fibration around the singular fiber can be read off from the asymptotic behavior for $h \to 0$ of the spectrum of the quantum-mechanical spherical pendulum.

It should be noted that the similarity between the toral fibration of the phase space of the spherical pendulum and elliptic fibrations is far from an isomorphism. For instance, at the stable equilibrium of the spherical pendulum the singular fiber is a point and the nearby fibers are small tori, whereas for an elliptic fibration a singular fiber never is a point, and the smooth fibers never become small. ⊘

Remark 7.7.13. For a singular fiber of type I_1, which in terms of the Hamiltonian system corresponds to a single focus–focus singularity, see Remarks 7.7.2 and 6.2.22, the asymptotic behavior of the Jacobian of the frequency map, which corresponds to our density $\mathrm{d}R_1 \wedge \mathrm{d}R_2$, has been determined by Rink [174, p. 352] and Dullin and Vũ Ngọc [50, Theorem 5], with the conclusion that it tends to infinity when approaching the singular value. The formula of Rink [174, p. 352] and the penultimate formula in the proof of Theorem 5 of Dullin and Vũ Ngọc [50] agree with our formula (7.7.13) for $b = 1$. The numbers $\partial F_0(0)/\partial H_2$ and α in Rink [174, p. 352] and Dullin and Vũ Ngọc [50, proof of Theorem 5], respectively, which are assumed to be nonzero, correspond to our number $\mathrm{Im}\, f(0)$. ⊘

Remark 7.7.14. It might be interesting to investigate whether the Picard–Fuchs equation $\mathcal{L}T = \mu$ in Proposition 2.5.20 can be used to obtain more information about the rotation map $(R_1,\ R_2)$ and the subset D of C^{reg} where $\mathrm{d}R_1 \wedge \mathrm{d}R_2 = 0$. In the real setting it has been used to obtain information about the zero-set of the derivative of the rotation function; see Section 2.6.3.

For quite a while I had not seen any example for which I could prove that the finite set D in Lemma 7.7.3 is not empty, and I worried whether the whole discussion about D could be void. In the presence of a real structure, the last statement in Lemma 8.1.5 implies that $c \in D$ if c is a regular real point where the derivative ρ' of the rotation function ρ is equal to zero. The first example in which this happens was shown to me by Jaap Eldering, by means of a computer picture of the rotation function of the square of the Lyness map for $a = -0.35$; see Remark 11.4.9. Also for the sine–Gordon map of Section 11.7, with $\vartheta \in \mathbb{R}$, $\pm = +$ and $|\vartheta| > 1$, there are two points $c \in C^{\mathrm{reg}} \cap C(\mathbb{R})$ where $\rho' = 0$; hence $c \in D$. ⊘

7.8 The Mordell–Weil Group in the Weierstrass Model

In this section we describe the set of all holomorphic sections of an elliptic surface in terms of the Weierstrass model in Theorem 6.3.6. We also use the terminology of Definition 7.4.3.

Proposition 7.8.1 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with a holomorphic section $o : C \to S$, and let Σ_φ denote the set of all holomorphic sections of φ. Let $f : S \to W$ be the Weierstrass model of S as in Theorem* 6.3.6*, in terms of the holomorphic line bundle $L = \mathfrak{f}^*$ over C. Then the relation $f(\sigma(c)) = (c, [1 : X(c) : Y(c)])$ defines a bijection from the set $\Sigma_\varphi \setminus \{o\}$ onto the set $\mathcal{R}$ of all pairs (X, Y) such that X and Y are meromorphic sections of L^2 and L^3, respectively, such that*

$$Y(c)^2 - 4\,X(c)^3 + g_2(c)\,X(c) + g_3(c) \equiv 0. \tag{7.8.1}$$

If $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is the element of the Mordell–Weil group such that $\sigma = \alpha \circ o$, then S_c is a fixed point fiber for α if and only if $\sigma(c) = o(c)$ if and only if X or Y has poles at c. If $m = \nu(\alpha, c)$ denotes the multiplicity of the fixed-point fiber for α at c, then the orders of the poles of X and Y at c are equal to $3\,m$ and $2\,m$, respectively.

Proof. If $\sigma \in \Sigma_\varphi \setminus \{o\}$, then $f \circ \sigma : C \to W$ is a holomorphic section of $p : W \to C$, not equal to the section $f \circ o : c \mapsto (c, \{0\} \times \{0\} \times {L_c}^3$ at infinity of p. It follows that there is a unique pair $(X, Y) \in \mathcal{R}$ such that $f(\sigma(c)) = (c, [1 : X(c) : Y(c)])$, for every $c \in C$ such that $\sigma(c) \neq o(c)$.

We have $\sigma(c) = o(c)$ if and only if X or Y has poles at c. If this is the case and z is a local holomorphic coordinate near c such that $z = 0$ corresponds to the point c, then the Weierstrass equation $Y(c)^2 - 4\,X(c)^3 + g_2(c)\,X + g_3(c)$ implies that $Y^2 \sim 4\,X^3$ near $z = 0$. If a and b are the orders of the poles of X and Y at c, respectively, we have $2\,b = 3\,a$, and therefore $a = 2\,m$, $b = 3\,m$ for some $m \in \mathbb{Z}_{>0}$. With a local trivialization of L we have $[1 : x : y] = [1/y : x/y : 1]$, where $1/y$ and x/y are local coordinates near the common point $[0 : 0 : 1]$ at infinity on the Weierstrass curves. Because $1/Y$ and and X/Y have zeros of order $3\,m$ and m, respectively, σ has a contact of order m with o at c. This proves the statements in the second part of the proposition.

Now assume conversely that $(X, Y) \in \mathcal{R}$. Then the mapping $c \mapsto (c, [1 : X(c) : Y(c)])$ extends to a holomorphic section $\tau : C \to W$ of the Weierstrass fibration $p : W \to C$, which is an isomorphism from C onto the compact curve $\tau(C)$ in W. Let $\widetilde{C}$ be the proper image in S of $\tau(C)$ under the resolution $f : S \to W$ of the singularities of W. Then $f|_{\widetilde{C}} : \widetilde{C} \to \tau(C)$ is an isomorphism, hence

$$\varphi|_{\widetilde{C}} = p|_{\tau(C)} \circ f|_{\widetilde{C}} : \widetilde{C} \to C$$

is an isomorphism, and its inverse $\sigma : C \to \widetilde{C}$ defines a holomorphic mapping from C to S such that $\varphi \circ \sigma$ is equal to the identity in C. That is, $\sigma : C \to S$ is a holomorphic section of φ, and $f(\sigma(c)) = (c, [1 : X(c) : Y(c)])$ for all $c \in C$ where X and Y have no poles. □

Remark 7.8.2. We have that α has no fixed-point fibers if and only if $\alpha(o(C)) \cap o(C) = \emptyset$ if and only if the sections X and Y both are holomorphic.

If $C \simeq \mathbb{P}^1$ then we have seen in Example 5 that there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $L \simeq \mathrm{O}(N)$, and the space of all holomorphic sections of L^m is isomorphic to the space of all homogeneous polynomials of degree $m\,N$ in two variables. That is, for given S, g_2, and g_3 there correspond homogeneous polynomials of degree $4\,N$ and $6\,N$, respectively, and elements of the Mordell–Weil group without fixed-point fibers correspond to solutions X and Y of (7.8.1) that are homogeneous polynomials in two variables of degree $2\,N$ and $3\,N$, respectively. Equation (7.8.1) then corresponds to $6\,N + 1$ polynomial equations for the $(2\,N + 1) + (3\,N + 1) = 5\,N + 2$ coefficients of X and Y.

The rational elliptic surfaces correspond to the above with $N = 1$, when the equation (7.8.1) represents seven polynomial equations for the seven coefficients of X and Y. It is a theorem of Oguiso and Shioda [155, Theorem 2.5] that for a rational elliptic surface the elements of the Mordell–Weil group that map a section to a disjoint one generate the Mordell–Weil group. That is, in this case the Mordell–Weil group is generated by the elements that correspond to polynomial solutions of (7.8.1). It follows from Theorem 4.3.3 that there are at most 240 elements of the Mordell–Weil group that map a section to a disjoint one, which implies that, still for a rational elliptic surface, equation (7.8.1) has at most 240 solutions X and Y that are homogeneous polynomials in two variables of degree 2 and 3, respectively. It follows from Corollary 4.5.6 that there are two isomorphism classes of rational elliptic surfaces, namely the ones described in Lemma 4.5.1, for which the Mordell–Weil group is trivial, hence for which equation (7.8.1) has no solutions X and Y at all, even if we allow X and Y to be homogeneous *rational* functions in two variables of degree 2 and 3, respectively. In all other cases, S is a QRT surface, meaning that it is the successive blowing up at base points of a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, when the QRT automorphism is an element of the Mordell–Weil group that maps any section to a disjoint one. According to Theorem 4.3.2, every element of the Mordell–Weil group that maps a section to a disjoint one is equal to the QRT map of a suitable pencil of biquadratic curves. Therefore, for given g_2 and g_3 that correspond to a rational elliptic surface, the polynomial solutions $(X,\ Y)$ of (7.8.1) correspond to the QRT maps, and they generate the rational solutions.

It may be remarked that the aforementioned facts about the Mordell–Weil group were *not* established by a direct inspection of the seven polynomials equations (7.8.1) for the seven coefficients of the polynomials X and Y.

We finally would like to emphasize that if the coefficients of the biquadratic polynomial $p = z_0\,p^0 + z_1\,p^1$ that define the pencil of biquadratic curves are given, then the homogeneous polynomials g_2, g_3, X, and Y in $(z_0,\ z_1)$ of degree 4, 6, 2, and 3, are explicitly given as in Corollaries 2.5.10 and 2.5.13, where the solution $(X,\ Y)$ of (7.8.1) corresponds to the QRT transformation of the given pencil of biquadratic curves. ⊘

Let K denote the field of all meromorphic functions on C. If C is compact, hence isomorphic to a complex projective algebraic curve, see Proposition 2.1.6, then K is equal to the field of all rational functions on C. Let λ be a nonzero meromorphic

section of the holomorphic line bundle $L = \mathfrak{f}^*$ over C. Note that for any divisor D on C, the Riemann–Roch formula (2.1.14) with L replaced by $L\,\delta(D)$ implies that the dimension of the space of meromorphic sections of L with at most poles at D is $\geq \deg(L) + \#(D) + 1 - \mathrm{g}(C)$, which implies that there are plenty of nonzero meromorphic sections of L.

Then the holomorphic sections g_2 and g_3 in Theorem 6.3.6 can be written as $g_2 = \widetilde{g}_2\,\lambda^4$ and $g_3 = \widetilde{g}_3\,\lambda^6$ for uniquely determined $\widetilde{g}_2,\ \widetilde{g}_3 \in K$. Furthermore, any meromorphic sections X and Y of L^2 and L^3 can be written as $X = \widetilde{X}\,\lambda^2$ and $Y = \widetilde{Y}\,\lambda^3$ for uniquely determined $\widetilde{X},\ \widetilde{Y} \in K$. Let E be the Weierstrass curve defined over K that is defined by the equation (2.3.6) with g_2 and g_3 replaced by $\widetilde{g}_2$ and $\widetilde{g}_3$, respectively. The following corollary corresponds to Silverman [186, Proposition 3.10].

Corollary 7.8.3 *With these conventions, Proposition* 7.8.1 *describes a bijection between the set $E(K)$ of all K-points of E and the set Σ_φ of all holomorphic sections of $\varphi : S \to C$, where the point $[0 : 0 : 1] \in E(K)$ at infinity corresponds to the section o. Combining this with the bijective mapping between $\mathrm{Aut}(S)^+_\varphi$ and Σ_φ in Lemma* 7.1.1*, this leads to a bijection between $E(K)$ and the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$.*

As explained in Remark 7.1.2, the name "Mordell–Weil group" comes from the above identifications between the set $E(K)$ of all K-points of E, the set Σ_φ of all holomorphic sections of the elliptic fibration $\varphi : S \to C$, and the group $\mathrm{Aut}(S)^+_\varphi$.

Remark 7.8.4. According to Lemma 4.2.1, the unique translation α on a Weierstrass curve W that sends the point $\infty = [0 : 0 : 1]$ at infinity to the point $b = [1 : X : Y]$ is equal to $\iota_\infty \circ \iota_b$. Here ι_b sends $w \in W$ to the third point of intersection with W of the projective line through b and w, and $\iota_\infty : [1 : x : y] \mapsto [1 : x : -y]$. It follows that $\alpha([1 : x : y]) = [1 : \widehat{x} : \widehat{y}]$, where, with the notation $d = (y - Y)/(x - X)$,

$$\widehat{x} = d^2/4 - x - X, \tag{7.8.2}$$

$$\widehat{y} = -d^3/4 + (x + 2\,X)\,d - Y. \tag{7.8.3}$$

These formulas can be obtained by first solving g_2 and g_3 in terms of $X,\ Y,\ x,\ y$ from the equations $Y^2 - 4\,X^3 + g_2\,X + g_3 = 0$, $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$, and then writing $\widehat{x} = (1 - t)\,X + t\,x$, $\widehat{y} = -((1 - t)\,Y + t\,y)$, where t is the solution, not equal to $t = 0$ or $t = 1$, of the equation $\widehat{y}^2 - 4\,\widehat{x}^3 + g_2\,\widehat{x} + g_3 = 0$.

It follows that if $(X,\ Y) \in E(K)$ and $(x,\ y) \in E(K)$ are the pairs of meromorphic functions that correspond to the respective elements α and β of the Mordell–Weil group as in Corollary 7.8.3, then the pair $(\widehat{x},\ \widehat{y}) \in E(K)$ of meromorphic functions that corresponds to the composition $\alpha \circ \beta$ is given by the formulas (7.8.2), (7.8.3).

Although the formula for the group law in $E(K)$ is completely explicit, the expressions for the iterates α^k of α rapidly become very complicated as k grows. For instance, because $d = (y - Y)/(x - X)$ converges to $D := (12\,X^2 - g_2)/2\,Y$ when the point $(x,\ y)$ on the curve $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$ converges to (X, Y), the formula for $(X_{\alpha^2},\ Y_{\alpha^2}) = \alpha^2(\infty)$ is given by (7.8.2), (7.8.3) with $x = X$ and $d = D$, that is,

$$X_{\alpha^2} = (12\,X^2 - g_2)^2/16\,Y^2 - 2\,X, \tag{7.8.4}$$

$$Y_{\alpha^2} = -(12\,X^2 - g_2)^3/32\,Y^3 + 3\,X\,(12\,X^2 - g_2)/2\,Y - Y. \tag{7.8.5}$$

We see in (7.8.4) and (7.8.5) the denominators Y^2 and Y^3, respectively. For $2 \le k \le 6$ the respective denominators ${\delta_k}^2$ and ${\delta_k}^3$ in the first and second coordinates of $\alpha^k(\infty)$ were displayed at the beginning of Section 5.2. Staring at the formulas for the pairs of meromorphic functions corresponding to the iterates α^k of α has *not* been a way to understand the behavior of the iterates for larger k. ⊘

Remark 7.8.5. The fiber S_c over $c \in C$ corresponds to the Weierstrass curve W_c in $\mathbb{P}^2$ defined by the equation $x_0\,{x_2}^2 - 4\,{x_1}^3 + g_2(c)\,{x_0}^2\,x_1 + g_3(c)\,x - 0^3 = 0$; see (6.3.4). The fiber S_c is reducible if and only if the Weierstrass surface W is singular at some point $w \in W_c$. If $f : S \to W$ is the minimal resolution of singularities of W in Theorem 6.3.6, then there is exactly one multiplicity-one irreducible component Θ of S_c such that f maps $\Theta \cap S^{\mathrm{reg}}$ diffeomorphically onto the nonsingular part of W_c, which contains the point $[x_0 : x_1 : x_1] = [0 : 1 : 1]$ at infinity. Furthermore, f maps all the other irreducible components of S_c to the point w; see the paragraphs preceding Theorem 6.3.6.

The map $\alpha \in \mathrm{Aut}(S)^+_\varphi$ leaves Θ invariant if and only if it leaves $\Theta \cap S^{\mathrm{reg}}$ invariant if and only if its conjugate by f on the nonsingular part of W_c maps the point at infinity to a nonsingular point of W_c. Because $[x_0 : x_1 : x_1]$ is a singular point of W_c if and only if $[x_0 : x_1 : x_1] = [1 : x : y]$, $y^2 - 4\,x^3 + g_2(c)\,x + g_3(c) = 0$, $y = 0$, and $12\,x^2 - g_2(c) = 0$, the mapping α leaves Θ invariant if and only if $Y(c) \neq 0$ or $12\,X(c)^2 - g_2(c) \neq 0$. Note that (7.8.4), (7.8.5) imply that if $Y(c) = 0$ and $12\,X(c)^2 - g_2(c) \neq 0$, then α^2 acts as the identity on Θ, hence on S_c, that is, S_c is a 2-periodic fiber for α.

According to Theorem 6.3.29 the mapping α leaves Θ invariant if and only if α leaves every irreducible component of S_c invariant if and only if the contribution $\mathrm{contr}_c(\alpha)$ is equal to zero. It follows that for any $\alpha \in \mathrm{Aut}(S)^+_\varphi$ and $c \in C$ the fiber S_c is reducible and α permutes its multiplicity-one irreducible components in a nontrivial way if and only if $Y(c) = 0$ and $12\,X(c)^2 - g_2(c) = 0$. According to Lemma 7.3.2, α acts as an Eichler–Siegel transformation on $\mathrm{NS}(S)$ if and only if it leaves all irreducible components of all fibers invariant, which therefore happens if and only if Y and $12\,X^2 - g_2$ do not have a common zero. Note that Y and $12\,X^2 - g_2$ are meromorphic sections of the respective line bundles L^3 and L^4.

If $\alpha = \tau$ is a QRT transformation on a rational elliptic surface S, then the Weierstrass data X, Y, and g_2 of τ are homogeneous polynomials in $(z_0,\ z_1)$ of respective degrees 2, 3, and 4, and there exists a reducible fiber whose irreducible components are permuted by τ in a nontrivial way if and only if the resultant R of Y and $12\,X^2 - g_2$ is equal to zero. Here R is a weighted homogeneous polynomial of degree 24 in the coefficients of X, Y, and g_2, provided with the respective weights 2, 3, and 4. It follows that the moduli space of isomorphism classes of QRT maps that permute the irreducible components of some reducible fiber in a nontrivial way form a codimension-one algebraic subset $R = 0$ of the moduli space of all QRT maps. I find it a bit surprising that this set is so big. ⊘

We now turn to the Picard–Fuchs equations of Section 2.5.3. We use a local trivialization λ of L as above, and omit the tildes in the notation for g_2 and g_3, which now are meromorphic functions on C. We also assume that the modulus function J is not constant.

Let v locally be a holomorphic vector field on C without zeros, viewed as a *derivation*. Then it follows from Lemma 2.5.15 that, still locally, there are unique meromorphic functions c_1 and c_2 on C such that with the notation

$$\mathcal{L} := v^2 + c_1\, v + c_2,$$

we have $\mathcal{L}\, p = 0$ for every period function p. Here, for generic c, $p(c)$ is the integral of $\lambda(c)$ over a closed path in the fiber S_c, where $\lambda(c) \in \mathfrak{f}_c^*$ is viewed as a holomorphic complex one-form on S_c. Note that p is a multivalued meromorphic function in the complement of the set of zeros of Δ.

If z is a local coordinate such that $v = \mathrm{d}/\mathrm{d}z$, and we use λ in order to trivialize L, then c_1 and c_2 can be expressed explicitly in terms of the discriminant function Δ and the modulus function J; see (2.5.17) and (2.5.14). The operator $\mathcal{L}$ is called the *Picard–Fuchs operator* defined by the local holomorphic section λ of $L = \mathfrak{f}^*$ over C and the local holomorphic vector field v on C.

For any $\alpha \in \mathrm{Aut}(S)_\varphi^+$ and generic $c \in C$, let $T_\alpha(c)$ denote the integral of $\lambda(c)$ over a path in S_c running from any initial point $s \in S_c$ to the point $\alpha(s)$. Modulo periods, this integral depends neither on the choice of the path nor on the choice of the initial point $s \in S_c$. Like p, it is a multivalued holomorphic function in the complement of the zero-set of Δ. On the other hand, the function μ_α defined by

$$\mu_\alpha(c) := (\mathcal{L}\, T_\alpha)(c) \tag{7.8.6}$$

locally is a single-valued, and actually meromorphic, function, locally on C.

If we replace v locally by $f\, v$, where f is a holomorphic function, then $\mathcal{L}$ is replaced by $M_{f^2}\, \mathcal{L}$, where M_g denotes multiplication by the function g. Therefore μ is replaced by $f^2\, \mu$. If, on the other hand, we replace λ locally by $a\, \lambda$, where a is a nonzero holomorphic function, when p and T are replaced by $a\, p$ and $a\, T$, respectively, then $\mathcal{L}$ is replaced by $M_a \circ \mathcal{L} \circ M_a^{-1}$. Therefore $\mu = \mathcal{L}\, T$ is replaced by $a\, \mu$. Because v and λ are sections of the line bundles $\mathrm{K}_C{}^*$ and $\mathfrak{f}^*$ over C, respectively, it follows that μ is invariantly defined as a global meromorphic section of the holomorphic line bundle $\mathrm{K}_C{}^2 \otimes \mathfrak{f}$ over C.

Remark 7.8.6. If C is compact, then it follows from the Riemann–Roch formula (2.1.15) that the degree of K_C is equal to $\deg(\mathrm{K}_C) = 2\ \mathrm{g}(C) - 2$. On the other hand, the degree of $\mathfrak{f}$ is equal to $\deg(\mathfrak{f}) = -\chi(S,\, \mathcal{O})$, see Lemma 6.2.34. It follows that the number of zeros of μ minus the number of poles of μ, all counted with their orders, is equal to $4\ \mathrm{g}(C) - 4 - \chi(S,\, \mathcal{O})$. If $\varphi : S \to C$ is a rational elliptic surface, when $\mathrm{g}(C) = 0$ and $\chi(S,\, \mathcal{O}) = 1$, then this number is equal to -5.

Lemma 7.4.1 implies the following result:

Corollary 7.8.7 *The meromorphic section $\mu = \mu_\alpha$ of $\mathrm{K}_C{}^2 \otimes \mathfrak{f}$ satisfies all the conclusions of Lemma* 2.6.8.

When α is the QRT automorphism defined by a pair of biquadratic polynomials, then the meromorphic section μ_α, Manin's function for the fiberwise translation α, can be explicitly computed; see Section 2.5.3. For arbitrary elements of the Mordell–Weil group, we have the following conclusions.

Proposition 7.8.8 *Let $\varphi : S \to C$ be an elliptic fibration with nonconstant modulus function J. Then the mapping $\alpha \mapsto \mu_\alpha$ defined by* (7.8.6) *is a homomorphism from the Mordell–Weil group* $\mathrm{Aut}(S)^+_\varphi$ *of $\varphi : S \to C$ to the additive group of all meromorphic sections of the holomorphic line bundle* $\mathrm{K}_C{}^2 \otimes \mathfrak{f}$ *over C.*

Assume that C is compact and $\varphi : S \to C$ has at least one singular fiber. Let $\alpha \in \mathrm{Aut}(S)^+_\varphi$. Then $\mu_\alpha = 0$ if and only if α is of finite order, that is, there exists a nonzero integer k such that α^k is the identity in S.

Proof. If $\alpha,\ \beta \in \mathrm{Aut}(S)^+_\varphi$, then the integral of λ over a path in S_c from s to $\alpha(\beta(s))$ is equal to the sum of the integral from s to $\beta(s)$ and the integral from $\beta(s)$ to $\alpha(\beta(s))$, that is,

$$T_{\alpha\circ\beta}(c) = T_\beta(c) + T_\alpha(c)$$

modulo periods. Because $\mathcal{L}$ is a linear operator that annihilates periods, we have that $\mu_{\alpha\circ\beta} = \mu_\beta + \mu_\alpha$, which proves that μ is a homomorphism.

Because $T_1 = 0$ modulo periods, we have $\mu_1 = \mathcal{L}\, T_1 = 0$, and therefore $k\, \mu_\alpha = \mu_{\alpha^k} = \mu_1 = 0$ if $\alpha^k = 1$, which implies that $\mu_\alpha = 0$ if $k \neq 0$.

We now prove the statement in the second paragraph of the proposition. Assume that $\alpha \in \mathrm{Aut}(S)^+_\varphi$, $\mu_\alpha = 0$, but α is not of finite order.

Because near regular points the kernel of $\mathcal{L}$ is spanned over $\mathbb{C}$ by two basic period functions p_1 and p_2, see lemma 2.5.15, the equation $0 = \mu_\alpha = \mathcal{L}\, T_\alpha$ implies that there are complex constants a_1 and a_2 such that $T_\alpha = a_1\, p_1 + a_2\, p_2$. If $a_1,\ a_2 \in \mathbb{Q}$, then there is a nonzero integer k such that $k\, T = 0$ modulo periods, or $\alpha^k = 1$ in a nonempty open subset of S. It then follows by analytic continuation that $\alpha^k = 1$ on S, in contradiction to our assumption. Therefore $a_1 \notin \mathbb{Q}$ or $a_2 \notin Q$.

Let $c_0 \in C^{\mathrm{sing}}$, and choose λ and D holomorphic and without zeros near c_0. It follows from Lemma 7.4.1 that there exist a nonzero integer k and integers k_1 and k_2 such that

$$l\, T_\alpha - l_1\, p_1 - l_2\, p_2 = (l\, a_1 - l_1)\, p_1 + (l\, a_2 - l_2)\, p_2$$

is holomorphic near c_0. The single-valuedness of this function implies that the vector $(l\, a_1 - l_1,\ l\, a_2 - l_2)$ is fixed under application of the monodromy matrix at c_0. If the singular fiber S_{c_0} is not of type I_b, $b \in \mathbb{Z}_{>0}$, then it follows from Table 6.2.40 that the monodromy has no nonzero fixed vectors; hence $l\, a_1 = l_1, l\, a_2 = l_2$, in contradiction to $(a_1,\ a_2) \notin \mathbb{Q} \times \mathbb{Q}$.

Therefore S_{c_0} is of type I_b, $b \in \mathbb{Z}_{>0}$. We can arrange that p_1 is holomorphic near c_0, whereas p_2 has logarithmic behavior. It follows that $l\, a_2 = l_2$, and therefore $a_2 \in \mathbb{Q}$, which in turn implies that $a_1 \notin \mathbb{Q}$. For the analytic continuation along any closed path in C^{reg} where λ is holomorphic and has no zeros, we have

$$T_\alpha^{\circlearrowleft} = a_1\, p_1^{\circlearrowleft} + a_2\, p_2^{\circlearrowleft} = (a_1\, M_1^1 + a_2\, M_2^1)\, p_1 + (a_1\, M_1^2 + a_2\, M_2^2)\, p_2,$$

where $M \in \mathrm{SL}(2, \mathbb{Z})$ is the monodromy matrix along the closed path. It follows from Lemma 7.4.1 with T_α replaced by $T_\alpha^{\circlearrowleft}$ that there are $m \in \mathbb{Z}_{\neq 0}$ and $m_1,\ m_2 \in \mathbb{Z}$, such that

$$m\, T_\alpha^{\circlearrowleft} - m_1\, p_1 - m_2\, p_2 = (m\,(a_1\, M_1^1 + a_2\, M_2^1) - m_1)\, p_1 + (m\,(a_1\, M_1^2 + a_2\, M_2^2) - m_2)\, p_2$$

is holomorphic in a neighborhood of c_0, which implies that $m\,(a_1\, M_1^2 + a_2\, M_2^2) - m_2 = 0$. Because $M_1^2,\ M_2^2 \in \mathbb{Z}$, $a_2 \in \mathbb{Q}$ and $a_1 \notin \mathbb{Q}$, it follows that $M_1^2 = 0$, and therefore $p_1^{\circlearrowleft} = \pm p_1$.

The period functions, in their dependence on λ, are invariantly defined as multivalued holomorphic sections of the holomorphic line bundle $L^{-1} = \mathfrak{f}$ over C. It follows either that p_1 defines a nonzero global holomorphic section over C of $\mathfrak{f}$, or that there is a twofold covering $\pi : \widetilde{C} \to C$ such that p_1 defines a nonzero global holomorphic section over $\widetilde{C}$ of $\pi^*\mathfrak{f}$. Recall that the degree $\deg(F)$ of a holomorphic line bundle F is equal to the number of zeros minus the number of poles of any meromorphic section of F, where the zeros and poles are counted with their multiplicities. Therefore $\deg(\mathfrak{f}) \geq 0$ and $\deg(\pi^*\mathfrak{f}) \geq 0$ in the first and second cases, respectively. In the second case, if q is a meromorphic section over C of $\mathfrak{f}$, then $q \circ \pi$ is a meromorphic section over $\widetilde{C}$ of $\pi^*\mathfrak{f}$ with twice the number of zeros and poles. Therefore $\deg(\pi^*\mathfrak{f}) = 2\ \deg(\mathfrak{f})$, hence $\deg(\mathfrak{f}) \geq 0$ again.

However, the assumptions that C is compact and $\varphi : S \to C$ has at least one singular fiber imply that $\deg(\mathfrak{f}) < 0$; see Lemma 6.2.34. This contradiction completes the proof that α is of finite order if $\mu_\alpha = 0$. □

As discussed Remark 7.8.4, the formulas for the meromorphic $X = X_k$ and $Y = Y_k$ which according to Proposition 7.8.1 correspond to the iterates α^k of $\alpha \in \mathrm{Aut}(S)_\varphi^+$ rapidly become very complicated for growing k. In contrast, the formula for μ_{α^k} remains of the same complexity as the formula for μ_α, because the homomorphism property implies that $\mu_{\alpha^k} = k\, \mu_\alpha$.

Remark 7.8.9. The homomorphism $\alpha \mapsto \mathcal{L}\, T_\alpha$ was introduced by Manin [128, pp. 190–192], and therefore it is called *Manin's homomorphism*. The characterization in Proposition 7.8.8 of the kernel of Manin's homomorphism as the torsion subgroup of the Mordell–Weil group $\mathrm{Aut}(S)_\varphi^+$ is a very special case of *Manin's kernel theorem* [128, Theorem 2 on p. 208]. A correction of Manin's proof of the kernel theorem has been given by Chai [33].

In an attempt to explain how the last statement in Proposition 7.8.8 follows from the formulation in [33] of Manin's kernel theorem, we specialize the latter to the case that U and $A_{/U}$ in [33] are our C and $\varphi : S \to C$, respectively. Then $\underline{M}$ and $\underline{M}^\sim$ correspond to integrals of holomorphic complex one-forms on the fibers over paths in the fibers that are closed and have their endpoints in the union of the sections o and $\alpha(o)$, respectively. Therefore the assumption for Manin's kernel theorem in [33] in our case means that $T(c) \in \mathbb{C}/P_c$, $c \in C_{\mathrm{reg}}$, locally is given by $T_\alpha(c) = a_1\, p_1(c) + a_2\, p_2(c)$, where a_1 and a_2 are complex constants. As shown in the proof of Proposition 7.8.8, this is obviously equivalent to $\mu := \mathcal{L}\, T = 0$. The remainder of the proof of Proposition 7.8.8 shows that our assumptions that J is not

constant, C is compact, and φ has at least one singular fiber imply that there are no nonzero globally defined single-valued period functions. This should mean that the $U_{/\mathbb{C}}$ trace of $A_{/U}$ mentioned in [33] is equal to zero. In this way the conclusion of Manin's kernel theorem in [33], stating that there is a nonzero integer k such that k times the section $\alpha(o)$ belongs to the trace, would mean that α is of finite order.

Our proof of the last statement in Proposition 7.8.8 resembles Chai's proof of Manin's kernel theorem in [33] in that we also used the global monodromy. ⊘

Chapter 8
Elliptic Fibrations with a Real Structure

Assume that the elliptic fibration $\varphi : S \to C$ has a real structure, and that the automorphism $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is real, in the sense of Definition 8.1.1 below. In this chapter we will analyze the action of α on the real domain, where the generic fibers are diffeomorphic to circles or pairs of circles, on each of which α, or α^2 if α interchanges two circles, is conjugate to a rigid rotation. We will give an asymptotic desciption, for $k \to \infty$, of the set of all real $c \in C$ such that the action of α on the real fiber over c is periodic with period k. This asymptotic description implies that the number of such real k-periodic fibers is asymptotically of order k, in contrast to the number of complex k-periodic fibers, which is of order k^2. See Lemma 8.1.5 and Corollary 8.4.5 below.

In Section 8.5 below, we show how, if we start with a pencil of real biquadratic curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, or a pencil of real cubic curves in $\mathbb{P}^2(\mathbb{R})$, the corresponding rational elliptic surface S inherits a real structure in a natural way. We note that not every real structure on S coming from a pencil of real biquadratic curves can be obtained from a pencil of real cubic curves.

The action of α on a generic real fiber over a real $c \in C$ is conjugate to a rotation $x + \mathbb{Z} \mapsto x + \rho(c) + \mathbb{Z}$ on the circle $\mathbb{R}/\mathbb{Z}$, where the rotation number $\rho(c) \in \mathbb{R}/\mathbb{Z}$ depends analytically on c. Here α has to be replaced by α^2 if the real fiber has two connected components that are switched by α. This description is simpler than the two rotation numbers $R_1(c)$ and $R_2(c)$ in Lemma 7.7.7 that describe the action of α on the complex fiber. On the other hand, the classification of the possibilities for the real fibration, and of the action of α on the real fibers, is less uniform than in the complex setting. We have taken some of the first steps in the investigation of the real fibrations, and many interesting questions remain.

8.1 Real Structures and Real Automorphisms

We recall that a complex conjugation in a complex n-dimensional complex analytic manifold M is defined as a real differentiable mapping $\gamma : M \to M$ such that $\gamma \circ \gamma$

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_8,
© Springer Science+Business Media, LLC 2010

is equal to the identity mapping and, for each $m \in M$, $\mathrm{T}_m\gamma$ is a complex antilinear mapping from $\mathrm{T}_m M$ to $\mathrm{T}_{\gamma(m)}M$. It follows from $\gamma \circ \gamma = 1$ that γ is bijective, with inverse $\gamma^{-1} = \gamma$, which is differentiable; hence γ is a diffeomorphism. In holomorphic local coordinates, the mapping γ followed by the ordinary complex conjugation is a holomorphic mapping, and it follows that the mapping γ is real analytic. The fixed points of γ are called the real points in M with respect to the complex conjugation γ. We will denote the set of all real points in M by $M(\mathbb{R})$. If f is a holomorphic function on an open subset U of M, then the formula $f^\gamma(m) = \overline{f(\gamma(m))}$, $m \in M$, defines a holomorphic function f^γ on $\gamma(U)$, and $(f^\gamma)^\gamma = f$. It follows that $g = (f + f^\gamma)/2$ is a holomorphic function on $V := U \cap \gamma(U)$ such that $g^\gamma = g$, that is, g is a real function in the sense that $g(\gamma(m)) = \overline{g(m)}$ for all $m \in V$. The operator $f \mapsto (f + f^\gamma)/2$ defines a linear projection from the space of all holomorphic functions on the γ-invariant open subset V of M onto the space of all real holomorphic functions on V. Let $m_0 \in M(\mathbb{R})$. Then there is a $\mathbb{C}$-basis e_j, $1 \le j \le n$, that is real in the sense that $\mathrm{T}_{m_0}\gamma(e_j) = e_j$ for every $1 \le j \le n$. There exist holomorphic functions ζ^j on a γ-invariant open neighborhood V of m_0 in M such that $(\mathrm{d}\zeta^j)(m_0)(e_k) = \delta_k^j$ for all $1 \le j,\ k \le n$. Then the total derivatives at m_0 of the holomorphic functions $z_j := (\zeta_j + \zeta_j{}^\gamma)/2$ are linearly independent over $\mathbb{C}$; hence, shrinking the neighborhood V of m_0 if neccesary, the z_j, $1 \le j \le n$, form a holomorphic system of local coordinates in V that consists of real functions. In other words, in V the mapping γ corresponds to the standard complex conjugation

$$(z^1, \ldots, z^n) \mapsto (\overline{z^1}, \ldots, \overline{z^n}).$$

This implies that

$$M(\mathbb{R}) \cap V = \{m \in V \mid z_j(m) \in \mathbb{R} \quad \text{for every} \quad 1 \le j \le n\}.$$

It follows in turn that $M(\mathbb{R})$, when nonempty, is a closed real n-dimensional real analytic submanifold of M.

Definition 8.1.1. A *real structure* on an elliptic fibration $\varphi : S \to C$ is a pair $(\gamma_S,\ \gamma_C)$ such that γ_S is a complex conjugation on S, γ_C is a complex conjugation on C, and $\varphi \circ \gamma_S = \gamma_C \circ \varphi$.

An automorphism α of S is called *real* if $\gamma_S \circ \alpha = \alpha \circ \gamma_S$. ⊘

Let $(\gamma_S,\ \gamma_C)$ be a real structure on $\varphi : S \to C$. If $s \in S(\mathbb{R})$, then $\gamma_C(\varphi(s)) = \varphi(\gamma_S(s)) = \varphi(s)$; hence $\varphi(s) \in C(\mathbb{R})$. That is, $\varphi(S(\mathbb{R})) \subset C(\mathbb{R})$, which implies in particular that $C(\mathbb{R}) \neq \emptyset$. The restriction to $S(\mathbb{R})$ of φ is a real analytic mapping

$$\varphi(\mathbb{R}) : S(\mathbb{R}) \to C(\mathbb{R}) \tag{8.1.1}$$

from the compact real two-dimensional real analytic manifold $S(\mathbb{R})$ to the compact real one-dimensional real analytic manifold $C(\mathbb{R})$.

Remark 8.1.2. For an elliptic fibration $\varphi : S \to C$, the mapping φ is surjective, meaning that for every $c \in C$ the fiber $S_c = \varphi^{-1}(\{c\})$ of φ over c is not empty. It may happen, even for rational elliptic surfaces coming from a pencil of biquadratic

curves, that the mapping $\varphi(\mathbb{R})$ in (8.1.1) is not surjective, meaning that there exist $c \in C(\mathbb{R})$ such that $S_c(\mathbb{R}) := S_c \cap S(\mathbb{R}) = \emptyset$. An example in which this happens is the elliptic billiard, see Section 11.2. ⊘

If $s \in S(\mathbb{R})$, $c = \varphi(s)$ then $\gamma_S(s) = s$, $\gamma_C(c) = c$, and $\mathrm{T}_s\varphi \circ \mathrm{T}_s\gamma_S = \mathrm{T}_c\gamma_c \circ \mathrm{T}_s\varphi$. Let J_s and J_c denote the multiplication by i in the tangent spaces, viewed as real linear mappings, with inverses equal to minus these linear mappings. Because $\mathrm{T}_s\gamma_S \circ J_s = -J_s \circ \mathrm{T}_s\gamma_S$, and similarly with S replaced by C, J_s and J_c map $\mathrm{T}_sS(\mathbb{R})$ and $\mathrm{T}_cC(\mathbb{R})$, the eigenspaces of $\mathrm{T}_S\gamma_S$ and $\mathrm{T}_c\gamma_C$ for the eigenvalue $+1$, bijectively onto the respective eigenspaces $\mathrm{T}_sS(\mathbb{R})^-$ and $\mathrm{T}_cC(\mathbb{R})^-$ for the eigenvalue -1, which are linear complements of the eigenspaces for the eigenvalue $+1$. Because furthermore $J_c \circ \mathrm{T}_s\varphi = \mathrm{T}_s\varphi \circ J_s$, the restriction of $\mathrm{T}_s\varphi$ to $\mathrm{T}_sS(\mathbb{R})^-$realperiodicss is equal to $J_c \circ \mathrm{T}_s\varphi(\mathbb{R}) \circ J_s^{-1}$, and therefore has the same rank as the restriction $\mathrm{T}_s\varphi(\mathbb{R})$ of $\mathrm{T}_s\varphi$ to $\mathrm{T}_sS(\mathbb{R})$. In particular, $\mathrm{T}_s\varphi(\mathbb{R}) = 0$ if and only if $\mathrm{T}_s\varphi = 0$. That is, the set of singular points for $\varphi(\mathbb{R}) : S(\mathbb{R}) \to C(\mathbb{R})$ is equal to $S(\mathbb{R}) \cap S^{\mathrm{sing}}$. It follows that the set of singular values of $\varphi(\mathbb{R})$ is equal to

$$C(\mathbb{R})^{\mathrm{sing}} := \varphi(S(\mathbb{R}) \cap S^{\mathrm{sing}}) \subset \varphi(S(\mathbb{R})) \cap C^{\mathrm{sing}}. \tag{8.1.2}$$

The set $C(\mathbb{R})^{\mathrm{sing}}$ is discrete because C^{sing} is discrete, and therefore $C(\mathbb{R})^{\mathrm{sing}}$ is finite if $C(\mathbb{R})$ is compact. Its complement

$$C(\mathbb{R})^{\mathrm{reg}} := C(\mathbb{R}) \setminus C(\mathbb{R})^{\mathrm{sing}} = C(\mathbb{R}) \setminus \varphi(S(\mathbb{R}) \cap S^{\mathrm{sing}}) \tag{8.1.3}$$

in $C(\mathbb{R})$, the set of all regular values of the mapping $\varphi(\mathbb{R}) : S(\mathbb{R}) \to C(\mathbb{R})$, is the complement in $C(\mathbb{R})$ of a discrete subset of $C(\mathbb{R})$.

The above conclusions hold for any complex analytic mapping φ that preserves the real structures. We now use that $\varphi : S \to C$ is an elliptic fibration. We have the Lie algebra bundle $\mathfrak{f}$, which is a holomorphic complex line bundle over C. See Section 6.2.7. If $c \in C^{\mathrm{reg}}$ and $v \in \mathfrak{v}_c$, then v is a holomorphic vector field on the elliptic curve S_c. The vector field $\gamma_* v$, defined by

$$(\gamma_* v)(\gamma_S(s)) = \mathrm{T}_s\gamma_S\, v(s), \quad s \in S_c,$$

is a holomorphic vector field on $\gamma_S(S_c) = S_{\gamma_C(c)}$, that is, $\gamma_* v \in \mathfrak{f}_{\gamma_C(c)}$. It follows from the description of the Lie algebra bundle $\mathfrak{f}$ in Section 6.2.7 that γ_* extends to an antiholomorphic involution of $\mathfrak{f}$, which we also denote by γ_*. Note that $\pi \circ \gamma_* = \gamma_C \circ \pi$, if $\pi : \mathfrak{f} \to C$ denotes the projection.

If $c \in C(\mathbb{R})$, meaning that $\gamma_C(c) = c$, we have the complex conjugation γ_* in $\mathfrak{f}_c$, and its fixed-point set is a real one-dimensional linear subspace $\mathfrak{f}_c(\mathbb{R})$ of $\mathfrak{f}_c$. The $\mathfrak{f}(\mathbb{R})_c := \mathfrak{f}_c(\mathbb{R})$, $c \in C(\mathbb{R}) \cap C^{\mathrm{reg}}$, form a real analytic rank-one real line bundle $\mathfrak{f}(\mathbb{R})$ over $C(\mathbb{R})$, a subbundle of the restriction to $C(\mathbb{R})$ of the complex line bundle $\mathfrak{f}$.

Remark 8.1.3. Let $\kappa : S \to \mathbb{P}^1$ be a rational elliptic surface with a real structure, where we can arrange that the complex conjugation in $\mathbb{P}^1$ is the standard one, and $\mathbb{P}^1(\mathbb{R})$ is the standard real projective line. In this case $\mathfrak{f}$ is the dual bundle of the bundle L whose holomorphic sections over $\mathbb{P}^1$ correspond to the homogeneous polynomials

of degree one on $\mathbb{C}^2$; see (v) in Lemma 9.1.2. Any nonzero real linear form λ whose restriction to $\mathbb{R}^2$ is a real-valued linear form on $\mathbb{R}^2$ defines a holomorphic section of L with only one zero, which is simple, at a point that we can arrange to be the point $[1:0] \in \mathbb{P}^1(\mathbb{R})$ at infinity. It follows that $v = \lambda^{-1}$ is a rational section of $\mathfrak{f}(\mathbb{R})$ without zeros and a simple pole at $[0:1]$. Because this section changes sign at the pole, the set of the connected components of the $\mathfrak{f}(\mathbb{R})_c$, $c \in \mathbb{P}^1(\mathbb{R})$, is connected, a twofold covering of the circle $\mathbb{P}^1(\mathbb{R})$. Therefore the line bundle $\mathfrak{f}(\mathbb{R})$ over the circle $\mathbb{P}^1(\mathbb{R})$ is topologically nontrivial, a *Möbius strip*. ⊘

In the same way as for the bundle $\mathfrak{f}$ of Lie algebras, the real structure of $\varphi : S \to C$ induces a complex conjugation in the fiber system $\mathfrak{F}$ of complex one-dimensional Lie groups defined in Section 6.3.6. The fixed points of this complex conjugation form a real analytic fiber system $\mathfrak{F}(\mathbb{R})$ of real one-dimensional Lie groups over $C(\mathbb{R})$, where for every $c \in C(\mathbb{R})$, $\mathfrak{F}(\mathbb{R})_c$ is equal to the real part $\mathfrak{F}_c(\mathbb{R})$ of $\mathfrak{F}_c$, and $\mathfrak{f}(\mathbb{R})_c$ is equal to the Lie algebra of $\mathfrak{F}(\mathbb{R})_c$. If $c \in C(\mathbb{R})$ and $S_c(\mathbb{R}) \cap S(\mathbb{R})^{\mathrm{reg}} = S_c \cap S(\mathbb{R}) \cap S^{\mathrm{reg}}$ is not empty, then $\mathfrak{F}(\mathbb{R})_c$ acts freely and transitively on $S_c(\mathbb{R}) \cap S(\mathbb{R})^{\mathrm{reg}}$. The last paragraph in the next lemma is about when the inclusion in (8.1.2) is not an equality.

Lemma 8.1.4 *Let $c_0 \in \varphi(S(\mathbb{R})) \setminus C(\mathbb{R})^{\mathrm{sing}}$, $v_0 \in \mathfrak{f}(\mathbb{R})_{c_0}$, $v_0 \neq 0$. Then there is a connected open neighborhood C_0 of c_0 in the complex curve C with the following properties.*

The real part $\varphi^{-1}(C_0) \cap S(\mathbb{R})$ of $\varphi^{-1}(C_0)$ is contained in S^{reg}. It is a real analytic real one-dimensional submanifold of $S(\mathbb{R})$ such that the restriction φ to it is a real analytic fibration of $\varphi^{-1}(C_0) \cap S(\mathbb{R})$ over $C_0 \cap C(\mathbb{R})$ with nonempty fibers.

There is a holomorphic section $v : C_0 \to \mathfrak{f}$ of $\mathfrak{f}$ over C_0 that is real in the sense that $\gamma_* v = v$, and such that $v(c_0) = v_0$. There is a unique holomorphic function $p : C_0 \to \mathbb{C}$ that is real and strictly positive on $C_0 \cap C(\mathbb{R})$ such that

$$(\mathbb{R}/\mathbb{Z}) \times (\varphi^{-1}(C_0) \cap S(\mathbb{R})) \ni (t,\, s) \mapsto \mathrm{e}^{t\, p(\varphi(s))\, v}(s) \in \varphi^{-1}(C_0) \cap S(\mathbb{R})$$

defines a free real analytic action of $\mathbb{R}/\mathbb{Z}$ on $\varphi^{-1}(C_0) \cap S(\mathbb{R})$. For each $c \in C_0 \cap C(\mathbb{R})$ the action is transitive on each connected component of the real fiber $S_c \cap S(\mathbb{R})$ over c.

We have $c_0 \in C^{\mathrm{sing}}$ if and only if S_{c_0} is a singular fiber of Kodaira type I_b with $b > 0$ and b even. The real part $S_{c_0} \cap S(\mathbb{R})$ of the fiber, contained in $S_{c_0} \cap S^{\mathrm{reg}}$, is one circle contained in one irreducible component or the union of two circles contained in two opposite irreducible components of the cycle of b irreducible components of S_{c_0}.

Proof. The assumption $c_0 \notin C(R)^{\mathrm{sing}}$ means that $S_{c_0} \cap S(\mathbb{R})$ is contained in the open subset S^{reg} of S. Using that the mapping φ is proper we can arrange, by taking C_0 sufficiently small, that $\varphi^{-1}(C_0) \cap S(\mathbb{R}) \subset S^{\mathrm{reg}}$, which in turn implies the remaining statements in the second paragraph of Lemma 8.1.4, shrinking C_0 further if necessary.

Because $\mathfrak{f}$ is a holomorphic complex line bundle over C, there exist an open neighborhood C_0 of c_0 in C and a holomorphic section $v : C_0 \to \mathfrak{f}$ of $\mathfrak{f}$ over C_0 such that $v(c_0) = v_0$. Replacing v by $(1/2)\,(v + \gamma_* v)$, we obtain that v is real, and still $v(c_0) = v_0$. Shrinking C_0 if necessary, we can arrange that v has no zeros in C_0.

We view v as a holomorphic vector field without zeros on $\varphi^{-1}(C_0) \cap S^{\mathrm{reg}}$. Its real flow $\mathbb{R} \ni t \mapsto \mathrm{e}^{t\,v}$ leaves $\varphi^{-1}(C_0) \cap S(\mathbb{R})$ of $\varphi^{-1}(C_0)$ invariant. Let $c \in C(\mathbb{R}) \cap C_0$. Then the real flow of v is a homomorphism h_c of real Lie groups from $(\mathbb{R}, +)$ to $\mathfrak{F}_c$. Because $\mathfrak{F}_c$ acts freely on $S_c \cap S^{\mathrm{reg}}$, see Theorem 6.3.29, it follows that for any $s \in S_c \cap S^{\mathrm{reg}}$ the kernel of h_c is equal to the set of all $t \in \mathbb{R}$ such that $\mathrm{e}^{t\,v}(s) = s$. Taking s in the compact real one-dimensional manifold $S_c \cap S(\mathbb{R})$, and using that v has no zeros, it follows that there is a unique $p(c) \in \mathbb{R}_{>0}$ such that $\ker h_c = \mathbb{Z}\, p(c)$. Using the implicit function theorem in the holomorphic setting, it follows that $p : C(\mathbb{R}) \cap S(\mathbb{R}) \to \mathbb{R}_{>0}$ extends to a holomorphic function $p : C_0 \to \mathbb{C}$, which is a period function for v. This completes the proof of the third paragraph in Lemma 8.1.4.

The condition $c_0 \in C^{\mathrm{sing}}$ means that S_{c_0} is a singular fiber. It follows from the last part of Section 6.3.6 that the identity component $\mathfrak{F}_{c_0}^{\mathrm{o}}$ of $\mathfrak{F}_{c_0}$ is isomorphic to the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers if S_{c_0} is of Kodaira type I_b, $b \in \mathbb{Z}_{\geq 1}$, and equal to the additive group $\mathbb{C}$ of all complex numbers in all other cases. Because H_{c_0} induces an isomorphism from $\mathbb{R}/p(c_0)\,\mathbb{Z}$ onto a compact one-dimensional Lie subgroup of $\mathfrak{F}_{c_0}$, and $\mathbb{C}$ has no nontrivial compact subgroups, it follows that S_{c_0} is of Kodaira type I_b, $b \in \mathbb{Z}_{\geq 1}$, where the irreducible components of S_{c_0} form a cycle of b complex projective lines, intersecting each other successively in single points and transversally.

The complex conjugation γ_S leaves S_{c_0} invariant, and therefore permutes the irreducible components of S_{c_0} and the singular points of S_{c_0}, preserving the incidence relations. Because in addition it is an involution, and none of the singular points of S_{c_0} is a fixed point, the conclusion is that b is even, and γ_S acts on the cycle $\mathbb{Z}/b\,\mathbb{Z}$ of irreducible components as $x + b\,\mathbb{Z} \mapsto -x + b\,\mathbb{Z}$. In particular, only two of the irreducible components, corresponding to $0 + b\,\mathbb{Z}$ and $((b/2) + b\,\mathbb{Z}$, are invariant under γ_S. The fixed points, which are the real points, are contained in these irreducible components, where we can arrange that the irreducible component corresponding to $0 + b\,\mathbb{Z}$ has real points. Furthermore, γ_S interchanges the two intersection points of the invariant irreducible components with its neighboring ones.

Every irreducible component A of S_{c_0} is isomorphic to $\mathbb{P}^1$, where $S_{c_0} \setminus S^{\mathrm{reg}}$ consists of the two intersection points with the neighboring irreducible components. We can arrange that 0 and ∞ are these intersection points, which leads to an identification of $A \cap S^{\mathrm{reg}}$ with $\mathbb{C}^\times$. Then $t \mapsto \gamma_S(\bar{t})$ is a complex analytic automorphism of $\mathbb{P}^1$ that interchanges 0 and ∞, and therefore of the form $t \mapsto a/t$ for a nonzero complex number a. It follows that $\gamma_S(t) = a/\bar{t}$. The condition that $\gamma \circ \gamma$ is equal to the identity is equivalent to the condition that $a \in \mathbb{R}$, whereas γ has fixed points if and only if $a > 0$, when the set of fixed points is equal to the circle with radius $\sqrt{a}$. □

An example of a complex singular fiber with real points, but without real singular points, as described in the last paragraph of Lemma 8.1.4, occurs in the discrete sine–Gordon map, see Section 11.7, when $\pm = +$ and $\vartheta < 0$.

The following lemma can be viewed as a real version of Lemma 7.7.7. In the last statement, we meet the possibility that c is a regular value of $\varphi(\mathbb{R}) : S(\mathbb{R}) \to C(\mathbb{R})$.

Lemma 8.1.5 *Let $\varphi : S \to C$ be an elliptic fibration with at least one section and at least one singular fiber. Let (γ_S, γ_C) be a real structure on $\varphi : S \to C$. Assume that $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is real, and not of finite order.*

Let I be a connected component of the set $C(\mathbb{R})^{\mathrm{reg}}$ of all regular values of $\varphi(\mathbb{R}) : S(\mathbb{R}) \to C(\mathbb{R})$, and suppose that $S(\mathbb{R})_I := S(\mathbb{R}) \cap \varphi^{-1}(I) \neq \emptyset$. Let

$$\varphi(\mathbb{R})_I : S(\mathbb{R})_I \to I \tag{8.1.4}$$

denote the restriction to $S(\mathbb{R})_I$ of $\varphi(\mathbb{R})$. Then (8.1.4) *exhibits $S(\mathbb{R})_I$ as a real analytic principal K-bundle over I, where the elements of the group act as restrictions to the real part of translations on the complex fibers. Here $K = \mathbb{R}/\mathbb{Z}$ or $K = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$ if the fibers have one or two connected components, and the action of K is unique up to the two or four automorphisms of K, respectively.*

If I is a circle, a full connected component of $C(\mathbb{R})$, then the above statement holds with the following provisos. If the restriction $\mathfrak{f}(\mathbb{R})_I$ to I of the real line bundle $\mathfrak{f}(\mathbb{R})$ is topologically trivial, then the $\mathbb{R}/\mathbb{Z}$-action on the fiber over $c \in I$ is single-valued as a function of c. In the other case, when $\mathfrak{f}(\mathbb{R})_I$ is a Möbius strip over I, the $\mathbb{R}/\mathbb{Z}$-action on the fiber over $c \in I$ is double-valued, and changes to its opposite when c runs once around the circle I. If the fibers of $\varphi(\mathbb{R})_I$ have two connected components, let n denote the number of all $c \in I$ such that $c \in C^{\mathrm{sing}}$ and S_c is of Kodaira type b with $b \in 4\,\mathbb{Z} + 2$. See the last paragraph of Lemma 8.1.4 *for the description of S_c when $c \in C^{\mathrm{sing}} \setminus C(\mathbb{R})^{\mathrm{sing}}$. For each $c \in I$, let $\epsilon(c)$ denote the action of the group element $(0 + \mathbb{Z},\ 1 + 2\,\mathbb{Z})$ on the fiber over c. If n is even, then $\epsilon(c)$ is single-valued as a function of c. If n is odd, then $\epsilon(c)$ is double-valued as a function of c and changes into the action of $(1/2 + \mathbb{Z},\ 1 + 2\,\mathbb{Z})$ when c runs once around I.*

For each $c \in I$, the automorphism α acts on the fiber $S_c \cap S(\mathbb{R})$ of $\varphi(\mathbb{R})_I$ over c by means of an element $k(c) \in K$. There is a real analytic map $\rho : I \to \mathbb{R}/\mathbb{Z}$ such that $k(c) = \rho(c)$ for every $c \in I$ if the fibers of $\varphi(\mathbb{R})_I$ are connected; otherwise $k(c) = (\rho(c),\ a)$ for a constant $a \in \mathbb{Z}/2\,\mathbb{Z}$, where $a = 0 + \mathbb{Z}$ if and only if α preserves the two connected components of each fiber. The mapping ρ may be multivalued, corresponding to the aforementioned multivaluedness of the action of K.

Assume that if the fibers have two connected components, then α preserves these, which always holds if α is replaced by $\alpha \circ \alpha$. If c belongs to the set D in Lemma 7.7.7, *then $\mathrm{d}\rho(c) = 0$. Conversely, if $\mathrm{d}\rho(c) = 0$ then $c \in D$ or $c \in C^{\mathrm{sing}} \setminus C(\mathbb{R})^{\mathrm{sing}}$. For the last case, see the last paragraph in Lemma* 8.1.4.

Proof. Note that I is an open interval or a circle. The second paragraph in Lemma 8.1.4 implies that the mapping (8.1.4) is a proper real analytic submersion from $S(\mathbb{R})_I$ to I. It follows that its image is a closed and open subset of I, and because I is connected and the image of $\varphi(\mathbb{R})_I$ is not empty, the image is equal to I and (8.1.4) is a locally trivial real analytic fiber bundle, where each fiber is compact and not empty. If $c \in I \cap C^{\mathrm{reg}}$, then S_c is an elliptic curve, and the fiber $S_c \cap S(\mathbb{R})$ of $\varphi(\mathbb{R})$ over $\mathbb{C}$ is the real part of S_c with respect to the restriction to S_c of the complex conjugation γ_S. It follows from Section 2.6.1 that $S_c \cap S(\mathbb{R})$ has 0, 1, or 2 connected components, each diffeomorphic to a circle. Because $I \setminus C^{\mathrm{reg}} \subset C^{\mathrm{sing}}$ is discrete and the fiber

bundle is locally trivial, we have that either every fiber is connected, or every fiber has two connected components.

Let $c_0 \in I$ and let v be as Lemma 8.1.4. If v is multiplied by a continuous real-valued function on $C_0 \cap I$ without zeros, then p is replaced by $p/|a|$, which shows that the $\mathbb{R}/\mathbb{Z}$-action does not depend on the choice of v, except that it passes to its opposite if $a < 0$. Therefore the $\mathbb{R}/\mathbb{Z}$-actions over neighborhoods of points in I automatically glue together to an $\mathbb{R}/\mathbb{Z}$-action as in the second and third paragraphs of Lemma 8.1.5.

Assume that each fiber has two connected components. Let $c \in I \cap C^{\mathrm{reg}}$. Because the real part $S_c \cap S(\mathbb{R})$ of the elliptic curve S_c has two connected components, it follows from Section 2.6.1 that the period lattice of S_c is generated by $p_1(c) \in \mathbb{R}_{>0}$ and $p_2(c) = \mathrm{i}\, q(z)$ with $q(z) \in \mathbb{R}_{>0}$. The elements of $\mathfrak{F}_c$ of order two that permute the two connected components of $S_c \cap S(\mathbb{R})$ are the $\mathrm{e}^{t\, v_c}$ with $t = (1/2)\, p_2(c)$ or $t = (1/2)\, p_1(c) + (1/2)\, p_2(c)$. For each open interval I_0 in $I \setminus C^{\mathrm{sing}}$, these elements of order two automatically piece together to an involution ϵ as in the third paragraph of Lemma 8.1.5. More precisely, there are an open neighborhood C_0 of I_0 in the complex curve C and two elements $\epsilon \in \mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ of order two such that for each $c \in I_0$, the involution ϵ interchanges the two connected components of $S_c \cap S(\mathbb{R})$. The two possibilities for ϵ correspond to $\epsilon = \mathrm{e}^{t\, v_c}$ on S_c with $t = (1/2)\, p_2(c)$ or $t = (1/2)\, p_1(c) + (1/2)\, p_2(c)$, respectively.

Let $c_0 \in I \cap C^{\mathrm{sing}}$, when S_{c_0} is a singular fiber of Kodaira type I_b, $b \in 2\,\mathbb{Z}_{>0}$, as in Lemma 8.1.4. Let z be a holomorphic coordinate function on an open neighborhood C_0 of c_0 in C such that $z(c_0) = 0$ and $z(c) \in \mathbb{R}$ when $c \in I \cap C_0$. It follows from Lemma 6.2.38 that, shrinking C_0 if necessary, we can choose one of the basic period functions $p_1(z)$ to be holomorphic on C_0 and such that $p_1(z(c)) \in \mathbb{R}_{>0}$ whenever $c \in I \cap C_0$. It follows from the previous paragraph that we can arrange that the second basic period function $p_2(z)$ takes values on the positive imaginary axis when $0 < z \ll 1$. Lemma 6.2.38 yields that

$$p_2(z) = p_1(z)\, (b/2\pi\, \mathrm{i})\, \log z + r(z),$$

where $r(z)$ is holomorphic in a neighborhood of $z = 0$. It follows that $r(z)$ is purely imaginary when $0 < z \ll 1$. The analytic continuation of $p_2(z)/2$ along a small circle in the complex plane with center at the origin leads to $p_2(z)^{\circlearrowleft}/2 = p_2(z)/2 + (b/2)\, p_1(z)$, and because b is even, it follows that $\epsilon = \mathrm{e}^{(p_2/2)\, v}$ defines a single-valued complex analytic diffeomorphism of $\varphi^{-1}(C_0')$, where $C_0' = C_0 \setminus \{c_0\}$. When $s \in \varphi^{-1}(C_0')$ approaches a point $s_0 \in S_{c_0} \cap S^{\mathrm{reg}}$, then it follows from the proof of Lemma 6.2.38 that $\epsilon(s)$ approaches a point in $S_{c_0} \cap S^{\mathrm{reg}}$ in the opposite irreducible component of S_{c_0} of the one that contains s_0. Therefore Riemann's theorem on removable singularities implies that ϵ has an extension to a complex analytic diffeomorphism of the complement in $\varphi^{-1}(C_0)$ of the b singular points of S_{c_0}. But then Hartogs's lemma implies that ϵ has an extension to a complex analytic diffeomorphism of $\varphi^{-1}(C_0)$. ϵ is an element of $\mathrm{Aut}(\varphi^{-1}(C_0))^+_\varphi$ of order two that for every $c \in I \cap C_0$, switches the two connected components of $S_c \cap S(\mathbb{R})$.

The analytic continuation of $p_2(z)$ along a circle in the complex upper half-plane with center at the origin, running from z to $-z$, yields

$$p_2(-z)/2 = (p_1(-z)\,(b/2\pi\,\mathrm{i})\,\log z)/2 + p_1(-z)\,b/4 + r(-z)/2.$$

It follows that the continuation of ϵ along I when passing through z_0 preserves and switches the choice of ϵ when $b \in 4\,\mathbb{Z}$ and when $b \in 2 + 4\,\mathbb{Z}$. This completes the proof of the third and fourth paragraphs in Lemma 8.1.5.

If $c_0 \in I \setminus C^{\mathrm{sing}}$, then we can arrange in the proof of Lemma 7.7.3 that $p_1(z) = p(z) \in \mathbb{R}$, $T(z) \in \mathbb{R}$, hence $R_2(z) = 0$ and $g(z) = R_1(z) = T(z)/p(z) = \rho(z)$ for z real and close to 0. Therefore $c_0 \in D$ implies that $\rho'(0) = \partial_1 R_1(0) = 0$. If conversely $\rho'(0) = 0$, then $\partial_1 R_1(0) = 0$, whereas $\partial_1 R_2(0) = 0$ because $R_2(z) = 0$ when z is real. Because $g = R_1 + R_2\,q$, where g, p_1, and p_2 are holomorphic functions, it follows that

$$0 = \partial_1(g - R_1 - R_2\,q)(0) = g'(0) - R_2(0)\,q'(0),$$

which according to the proof of Lemma 7.7.3 means that $c_0 \in D$. This completes the proof of the lemma. □

The number $\rho(c) \in \mathbb{R}/\mathbb{Z}$, which appears in Lemma 8.1.5, is called the *rotation number* of α on the real fiber $S_c(\mathbb{R})$. The function $\rho : I \to \mathbb{R}/\mathbb{Z} : c \mapsto \rho(z)$ is called the *rotation function* of α on the interval I. If the fibers of $\varphi(\mathbb{R})_I$ are not connected, and α switches the connected components of the real fibers, then the rotation function of α is not defined on I, but the rotation function of α^2 on I is always well defined. Also, if the line bundle $\mathfrak{f}(\mathbb{R})_I$ is not topologically trivial when I is a circle and $\mathfrak{f}(\mathbb{R})_I$ is a Möbius strip over I, then ρ is a two-valued function on I, which changes to its opposite if we run once along I.

8.2 The Real Periods near the Singular Fibers

In order to understand the asymptotic behavior of the rotation function $\rho(c)$ when c approaches an endpoint $c_0 \in C(\mathbb{R})^{\mathrm{sing}}$ of an interval I as in Lemma 8.1.5, we first investigate the asymptotic behavior, as $c \in I$ approaches c_0, of the real period function $p(c)$ discussed in Lemma 8.1.4. Note that Proposition 8.5.1 below implies that if the surface S is obtained from a pencil of biquadratic curves defined as in (2.5.3) with real matrices A^0 and A^1, and $S(\mathbb{R}) \neq \emptyset$, then $S(\mathbb{R}) \cap S^{\mathrm{sing}} \neq \emptyset$, that is, $C(\mathbb{R})^{\mathrm{sing}} \neq \emptyset$.

It follows from Theorem 6.2.18 that even if $c_0 \in C(\mathbb{R})^{\mathrm{sing}}$, there are an open neighborhood C_0 of c_0 in C and a holomorphic section $v : C_0 \to \mathfrak{f}$ of $\mathfrak{f}$ over C_0 such that $v_{c_0} \neq 0$. As in the proof of Lemma 8.1.4 we can arrange that v is real in the sense that $\gamma_*\,v = v$, which implies that the restriction of v to $C_0 \cap C(\mathbb{R})$ is a real analytic section of $\mathfrak{f}(\mathbb{R})$ over $C(\mathbb{R}) \cap C_0$. Let z denote a holomorphic local coordinate function on C_0 such that $z(c_0) = 0$, where we can arrange that z is real in the sense

that $z(\gamma_S(c)) = \overline{z(c)}$, or equivalently $z(c) \in \mathbb{R}$ when $c \in C(\mathbb{R}) \cap C_0$. We assume that for $0 < z = z(c) \ll 1$ the real fiber $S_c \cap S(\mathbb{R})$ over c is not empty.

The descriptions in Lemma 6.2.38 and Section 2.6.1 of the basic period functions p_1, p_2 and of the period lattice of real fibers, respectively, imply that only the following cases in the lemmas 8.2.1—8.2.4 below can occur. The proofs, given after Lemma 8.2.4, are a bit long, but yield additional information about the complex periods when $c \in C(\mathbb{R})$ approaches c_0 from either side.

Whether and where the cases described in the lemmas can occur is an entirely different question. The examples in Chapter 11, of QRT surfaces that occur in the literature show quite a large number of the cases.

Lemma 8.2.1 *If the complex singular fiber S_{c_0} is of Kodaira type I_b, $b \in \mathbb{Z}_{>0}$, then we have the following two cases:*

(1) *The period function $p(z)$ for $0 < z \ll 1$ extends to a real analytic function on a neighborhood of $z = 0$, with $p(0) > 0$. If the real fibers over $-1 \ll z < 0$ are nonempty, then the period function $p(z)$ for $-1 \ll z < 0$ is equal to the analytic extension of the period function $p(z)$ for $0 < z \ll 1$, and the number of connected components of the real fibers are the same or different if b is even or odd, respectively.*

(2) *There are real analytic functions $a(z)$ and $b(z)$ on a neighborhood of $z = 0$, with $a(0) > 0$, such that $p(z) = a(z)\ \log(1/z) + b(z)$ for $0 < z \ll 1$. If the real fibers for $-1 \ll z < 0$ are nonempty, then we have the following cases. If b is even then the real fiber for $-1 \ll z < 0$ has the same number of connected components as the real fiber for $0 < z \ll 1$, and $p(z) = a(z)\ \log(1/|z|) + b(z)$ for $-1 \ll z < 0$. If b is odd, then the real fiber for $-1 \ll z < 0$ has two or one connected components when the real fiber for $0 < z \ll 1$ has one or two components, and $p(z) = (a(z)\ \log(1/|z|) + b(z))/2$ or $p(z) = 2\,(a(z) \log(1/|z|) + b(z))$ for $-1 \ll z < 0$, respectively.*

Lemma 8.2.2 *Assume that S_{c_0} is of Kodaira type I_0^*.*

Then there is a real analytic function $a(z)$ on a neighborhood of $z = 0$ such that $a(0) > 0$ and $p(z) = a(z)\, z^{-1/2}$ for $0 < z \ll 1$. If the real fibers over $-1 \ll z < 0$ are nonempty, then the real fibers over $-1 \ll z < 0$ have the same number of connected components as the real fibers over $0 < z \ll 1$, and there is a real analytic function $a^-(z)$ on a neighborhood of $z = 0$ such that $a^-(0) > 0$ and $p(z) = a^-(z)\, |z|^{-1/2}$ for $-1 \ll z < 0$. If the real fibers are connected, then $p_1(z) = a(z)\, z^{-1/2}$ and $p_2(z) = (a(z) + a^-(z)\ \mathrm{i})\, z^{-1/2}/2$ form a $\mathbb{Z}$-basis of the period lattice. If each real fiber has two connected components, then $p_1(z) = a(z)\, z^{-1/2}$ and $p_2(z) = a^-(z)\, z^{-1/2}\ \mathrm{i}$ form a $\mathbb{Z}$-basis of the period lattice.

Lemma 8.2.3 *If S_{c_0} is of Kodaira type I_b^*, $b \in \mathbb{Z}_{>0}$, then we have the following cases:*

(1) *There is a real analytic function $a(z)$ on a neighborhood of $z = 0$ such that $a(0) > 0$ and $p(z) = a(z)\, z^{-1/2}$ for $0 < z \ll 1$. If the fibers over $-1 \ll z < 0$ are nonempty, then there are real analytic functions $a^-(z)$, $b^-(z)$ in a neighborhood of $z = 0$ such that $a^-(0) > 0$ and $p(z) = a^-(z)\, |z|^{-1/2}\ \log(1/|z|) + b^-(z)$*

for $-1 \ll z < 0$. We have $a^-(z) = a(z)\, b/2\pi$ or $a^-(z) = a(z)\, b/\pi$ when the real fiber for $-1 \ll z < 0$ has two or one connected component, respectively. If b is even then the numbers of connected components of the real fibers for $z < 0$ and $z < 0$ are the same, whereas these numbers differ by one if b is odd, still under the assumption that the real fibers for $-1 \ll z < 0$ are nonempty.

(2) *As in (1), but with the roles of $z > 0$ and $z < 0$ interchanged.*

Lemma 8.2.4 *Assume that the complex singular fiber S_{c_0} is of one of the remaining Kodaira types* II, II*, III, III*, IV, *and* IV*, *where we write* (α, β) *equal to* $(-1/6, 1/6)$, $(-5/6, -1/6)$, $(-1/4, 1/4)$, $(-3/4, -1/4)$, $(-1/3, 1/3)$, *and* $(-2/3, -1/3)$, *respectively.*

Then there are real analytic functions $A(z)$ and $B(z)$ in a neighborhood of $z = 0$, with $A(0) > 0$, such that the real period function is equal to $p(z) = A(z)\, z^\alpha + B(z)\, z^\beta$ for $0 < z \ll 1$. If the real fibers for $-1 \ll z < 0$ are nonempty, then there is a constant $R \in \mathbb{R}_{>0}$ such that the real period function for $-1 \ll z < 0$ is of the form $p^-(z) = R\,(A(z)\,|z|^\alpha + B(z)\,|z|^\beta)$ for the unstarred types and $p^-(z) = R\,(A(z)\,|z|^\alpha - B(z)\,|z|^\beta)$ for the starred types.

For the types II, II*, IV, IV*, *the real fibers for $0 < |z| \ll 1$, when nonempty, are connected. We have $R = \sqrt{3}$ or $R = 1/\sqrt{3}$ for types* II *and* II*, *whereas $R = 1$ for types* IV *and* IV*.

For types III *and* III*, *if the real fibers both for $0 < z \ll 1$ and for $-1 \ll z < 0$ are nonempty, then the number of connected components on the two sides are different. We have $R = \sqrt{2}$ and $R = 1/\sqrt{2}$ if the number of connected components for $0 < z \ll 1$ is two and one, respectively.*

Proof. Because $p(z)$ is a primitive period, there exist $k_1,\ k_2 \in \mathbb{Z}$ such that $p(z) = k_1\, p_1(z) + k_2\, p_2(z)$ and $\gcd(k_1,\ k_2) = 1$. Let $l_1,\ l_2 \in \mathbb{Z}$ be such that $k_1\, l_2 - k_2\, l_1 = 1$. It follows from Section 2.6.1 that we can arrange that there exists $b \in \mathbb{R}_{>0}$ such that $l_1\, p_1 + l_2\, p_2 = \epsilon\, p + q$ i, where $\epsilon = 1/2$ or $\epsilon = 0$ if the real fiber has one or two connected components, respectively. If the real fiber for $-1 \ll z < 0$ is nonempty, then we provide these data with a minus sign as a superscript.

Type I_b, $b \in \mathbb{Z}_{>0}$

Lemma 6.2.38 yields that $p_1(z)$ is holomorphic in a neighborhood of $z = 0$ with $p_1(0) \neq 0$ and $p_2(z) = p_1(z)\,(b/2\pi\ \mathrm{i})\,\log z + r(z)$, where $r(z)$ is holomorphic in a neighborhood of $z = 0$. For $-1 \ll z < 0$ we use the analytic continuation of the logarithm via the complex upper half-plane, that is,

$$p_2^-(z) = p_1(z)\,((b/2\pi\ \mathrm{i})\,\log|z| + b/2) + r(z).$$

Let $k_2 = 0$. Then $k_1 = \pm 1$, $l_2 = \pm 1$, $p(z) = \pm p_1(z)$, which implies that $p(z)$ has a holomorphic extension to a neighborhood of $z = 0$ and $\pm p_1(z)$ is real and positive. The real part of $\epsilon\, p + q$ i $= l_1\, p_1 \pm p_2$ yields that $\pm\epsilon\, p_1 = l_1\, p_1 \pm \mathrm{Re}\, q$.

If the real fibers for $-1 \ll z < 0$ are nonempty, then

$$p^-(z) = k_1^-\, p_1(z) + k_2^-\,(p_1(z)\,((b/2\pi\ \mathrm{i})\,\log|z| + b/2) + r(z)), \qquad (8.2.1)$$

because in the asymptotic expansion for $z \uparrow 0$ the logarithmic times power terms cannot be canceled by power terms, it follows from the fact that p^- and p_1 are real that $k_2^- = 0$. Hence $k_1^- = \pm 1$, with the same sign as k_1 because $p^-(z)$ is positive, and $p^-(z)$ is equal to the analytic continuation of $p(z)$. Furthermore, $l_2^- = \pm 1$. The real part of $\epsilon^- p^- + q^- \,\mathrm{i} = l_1^- p_1 \pm p_2^-$ yields that $\pm\epsilon^- p_1 = l_1^- p_1 \pm p_1 b/2 \pm \operatorname{Re} q$, which in view of $\pm\epsilon p_1 = l_1 p_1 \pm \operatorname{Re} q$ implies that $\epsilon^- = \epsilon \pm (l_1^- - l_1) + b/2$. Therefore $\epsilon^- = \epsilon$ or $\epsilon^- \neq \epsilon$ if b is even or odd, respectively. This completes the proof of (1) for type I_b, $b > 0$.

Let $k_2 \neq 0$. Then the fact that in the asymptotic expansion for $z \downarrow 0$ of the positive real function $p(z) = k_1 p_1(z) + k_2 (p_1(z) (b/2\pi \,\mathrm{i}) \log z + r(z))$ the logarithmic times power terms cannot be canceled by power terms implies that $p_1(z)$ is purely imaginary, $k_2 \operatorname{Im} p_1(0) < 0$, and $k_1 \operatorname{Im} p_1(z) + k_2 \operatorname{Im} r(z) = 0$. The real part of $\epsilon p + q \,\mathrm{i} = l_1 p_1 + l_2 p_2$ yields $\epsilon k_2 = l_2$, hence $1 = k_1 l_2 - k_2 l_1 = (\epsilon k_1 - l_1) k_2$. If $\epsilon = 0$ we have that $k_2 = \pm 1$. When $\epsilon = 1/2$ we have $k_2 = 2 l_2$ is even, $k_2 = \pm 2$, and k_1 is odd.

If the real fibers for $-1 \ll z < 0$ are nonempty, then (8.2.1) with p_1 purely imaginary implies that $k_1^- \operatorname{Im} p_1(z) + k_2^- (\operatorname{Im} p_1 \, b/2 + \operatorname{Im} r(z)) = 0$, which in view of $k_1 \operatorname{Im} p_1(z) + k_2 \operatorname{Im} r(z) = 0$ is equivalent to $k_1^- + k_2^- b/2 - k_2^- k_1/k_2 = 0$. When $\epsilon^- = 0$ we have $k_2^- = \pm 1$ and when $\epsilon^- = 1/2$ then $k_2^- = \pm 2$ and k_1^- is odd. This leads to the following cases: (i) $\epsilon = 0$, $\epsilon^- = 0$, $k_2 = \pm 1$, $k_2^- = \pm 1$, b is even; (ii) $\epsilon = 0$, $\epsilon^- = 1/2$, $k_2 = \pm 1$, $k_2^- = \pm 2$, b is odd; (iii) $\epsilon = 1/2$, $\epsilon^- = 0$, $k_2 = \pm 2$, $k_2^- = \pm 1$, b is odd; and (iv) $\epsilon = 1/2$, $\epsilon^- = 1/2$, $k_2 = \pm 2$, $k_2^- = \pm 2$, b is even. This completes the proof of (2) for type I_b, $b > 0$.

Type I_0^*

It follows from Lemma 6.2.38 that every period function is a unit times $z^{1/2}$. We can therefore arrange that $p(z) = p_1(z) = a(z) z^{1/2}$ and $\epsilon p(z) + q(z) \,\mathrm{i} = p_2(z) = b(z) z^{1/2}$, where $a(z)$ and $b(z)$ are holomorphic in a neighborhood of $z = 0$, a is real, $a(0) > 0$, and $\operatorname{Im} b(0) > 0$. Note that $\operatorname{Re} b(z) = \epsilon a(z)$ and $\operatorname{Im} b(z) = q(z)$.

We use the analytic continuation of $z^{-1/2}$ via the complex upper half-plane, that is, $z^{-1/2} = |z|^{-1/2}/\mathrm{i}$ when $z < 0$. If the real fibers for $-1 \ll z < 0$ are nonempty, then

$$p^-(z) = k_1^- p_1(z) + k_2^- p_2(z) = (k_1^- a(z) + k_2^- b(z)) |z|^{-1/2}/\mathrm{i}\,.$$

Because $p^-(z)$ is real we have $0 = k_1^- a(z) + k_2^- \operatorname{Re} b(z) = (k_1^- + k_2^- \epsilon) a(z)$, hence $k_1^- + k_2^- \epsilon = 0$. On the other hand, the real part of $\epsilon^- p^- + q^- \,\mathrm{i} = l_1^- p_1(z) + l_2^- p_2(z)$ yields $\epsilon^- k_2^- \operatorname{Im} b(z) = l_2^- \operatorname{Im} b(z)$, that is, $\epsilon^- k_2^- = l_2^-$. Therefore, $\epsilon = 0 \Rightarrow k_1^- = 0 \Rightarrow 1 = k_1^- l_2^- - k_2^- l_1^- = -k_2^- l_1^- \Rightarrow k_2^- = \pm 1 \Rightarrow \epsilon^- = \pm l_2 \in \mathbb{Z} \Rightarrow \epsilon^- = 0 \Rightarrow l_2^- = 0 \Rightarrow 1 = k_1^- l_2^- - k_2^- l_1^- = -k_2^- l_1^- \Rightarrow k_2^- = \pm 1 \Rightarrow \epsilon = \mp k_1^- \in \mathbb{Z} \Rightarrow \epsilon = 0$. This completes the proof for type I_0^*.

Type I_b^*, $b \in \mathbb{Z}_{>0}$

Let S_{c_0} be of type I_b^*, $b > 0$. Lemma 6.2.38 yields

$$p_1(z) = a(z) z^{-1/2} + a(z) z^{-1/2} ((b/2\pi \,\mathrm{i}) \log z + r(z)),$$

where $a(z)$ and $r(z)$ are holomorphic in a neighborhood of $z = 0$ and $a(0) \neq 0$. If in $p = k_1\, p_1 + k_2\, p_2$, $\epsilon\, p + q\ \mathrm{i} = l_1\, p_1 + l_2\, p_2$, $k_1\, l_2 - k_2\, l_1 = 1$ we have $k_2 = 0$, then $k_1 = l_2 = \pm 1$, $a(z)$ is real, $\pm a(0) > 0$, $\pm\epsilon = l_1 \pm \operatorname{Re} r(z)$. If the real fibers for $-1 \ll z < 0$ are nonempty, then we use the analytic continuation of $z^{-1/2}$ and $\log z$ via the complex upper half plane, where $z^{-1/2} = |z|^{-1/2}/\mathrm{i}$ and $\log z = \log|z| + \pi\ \mathrm{i}$ for $z < 0$. Then the reality of $p^- = k_1^-\, p_1^- + k_2^-\, p_2^-$ implies that $0 = k_1^- + k_2^-\, (b/2 + \operatorname{Re} r) = k_1^- + k_2^-\, (b/2 + \epsilon \mp l_1)$, and then $p^- = k_2^-\ \operatorname{Re} p_2^-$, which in turn implies that $k_2^- \neq 0$. On the other hand, the real part of $\epsilon^-\, p^- + q^-\ \mathrm{i} = l_1^-\, p_1 + l_2^-\, p_2^-$, where $k_2^-\, l_2^- - k_1^-\, l_2^- = 1$, implies that $\epsilon^-\, k_2^- = l_2^-$.

If b is even, then $\epsilon^- = 0 \Rightarrow l_2^- = 0 \Rightarrow k_2^- = \pm 1 \Rightarrow 0 = k_1^- \pm (b/2 + \epsilon \mp l_1)$ $\Rightarrow \epsilon = 0 \Rightarrow k_1^- = -k_2^-\, (b/2 \mp l_1)$ is an integral multiple of k_1^-, which in view of $k_2^-\, l_2^- - k_1^-\, l_2^- = 1$ implies that $k_2 = \pm 1$ and then $\epsilon\, k_2^- = l_2^-$ implies that $\epsilon = 0$. This leads to the equivalences $\epsilon^- = 0 \Leftrightarrow k_2^- = \pm 1 \Leftrightarrow \epsilon = 0$ and $\epsilon^- = 1/2 \Leftrightarrow k_2^- = \pm 2 \Leftrightarrow \epsilon = 1/2$ when b is even.

If b is odd, then $\epsilon^- = 0 \Rightarrow l_2^- = 0 \Rightarrow k_2^- = \pm 1 \Rightarrow 0 = k_1^- \pm (b/2 + \epsilon \mp l_1)$ $\Rightarrow \epsilon = 1/2 \Rightarrow k_1^- = -k_2^-\, (b/2 + \epsilon \mp l_1)$ is an integral multiple of $k_2^- \Rightarrow k_2^- = \pm 1$ $\Rightarrow \pm\epsilon^- = l_2^- \Rightarrow \epsilon^- = 0$. This leads to the equivalences $\epsilon^- = 0 \Leftrightarrow k_2^- = \pm 1 \Leftrightarrow$ $\epsilon = 1/2$, and $\epsilon^- = 1/2 \Leftrightarrow k_2^- = \pm 2 \Leftrightarrow \epsilon = 0$ when b is odd.

If $k_2 \neq 0$, then the reality of $p(z) = k_1\, p_1(z) + k_2\, p_2(z)$, together with the fact that in the asymptotic expansion for $z \downarrow 0$ logarithmic times power terms cannot be canceled by power terms, implies that $a(z)$ is purely imaginary. If the real fibers for $-1 \ll z < 0$ are nonempty, then the reality of $p^- = k_1^-\, p_1 + k_2^-\, p_2^-$, where the logarithmic period p_2^- has obtained an additional factor $1/\mathrm{i}$, together with the fact that in the asymptotic expansion for $z \uparrow 0$ logarithmic times power terms cannot be canceled by power terms, implies that $k_2^- = 0$. That is, we are in the previous situation with $z > 0$ and $z < 0$ interchanged. This completes the proof for the type I_b^*, $b > 0$.

Types II, II*, III, III*, IV, **and** IV*

Lemma 6.2.38 yields

$$p_1(z) = a(z)\, z^\alpha + b(z)\, z^\beta, \quad p_2(z) = a(z)\, z^\alpha\, \omega + b(z)\, z^\beta\, \overline{\omega},$$

where $a(z)$, $b(z)$ are holomorphic in a neighborhood of $z = 0$, $a(0) \neq 0$, where the exponents α and β are as in the statement. Note that $-1 < \alpha < 0 < \beta < 1$, $\alpha + \beta = 0$ for the unstarred types and $\alpha + \beta = -1$ for the starred types. We will also use that for each type, $\beta - \alpha$ is not an integer, which implies that for power series $f(z)$ and $g(z)$, we have $f(z)\, z^\alpha + g(z)\, z^\beta = 0$ if and only if $f(z) = 0$ and $g(z) = 0$. Finally, $\omega = \mathrm{e}^{2\pi\,\mathrm{i}}/4 = \mathrm{i}$ for types IV and IV*, whereas $\omega = \mathrm{e}^{2\pi\,\mathrm{i}}/6 = (1+\mathrm{i}\,\sqrt{3})/2$ for the other types.

We have $p(z) = k_1\, p_1(z) + k_2\, p_2(z) = A(z)\, z^\alpha + B(z)\, z^\beta$ with $A(z) = a(z)\, (k_1 + k_2\, \omega)$ and $B(z) = b(z)\, (k_1 + k_2\, \overline{\omega})$. Because $p(z)$ is real and positive for $0 < z \ll 1$, it follows that the holomorphic functions $A(z)$ and $B(z)$ are real. Because 1 and ω are linearly independent over $\mathbb{Q}$, we have $k_1 + k_2\, \omega \neq 0$, hence $A(0) \neq 0$, and the positivity of $p(z)$ implies that $A(0) > 0$.

We will use the analytic continuation of z^γ via the complex upper half-plane, that is, $z^\gamma = |z|^\gamma\ \mathrm{e}^{\mathrm{i}\,\pi\,\gamma}$ if $z < 0$. If the real fiber for $-1 \ll z < 0$ is nonempty, then the period function there is of the form

$$p^-(z) = k_1^-\ p_1^-(z) + k_2^-\ p_2^-(z) = A^-(z)\,|z|^\alpha + B^-(z)\,|z|^\beta,$$

where $A^-(z) = a(z)\,(k_1^- + k_2^-\,\omega)\ \mathrm{e}^{\mathrm{i}\,\pi\,\alpha}$, $B^-(z) = b(z)\,(k_1^- + k_2^-\,\overline{\omega})\ \mathrm{e}^{\mathrm{i}\,\pi\,\beta}$ are real analytic functions and $A^-(0) > 0$. Substituting $a(z) = A(z)/(k_1 + k_2\,\omega)$, $b(z) = B(z)/(k - 1 + k_2\,\overline{\omega})$, writing

$$R := (k_1^- + k_2^-\,\omega)\ \mathrm{e}^{\mathrm{i}\,\pi\,\alpha}\,/(k_1 + k_2\,\omega), \tag{8.2.2}$$

and using that $\mathrm{e}^{\mathrm{i}\,\pi\,\beta}$ is equal to plus or minus the complex conjugate of $\mathrm{e}^{\mathrm{i}\,\pi\,\alpha}$, we obtain that $p^-(z) = R\,A(z)\,|z|^\alpha + \overline{R}\,B(z)\,|z|^\beta$ or $p^-(z) = R\,A(z)\,|z|^\alpha - \overline{R}\,B(z)\,|z|^\beta$ for the unstarred or starred types, respectively. Because R must be real and positive, this proves the first statement about the period functions for the types other than I_b and I_b^*.

In order to further restrict the possibilities, we solve p_1 and p_2 from the equations $p = k_1\,p_1 + k_2\,p_2$, $\epsilon\,p + q\ \mathrm{i} = l_1\,p_1 + l_2\,p_2$, using $k_1\,l_2 - k_2\,l_1 = 1$. This yields $p_1 = l_2\,p - k_2\,(\epsilon\,p + q\ \mathrm{i}) = (l_2 - k_2\,\epsilon)\,p - k_2\,q\ \mathrm{i}$ and $p_2 = -l_1\,p + k_1\,(\epsilon\,p + q\ \mathrm{i}) = (-l_1 + k_1\,\epsilon)\,p + k_1\,q\ \mathrm{i}$. With the notation $r = p_2/p_1 = p_2\,\overline{p_1}/|p_1|^2$ and $Q = q/p$, this yields

$$((l_2 - k_2\,\epsilon)^2 + {k_2}^2\,Q^2)\,r = -(l_1 + k_1\,\epsilon)\,(l_2 - k_2\,\epsilon) - k_1\,k_2\,Q^2 + Q\ \mathrm{i},$$

where we note that $r \to \omega$ if $z \downarrow 0$. The fact that Q is a real solution of the quadratic equation

$$((l_2 - k_2\,\epsilon)^2 + {k_2}^2\,Q^2)\ \mathrm{Im}\,r = Q$$

implies the discriminant inequality $0 \le 1 - 4\,(l_2 - k_2\,\epsilon)^2\,{k_2}^2\,(\mathrm{Im}\,r)^2$. Because the limit value of $(\mathrm{Im}\,r)^2$ is $3/4$ and 1 for the types II, II*, IV, IV*, and III, III*, respectively, and $(l_2 - k_2\,\epsilon)^2 = 1/4$, 1, or larger and ${k_2}^2 = 1$, 4, or larger, it follows that $k_2 = 0$, or $k_2 \neq 0$ and $l_2 = k_2\,\epsilon$, or $k_2 = \pm 1$ and $l_2 \mp \epsilon = 1/2$ up to sign.

If $k_2 = 0$, then $1 = k_1\,l_2 - k_2\,l_1 = k_1\,l_2$ implies that $k_1 = l_2 = \pm 1$, $r = \mp(l_1 \pm \epsilon) + Q\ \mathrm{i}$. For types II, II*, IV, IV*, when $r \to \omega = (1 + \sqrt{3}\ \mathrm{i})/2$, we conclude that $1/2 = \mp(l_1 \pm \epsilon)$, hence $\epsilon = 1/2$, and $Q = \mathrm{Im}\ r \to \sqrt{3}/2$. For types III, III*, when $r \to \mathrm{i}$, we obtain $0 = l_1 \pm \epsilon$, hence $\epsilon = 0$, $r = Q\ \mathrm{i}$, and $Q \to 1$.

If $k_2 \neq 0$ and $l_2 - k_2\,\epsilon = 0$, then ${k_2}^2\,Q^2\,r = -k_1\,k_2\,Q^2 + Q\ \mathrm{i}$, that is, $k_2\ \mathrm{Re}\,r = -k_1$ and ${k_2}^2\,Q\ \mathrm{Im}\,r = 1$. For types II, II*, IV, IV*, we conclude that $k_2 = -2\,k_1$, $1 = k_1\,l_2 - k_2\,l_1 = k_1\,(l_2 + 2\,l_1)$, hence $k_1 = \pm 1$, $k_2 = \mp 2$, $l_2 + 2\,l_1 = \pm 1$, hence $l_2 \neq 0$, and therefore $l_2 - k_2\,\epsilon = 0$ implies that $\epsilon = 1/2$. Furthermore, $Q = 1/(4\ \mathrm{Im}\,r) \to 1/(2\,\sqrt{3}) = \sqrt{3}/6$. For types III, III* we have $\mathrm{Re}\,r \to 0$, hence $k_1 = 0$, $k_2 = \mp 1$, and therefore $l_2 - k_2\,\epsilon = 0$ implies that $\epsilon = 0$. Furthermore, $Q = 1/\mathrm{Im}\,r \to 1$.

Now assume that $k_2 = \pm 1$ and $l_2 \mp \epsilon = 1/2$ up to sign. Because $l_2 \in \mathbb{Z}$, it follows that $\epsilon = 1/2$. In the limit for $z \downarrow 0$ the equation $(1/4 + Q^2)\ \mathrm{Im}\,r = Q$ for types

II, II*, IV, IV* has the form $(1/4 + Q^2)\,\sqrt{3}/2 - Q = 0$, which has solutions, and it follows that $Q \to \sqrt{3}/2$ or $Q \to \sqrt{3}/6$. For types III, III* the limit equation is $1/4 + Q^2 - Q = 0$, and we conclude that $Q \to 1/2$.

As in the proof of Lemma 6.2.38, we arrive at a $\mathbb{Z}$-basis of the period lattice of the form $P_1(z) := p(z) = A(z)\,z^\alpha + B(z)\,z^\beta$, $P_2(z) := \epsilon\,p(z) + q(z)\,\mathrm{i} = A(z)\,z^\alpha\,\nu + B(z)\,z^\beta\,\theta$, where ν and θ are complex constants with $\operatorname{Im}\,\nu > 0$. Running with z once around the origin in the positive direction, we obtain the equations $M_{11} + M_{12}\,\nu = \mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}$, $M_{11} + M_{12}\,\theta = \mathrm{e}^{2\pi\,\mathrm{i}\,\beta}$, $M_{21} + M_{22}\,\nu = \mathrm{e}^{2\pi\mathrm{i}\,\alpha}\;\nu$, $M_{21} + M_{22}\,\theta = \mathrm{e}^{2\pi\mathrm{i}\,\beta}\;\theta$, where $M \in \mathrm{SL}(2, \mathbb{Z})$ is the monodromy matrix. Because $\mathrm{e}^{2\pi\mathrm{i}\,\beta}$ is equal to the complex conjugate of $\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}$, it follows that $\theta = \overline{\nu}$.

For types II, II*, IV, IV*, we have $\nu = 1/2 + \mathrm{i}\,\sqrt{3}/2 = \omega = \mathrm{e}^{2\pi\,\mathrm{i}/6}$ or $\nu = 1/2 + \mathrm{i}\,\sqrt{3}/6 = \mathrm{e}^{\pi\,\mathrm{i}/6}/\sqrt{3}$, whereas always $\epsilon = 1/2$, that is, the real fibers are connected. For types III, III*, we have $\nu = \mathrm{i} = \omega$ when $\epsilon = 0$, that is, each real fiber for $0 < z \ll 1$ has two connected components, whereas $\nu = 1/2 + \mathrm{i}\,/2 = \mathrm{e}^{\pi\,\mathrm{i}/4}/\sqrt{2}$ when $\epsilon = 1/2$, that is, each real fiber for $0 < z \ll 1$ is connected.

If we replace the $\mathbb{Z}$-basis p_1, p_2 of the period lattice by the $\mathbb{Z}$-basis P_1, P_2, then we have to replace k_1, k_2, and ω by 1, 0, and ν, respectively. Then the formula (8.2.2) for R takes the form

$$R = (k_1^- + k_2^-\,\nu)\;\mathrm{e}^{\pi\,\mathrm{i}\,\alpha}\,.$$

In combination with $k_1^-,\ k_2^- \in \mathbb{Z}$, $\gcd(k_1^-,\ k_2^-) = 1$, and $R \in \mathbb{R}_{>0}$, this leads to a unique determination of $k_1^-,\ k_2^-$, hence of R, for each of the twelve possibilities for α and ν. For instance, for type II with $\alpha = -1/6$, $\nu = \mathrm{e}^{2\pi\,\mathrm{i}/6}$, $\operatorname{Im} R = 0$ implies that $k_1^- = k_2^-$, hence $k_1^- = k_2^- = \pm 1$, $R = \pm\sqrt{3}$, and the positivity of R implies that $\pm = +$, and therefore $k_1^- = k_2^- = 1$ and $R = \sqrt{3}$. We have summarized the results, except for the integers $k_1^-,\ k_2^-$, in Lemma 8.2.4. □

The computer plot in Figure 11.4.10 of the rotation function for the square of the Lyness map for $a = -1/4$ appears to exhibit the $\sqrt{3}$ behavior near a real singular fiber of Kodaira type II.

Remark 8.2.5. Suppose that an elliptic surface with real structure is obtained from a real pencil of biquadratic curves, when we have the explicitly computable Weierstrass invariant $\Delta = {g_2}^2 - 27\,{g_3}^2$, the zeros of which correspond to the singular fibers. According to Lemma 2.6.3, the real fiber is nonempty and connected, a circle, if $\Delta < 0$. If $\Delta > 0$, then the real fiber is empty or has two connected components.

If the order of a zero of Δ is even or odd, then the sign of Δ at both sides of the zero in $C(\mathbb{R})$ is the same or the opposite, respectively. According to Table 6.3.2, the order of the zero of Δ, which is equal to the Euler number of the singular fiber, is odd when the singular fiber is of type I_b or I_b^* with b odd, or of type III or III*. The order of the zero of Δ is even in all other cases. This agrees with the determination of the numbers of connected components of the real fibers at the different sides of c_0 in $C(\mathbb{R})$ in the above description of the real period functions.

For general elliptic fibrations with a real structure, the description in Section 8.3 below of the singular fibers of Kodaira type I_b, $b > 0$, yields the following information about the number of connected components of the nearby real fibers. If the singular fiber is of hyperbolic type I_b, $b > 0$, with respect to the real structure, then

for both signs of z, $0 < |z| \ll 1$, the real fibers are nonempty, and their numbers of connected components are the same and differ by one if b is even and odd, respectively. If the singular fiber is elliptic with respect to the real structure and b is even, then either for both signs of z, $0 < |z| \ll 1$, the real fibers are nonempty and have the same number of connected components, or the real fibers for one sign of z have two connected components, but are empty for the other sign of z. If the singular fiber is elliptic and b is odd, then the numbers of connected components of the real fibers differ by one for the different signs of z.

We have not investigated, for general elliptic fibrations with a real structure, whether if on one side each real fiber is nonempty and connected, and the complex fiber is of type I_b^*, b even, II, II*, IV, or IV*, then the real fiber at the other side is also nonempty, and then connected as well. It follows from Lemma 2.6.3 that this conclusion holds for real QRT surfaces. And also for type I_b, b even, as follows from the above description of the hyperbolic and elliptic types. ⊘

8.3 Hyperbolic and Elliptic I_b, $b > 0$

The information in Section 8.2 about the real period functions has been obtained from the information in Lemma 6.2.38 about the complex period functions, and the description in Section 2.6.1 of the period lattice of an elliptic curve with a real structure. More information can be obtained by a determination of the geometry of the real fibers when approaching the singular one, and the way in which the real fibers are contained in the complex fibers.

As an example we dicuss the case that s_0 is a real singular point of a complex fiber S_{c_0} of Kodaira type I_b, $b > 0$. We have holomorphic local coordinates (x, y) in a neighborhood of s_0 in S such that $(0, 0)$ corresponds to s_0, and a holomorphic coordinate z in a neighborhood of c_0 in C such that $z = 0$ corresponds to c_0, and moreover $\gamma_S : (x, y) \mapsto (\overline{x}, \overline{y})$ and $\gamma_C : z \mapsto \overline{z}$. If $f(x, y)$ is a holomorphic function, then its complex conjugate is the holomorphic function $\overline{f}$ defined by $\overline{f}(x, y) = \overline{f(\overline{x}, \overline{y})}$. Our assumption that $\varphi : S \to C$ is real implies that $\overline{\varphi} = \varphi$.

Because each singular point of fibers of type I_b, $b > 0$, is a normal crossing of two complex projective lines, each of multiplicity one, we have $\varphi = \psi\,\chi$, where ψ and χ form a complex analytic coordinate system near the origin. Because $\psi\,\chi = \varphi = \overline{\varphi} = \overline{\psi}\,\overline{\chi}$, we have a unit u such that $\overline{\psi} = u\,\psi$ and $\overline{\chi} = u^{-1}\,\chi$, or $\overline{\psi} = u\,\chi$ and $\overline{\chi} = u^{-1}\,\psi$.

8.3.1 The Hyperbolic Case

Assume that $\overline{\psi} = u\,\psi$. Then $\psi = \overline{\overline{\psi}} = \overline{u}\,\overline{\psi} = \overline{u}\,u\,\psi$ shows that $u\,\overline{u} = 1$, that is, locally $u = e^{i\,\theta}$ for a real holomorphic function θ. If $v := e^{i\,\theta/2}$, then $\overline{v}\,u = v$; hence $\overline{v\,\psi} = \overline{v}\,\overline{\psi} = \overline{v}\,u\,\psi = v\,\psi$. That is, replacing ψ and χ by $v\,\psi$ and $v^{-1}\,\chi$, respectively, we have $\varphi = \psi\,\chi$, where ψ and χ form a real analytic coordinate

system. In particular, the real part $S_{c_0} \cap S(\mathbb{R})$ of S_{c_0} near $s_0 \simeq (0,\ 0)$ is the normal crossing of two real analytic curves, and the nearby real curves $\varphi = z$, with $z \in \mathbb{R}$ and $0 < |z| \ll 1$, are hyperbolas approaching the normal crossing. It follows from the proof of Lemma 6.2.38 that the solution curves of the vector field v spend time of order $\delta\ \log(1/z)$, asymptotically for $z \downarrow 0$, near each singular point s_0, where the positive constant δ is the same for all the b singular points of S_{c_0}. Therefore we are in case (2) of the description of the real period function for type I_b, $b > 0$. Because of the hyperbolic nature of the real curves near the singular point, this is called a *hyperbolic singular fiber of type* I_b, $b > 0$, with respect to the real structure.

Assume that $b \geq 2$, where the case $b = 1$ is different but simpler. For the hyperbolic singular fiber of type I_b, $b > 0$, each of the two irreducible components A_- and A_+ of the complex curve S_{c_0} that intersect each other at s_0 contains a whole curve in $A_\pm \cap S(\mathbb{R}) \cap S^{\mathrm{reg}}$. It follows that both A_+ and A_- are invariant under the complex conjugation γ_S. Then S_{c_0} is a cycle of b complex projective lines A_i, $i \in \mathbb{Z}/b\,\mathbb{Z}$, intersecting each other consecutively in points a_i such that $\{a_i\} = A_{i-1} \cap A_i$. Because γ_S permutes the irreducible components A_i of S_{c_0}, preserving the intersection structure, and preserves s_0, which is one of the intersection points, it follows that γ_S preserves each of the A_i and each of the intersection points a_i, which are the singular points of S_{c_0}.

As in the proof of Lemma 8.1.4, we identify $A_i \cap S^{\mathrm{reg}}$ with the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers. Then the complex conjugation γ_S preserves 0 and ∞, and therefore $t \mapsto \gamma_S(\overline{t})$ defines a complex analytic automorphism of $\mathbb{P}^1$ preserving 0 and ∞, which is equal to multiplication by a nonzero complex number a. Therefore $\gamma_S(t) = a\,\overline{t}$, and because $\gamma_S \circ \gamma_S$ is the identity, we have $a\,\overline{a} = 1$. We have $\gamma_S(b\,t) = a\,\overline{b\,t} = a\,\overline{b}\,t = b\,\overline{t}$ if we choose $b \in \mathbb{C}^\times$ such that $\overline{b}/b = a$. Therefore, replacing t by $b\,t$ we can arrange that $a = 1$. Then $\gamma_S(t) = t$ if and only if $t \in \mathbb{R}^\times$, the multiplicative group of all nonzero real numbers. It follows that for each irreducible component A_i of S_{c_0} the real part $A_i(\mathbb{R}) := A_i \cap S(\mathbb{R})$ of A_i is a real projective line, a circle, and its regular part $A_i \cap S^{\mathrm{reg}} = A_i \setminus \{a_i,\ a_{i+1}\}$ is isomorphic to the multiplicative group $\mathbb{R}^\times$ of all nonzero real numbers, which is the union of two intervals. $A_{i-1}(\mathbb{R})$ and $A_i(\mathbb{R})$ intersect each other at a_i, and the intersection is transversal in the real surface $S(\mathbb{R})$. This implies that each of the singular points a_i of S_{c_0} is hyperbolic with respect to the real structure. Furthermore, the real part of S_{c_0} of a hyperbolic singular fiber of type I_b is a *garland*, that is, a cycle of b real projective lines, or equivalently circles, in the cycle of complex projective lines.

The real vector field v is tangent to the $A_i(\mathbb{R})$ and has only zeros at the intersection points, that is, v has no zeros on $A_i(\mathbb{R}) \setminus \{a_i,\ a_{i+1}\}$. On both connected components $A_i(\mathbb{R})^\pm$ of $A_i(\mathbb{R}) \setminus \{a_i,\ a_{i+1}\}$, the flow of v is in the same direction, either from a_i to a_{i+1} or vice versa. In the first case the flow on $A_i(\mathbb{R})$ is from a_i to a_{i+1}, and since the flow of v on $A_{i+1}(\mathbb{R})$ is away from a_{i+1}, it follows by induction that the flow is from a_j to a_{j+1} for all i. By reversing the numbering of the circles in the garland if necessary, we can arrange that the flow of v on $A_i(\mathbb{R}) \setminus \{a_i,\ a_{i+1}\}$ is from a_i to a_{i+1} for all i.

For $0 < z \ll 1$, the real fiber near the intersection point a_{i+1} of A_i and A_{i+1} consists of two branches of an approximate hyperbola near a normal crossing. After

a time of order $\delta\ \log(1/z)$, the flow of v connects the part of the real fiber close to one of the connected components $A_i(\mathbb{R})^{\pm}$ of $A_i(\mathbb{R}) \setminus \{a_i,\ a_{i+1}\}$ with the part of the real fiber close to one of the connected components $A_{i+1}(\mathbb{R})^{\pm}$ of $A_{i+1}(\mathbb{R}) \setminus \{a_{i+1},\ a_{i+2}\}$. Continuing this along the cycle, we have after b steps that the part of the real fiber close to $A_i(\mathbb{R})^{\pm}$ is connected either to itself or to the part of the real fiber close to $A_i(\mathbb{R})^{\mp}$, the nearby nonsingular real fiber has two connected components or is connected, and the period $p(z)$ is of order $b\ \log(1/z)$ or $2\,b\ \log(1/z)$, respectively.

For $-1 \ll z < 0$, the real fiber near a_{i+1} consists of the two branches of an approximate hyperbola in the opposite quadrants of the normal crossing, as compared to the case that $0 < z \ll 1$. After time of order $\delta\ \log(1/|z|)$, the flow of v connects the part of the real fiber for $-1 \ll z < 0$ close to $A_i(\mathbb{R})^{\pm}$ to the part of the real fiber close to $A_i(\mathbb{R})^{\mp}$. If b is even or odd, then after b steps we arrive at the part of the real fiber near the same or the other connected component of $A_i(\mathbb{R}) \setminus \{a_i,\ a_{i+1}\}$ as for $0 < z \ll 1$, respectively. It follows that the real fibers for $-1 \ll z < 0$ have the same or a different number of connected components as those for $0 < z \ll 1$ if b is even or odd, respectively. In all cases, the period $p(z)$ for $0 < |z| \ll 1$ is of order $b\,\delta\ \log(1/|z|)$ and $2\,b\,\delta\ \log(1/|z|)$ if the real fiber has two connected components and is connected, respectively. This confirms the description in case (2) in Lemma 8.2.1, with the additional information that for both signs of z, $0 < |z| \ll 1$, the real fibers are nonempty near a hyperbolic singular fiber of type I_b.

Figure 8.3.1 If we attach the right-hand side to the left hand side in Figure 8.3.1, then we have $b = 4$, and for both signs of z, where $|z| \ll 1$, the real curve over z has two connected components. If we do the same after adding one more circle to the garland, then we have $b = 5$, and the real curve over z has two components for one sign of z and is connected for the other sign of z.

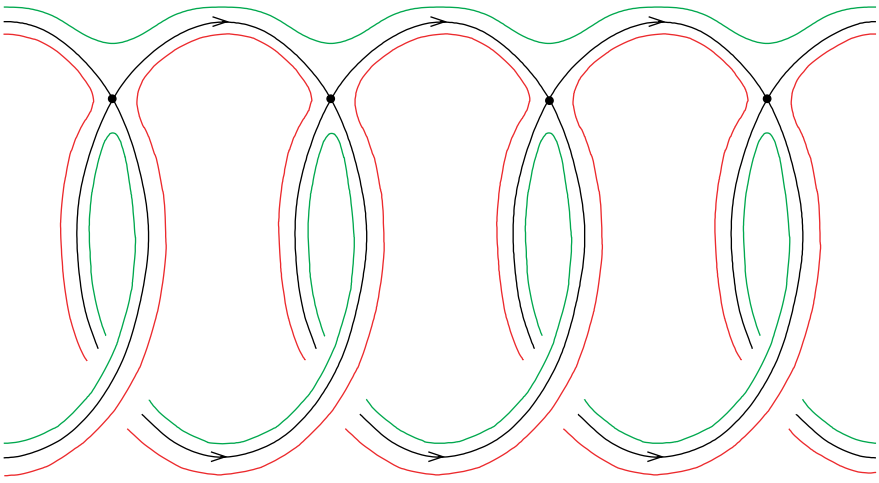


Fig. 8.3.1 Real fibers near a hyperbolic singular fiber of type I_b.

8.3.2 The Elliptic Case

Now assume that $\overline{\psi} = u\,\chi$, $\overline{\chi} = u^{-1}\,\psi$. Then $\psi = \overline{\overline{\psi}} = \overline{u}\,\overline{\chi} = \overline{u}\,u^{-1}\,\psi$ shows that u is real in the sense that $\overline{u} = u$. We have $\overline{v\,\psi} = \overline{v}\,\overline{\psi} = \overline{v}\,u\,\chi = \overline{v}\,v\,u\,(v^{-1}\,\chi)$. Choosing v such that $\overline{v}\,v\,u = \pm 1$, we obtain that by replacing ψ and χ by $v\,\psi$ and $v^{-1}\,\chi$, respectively, we can arrange that $\overline{\psi} = \pm\chi$, that is, $\varphi = \pm\psi\,\overline{\psi}$. In other words, $\pm\varphi$ has a nondegenerate stationary point at s_0, with a positive definite Hessian = second-order derivative matrix. It follows that s_0 is an isolated point of $S_{c_0} \cap S(\mathbb{R})$, and for $0 < \pm c \ll 1$ the level sets $\varphi = c$, the real fibers $S_c \cap S(\mathbb{R})$, near s_0 are small ellipses that shrink to the singular point at $\pm c \downarrow 0$. For $-1 \ll \pm c < 0$, there are no real points of S_c near s_0. Because the periods of the solutions of the Hamiltonian system of the Hamiltonian function φ on the curves $\varphi = c$ remain bounded as $\pm c \downarrow 0$, we are in case (1) of the description of the real period function for type I_b, $b > 0$. Because of the elliptic nature of the real curves near the singular point, this is called an *elliptic singular fiber of type* I_b, $b > 0$, with respect to the real structure. Because in the case that $c_0 \in C^{\mathrm{sing}} \setminus C(\mathbb{R})^{\mathrm{sing}}$, described at the end of Lemma 8.1.4, we also have that the real period function $p(z)$ extends to a real analytic function on a neighborhood of $z = 0$, we also in this case call S_{c_0} an elliptic singular fiber of type I_b, $b > 0$, with respect to the real structure.

For the elliptic singular fiber of type I_b, $b > 0$, the two irreducible components A_b and A_1 of the complex curve S_{c_0} that intersect each other at s_0 locally are the zero-sets of ψ and χ, respectively, and therefore are interchanged by the complex conjugation γ_S. Because γ_S permutes the irreducible components of S_{c_0}, preserving the intersection structure, and the irreducible components of S_{c_0} form a cycle A_k, $k \in \mathbb{Z}/b\,\mathbb{Z}$, of b complex projective lines intersecting each other consecutively, it follows that γ_S maps A_k to A_{b-k+1} for each $k \in \mathbb{Z}/b\,\mathbb{Z}$.

If $b = 2\,a$ is even, then none of the A_k is invariant under γ_S, and the only other singular point of S_{c_0} that is invariant under γ_S, and therefore is the only other real point of S_{c_0}, is the intersection point of A_a and $\gamma_S(A_a) = A_{2\,a-a+1} = A_{a+1}$. When $c \in C(\mathbb{R})$ passes through c_0 we have either two small ellipses near the opposite singular points of S_{c_0} that shrink to a point and disappear, or one small ellipse near one of the singular points of S_{c_0} that shrinks to a point and disappears when the other singular point of S_{c_0} has appeared as an isolated real point of S_{c_0}, and after which a small ellipse in $S_c \cap S(\mathbb{R})$ grows around that other singular point of S_{c_0}. Quite amusing. Note that this implies that if b is even and the real fiber over $c \in C(\mathbb{R})$ at one side of c_0 is nonempty and connected, then the real fiber over $c \in C(\mathbb{R})$ at the other side of c_0 is nonempty and connected as well. Also note that if the singular fiber S_{c_0} of type I_b is elliptic with respect to the real structure and b is even, then $S_{c_0} \cap S(\mathbb{R}) \cap S^{\mathrm{reg}} = \emptyset$, and it follows that the fibration has no real smooth section σ over any neighborhood of c_0 in $C(\mathbb{R})$, since this would imply that $\sigma(c_0) \in S_{c_0} \cap S(\mathbb{R}) \cap S^{\mathrm{reg}}$. This in turn implies that the elliptic fibration over any open neighborhood of c_0 in C, provided with its real structure, is not isomorphic to a real Weierstrass model.

If $b = 2a - 1$ is odd, then $\gamma_S(A_a) = A_{b-a+1} = A_{2a-1-a+1} = A_a$, where the complex conjugation on $A_a \simeq \mathbb{P}^1$ interchanges the two intersection points with the other irreducible components. As in the proof of Lemma 8.1.4, we obtain that $S_{c_0} \cap S(\mathbb{R}) \setminus \{s_0\}$ is either empty or a circle in $A_a \cap S^{\mathrm{reg}}$. Away from s_0, the topological description of $S_c \cap S(\mathbb{R})$ does not change when $c \in C(\mathbb{R}) \setminus \{c_0\}$ is close to c_0. These conclusions also hold if $b = 1$, when the complex curve S_{c_0} is irreducible with one singular point, a double point, and is isomorphic to a complex projective line with two points on it identified with transversal intersection of the local pieces of the complex projective line.

Remark 8.3.1. For an elliptic fibration with a real structure that is obtained from a real pencil of biquadratic curves, we have the explicitly computable Weierstrass invariants g_2, g_3. Then the singular fiber is elliptic or hyperbolic if and only if $g_3 > 0$ or $g_3 < 0$, respectively; see Remark 2.6.5. ⊘

8.4 Singularities of the Real Rotation Function

We keep the notation of Section 8.2. Assume that $\alpha \in \mathrm{Aut}(S)^+_\varphi$ is real and, in the case that the real fibers for $0 < z \ll 1$ have two connected components, α preserves these. We also assume that α is not of finite order when the rotation number is a constant in $\mathbb{Q}/\mathbb{Z}$.

On the fiber over z, α acts as $\mathrm{e}^{T(z)\,v}$, where $T(z) \in \mathbb{R}/p(z)\,\mathbb{Z}$ depends analytically on z for $0 < z \ll 1$. The rotation function $\rho(z) \in \mathbb{R}/\mathbb{Z}$ of Lemma 8.1.5 is equal to $\rho(z) = T(z)/p(z)$. About the time function $T(z)$ we have the following consequence of Lemma 7.4.1.

Lemma 8.4.1 *Let l be the smallest positive integer such that α^l leaves the connected components of $S_{c_0} \cap S^{\mathrm{reg}}$ invariant. Then there exist an $m \in (1/2)\,\mathbb{Z}$ and a real analytic function $f(z)$ on a neighborhood of $z = 0$ such that $T(z) = (m/l)\,p(z) + f(z)$ for $0 < z \ll 1$. Here $m \in \mathbb{Z}$ unless the complex singular fiber if of hyperbolic type I_b, $b > 0$, with respect to the real structure, and the real fibers for $0 < z \ll 1$ are connected.*

Assume that the complex singular fiber S_{c_0} is not of elliptic Kodaira type I_b, $b > 0$, with respect to the real structure. Then $\rho(z) = m/l + f(z)/p(z) \to m/l$ as $z \downarrow 0$, and ρ does not extend to an analytic function on a neighborhood of $z = 0$. On the other hand, if S_{c_0} is of elliptic Kodaira type I_b, $b > 0$, then $\rho(z)$ extends to an analytic function on a neighborhood of $z = 0$.

Proof. We apply Lemma 7.4.1 with $p_1(z)$, $p_2(z)$ replaced by the $\mathbb{Z}$-basis $p(z)$, $\epsilon\,p(z) + q(z)\,\mathrm{i}$ of the period lattice. Here $p(z)$ and $q(z)$ are real and positive for $0 < z \ll 1$, and $\epsilon = 0$ and $\epsilon = 1/2$ if the real fiber for $0 < z \ll 1$ has two and one connected components, respectively. It follows that there are $l_1,\ l_2 \in \mathbb{Z}$ such that $h(z) := l\,T(z) - l_1\,p(z) - l_2\,(\epsilon\,p(z) + q(z)\,\mathrm{i}) = l\,T(z) - (l_1 + l_2\,\epsilon)\,p(z) - l_2\,q(z)\,\mathrm{i}$ extends to a holomorphic function on a neighborhood of $z = 0$. The first conclusion follows with $m = l_1 + l_2\,\epsilon$ and $f(z) = \mathrm{Re}\,h(z)/l$.

Assume that $m \notin \mathbb{Z}$, which means that $\epsilon = 1/2$, that is, the real fibers for $0 < z \ll 1$ are connected, and l_2 is odd, which implies that $l_2 \neq 0$. Because $l_2\, q(z) = \operatorname{Im} h(z)$ is real analytic in a neighborhood of $z = 0$, it follows that $q(z)$ extends to a real analytic function on a neighborhood of $z = 0$, and it follows from the proof of Lemmas 8.2.1—8.2.4 that the complex singular fiber is of hyperbolic Kodaira type I_b, $b > 0$, with respect to the real structure. This proves the second statement in the lemma.

If S_{c_0} is an elliptic singular fiber of type $\mathrm{I}_b, b > 0$, with respect to the real structure, that is, we are in case (1) in Lemma 8.2.1, then $p(z)$ extends to an analytic function on a neighborhood of $z = 0$ with $p(0) > 0$, and therefore $\rho(z) = m/l + f(z)/p(z)$ extends to an analytic function on a neighborhood of $z = 0$.

If $f(z)$ is identically zero, then $\rho(z) \equiv m/l$, and therefore $\alpha^{2l} = 1$ on the real, hence on the complex, fiber for $0 < z \ll 1$. It follows by complex analytic contuation that $\alpha^{2l} = 1$ everywhere, in contradiction to our assumption that α is not of finite order. We conclude that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f(z) = u(z)\, z^n$, where $u(z)$ is analytic on a neighborhood of $z = 0$ and $u(0) \neq 0$. Because if we are not in case (1) in Lemma 8.2.1, it follows from Lemmas 8.2.1—8.2.4 that $p(z) \to \infty$ as $z \to \infty$ in a nonmeromorphic way, and therefore $f(z)/p(z) = u(z)\, z^m/p(z) \to 0$ as $z \downarrow 0$ in a nonanalytic way. □

Remark 8.4.2. Assume that the complex singular fiber for $z = 0$ is of hyperbolic Kodaira type I_b, $b > 0$, with respect to the real structure.

Then it follows from Lemma 8.2.1 that there are real analytic functions $a(z), b(z)$ in a neighborhood of $z = 0$, with $a(0) > 0$, such that $p(z) = a(z)\, \log(1/z) + b(z)$ for $0 < z \ll 1$. Assume that $f(0) \neq 0$. Then the remainder term $r(z) = f(z)/p(z)$ in $\rho(z) = m/l + f(z)/p(z)$ is asymptotically of the form $r(z) \sim c/\log(1/z)$ for $z \downarrow 0$, where $c := f(0)/a(0) \neq 0$ and c has the same sign as $f(0)$. When $c > 0$ and $c < 0$, this remainder term decreases and increases to zero as $z \downarrow 0$, respectively, where the decrease and increase is *infinitely steep*, in the following sense. Let $c > 0$, where the discussion for $c < 0$ is analogous. Then the equation $r = c/\log(1/z)$ is equivalent to $z = \mathrm{e}^{-c/r}$, $0 < r \ll 1$, where the function $r \mapsto \mathrm{e}^{-c/r}$ extends to a C^∞ function on the real axis that is infinitely flat in the sense that all its derivatives at $r = 0$ are equal to zero. In other words, at the origin the graph of the remainder term has an infinite order of contact with the vertical axis.

If $f(z)$ has a zero of order $l \geq 1$ at $z = 0$, the in the above, $\log(1/z)$ has to be replaced by $z^l\, \log(1/z)$, whose derivative with respect to z converges to 0 as $z \to 0$. Therefore, if the approch of the rotation function to its limit value is not infinitely steep, then it is horizontal. ⊘

Remark 8.4.3. Recall the description of the hyperbolic case in Section 8.3. In particular, the flow of v on the real fiber for $0 < z \ll 1$, after time of order $l\, c\, \log(1/|z|)$, $l \in \mathbb{Z}$, $0 \leq l < b$, maps the part of the real fiber near $A_i(\mathbb{R})^+$ to the part of the real fiber near $A_{i+l}(\mathbb{R})^+$. For $-1 \ll z < 0$, we have the same if l is even, but $A_{i+l}(\mathbb{R})^+$ has to be replaced by $A_{i+l}(\mathbb{R})^-$ if l is odd.

There is a minimal $k \in \mathbb{Z}$, $0 \leq k < b$, such that α maps A_i to A_{i+k}. Assume that for all $0 < |z| \ll 1$, α *preserves the connected components of the real fibers*. We

write $\rho^{\pm}$ for the limit of $\rho(z)$ as $\pm z \downarrow 0$, where we can arrange that $0 \leq \rho^+ < 1$, because the rotation number is determined only modulo integers.

Assume that for $0 < z \ll 1$ the real fibers have two connected components. Then $p(z) \sim b\,c\,\log(1/z)$, $\alpha(A_i(\mathbb{R})^+) = A_{i+k}(\mathbb{R})^+$, $T(z) \sim k\,c\,\log(1/z)$, and $\rho^+ = k/b$. If b is even, then the real fibers for $-1 \ll z < 0$ also have two connected components, k is even, and $\rho^- = k/b = \rho^+$. If b is odd, then the real fibers for $-1 \ll z < 0$ are connected, and it follows from the next paragraph that $\rho^- = k/2\,b = \rho^+/2$ if k is even and $\rho^- = (k+b)/2\,b = \rho^+/2 + 1/2$ if k is odd.

Assume that for $0 < z \ll 1$ the real fibers are connected. Then $p(z) \sim 2\,b\,c\,\log(1/z)$. If $\alpha(A_i(\mathbb{R})^+) = A_{i+k}(\mathbb{R})^+$ then $T(z) \sim k\,c\,\log(1/z)$, hence $\rho^+ = k/2\,b$. If $\alpha(A_i(\mathbb{R})^+) = A_{i+k}(\mathbb{R})^-$, then $T(z) \sim (k+b)\,c\,\log(1/z)$, and $\rho^+ = (k+b)/2\,b = k/2\,b + 1/2$. If b is odd then the real fibers for $-1 \ll z < 0$ have two connected components, for which we refer to the previous paragraph with the sign of z reversed, and $\rho^- = k/b$. If b is even, then the real fibers for $-1 \ll z < 0$ are also connected. If $\alpha(A_i(\mathbb{R})^+) = A_{i+k}(\mathbb{R})^+$, then $\rho^- = k/2\,b = \rho^+$ if k is even and $\rho^- = (k+b)/2\,b = \rho^+ + 1/2$ if k is odd. If $\alpha(A_i(\mathbb{R})^+) = A_{i+k}(\mathbb{R})^-$, then $\rho^- = (k+b)/2\,b = \rho^+$ if k is even and $\rho^- = k/2\,b = \rho^+ - 1/2$ if k is odd.

Summarizing, we have the following cases:

(i) b is even, for both signs of z the real fibers have two connected components, k is even, and $\rho^+ = \rho^- = k/b$.

(ii) b is even and for both signs of z the real fiber is connected. Both ρ^+ and ρ^- are equal to $k/2\,b$ or to $(k+b)/2\,b = k/2\,b + 1/2$. If k is even, then $\rho^- = \rho^+$. If k is odd, then ρ^- and ρ^+ differ by $1/2$.

(iii) b is odd where for one sign of z, say the positive one, the real fibers have two connected components, and for the other sign of z the real fibers are connected. Then $\rho^+ = k/b$, and $\rho^- = k/2\,b = \rho^+/2$ when k is even, whereas $\rho^- = (k+b)/2\,b = \rho^+/2 + 1/2$ when k is odd.

The Lyness map shows a number of cases in which $\rho^- \neq \rho^+$, that is, where the rotation function ρ has a jump discontinuity at the point in $C(\mathbb{R})$ over which the fiber is singular. See Table 11.4.4. Actually, one of the computer pictures of the rotation function as given in Section 11.4 cured me of the too naive conjectures that I first had about the rotation function, and stimulated the above analysis of its limit behavior when approaching singular fibers.

If, for $0 < z \ll 1$ or for $-1 \ll z < 0$, *the real fiber over z has two connected componenents that are interchanged by α*, then the above can be applied with α replaced by α^2, where $\rho_{\alpha^2} = 2\,\rho_\alpha$ modulo $\mathbb{Z}$, and hence $\rho_\alpha = (1/2)\,\rho_{\alpha^2}$ modulo $(1/2)\,\mathbb{Z}$. Jogia's example in Section 11.8 exhibits a limit value $1/2$ of the rotation number of the QRT map near a hyperbolic singular fiber of type I_1, where $b = 1$, hence $k = 0$. The example in Section 9.2.5 for $a = 0.538$, $b = 0.038$ appears to exhibit the same behavior; see Figures 9.2.4 and 9.2.5. ⊘

Remark 8.4.4. Assume that the complex singular fiber for $z = 0$ is of elliptic Kodaira type I_b, $b > 0$, with respect to the real structure.

In the Weierstrass normal form, taking the limit for $z \downarrow 0$, this corresponds to $\Delta = {g_2}^3 - 27\,{g_3}^2 = 0$, and $g_3 > 0$, hence $g_2 > 0$ as well. It follows that

$$f(x) := 4x^3 - g_2 x - g_3 = 4(x - x_-)^2 (x - x_+),$$

where $x_- := -(g_2/12)^{1/2} < 0 < x_+ := (g_2/3)^{1/2}$. It is classically known how to integrate $f(x)^{-1/2}$ if $f(x)$ has a multiple zero. To be specific, the substitution of variables $x = x_+ + (x_+ - x_-)\,\xi^2$ leads to the formula

$$T = \int^X f(x)^{-1/2}\,\mathrm{d}x = (x_+ - x_-)^{-1/2} \arctan t + C,$$
$$t = \pm(X - x_+)^{1/2}\,(x_+ - x_-)^{-1/2},$$

for the integral (2.5.15), where the constant C and the sign of t will be determined now. Note that $X > x_+$ because $(X,\ Y)$ is not equal to the isolated singular point $(x_-,\ 0)$ of the curve $y^2 - f(x) = 0$ in the real $(x,\ y)$-plane. The equation $Y^2 = f(X)$ allows us to replace $(X - x_+)^{1/2}$ by $Y/2\,(X - x_-)^{1/2}$ up to sign. Since the orientation on the unbounded part of the Weierstrass curve $y^2 - f(x) = 0$ is such that y is increasing, t must have the same sign as Y. The period is equal to

$$p = 2\int_0^\infty f(x)^{-1/2}\,\mathrm{d}x = (x_+ - x_-)^{-1/2}\,\pi,$$

and $Y = 0$ corresponds to half the period. It follows that $\rho = T/p = 1/2 + (1/\pi)\arctan t$.

Because $\arctan t = -\pi/2 + \arccos(-t/(t^2+1)^{1/2})$, this leads to the formula

$$\lim_{z \downarrow 0} \rho(z) = \frac{1}{\pi} \arccos \frac{-Y}{2\,(X + (g_2/12)^{1/2})^{3/2}} \tag{8.4.1}$$

for the limit value of the rotation number if the complex singular fiber is of elliptic type I_b, $b > 0$. The convention in (8.4.1) is that $0 \le \arccos \le \pi$, and therefore $0 \le \rho(0) \le 1$.

For the QRT map, we have explicit formulas for g_2, g_3, X, and Y; see Corollary 2.4.7 and Proposition 2.5.6. Actually, X, Y, and g_2 are homogeneous polynomials of degree 2, 3, and 4 in $(z_0,\ z_1)$. It follows that for QRT maps (8.4.1) for which we can explicitly determine the point $[z_0 : z_1]$ over which we have the singular real fiber of elliptic type, we have an explicit formula for the limit value of the rotation number when approaching that singular fiber.

For the square of the billiard map, (8.4.1) is equal to the limit (11.2.14) of the rotation number when the billiard ball is bouncing up and down along the short axis of the billiard. The formula (11.2.14) shows that this limit rotation number can take any real value between 0 and 1, depending on the parameter λ_0 of the elliptic billiard. ⊘

The following corollary is a real analogue of Corollary 7.7.10. The assumptions are as in Lemma 8.1.5. Corollary 7.7.10 implies that the number of k-periodic real fibers for α is asymptotically of order k as $k \to \infty$. In the real case we have no formula for the leading term in the asymptotic behavior that is analogous to the formula in Corollary 7.7.10 for the limit of $\nu(\alpha^k)/k^2$ as $k \to \infty$.

Corollary 8.4.5 *For every $k \in \mathbb{Z}_{>0}$, let $\mu_k(\mathbb{R})$ denote the measure on $\varphi(S(\mathbb{R})) \subset C(\mathbb{R})$ that is equal to $1/k$ times the sum of the point masses at the points $c \in \varphi(S(\mathbb{R}))$ such that $S_c(\mathbb{R}) \cap S^{\mathrm{reg}}$ is k-periodic, counted with multiplicities. For k is even or odd and as $k \to \infty$, the measures $\mu_k(\mathbb{R})$ converge to a finite measure $\mu(\mathbb{R})^{\mathrm{even}}$ or $\mu(\mathbb{R})^{\mathrm{odd}}$ on $\varphi(S(\mathbb{R}))$, respectively. Both limit measures have a density, equal to $|\,\mathrm{d}\rho(c)|$, where $\rho(c) \in \mathbb{R}/\mathbb{Z}$ is the rotation number, uniquely determined up to its sign. Whether k is even or odd, $1/k$ times the total number of k-periodic real fibers converges to the total variation of the rotation function ρ, the integral over $C(\mathbb{R})$ of the density $|\,\mathrm{d}\rho(c)|$, where this integral is a finite positive real number.*

If α preserves the connected components of the fibers of $\varphi(\mathbb{R})_I$, then $\mu(\mathbb{R})^{\mathrm{even}} = \mu(\mathbb{R})^{\mathrm{odd}}$ on I, in which case we denote the limit measure by $\mu(\mathbb{R})$. In the other case, in which α interchanges the two connected components of each fiber of $\varphi(\mathbb{R})_I$, we have $\mu_k = 0$ on I for each odd k, hence $\mu(\mathbb{R})^{\mathrm{odd}} = 0$ on I.

The density of $\mu(\mathbb{R})^{\mathrm{even}}$ is real analytic and strictly positive on

$$I \setminus (D \cup (I \cap C^{\mathrm{sing}})).$$

It is continuous on I and equal to zero at D.

If c_0 is a boundary point of I, then $c_0 \in C(\mathbb{R})^{\mathrm{sing}} \subset C^{\mathrm{sing}} \cap C(\mathbb{R})$, and we have the following two cases. The density of μ has an analytic extension to a neighborhood of $c = c_0$ if and only if S_{c_0} is of elliptic Kodaira type I_b, $b > 0$, with respect to the real structure, and the limit density is equal to zero if and only if S_{c_0} is a periodic fiber for α. In the other case, the density converges to $+\infty$ unless S_{c_0} is a periodic fiber for α, when again the density converges to 0.

Proof. Only the statement about the asymptotic behavior of the total number of k-periodic real fibers still needs a proof. Because ρ is real analytic in each maximal open interval I that does not contain points of $C(\mathbb{R})^{\mathrm{sing}}$, its derivative has zeros only in isolated points. The endpoints of I belong to the finite set $C(\mathbb{R})^{\mathrm{sing}}$. It follows from the description in Lemmas 8.2.1–8.2.4 of the behavior of the period function near points of $C(\mathbb{R})^{\mathrm{sing}}$, in combination with Lemma 8.4.1, that the zeros of $\mathrm{d}\rho$ do not accumulate at the endpoints of I, and that ρ has finite limits at the endpoints of I. Therefore the support of $|\,\mathrm{d}\rho|$ is equal to the union of finitely many intervals I_j, on each of which ρ is monotonic, whereas at the left and right endpoints of I_j the function ρ has a finite limit ρ_j^- and ρ_j^+, respectively. Because the fiber over $c \in I_j$ is k-periodic for α if and only if $\rho(c) = l/l$ for some $l \in \mathbb{Z}$, the number of these $c \in I_j$ differs from $k\,|\rho_j^+ - \rho_j^-|$ by at most two. It follows that $1/k$ times the total number of k-periodic real fibers converges to

$$\sum_j |\rho_j^+ - \rho_j^-| = \int_{C(\mathbb{R})} |\,\mathrm{d}\rho|. \tag{8.4.2}$$

The number (8.4.2), which is a finite positive real number, is called the *total variation of the rotation function* ρ. □

Note that Lemma 2.6.8 implies that there is an absolute upper bound on the total variation of the rotation for QRT transformations.

The elliptic billiard, see the paragraph with the heading $0 < \lambda < 1$ in Section 11.2.2, exhibits the phenomenon of switching of the connected components of real fibers, mentioned in Lemma 8.1.5 and Corollary 8.4.5.

Remark 8.4.6. Let $c_0 \in C^{\text{sing}} \cap C(\mathbb{R})$ and assume that the fiber over c_0 is of elliptic type I_b with respect to the real structure, where we also allow that $S_{c_0} \cap S(\mathbb{R}) \subset S^{\text{reg}}$ as in the last paragraph of Lemma 8.1.4. Then the density $\delta_\alpha(c)$ of α, see Definition 7.7.1, converges to 0 as c in the complex curve C approaches c_0.

For the proof we observe that if $s_0 \in S_{c_0} \cap S(\mathbb{R})$ is a real singular point of S_{c_0}, then, replacing α by its square if necessary, s_0 it is a fixed point of α, and the tangent map $\mathrm{T}_{c_0}\alpha$ of α is a linear transformation in $\mathrm{T}_{c_0}S$ that leaves the real subspace $\mathrm{T}_{c_0}S(\mathbb{R})$ invariant, and on it leaves the quadratic approximation of $\varphi(c) - \varphi(c_0)$ invariant. Because this quadratic approximation is a positive or negative definite quadratic form, it follows that the eigenvalues of $\mathrm{T}_{c_0}\alpha$ lie on the unit circle, and therefore the action of α on $S_{c_0} \cap S^{\text{reg}}$ belongs to the maximal compact subgroup of $\mathfrak{F}_{c_0}$. In the case of the last paragraph of Lemma 8.1.4, the fact that α^2 leaves a circle in $S_{c_0} \cap S^{\text{reg}}$ invariant implies that the action of α on $S_{c_0} \cap S^{\text{reg}}$ belongs to the maximal compact subgroup of $\mathfrak{F}_{c_0}$. In both cases it follows from Corollary 7.7.10 that $\delta_\alpha(c) \to 0$ as $c \to c_0$.

I was prompted to this remark by seeing the prominent zero density in the complex λ-plane near $\lambda = 0$ for the elliptic billiard. Over $\lambda = 0$ we have the singular fiber of Kodaira type I_2, which is elliptic with respect to the real structure. See Figure 11.2.5, where $\lambda = 0$ is the left boundary point of the real interval $[0,\ \lambda_0]$ over which we have nonempty real fibers, marked by the gray horizontal segment in the middle of the picture. ⊘

8.5 Real Pencils

In this subsection we show how, if we start with a pencil of real biquadratic curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, or a pencil of real cubic curves in $\mathbb{P}^2(\mathbb{R})$, the corresponding rational elliptic surface S inherits a real structure in a natural way.

If γ is a complex conjugation in a complex analytic surface S and ω is a holomorphic complex two-form on an open subset $\gamma(U)$ of S, then the formula

$$(\omega^\gamma)_s(v_1,\ v_2) = \text{ the complex conjugate of } \omega_{\gamma(s)}((\mathrm{T}_s\gamma)(v_1),\ (\mathrm{T}_s\gamma)(v_2))$$

defines a holomorphic complex two-form ω^γ on U. The transpose of the mapping $\omega \mapsto \omega^\gamma$ assigns to every holomorphic exterior two-vector field w on S a holomorphic exterior two-vector field w^γ on S, and therefore defines a complex conjugation in the space $\mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S^*))$ of all holomorphic exterior two-vector fields on S.

Assume that S is compact and that W is a complex two-dimensional complex linear subspace of $\mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S^*))$ that is real in the sense that $w^\gamma \in W$ for every $w \in W$. If $s \in S$, $w \in W$, and $w(s) = 0$, then $w^\gamma(\gamma(s)) = 0$. Therefore, if s is a base point of the anticanonical pencil defined by W, then $\gamma(s)$ is a base point of

the same anticanonical pencil. If $\gamma(s) = s$, that is, $s \in S(\mathbb{R})$, and $\pi : \widetilde{S} \to S$ is the blowing up of S at s, then there is a unique complex conjugation $\widetilde{\gamma}$ in $\widetilde{S}$ such that $\pi \circ \widetilde{\gamma} = \gamma \circ \pi$, and

$$\widetilde{W} := \{\widetilde{w} \in \mathrm{H}^0(\widetilde{S},\ \mathrm{O}(\mathrm{K}_{\widetilde{S}})) \mid \pi_* \widetilde{w} \in W\}$$

is a complex two-dimensional vector space of holomorphic complex two-forms on $\widetilde{S}$ that is real in the sense that $\widetilde{W}^{\widetilde{\gamma}} = \widetilde{W}$. If $\gamma(s) \neq s$, then we arrive at the same conclusion after blowing up at the two points s and $\gamma(s)$.

Continuing the process of blowing up at the base points of the anticanonical pencils, as in Sections 3.3.3, 3.3.4, we arrive at an S, γ, and W as above, but with the additional condition that the anticanonical pencil defined by W has no base points. It follows that we have a well-defined holomorphic mapping

$$\kappa : S \to \mathrm{P}(W) : s \mapsto \{w \in W \mid w(s) = 0\}.$$

Because $w(s) = 0$ implies that $w^\gamma(\gamma(s)) = 0$, we have $\kappa(\gamma(s)) = \kappa(s)^\gamma$ for every $s \in S$. In other words, $\kappa \circ \gamma_S = \gamma_P \circ \kappa$ if γ_S denotes the complex conjugation γ in S and $\gamma_P : [w] \mapsto [w^\gamma]$ denotes the induced complex conjugation in $P = \mathrm{P}(W)$.

If the exterior two-vector field w is as in (3.3.2), and $\gamma : (x,\ y) \mapsto (\overline{x},\ \overline{y})$, then

$$w^\gamma = \overline{q(\overline{x},\ \overline{y})}\ \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Let A^0 and A^1 be *real-valued* 3×3 matrices. If $p_z(x,\ y)$ denotes the left-hand side of (2.5.3), then $p_{\overline{z}}(\overline{x},\ \overline{y}) = \overline{p_z(x,\ y)}$. Therefore, if w_z is the holomorphic exterior two-vector field (3.3.2) with $q = p_z$, and γ is the ordinary complex conjugation in $\mathbb{P}^1 \times \mathbb{P}^1$, then $(w_z)^\gamma = w_{\overline{z}}$. If $\kappa : S \to P$ is the rational elliptic surface obtained by successively blowing up at a base point of the anticanonical pencil, eight times, starting with the pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by (2.5.3), then the above procedure leads to a real structure $(\gamma_S,\ \gamma_P)$ on $\kappa : S \to P$, where the complex conjugation γ_P on $P \simeq \mathbb{P}^1$ is the standard complex conjugation, for which the set $P(\mathbb{R})$ of real points is the real projective line $\mathbb{P}^1(\mathbb{R})$, which is diffeomorphic to a circle. Also, if $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing-up map, then $\pi \circ \gamma_S = \gamma \circ \pi$, in which γ denotes the standard complex conjugation in the Cartesian product of the two complex projective lines. This determines γ_S uniquely in terms of γ and π. Finally, the QRT automorphism $\tau^S \in \mathrm{Aut}(S)^+_\kappa$, defined by $\mathfrak{B}$, is real.

Now consider a pencil of cubic curves in the complex projective plane $\mathbb{P}^2$ defined by (4.1.9), where q^0 and q^1 are homogeneous polynomials of degree 3 with *real* coefficients. If $\kappa : S \to P$ is the rational elliptic surface obtained by successively blowing up at a base point of the anticanonical pencil, nine times, starting with pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ defined by (2.5.3), then in the same way we obtain a real structure $(\gamma_S,\ \gamma_P)$ on $\kappa : S \to P$, where the complex conjugation γ_P on $P \simeq \mathbb{P}^1$ is the standard one. If $\pi : S \to \mathbb{P}^2$ is the blowing-up map, then $\pi \circ \gamma_S = \gamma \circ \pi$, in which γ denotes the standard complex conjugation in the complex projective plane. This determines γ_S uniquely in terms of γ and π.

If $\mathfrak{C}$ is a real pencil of cubic curves in $\mathbb{P}^2$, then for every $[z_0 : z_1] \in \mathbb{P}^1(\mathbb{R})$ the member $C_{[z]}$ defined by (2.5.3) meets every real projective line, because a homogeneous polynomial of degree 3 in two variables always has nontrivial real zeros. It follows that always $\kappa(S(\mathbb{R})) = \mathbb{P}^1(\mathbb{R})$ for the rational elliptic surfaces $\kappa : S \to \mathbb{P}^1$ defined by a real pencil of cubic curves in $\mathbb{P}^2$. Also, $\mathfrak{C}$ has at least one real base point, which can be proved, for instance, by observing that the complex conjugation is an involution on the set of all base points, and because counted with multiplicity, there are nine base points, which is an odd number, at least one of the base points is fixed under the complex conjugation.

The billiard map is isomorphic to a QRT root for a real pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, where for $\lambda \notin [0, \lambda_0]$ the corresponding member of the pencil has no real points. See Section 11.2.2. In general, we have $\kappa(S(\mathbb{R})) \neq P^1(\mathbb{R})$ if and only if not all real members of the pencil of biquadratic curves have real points. Therefore, if the real structure on a given rational elliptic surface comes from a real pencil of biquadratic curves of which some real member has no real points, then this real structure cannot come from a real pencil of cubic curves in $\mathbb{P}^2$. Also note that if some real member of the pencil of biquadratic curves has no real points, then the pencil cannot have a real base point, because a real base point lies on every real member.

At each blowing up, the real locus of the blown up surface is isomorphic to the real locus of the surface we started out with, unless the base point at which we blow up is real. As we have seen in the proof of Corollary 8.4.5, in this case the point is replaced by a real projective line, which is diffeomorphic to a circle C, and a small disk around the point is replaced by a Möbius strip along this circle. See Figure 3.2.1 and its explanation. In particular, C can be deformed to a nearby curve that intersects C once and transversally, whereas the double loop can be perturbed to a curve that does not intersect C at all. It follows that the dimension over $\mathbb{Z}/2\,\mathbb{Z}$ of $\mathrm{H}_1(S(\mathbb{R}),\ \mathbb{Z}/2\,\mathbb{Z})$ increases by one at each blowing up at a real base point. This also shows that $S(\mathbb{R})$ is not orientable, unless we start out with a real pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ without any real base points, in which case $S(\mathbb{R})$ is diffeomorphic to the real two-dimensional torus $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$.

Proposition 8.5.1 *Assume that the matrices A^0 and A^1 in (2.5.3) are real. If there are $[z] \in \mathbb{P}^1(\mathbb{R})$ for which the curve defined by (2.5.3) has real points, then there exist $[z] \in \mathbb{P}^1(\mathbb{R})$ such that the curve defined by (2.5.3) has a real singular point, which implies that the corresponding real fiber $S(\mathbb{R})_{[z]}$ in the elliptic surface S has a singular point.*

Proof. Assume that the restriction to $S(\mathbb{R})$ of the elliptic fibration $\kappa : S \to \mathbb{P}^1$ defines a smooth locally trivial fibration $\kappa(\mathbb{R}) : S(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$, which we assume from now on.

Choose an orientation on $\mathbb{P}^1(\mathbb{R}) \simeq$ a circle. A nonzero element of $\mathfrak{f}(\mathbb{R})_{[z]}$ is a vector field without zeros on the fiber $S(\mathbb{R})_{[z]}$ of κ over $[z]$, which leads to an orientation of $S(\mathbb{R})_{[z]}$. Together with the pullback by κ of the orientation on $\mathbb{P}^1(\mathbb{R})$, this leads to an orientation of the real twodimensional surface $S(\mathbb{R})$ along $S(\mathbb{R})_{[z]}$. If $O_{[z]}$ denotes the set of the two connected components of $\mathfrak{f}_{[z]}$, then this defines a

mapping from $O_{[z]}$ to the set of the orientations of $S(\mathbb{R})$ along $S(\mathbb{R})_{[z]}$, which depends continuously on $[z] \in \mathbb{P}^1(\mathbb{R})$. Because any continuous section $[z] \mapsto o_{[z]} \in O_{[z]}$ of O returns to its opposite if $[z]$ runs once around $\mathbb{P}^1(\mathbb{R})$, see Remark 8.1.3, the corresponding orientation of $S(\mathbb{R})$ along $S(\mathbb{R})_{[z]}$ returns to its opposite, which implies that $S(\mathbb{R})$ is not orientable. Because the restriction to $S(\mathbb{R})$ of the blowing-up map $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is an isomorphism from $S(\mathbb{R})$ onto $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ if there are no real base points, and $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ is orientable, it follows that there is at least one real base point b.

The real projective line $E(\mathbb{R})$ that appears in $S(\mathbb{R})$ at the last blowing up over b is a real analytic global section of the fibration $\kappa(\mathbb{R}) : S(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. $E(\mathbb{R})$ intersects each fiber $S(\mathbb{R})_{[z]}$ exactly once and transversally. Let U_+ and U_- denote the union of the connected components of the fibers which are and are not intersected by $E(\mathbb{R})$, respectively. Then U_+ and U_- are disjoint open subsets of $S(\mathbb{R})$ with union equal to $S(\mathbb{R})$, and because $S(\mathbb{R})$ is connected as a real blowing up of the connected surface $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, it follows that $U_- = \emptyset$, that is, the real fibers are connected. If $(g,\ s) \mapsto g \cdot s$ denotes the double-valued action of $\mathbb{R}/\mathbb{Z}$ on $S(\mathbb{R})$ in Lemma 8.1.5, then $(g,\ e) \mapsto g \cdot e$ defines a double-valued action from $(\mathbb{R}/\mathbb{Z}) \times E$ onto $S(\mathbb{R})$, which, if we identify E with $\mathbb{R}/\mathbb{Z}$, leads to an identification of $S(\mathbb{R})$ with $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/2\,\mathbb{Z})$ modulo the involution $(t + \mathbb{Z},\ s + 2\,\mathbb{Z}) \mapsto (-t + \mathbb{Z},\ s + 1 + 2\,\mathbb{Z})$. In other words, $S(\mathbb{R})$ is a Klein bottle. Therefore the fundamental group of $S(\mathbb{R})$ has two generators c_1 and c_2 with the single relation ${c_1}^2\,{c_2}^2 = 1$, see for instance Spanier [190, p. 149], which implies that the $\mathbb{Z}/2\mathbb{Z}$-dimension of $\mathrm{H}_1(S(\mathbb{R}),\ \mathbb{Z}/2\,\mathbb{Z})$ is equal to two. On the other hand, at each real blowing up, the $\mathbb{Z}/2\,\mathbb{Z}$-dimension of the H_1 with coefficients in $\mathbb{Z}/2\mathbb{Z}$ increases by one, and because this dimension is equal to two for the surface $\mathbb{P}^(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ we start out with, we arrive at a contradiction.

We conclude that at least one of the fibers in S has a real singular point, which implies that at least one of the biquadratic curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ of the pencil has a singular point. □

I have not seen any attempt in the literature to classify all possible real structures on rational elliptic surfaces $\kappa : S \to P$, and all possible real structure of the corresponding real elements of $\mathrm{Aut}(S)^+_\kappa$.

Chapter 9
Rational elliptic surfaces

9.1 Equivalent Characterizations of Rational Elliptic Surfaces

Let S be a compact smooth complex analytic surface, and assume that the space $W = \mathrm{H}^0(S,\ \mathcal{O}(\mathrm{K}_S^*))$ of all global holomorphic two-vector fields on S is two-dimensional, and the pencil defined by W has no base points, that is, there are no $s \in S$ such that $w(s) = 0$ for every $w \in W$. According to Lemma 6.1.6, the mapping $\kappa : S \to \mathrm{P}(W) \simeq \mathbb{P}^1 : s \mapsto \{w \in W \mid w(s) = 0\}$ is a fibration of S over a complex projective line, and we will say that *the anticanonical system of S defines a fibration*. Actually, the anticanonical system defines a fibration if and only if $\dim W = 2$, the anticanonical line bundle K_S^* of S has no fixed part, and $\mathrm{K}_S^* \cdot \mathrm{K}_S^* = \mathrm{K}_S \cdot \mathrm{K}_S = 0$.

If A is an irreducible component of a fiber of κ, then $\mathrm{K}_S \cdot A = -\mathrm{K}_S^* \cdot A = 0$; hence (6.2.8) yields $A \cdot A = 2\ \mathrm{vg}(A) - 2$. It follows that A cannot be a -1 curve, that is, the fibration κ is relatively minimal. If A is a smooth connected component of F, then A is homologous to a connected component of a nearby fiber that is disjoint from F, and therefore $A \cdot A = 0$, $\mathrm{g}(A) = \mathrm{vg}(A) = 1$, and A is an elliptic curve. The Stein factorization, Lemma 6.1.3, with $\varphi = \kappa$ yields a compact complex analytic curve C, a fibration $\psi : S \to C$ with connected fibers, and a branched covering $\pi : C \to \mathbb{P}^1$ such that $\kappa = \psi \circ \pi$. The previous observations imply that $\psi : S \to C$ is a relatively minimal elliptic fibration. If F is a fiber of ψ over $c \in C$, then $\pi(c) = \mathbb{C}\, w$ for a nonzero $w \in W$. There exists $w' \in W \setminus \{0\}$ such that $\mathbb{C}\, w' \neq \mathbb{C}\, w$, which implies that $1/w'$ is a holomorphic two-form without zeros on an open neighborhood of F, when Corollary 6.2.47 implies that F is not a multiple singular fiber. Therefore the fibration $\psi : S \to C$ has no multiple singular fibers. All fibers of $\kappa : S \to \mathrm{P}(W)$ are connected if and only if there is no irreducible curve A in S and $d \in \mathbb{Z}_{>1}$ such that the common homology class of the fibers of κ is equal to d times the homology class of C. A variation on this theme appears in Section 11.9.1, where the elliptic fibration is defined by a two-dimensional vector space of holomorphic sections of $(\mathrm{K}_S^*)^2$ instead of K_S^*. Eduard Looijenga told me that there is quite a number of interesting examples of elliptic fibrations defined by suitable integral powers of the (anti)canonical bundle.

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_9,
© Springer Science+Business Media, LLC 2010

The following lemma is a stepping-stone for Theorem 9.1.3 below, which implies that κ admits a holomorphic section. The line bundle $\mathrm{O}(1)$ over $\mathrm{P}(W)$ is the one for which the fiber over the one-dimensional linear subspace l of W is equal to the one-dimensional vector space l^* of all linear forms on l.

Lemma 9.1.1 *Assume that S is a compact smooth complex analytic surface, the anticanonical system of S defines a fibration $\kappa : S \to \mathrm{P}(W)$, and all anticanonical curves in S = fibers of κ are connected. Then $\mathrm{K}_S^* \simeq \kappa^*(\mathrm{O}(1))$, $\kappa : S \to \mathrm{P}(W)$ is a relatively minimal elliptic fibration without multiple singular fibers, $\mathrm{q}(S) = \mathrm{p_g}(S) = 0$, and the surface S is projective algebraic.*

Proof. We have already shown that $\kappa : S \to \mathrm{P}(W)$ is a relatively minimal elliptic fibration without multiple singular fibers. Therefore Theorem 6.2.18 with $C = \mathrm{P}(W)$ and $\varphi = \kappa$ can be applied in order to conclude that $\mathrm{K}_S^* \simeq \kappa^* \simeq \varphi^*(\mathrm{K}_C^* \otimes \mathfrak{f})$, when Lemma 2.1.3 implies that $\varphi^* : \mathrm{H}^0(C, \mathcal{O}(\mathrm{K}_C^* \otimes \mathfrak{f})) \to \mathrm{H}^0(S, \mathcal{O}(\varphi^*(\mathrm{K}_C^* \otimes \mathfrak{f}))) \simeq \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S^*)) = W$ is a linear isomorphism. Because $C = \mathrm{P}(W) \simeq \mathbb{P}^1$, every holomorphic line bundle L over C is isomorphic to $\mathrm{O}(d)$, where $d := \deg(L)$, $\dim \mathrm{H}^0(C, \mathcal{O}(\mathrm{O}(d))) = d + 1$ if $d \geq 0$, and $\mathrm{H}^0(C, \mathcal{O}(\mathrm{O}(d))) = 0$ if $d < 0$. See Example 5 after Remark 6.3.11 with $N = 1$. Because $\dim W = 2$, it follows that $1 = \deg(\mathrm{K}_C^* \otimes \mathfrak{f}) = -\deg(\mathrm{K}_C) + \deg(\mathfrak{f})$. Because $\deg(\mathrm{K}_C) = 2\,\mathrm{g}(C) - 2 = -2$, see (2.1.15), it follows that $\deg(\mathfrak{f}) = -1$, $\mathfrak{f} \simeq \mathrm{O}(-1)$, $\mathrm{K}_C^* \otimes \mathfrak{f} \simeq \mathrm{O}(1)$, $\mathrm{K}_S^* \simeq \kappa^*(\mathrm{O}(1))$, and $\mathrm{K}_S \simeq \kappa^*(\mathrm{O}(-1))$, when another application of Lemma 2.1.3 yields that $\mathrm{p_g}(S) = \dim \mathrm{H}^0(S, \mathcal{O}(\mathrm{K}_S)) = 0$.

Because $\deg(\mathfrak{f}) = -1$, Theorem 6.2.31 yields $1 = \chi(S, \mathcal{O}) =: 1 - \mathrm{q}(S) + \mathrm{p_g}(S)$, and therefore $\mathrm{q}(S) = 0$ because $\mathrm{p_g}(S) = 0$. Theorem 6.2.23 (iv) implies that if $\mathrm{b}_1(S)$ is odd, then $\mathrm{b}_1(S) = 2\,\mathrm{q}(S) - 1 = -1$, in contradiction to the fact $\mathrm{b}_1(S) := \dim \mathrm{H}_1(S, \mathbb{R}) \geq 0$. Therefore $\mathrm{b}_1(S)$ is even, when Theorem 6.2.23 (iv) implies that $\mathrm{b}_1(S) = 2\,\mathrm{q}(S) = 0$ and $\mathrm{b}_2^+(S) = 2\,\mathrm{p_g}(S) + 1 = 1$ because $\mathrm{p_g}(S) = 0$. On the other hand $0 = \mathrm{q}(S) := \dim \mathrm{H}^1(S, \mathcal{O})$ and $0 = \mathrm{p_g}(S) := \dim \mathrm{H}^2(S, \mathcal{O})$ imply that the Chern homomorphism $\mathrm{c} : \mathrm{Pic}(S) \to \mathrm{H}^2(S, \mathbb{Z})$ in the exact sequence (2.1.8) is an isomorphism, and $\mathrm{b}_2^+(S) > 0$ is equivalent to the existence of a holomorphic line bundle L over S such that $L \cdot L = \mathrm{c}(L) \cdot \mathrm{c}(L) > 0$. Corollary 6.2.26 therefore implies that the surface S is projective algebraic. □

The following lemma was inspired by Miranda [136, Corollary (2.4) and the discussion of the case $N = 1$ on p. 386]. Note that according to Definition 9.1.4 with the choice of (c) in Theorem 9.1.3, an elliptic fibration $\varphi : S \to C$ is a rational elliptic surface if and only if it satisfies the conditions in Lemma 9.1.2, together with any, hence each of, the conditions (i)–(viii).

Lemma 9.1.2 *Let $\varphi : S \to \mathbb{P}^1$ be a relatively minimal elliptic fibration which admits a holomorphic section, that according to Corollary* 6.2.28 *implies that S is projective algebraic, hence Kähler. Then the following conditions are equivalent:*

(i) *Any holomorphic section of φ is a -1 curve.*
(ii) *The degree of the Lie algebra bundle $\mathfrak{f}$ of $\varphi : S \to \mathbb{P}^1$ is equal to -1.*
(iii) *The holomorphic Euler number $\chi(S, \mathcal{O})$ of S is equal to* 1.

(iv) *The topological Euler number* $\chi_{\text{top}}(S)$ *of* S *is equal to* 12.
(v) $\varphi : S \to \mathbb{P}^1$ *is the minimal resolution of singularities* $f : S \to W$ *of a Weierstrass model* $p : W \to \mathbb{P}^1$, *defined by holomorphic sections* g_2 *and* g_3 *of* $\mathrm{O}(4)$ *and* $\mathrm{O}(6)$, *respectively, where* $\mathrm{O}(N)$ *denotes the holomorphic line bundle over* $\mathbb{P}^1$ *of degree* N, ${g_2}^3 - 27\,{g_3}^2$ *is not identically equal to zero, and, for each* $c \in \mathbb{P}^1$, $\mathrm{ord}_c(g_2) < 4$ *or* $\mathrm{ord}_c(g_3) < 6$.
(vi) *The anticanonical system of* S *defines the fibration* φ.
(vii) $\mathrm{q}(S) = \mathrm{p_g}(S) = 0$.
(viii) *There are no nonzero holomorphic complex one-forms or two-forms on* S.

Proof. (i) $\Leftrightarrow$ (ii) follows from Lemma 6.2.34. (ii) $\Leftrightarrow$ (iii) follows from Theorem 6.2.31. (iii) $\Leftrightarrow$ (iv) follows from Lemma 6.2.30. (ii) $\Leftrightarrow$ (v) follows from Example 5. (vi) $\Rightarrow$ (vii) follows from Lemma 9.1.1, and (ii) $\Leftrightarrow$ (vi) follows from its proof. (vii) $\Leftrightarrow$ (viii) follows from (6.2.27) and (6.2.36) with $M = S$, $p = 1$, and $q = 0$. (vii) $\Rightarrow$ (iii) follows from (6.2.30) with $M = S$ and $n = 2$. □

A surface S is called *rational* if there is a birational transformation from $\mathbb{P}^2$ to S. The following theorem was inspired by the introduction and Section 2.1 in Heckman and Looijenga [81], and by Wall [206, §1]. According to Definition 9.1.4, an elliptic fibration $\varphi : S \to C$ is a rational elliptic surface if and only if it satisfies any, hence each, of the conditions (a)–(d) in Theorem 9.1.2.

Theorem 9.1.3 *Let* $\varphi : S \to C$ *be a fibration over a compact curve* C. *Then the following statements are equivalent:*

(a) *The fibration* φ *is equal to the elliptic fibration over* $\mathbb{P}^1$ *obtained from a pencil of cubic curves in* $\mathbb{P}^2$ *with at least one smooth member, by nine times successively blowing up at base points, until one reaches a surface on which the pencil has no base points.*
(b) *The surface* S *is rational, the fibration* φ *is elliptic, relatively minimal and admits a holomorphic section.*
(c) *The curve* C *is isomorphic to* $\mathbb{P}^1$, *the fibration* φ *is elliptic, relatively minimal and admits a holomorphic section, and any, hence each, of the conditions (i)–(viii) in Lemma* 9.1.2 *holds.*
(d) *The anticanonical system of* S *defines the fibration* φ, *and the fibers of* φ *are connected.*

Proof. (a) $\Rightarrow$ (b) follows from Corollary 3.3.9. Assume that b) holds, when both Lemma 3.2.3 and Corollary 6.2.28 imply that S is a complex projective algebraic surface, hence compact Kähler. According to Corollary 6.2.55, the irregularity $\mathrm{q}(S)$ and the arithmetic genus $\mathrm{p_g}(S)$ of S are birational invariants. Now it follows from Griffiths and Harris [74, p. 118] that for any $p > 0$ there are no nonzero holomorphic p-forms on $\mathbb{P}^n$, which for $n = 2$, $p = 2$, and $p = 1$ implies that $\mathrm{q}(\mathbb{P}^2) = 0$ and $\mathrm{p_g}(\mathbb{P}^2) = 0$. Therefore $\mathrm{q}(S) = 0$ and $\mathrm{p_g}(S) = 0$. Because $\varphi^*\omega$ is a nonzero holomorphic one-form on S if ω is a nonzero holomorphic one-form on C, it follows that there are no nonzero holomorphic one-forms on C. That is, $\mathrm{g}(C) = 0$, and hence

$C \simeq \mathbb{P}^1$; see the third paragraph in Section 2.1.5. This completes the proof of (b) $\Rightarrow$ (c), with the choice of condition (vii) or (viii) in Lemma 9.1.2. If (c) holds, then (vi) in Lemma 9.1.2 implies (d). It therefore remains to prove (d) $\Rightarrow$ (a).

Assume that (d) holds, when Lemma 9.1.1 implies that $\mathrm{q}(S) = \mathrm{p_g}(S) = 0$ and S is projective algebraic. Starting with $S_0 = S$, let S_i be surfaces obtained by induction on i in the following way. If $i > 0$ and E_i is a -1 curve in S_{i-1}, then the Castelnuovo–Enriques criterion, Theorem 3.2.4, yields a nonsingular complex analytic surface S_i and a blowing up $\pi_i : S_{i-1} \to S_i$ of S_i at a point $b_i \in S_i$ such that $E_i = {\pi_i}^{-1}(\{b_i\})$. Because Lemma 3.2.5 implies that $\operatorname{rank} \mathrm{H}_2(S_i, \mathbb{Z}) = \operatorname{rank} \mathrm{H}_2(S_{i-1}, \mathbb{Z}) - 1$, this process has to stop at some $i = m \leq \operatorname{rank} \mathrm{H}_2(S, \mathbb{Z})$, when $M := S_m$ is a minimal surface in the sense that it cannot be blown down to a smooth surface. It follows by induction on i from Corollary 6.2.55 and Lemma 3.2.3 that $\mathrm{q}(S_i) = \mathrm{p_g}(S_i) = 0$ and S_i is projective algebraic; hence $\mathrm{q}(M) = \mathrm{p_g}(M) = 0$ and M is projective algebraic.

Let A be a member of the pencil $\mathfrak{A}_m$ of anticanonical curves A in $M = S_m$ and $a \in A$. Then there exists a smooth and connected hence irreducible hyperplane section H such that $a \in H$ and H is not equal to any of the irreducible components of A, and therefore $-\mathrm{K}_M \cdot H = \mathrm{K}_M^* \cdot H = A \cdot H > 0$. Therefore M satisfies condition (i) in Theorem 6.2.9; hence Mori's theorem implies that M is isomorphic to $\mathbb{P}^2$ or a relatively minimal ruled surface $\neq \Sigma_1$. Suppose that M is ruled. Theorem 6.2.6 implies that $0 = \mathrm{q}(M)$ is equal to the genus of the curve over which M is ruled, when Theorem 6.2.8 implies that $M \simeq \Sigma_n$ for a nonnegative integer $n \neq 1$. It follows from Corollary 3.3.9 that $n = 0$ or $n = 2$, that is, M is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or Σ_2.

If $M \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then the zigzag at the base point b_m of $\mathfrak{A}_m$ leads to Σ_1. If $M \simeq \Sigma_2$, then the base point b_m cannot lie on the curve $T_2 \simeq \mathbb{P}^1$ in Σ_2 such that $T_2 \cdot T_2 = -2$, because (3.2.8) implies that $\pi'(C) \cdot \pi'(C) < C \cdot C$ if π is the blowup at a point on a curve C and C' is the proper transform of C, and Corollary 3.3.9 implies that S has no irreducible curves C with $C \cdot C < -2$. We conclude in view of Theorem 6.2.8 that the zigzag at b_m again leads to Σ_1. Therefore, in each case that M is not isomorphic to $\mathbb{P}^2$, we can replace the last blowdown by one to Σ_1. It follows from Theorem 6.2.8 that if we blow down the unique -1 curve T_1 in Σ_1, we arrive at $\mathbb{P}^2$. Because every anticanonical pencil in $\mathbb{P}^2$ is a pencil of cubic curves in $\mathbb{P}^2$, see Lemma 4.1.1, we have proved that $\kappa : S \to \mathrm{P}(W)$ is obtained from a pencil of cubic curves in $\mathbb{P}^2$ by successively blowing up the base points. Because a pencil of cubic curves in $\mathbb{P}^2$ has nine base points when counted with multiplicities, see the paragraph after (4.1.9), this completes the proof of (d) $\Rightarrow$ (a). We have $m = 9$ if $M \simeq \mathbb{P}^2$, and $m = 8$ if $M \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $M \simeq \Sigma_2$. □

Definition 9.1.4. Let $\varphi : S \to C$ be an elliptic fibration over a compact curve C. Then $\varphi : S \to C$ is called a *rational elliptic surface* if it satisfies any, hence each, of the equivalent conditions (a)–(d) in Theorem 9.1.3. ⊘

Remark 9.1.5. The birational transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$, performed in the proof of (c) $\Rightarrow$ (a) in Theorem 9.1.3 when $M \simeq \mathbb{P}^1 \times \mathbb{P}^1$, is the one discussed in Section 4.1.

At each of the blowdowns $\pi_j : S_{j-1} \to S_j$ in the proof of c) $\Rightarrow$ a) in Theorem 9.1.3, the -1 curve in S_{j-1} that is blown down to a point in S_j can be chosen

freely from the set of all -1 curves in S_{j-1}. After the seventh blowdown, we arrive at one of the two surfaces Σ_{01} and Σ_{12} mentioned in the diagram at the end of Section 6.2.3, where Σ_{01} is the del Pezzo surface of degree 7; see Definition 9.2.21. In Σ_{01} the irreducible curves C such that $C \cdot C < 0$ form a chain A_1, A_2, A_3 of -1 curves, where A_1 and A_2 intersect each other transversally at a point a_{12}, A_2 and A_3 intersect each other transversally at a point $a_{23} \neq a_{12}$, and there are no other intersections. The only ways to blow down Σ_{01} to $\mathbb{P}^2$ are to first blow down A_1 or A_3, which leads to Σ_1, and then the image in Σ_1 of A_3 or A_1, respectively. If A_2 is blown down, we arrive at $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In Σ_{12} the irreducible curves C such that $C \cdot C < 0$ form a chain of embedded rational curves A_1, A_2, A_3 with the same intersection properties, except that $A_3 \cdot A_3 = -1$ is replaced by $A_3 \cdot A_3 = -2$. The only way to blow down Σ_{12} to $\mathbb{P}^2$ is to first blow down A_2, which leads to Σ_1, and then the image in Σ_1 of A_3, which is the unique -1 curve in Σ_1. If A_1 is blown down, we arrive at Σ_2. The fact that every choice of blowdowns lead to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or Σ_2 might have interesting consequences for the configurations of the -1 curves and the reducible singular fibers in S.

Different choices of the blowdowns may lead to pencils of cubic curves in $\mathbb{P}^2$ that are not isomorphic by means of a projective linear transformation in $\mathbb{P}^2$. For instance, Lemma 4.5.3 implies that the rational elliptic surface with the configuration of singular fibers $\mathrm{I}_9\ 3\,\mathrm{I}_1$, which in view of Proposition 9.2.19 is unique up to isomorphism, can arise from a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with one base point, and also from a pencil $\mathfrak{C}'$ of cubic curves in $\mathbb{P}^2$ with three base points, when $\mathfrak{C}$ is not isomorphic to $\mathfrak{C}'$ by means of a projective linear transformation. On the other hand, the choice of blowdowns π_j, $2 \leq j \leq 9$, is unique if S has a singular fiber of type II^*, as in Lemma 4.5.1. It might be an interesting project to determine the isomorphism classes for some of the examples in Chapter 11. By contrast, given a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with at least one smooth member, the successive blowing-up procedure in the base points of the anticanonical pencils is unique in the sense that the resulting rational elliptic surfaces S are isomorphic to each other.

There are only finitely many isomorphism classes of pencils $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ that correspond to the given rational elliptic surface S. This implies the finiteness of the morphism $\mathrm{AM} : (\mathrm{MCP})^s \to (\mathrm{MRES})^s$ discussed in Section 6.3.3.

Proof The -1 curves E in the rational elliptic surface S are the holomorphic sections of $\kappa : S \to \mathbb{P}^1$, see Corollary 3.3.9; and Lemma 7.1.1 implies that the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ acts freely and transitively on the set of all holomorphic sections $= -1$ curves in S. It follows that we have infinitely or finitely many choices for the first blowdown $\pi_1 : S = S_0 \to S_1$ of the -1 curve E_1 in S, depending on whether the rank of the Mordell–Weil group is > 0 or zero, respectively. On the other hand, since the -1 curves in S are mapped to each other by automorphisms of S, the resulting surfaces S_1 for the different choices are isomorphic to each other. The proper transform of a -1 curve in S_i under $\pi_i \circ \cdots \circ \pi_1 : S \to S_i$ is an irreducible curve C in S such that $C \cdot C < -1$ or $C \cdot C = -1$, depending on whether a base point of any of the successive blowups lies on the proper transform of C or not. It follows from Corollary 3.3.9 that in the first case C is an irreducible component of a reducible

fiber of κ, whereas in the second case C is a -1 curve in S that is disjoint from E_1. The -1 curves disjoint from E_1 are in bijective correspondence with the Manin QRT automorphisms, see Theorem 4.3.2, and Theorem 4.3.3 implies that there are at most 240 Manin QRT automorphisms of S. Because there are also finitely many irreducible components of reducible fibers, this proves that there are only finitely many isomorphism classes of pencils $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ that correspond to the given rational elliptic surface S. ⊘

If $\varphi : S \to \mathbb{P}^1$ is a relatively minimal elliptic fibration, where the surface S is rational, but not a rational elliptic surface, then (b) in Theorem 9.1.3 implies that φ has no holomorphic section. An example of such $\varphi : S \to \mathbb{P}^1$ appears in Section 11.9.1, where the existence of a multiple singular fiber of Kodaira type ${}_2\mathrm{I}_3$ precludes the existence of a holomorphic section, and therefore implies that S is not a rational elliptic surface.

9.2 Properties of Rational Elliptic Surfaces

In this section we collect a number of properties of rational elliptic surfaces $\kappa : S \to \mathbb{P}^1$, in addition to the properties mentioned in Theorem 9.1.3, and Theorems 4.3.2 and 4.3.3 about Manin QRT automorphisms of S.

I found the following very useful and striking lemma, which follows from Corollary 3.3.9, in Wall [206, pp. 375, 376]. That on a rational elliptic surface every section has self-intersection number -1 had been observed before, in the setting of Manin elliptic surfaces, by Manin [129, (13) on p. 95].

Lemma 9.2.1 *If C is an irreducible curve in S, then $C \cdot C < 0$ if and only if either C is a -1 curve in S, or C is an irreducible component of a reducible fiber of κ, when C is an embedded complex projective line and $C \cdot C = -2$. A curve E in S is a -1 curve if and only if it is a holomorphic section of κ.*

Lemma 9.2.2 *Let $\kappa : S \to P$ be a rational elliptic surface. Then S is simply connected. The sum of the topological Euler numbers of the singular fibers of κ is equal to 12. The degree j of the modulus function $J : P \simeq \mathbb{P}^1 \to \mathbb{P}^1$ is equal to (6.2.48). We have $j \leq 12$, with equality if and only if every singular fiber is of Kodaira type I_k for some $k > 0$.*

For every pair g_2 and g_3 of homogeneous polynomials on $\mathbb{C}^2$ of degree 4 and 6, respectively, such that at every point of $\mathbb{P}^1$, the order of a zero of g_2 and of g_3 is < 4 and < 6, respectively, and $\Delta := {g_2}^3 - 27\,{g_3}^2$ is not identically equal to zero, there is a rational elliptic surface $\kappa : S \to \mathbb{P}^1$ with modulus function $J : \mathbb{P}^1 \to \mathbb{P}^1$ equal to ${g_2}^3/\Delta$.

Proof. The projective plane $\mathbb{P}^2$ is simply connected; see Griffiths and Harris [74, p. 60]. Because S is obtained from $\mathbb{P}^2$ by successive blowing-up transformations, it follows from Lemma 3.2.5 that S is simply connected as well. The second statement follows from Lemma 6.2.30 and (iv) in Lemma 9.1.2. The last statement follows from (v) in Lemma 9.1.2. □

Lemma 9.2.3 *Let $\kappa : S \to P$ be a rational elliptic surface, which according to (a) in Theorem* 9.1.3 *can be obtained by successively blowing up, nine times, $\mathbb{P}^2$ in the base points of a pencil of cubic curves with a smooth member. Let e_i, $1 \le i \le 9$, be the classes of the total transforms of the -1 curves E_i that appear at the blowing-up transformations, and let l be the class of the total transform of any projective line in $\mathbb{P}^2$. Then*

$$\mathrm{H}_2(S, \mathbb{Z}) = \mathbb{Z}\, l \oplus \bigoplus_{i=1}^{9} \mathbb{Z}\, e_i, \tag{9.2.1}$$

where the generators l and e_i are mutually orthogonal with respect to the intersection form, $l \cdot l = 1$ and $e_i \cdot e_i = -1$ for $1 \le i \le 9$. It follows that $L := \mathrm{H}_2(S, \mathbb{Z})$, together with the intersection form, is a unimodular integral lattice of rank 10. *See Definition* 7.2.6 *for the terminology of lattices.*

If $f \in L$ is the class of a fiber, where all fibers are homologous, then $f \cdot f = 0$ and

$$f = 3\, l - \sum_{i=1}^{9} e_i\,. \tag{9.2.2}$$

If $f^\perp := \{c \in L \mid c \cdot f = 0\}$, then the restriction to $f^\perp$ of the intersection form is negative semidefinite, with kernel equal to $\mathbb{Z}\, f$. Moreover, $c \cdot c \in 2\mathbb{Z}$ for every $c \in f^\perp$. It follows that, provided with the intersection form, the quotient $Q := f^\perp/\mathbb{Z}\, f$ is an even and negative definite lattice of rank 8. The lattice Q is of type E_8 *in the sense that the intersection matrix of a $\mathbb{Z}$-basis of Q is equal to minus the Cartan matrix of type* E_8. *The lattice Q is unimodular, and therefore the conclusions of Theorem* 7.6.6 *hold for every rational elliptic surface S.*

Proof. We have $\mathrm{H}_2(\mathbb{P}^2) \simeq \mathbb{Z}$, with the homology class λ of any complex projective line in $\mathbb{P}^2$ as a generator, where $\lambda \cdot \lambda = 1$. See Griffiths and Harris [74, p. 60]. The formula (9.2.1) therefore now follows from (3.2.4), with ϵ_0 equal to the total transform under $\pi : S \to \mathbb{P}^2$ cohomology class of any complex projective line in $\mathbb{P}^2$, and the e_i, $1 \le i \le 9$, equal to the total transforms of the -1 curves that appear at each of the nine successive blowing-up transformations. For the intersection numbers, we use (3.2.6). Applying (3.2.6) at every blowing up, we obtain that $l \cdot l = 1$. The formula (9.2.1) follows from (3.2.4), where we use Poincaré duality and the fact that no torsion appears at blowing up. It follows from (3.2.6) and the definition of the e_i that $e_i \cdot e_i = -1$ for $1 \le i \le 9$, whereas all the other intersection numbers are equal to zero because the curves can be chosen to be disjoint.

We have $f \cdot f = 0$ because distinct fibers are disjoint. The intersection number of each smooth member of the pencil of cubic curves in $\mathbb{P}^2$ with a projective line is equal to 3, the degree of the cubic curve, which in view of (3.2.6) implies that $f \cdot l = 3$. On the other hand, it follows from Corollary 3.3.8 = Lemma 3.2.8 that the intersection number of any smooth member of the pencil with the -1 curve that appears is equal to 1, and again using (3.2.6) we obtain that $f \cdot e_i = 1$ for $1 \le i \le 9$. This proves (9.2.2).

If we write $c \in L$ as $c = \lambda\, l + \sum_{i=1}^{9} \varepsilon_i\, e_i$ for uniquely determined integers α and ε_i, $1 \le i \le 9$, then $c \in f^\perp$ if and only if $3\,\alpha + \sum_{i=1}^{9} \varepsilon_i = 0$. Let $\varepsilon = (\varepsilon_i)_{1 \le i \le 9} \in \mathbb{Z}^9$, and let σ denote the vector in $\mathbb{Z}^9$ with $\sigma_i = 1$ for every $1 \le i \le 9$. If $\langle \cdot,\, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^9$, then this condition means that $3\,\lambda = -\langle \varepsilon,\, \sigma \rangle$, and

$$c \cdot c = \lambda^2 - \sum_{i=1}^{9} {\varepsilon_i}^2 = \langle \varepsilon,\, \sigma \rangle^2 / \langle \sigma,\, \sigma \rangle - \langle \varepsilon,\, \varepsilon \rangle,$$

which in view of the Cauchy–Schwarz inequality for the standard inner product is ≤ 0, with equality if and only if ε is a multiple of σ if and only if there exists an integer k such that $\varepsilon_i = k$ for all $1 \le i \le 9$, when $\lambda = -3k$, hence $c = -k\, f$. In other words, the restriction to $f^\perp \times f^\perp$ of the intersection form is negative semidefinite and has kernel equal to $\mathbb{Z}\, f$.

Following Wall [206, p. 392], we introduce the classes $\alpha_0 := l - e_1 - e_2 - e_3$ and $\alpha_j := e_j - e_{j+1}$ for $1 \le j \le 8$. They belong to $f^\perp$, and satisfy $\alpha_j \cdot \alpha_j = -2$.

Furthermore, we have, for $\lambda,\ \varepsilon_i,\ a_j \in \mathbb{Z}$, that

$$c := \lambda\, l + \sum_{i=1}^{9} \varepsilon_i\, e_i = \sum_{j=0}^{9} a_j\, \alpha_j$$

if and only if $\lambda = a_0$, $\varepsilon_1 = a_1 - a_0$, $\varepsilon_2 = a_2 - a_1 - a_0$, $\varepsilon_3 = a_3 - a_2 - a_0$, and $\varepsilon_i = a_i - a_{i-1}$ for $4 \le i \le 9$, if and only if $a_0 = \lambda$, $a_1 = \varepsilon_1 + a_0 = \varepsilon_1 + \lambda$, $a_2 = \varepsilon_2 + a_1 + a_0 = \varepsilon_1 + \varepsilon_2 + 2\,\lambda$, and $a_j = \sum_{i=1}^{j} \varepsilon_i + 3\,\lambda$ for $3 \le j \le 9$. Furthermore, (9.2.2) implies that $c \cdot f = a_9$; hence $a_9 = 0$ if and only if $c \in f^\perp$. It follows that the α_j, $0 \le j \le 8$, form a $\mathbb{Z}$-basis of $f^\perp$.

For any $c \in f^\perp$ we have $c = \sum_{i=0}^{8} a_j\, \alpha_j$ for uniquely determined $a_j \in \mathbb{Z}$, and therefore

$$c \cdot c = \sum_{i=0}^{8} {a_j}^2\, \alpha_j \cdot \alpha_j + 2 \sum_{0 \le i < j \le 8} a_i\, a_j \alpha_i \cdot \alpha_j = -2 \sum_{i=0}^{8} {a_j}^2 + 2 \sum_{0 \le i < j \le 8} a_i\, a_j \alpha_i \cdot \alpha_j,$$

which implies that $c \cdot c \in 2\,\mathbb{Z}$. This completes the proof of the statement that $Q = f^\perp / \mathbb{Z}\, f$ is a negative definite even lattice of rank 8.

We have $e_i = \sum_{j=i}^{8} \alpha_j + e_9$ for $9 \ge i \ge 1$, and

$$l = \alpha_0 + e_1 + e_2 + e_3 = \alpha_0 + \alpha_1 + 2\,\alpha_2 + 3 \sum_{j=3}^{8} \alpha_j + \epsilon_9,$$

hence

$$f = 3\,e - \sum_{i=1}^{9} e_i = 3\,\alpha_0 + 2\,\alpha_1 + 4\,\alpha_2 + \sum_{j=3}^{8} (9 - j)\,\alpha_j.$$

Therefore $\alpha_8 = f$ modulo integral linear combinations of the α_i with $0 \le i \le 7$, and it follows that the $\alpha_i + \mathbb{Z} f$ for $0 \le i \le 7$ form a $\mathbb{Z}$-basis of Q. When $1 \le k < l \le 7$ we have that $\alpha_j \cdot \alpha_k = (e_j - e_{j+1}) \cdot (e_k - e_{k+1}) = -e_{j+1} \cdot e_k = 1$ if $j + 1 = k$, and $\alpha_j \cdot \alpha_k = 0$ otherwise. On the other hand, if $1 \le k \le 7$ then $\alpha_0 \cdot \alpha_k = (l - e_1 - e_2 - e_3) \cdot (e_k - e_{k+1})$ is equal to 0 when $1 \le k \le 2$ or $k \ge 4$ and equal to 1 if $k = 3$. It follows that the matrix $-\alpha_i \cdot \alpha_j$, $0 \le i,\ j \le 7$, is a Cartan matrix of type E_8, where our α_0, α_1, α_2, and α_i, $3 \le i \le 7$ correspond to the nodes α_2, α_1, α_3, and α_{i+1}, $3 \le i \le 7$, in the Dynkin diagram in Bourbaki [23, Chapter VI, Planche VII, p. 269]. The determinant of the intersection matrix is equal to the "indice de connexion" = determinant of the Cartan marix in [27], which is equal to 1, and therefore the lattice Q is unimodular. □

Recall from Section 2.1.3 that isomorphism classes of holomorphic line bundles over S are classified by the multiplicative commutative group $\mathrm{H}^1(S, \mathcal{O}^\times)$, called the Picard group $\mathrm{Pic}(S)$ of S. Also recall the Néron–Severi group $\mathrm{NS}(S)$ introduced in Section 7.5.

Theorem 9.2.4 *Let S be a rational elliptic surface. Then the Néron–Severi group $\mathrm{NS}(S) := \mathrm{H}(\mathrm{Div}(S))$, the Picard group $\mathrm{Pic}(S) := \mathrm{H}^1(S, \mathcal{O}^\times)$, the cohomology group $\mathrm{H}^2(S, \mathbb{Z})$, and the homology group $\mathrm{H}_2(S, \mathbb{Z})$ are canonically isomorphic to each other. Every real two-dimensional cycle in S is homologous to an algebraic cycle, a linear combination with integral coefficients of complex algebraic curves in S, where two algebraic cycles are homologous in the topological sense if and only if they are algebraically equivalent.*

For every $x \in \mathrm{H}^2(S, \mathbb{Z})$ there is a holomorphic line bundle L over S with Chern class equal to x, and two holomorphic line bundles over S are isomorphic if and only if they have the same Chern class. The line bundle L has a nonzero meromorphic section λ, and for every such section we have $[L] = \delta(\mathrm{Div}(s))$, and $c = \mathrm{c}(L) \in \mathrm{H}^2(S, \mathbb{Z})$ is the Poincaré dual of the homology class $\mathrm{H}(\mathrm{Div}(\lambda)) \in \mathrm{H}_2(S, \mathbb{Z})$.

Proof. It follows from (viii) in Lemma 9.1.2 that $\mathrm{H}^1(S, \mathcal{O}) = 0$ and $\mathrm{H}^2(S, \mathcal{O}) = 0$. In view of the exact sequence (2.1.8) this implies that the Chern homomorphism $\mathrm{c} : \mathrm{Pic}(S) := \mathrm{H}^1(S, \mathcal{O}^\times) \to \mathrm{H}^2(M, \mathbb{Z})$ is an isomorphism. That is, for every $x \in \mathrm{H}^2(S, \mathbb{Z})$ there exists a holomorphic line bundle L on S such that $\mathrm{c}(L) = x$, and L is unique up to isomorphisms of line bundles.

Because S is projective algebraic, it follows from Lemma 6.2.4 that $\delta : \mathrm{Div}(S) \to \mathrm{Pic}\, S$ is surjective and that each holomorphic line bundle L admits nonzero meromorphic sections λ. The Poincaré duality isomorphism $\mathrm{pd} : \mathrm{H}_2(S, \mathbb{Z}) \to \mathrm{H}^2(S, \mathbb{Z})$ in (2.1.17) with $n = 4$ and $a = b = 2$ is an isomorphism, and (2.1.24) implies that $\mathrm{pd} \circ \mathrm{H} = \mathrm{c} \circ \delta$; hence $\mathrm{H} : \mathrm{Div}(S) \to \mathrm{H}_2(S, \mathbb{Z})$ is surjective. The last statement in the theorem follows from Lemma 2.1.2. □

Remark 9.2.5. It follows from Theorem 9.2.4 and Lemma 9.2.3 that if S is a rational elliptic surface, then $\mathrm{NS}(S) \simeq \mathrm{H}_2(S, \mathbb{Z})$, provided with the intersection form, is a lattice isomorphic to $\mathrm{U} \oplus \mathrm{E}_8^-$ in the notation of (6.3.8). There is no such uniform behavior of the Néron–Severi group $\mathrm{NS}(S)$ of elliptic K3 surfaces S.

Morrison [141, Corollary 1.9] proved that a lattice Λ is isomorphic to the Néron–Severi lattice of a projective algebraic K3 surface if and only if it has signature $(1,\ \rho-1)$ and is isomorphic to a sublattice of the K3 lattice Λ_{K3} such that $\Lambda_{\mathrm{K3}}/\Lambda$ is torsion-free. If $\rho \in \mathbb{Z}_{\geq 0}$, then ρ is equal to the rank of a projective algebraic K3 surface if and only if $\rho \leq 20$. If $\rho \leq 11$, then every even lattice of signature $(1,\ \rho-1)$ is isomorphic to the Néron–Severi group of a projective algebraic K3 surface S. See Morrison [141, p. 112]. Because S is projective algebraic, $\operatorname{Pic} S = \delta(\operatorname{Div}(S))$. Therefore Lemmas 6.3.15 and 6.3.14 imply that S admits a relatively minimal elliptic fibration $\varphi : S \to \mathbb{P}^1$ with a holomorphic section if and only if there exist $a,\ b \in \Lambda$ such that $a \cdot a = b \cdot b = 0$ and $a \cdot b = 1$. The sublattice $\mathbb{Z}\,a + \mathbb{Z}\,b$ of Λ is isomorphic to U, and because every $x \in \Lambda$ has a unique decomposition $x = (x \cdot b)\,a + (x \cdot a)\,b + y$ with $y \in \Lambda$ and $y \cdot a = y \cdot b = 0$, the condition is equivalent to the statement that $\Lambda = \mathrm{U} \oplus \Lambda'$ for some lattice Λ'. For example, there is an elliptic K3 surface S with a holomorphic section and $\mathrm{NS}(S) \simeq \mathrm{U} \oplus \mathbb{Z}\,v \oplus \mathbb{Z}\,w \oplus 2\ \mathrm{E}_8^-$, where $v \cdot v = w \cdot w = -2$. In this case rank $\mathrm{NS}(S) = 20$. At the other extreme there exists an elliptic K3 surface S with a holomorphic section such that $\mathrm{NS}(S) \simeq \mathrm{U}$, when rank $\mathrm{NS}(S) = 2$.

The following example is due to Oguiso and Shioda [155, Appendix 1]. Let $m \in \mathbb{Z}_{>0}$ and let $\Lambda \simeq \mathrm{U} \oplus \mathbb{Z}\,c$ with $c \cdot c = -2\,m$. Then Λ is an even lattice of signature $(2,\ 1)$; hence there exists an elliptic K3 surface $\varphi : S \to \mathbb{P}^1$ with a holomorphic section such that $\mathrm{NS}(S) \simeq \Lambda$. The lattice $Q = f^{\perp}/\mathbb{Z}\,f$ is generated by a single element q, see Theorem 7.2.7, and because the determinant of the intersection matrix is equal to $2\,m$, we have $q \cdot q = -2\,m$. Since Q^{irr} is generated by elements with self-intersection number -2, we have $Q^{\mathrm{irr}} = 0$ if $m > 1$, which we assume from now on. It follows from Theorem 7.2.7 that the mapping $\alpha \mapsto \alpha^{\Lambda}(e_0) - e_0 + \mathbb{Z}\,f$ is an isomorphism from the Mordell–Weil group $\operatorname{Aut}(S)^+_{\varphi}$ onto the additive group $Q = \mathbb{Z}\,q$. Let $\alpha \in \operatorname{Aut}(S)^+_{\varphi}$ denote the generator of the Mordell–Weil group that corresponds to q. If $k \in \mathbb{Z}$ and we drop the superscript Λ from the notation, then $-2\,m\,k^2 = (k\,q) \cdot (k\,q) = (\alpha^k(e_0) - e_0) \cdot (\alpha^k(e_0) - e_0) = \alpha^k(e_0) \cdot \alpha^k(e_0) - 2\,\alpha^k(e_0) \cdot e_0 + e_0 \cdot e_0$ and $\alpha^k(e_0) \cdot \alpha^k(e_0) = e_0 \cdot e_0 = -\chi(S,\ \mathcal{O}) = -2$ imply that $\nu(\alpha^k) = \alpha^k(e_0) \cdot e_0 = m\,k^2 - 1$. Here $\nu(\alpha^k)$ is the number of fixed-point fibers of α^k, the k-periodic fibers of α, counted with multiplicities, and $\nu(\alpha^k) = \alpha^k(E) \cdot E$ for any holomorphic section E. Since $m\,k^2 - 1 \geq m - 1 > 0$ for each nonzero integer k, this is in strong contrast to the statement in Theorem 4.3.3 that the Mordell–Weil group of a rational elliptic surface is generated by elements that map a holomorphic section to a disjoint one. ⊘

The positive definite bilinear form on the root lattice Q of type E_8, see Lemma 9.2.3, is equal to minus the intersection form, where we use the bracket notation $(\alpha\,|\,\beta) = -\,\alpha \cdot \beta$ as in Bourbaki [23, chapter VI]. The root system is the set R of all $q \in Q$ such that $(q\,|\,q) = 2$, see Bourbaki [23, p. 213], where the root lattice is $Q = \mathbb{Z}\,R$, the group generated by R. For each reducible fiber S_r the classes of the irreducible components of S_r generate a subgroup Q_r of Q, where $Q_r \cap R$ is a root system of type A_l, D_l, or E_l, if the intersection diagram of S_r is $A_l^{(1)}$, $D_l^{(l)}$, or $E_l^{(1)}$, respectively. See Remark 6.2.12. It follows that the orthogonal direct sum Q^{irr} of the Q_r is a root sublattice of Q, with the corresponding root subsytem $Q^{\mathrm{irr}} \cap R$ of R. Each root

sublattice of E_8 is an orthogonal direct sum of root lattices of type A_l, D_l, or E_l. The root subsystems of the root system of type E_8, modulo automorphisms of R, have been classified by Dynkin [54, Table 11].

The *list of Oguiso and Shioda* [155, pp. 84–86] describes, for each root sublattice T of Q, the orthogonal complement $T^\perp \cap Q$ of T in Q and the torsion subgroup of Q/T, where in the case that $T = Q^{\mathrm{irr}}$, the Mordell–Weil group $E(K)$ in the notation of [155] is isomorphic to the group Q/T, see Theorem 7.2.7, and $E(K)^0 = T^\perp \cap Q$ is the narrow Mordell–Weil lattice, denoted by Q^0 in Corollary 7.3.3. In their list, Oguiso and Shioda have left out the orthogonal direct sums T of root lattices of type A_l, D_l, or E_l when the sum of the Euler numbers is > 12. Here the Euler number of A_l, D_l, E_6, E_7, and E_8 is defined as $l+1$, $l+2$, 8, 9, and 10, which according to Section 6.2.6 and Lemma 6.2.29 are the Euler numbers of the reducible fibers of the corresponding Kodaira types I_{l+1}, I^*_{l-4}, IV^*, III^*, and II^*, respectively. Note that a singular fiber of Kodaira type III or IV has Euler number 3 or 4, and intersection diagram of type $\mathrm{A}_1^{(1)}$ or $\mathrm{A}_2^{(1)}$, respectively, when the Euler number of the singular fiber is one higher than the aforementioned Euler number of the root system. Because the sum of the Euler numbers of all the singular fibers is equal to $\chi_{\mathrm{top}}(S) = 12$, see Lemmas 6.2.29 and 9.1.2, the T with sum of the Euler numbers > 12 cannot occur as a Q^{irr} for a rational elliptic fibration.

If α_i, $1 \le i \le l$, is a basis of a root system of type A_l, D_l, or E_l, then these form the vertices of the corresponding Dynkin diagram, when the extended Dynkin diagram of the respective type $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(1)}$, or $\mathrm{E}_l^{(1)}$ is obtained by adding $\alpha_0 = -\sum_{i=1}^{l} \mu_i \alpha_i$, as in Remark 6.2.12. Deleting one of the α_j, $1 \le j \le l$, from the extended Dynkin diagram, one obtains a basis of a root sublattice of the same rank l, which happens to be the same if and only if we started with type A_l. Performing this to any indecomposable summand of a root sublattice T of Q, one obtains a root sublattice T_1 of T with rank $T_1 = \operatorname{rank} T$. Dynkin [54, Theorem 5.3] proved that every root sublattice S of T with $\operatorname{rank} S = \operatorname{rank} T$ is obtained from T by means of a sequence of such "elementary transformations." Furthermore Dynkin [54, Theorem 5.2] implies that up to automorphisms of T all root sublattices of lower rank are generated by subsets of the Dynkin diagram of T.

The rank-8 root sublattices of E_8 are E_8 and those obtained from E_8 by elementary transformations: A_8, D_8, $\mathrm{A}_7+\mathrm{A}_1$, $\mathrm{A}_5+\mathrm{A}_2+\mathrm{A}_1$, $2\,\mathrm{A}_4$, $4\,\mathrm{A}_2$, $\mathrm{E}_6+\mathrm{A}_2$, $\mathrm{E}_7+\mathrm{A}_1$, $\mathrm{D}_6+2\,\mathrm{A}_1$, $\mathrm{D}_5+\mathrm{A}_3$, $2\,\mathrm{D}_4$, $\mathrm{D}_4+4\,\mathrm{A}_1$, $2\,\mathrm{A}_3+2\,\mathrm{A}_1$, and $8\,\mathrm{A}_1$, where the sum sign means orthogonal direct sums. See Dynkin [54, Table 10, 11], where in [54, Table 11] the subsequent $\mathrm{A}_6+\mathrm{A}_2$ and $\mathrm{A}_7+\mathrm{A}_1$ have to be replaced by $\mathrm{E}_6+\mathrm{A}_2$ and $\mathrm{E}_7+\mathrm{A}_1$, respectively. These root sublattices, and the root sublattices of lower rank obtained by deleting vertices of Dynkin diagrams, are uniquely determined modulo automorphisms of Q by the types of their summands, except for A_7, $\mathrm{A}_5+\mathrm{A}_1$, $2\,\mathrm{A}_3$, $\mathrm{A}_3+2\,\mathrm{A}_1$, and $4\,\mathrm{A}_1$, where in each case there are two isomorphism classes, depending on whether the indecomposable summands are summands of a rank-8 root sublattice of Q or not. In the first case the torsion subgroup $\mathrm{MW}_{\mathrm{tor}}$ of the Mordell–Weil group is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$, whereas $\mathrm{MW}_{\mathrm{tor}} \simeq \{0\}$ in the second case.

The strategy to determine the narrow Mordell–Weil lattice $T^\perp \cap Q$ is as follows. In the case that the indecomposable summands in T are summands in one of the

rank-8 root sublattices of Q, as listed above, we write $Q = T + S$ where S is the sum of the remaining summands. If there are more ways of doing so we choose S to be maximal in the sense that another one cannot be obtained from S by elementary transformations. Then $T^\perp \cap Q = S'$, where $S' \supset S$ is defined by (7.6.6) with $\Lambda = Q$ and $M = S$. Then S' is a root sublattice of Q and orthogonal to T, hence $T + S'$ is a root sublattice of Q of rank 8. Because S is a root sublattice of S' with rank S' = rank S, the maximal choice of S implies that $S' = S$. This takes care of all 44 of the 74 items in the list of Oguisa and Shioda where the narrow Mordell–Weil lattice is a root sublattice of Q, except no. 14, $T = 4\ \mathrm{A}_1$, where not all summands are summands in a rank 8 sublattice of Q, but nevertheless $T^\perp \cap Q \simeq 4\ \mathrm{A}_1$ is a root sublattice of Q. In all other cases one can write $T = T_1 + T_2$, where T_1 is as in the previous paragraph. That is, T_1 is a root sublattice of a rank-8 root sublattice $T_1 + S_1$, $S_1 = T_1^\perp \cap Q$, and T_2 is an indecomposable summand in T which at the same time is contained in one of the indecomposable summands S_2 of S_1. It follows that $T^\perp \cap Q = (T_1^\perp \cap Q) \cap (T_2^\perp \cap Q) = S_1 \cap T_2^\perp \cap Q = T_2^\perp \cap S_1 = (T_2^\perp \cap S_2) + S_3$ if we have the orthogonal sum $S_1 = S_2 + S_3$ of root sublattices. In each case the determination of the lattice $T_2^\perp \cap S_2$ is a straightforward Diophantine linear algebra computation.

For example, in no. 12 in the list of Oguisa and Shioda, we have $T = 2\ \mathrm{A}_1 + \mathrm{A}_2$, where $(\mathrm{A}_1 + \mathrm{A}_2)^\perp \cap Q = \mathrm{A}_5$ hence $T^\perp \cap Q \simeq \mathrm{A}_1^\perp \cap \mathrm{A}_5$. If α_j, $1 \le j \le 5$ are the roots in the A_5-chain, and $\mathrm{A}_1 = \mathbb{Z}\,\alpha_1$, then $\mathrm{A}_1^\perp \cap \mathrm{A}_5$ is the set of all $q = \sum_{j=1}^5 n_j\,\alpha_j$ such that $n_j \in \mathbb{Z}$ and $0 = (q|\alpha_1) = 2\,n_1 - n_2$. That is, $\mathrm{A}_1^\perp \cap \mathrm{A}_5$ is the lattice generated by $\beta_1 = \alpha_1 + 2\,\alpha_2$ and the $\alpha_j, 3 \le j \le 5$. The element $\beta_2 = \beta_1 + \alpha_3 = \alpha_1 + 2\,\alpha_2 + \alpha_3$ is shorter than β_1, and with respect to the $\mathbb{Z}$-basis $\beta_2, \alpha_3, \alpha_4, \alpha_5$ the intersection matrix of the narrow Mordell–Weil lattice $T^\perp \cap Q$ is equal to

$$\begin{pmatrix} -4 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

which has determinant equal to 12. In the list of Oguisa and Shioda, the elements of the $\mathbb{Z}$-basis of the narrow Mordell–Weil lattice are chosen as short as possible, but even with this requirement the presentation of the lattice is not unique. For instance, the above intersection matrix is not equal to minus the matrix in no. 12 of the list of Oguiso and Shioda, where the second entry in the first row should be -1 instead of 1.

The Mordell–Weil lattice is equal to the dual $(Q^0)^*$ of the narrow Mordell–Weil lattice $Q^0 = T^\perp \cap Q$; see the definition preceding Lemma 7.6.2 and Theorem 7.6.6. The structure of $(Q^0)^*$ is easily determined from the structure of Q^0. In contrast to the narrow Mordell–Weil lattice Q^0, which is an even integral lattice, the Mordell–Weil lattice $(Q^0)^*$ is not integral, except when $Q^{\mathrm{irr}} = \{0\}$ or $Q^{\mathrm{irr}} = Q$. The proof of the fact that the Manin QRT automorphisms generate the Mordell–Weil group uses the description of the lattice structure of $(Q^0)^*$ in each case of the list; see the end

of the proof of Theorem 4.3.3. The group structure of the torsion subgroup of the Mordell–Weil group is described in the next lemma.

Lemma 9.2.6 *The Mordell–Weil group* MW *is isomorphic to* $\mathbb{Z}^\rho \times \mathrm{MW}_{\mathrm{tor}}$, *where* $\rho = \operatorname{rank} \mathrm{MW} = \operatorname{rank} Q - \operatorname{rank} Q^{\mathrm{irr}} = 8 - \sum_r (n_r - 1)$, *in which* n_r *is the number of irreducible components of* S_r. *The torsion subgroup* $\mathrm{MW}_{\mathrm{tor}}$ *of* MW *is isomorphic to the group at the left in the list below, where* $\mathbb{Z}_m := \mathbb{Z}/m\,\mathbb{Z}$, *if the lattice* Q^{irr} *is isomorphic to one of the lattices at the right.*

In all other cases, $\mathrm{MW}_{\mathrm{tor}}$ *is trivial. For each direct sum of sublattices there is exactly one isomorphism class of* $\mathrm{MW}_{\mathrm{tor}}$, *except that* $\mathrm{MW}_{\mathrm{tor}}$ *is trivial or isomorphic to* $\mathbb{Z}_2$ *when* Q^{irr} *is of type* $4\,\mathrm{A}_1$, $\mathrm{A}_3+2\,\mathrm{A}_1$, $2\,\mathrm{A}_3$, $\mathrm{A}_5+\mathrm{A}_1$, *or* A_7.

$\mathbb{Z}_2$	$4\,\mathrm{A}_1,\ 5\,\mathrm{A}_1,\ \mathrm{A}_3+2\,\mathrm{A}_1,\ \mathrm{A}_2+4\,\mathrm{A}_1,\ \mathrm{A}_3+3\,\mathrm{A}_1,\ 2\,\mathrm{A}_3,\ \mathrm{D}_4+2\,\mathrm{A}_1,$ $\mathrm{A}_5+\mathrm{A}_1,\ \mathrm{A}_3+\mathrm{A}_2+2\,\mathrm{A}_1,\ \mathrm{D}_4+\mathrm{A}_3,\ \mathrm{A}_5+2\,\mathrm{A}_1,\ \mathrm{D}_5+2\,\mathrm{A}_1,\ \mathrm{D}_6+\mathrm{A}_1,$ $\mathrm{A}_7,\ \mathrm{E}_7+\mathrm{A}_1,\ \mathrm{D}_8$.
${\mathbb{Z}_2}^2$	$6\,\mathrm{A}_1,\ \mathrm{A}_3+4\,\mathrm{A}_1,\ \mathrm{D}_4+3\,\mathrm{A}_1,\ 2\,\mathrm{D}_4,\ \mathrm{D}_6+2\,\mathrm{A}_1$.
$\mathbb{Z}_3$	$3\,\mathrm{A}_2,\ 3\,\mathrm{A}_2+\mathrm{A}_1,\ \mathrm{A}_5+\mathrm{A}_2,\ \mathrm{E}_6+\mathrm{A}_2,\ \mathrm{A}_8$.
${\mathbb{Z}_3}^2$	$4\,\mathrm{A}_2$.
$\mathbb{Z}_4$	$2\,\mathrm{A}_3+\mathrm{A}_1,\ \mathrm{D}_5+\mathrm{A}_3,\ \mathrm{A}_7+\mathrm{A}_1$.
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2\,\mathrm{A}_3+2\,\mathrm{A}_1$.
$\mathbb{Z}_5$	$2\,\mathrm{A}_4$.
$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathrm{A}_5+\mathrm{A}_2+\mathrm{A}_1$.

Proof. The formula for the rank of the Mordell–Weil group follows from (7.2.4), where $\operatorname{rank} Q = 8$ follows from Lemma 9.2.3.

It follows from Theorem 4.3.3 that for every nontrivial $\alpha \in \mathrm{MW}_{\mathrm{tor}}$ the sum of the contributions of the reducible fibers to α is equal to two. This also implies that the restriction to $\mathrm{MW}_{\mathrm{tor}}$ of the homomorphism (7.3.2), the mapping that assigns to each $\alpha \in \mathrm{MW}$ its action of permuting the irreducible components of the reducible fibers, is injective. See the end of Section 6.3.6 for the action of $\mathfrak{F}_r/\mathfrak{F}_r^\circ$ on the set $S_r^{\mathrm{irr},\,1}$ of all multiplicity-one irreducible components of a reducible fiber S_r, and Lemma 7.5.3 for the contribution $\mathrm{contr}_r(\alpha)$ of S_r to such an action. See Remark 6.3.32 for the description of the action of $\mathfrak{F}_r/\mathfrak{F}_r^\circ$ on $S_r^{\mathrm{irr},\,1}$ in terms of root lattices. For each configuration of types $\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_n$ of reducible fibers, the T's in the list of Oguiso and Shioda [155, pp. 84–86], it is a straightforward exercise to determine the maximal subgroups H of $\prod_r \mathfrak{F}_r/\mathfrak{F}_r^\circ$ such that $\sum_r \mathrm{contr}_r(\alpha) = 2$ for every nontrivial element α of H.

In each case, H is isomorphic to the group mentioned in Lemma 9.2.6. The formula (7.6.10) can be used to read off the number of elements of $\mathrm{MW}_{\mathrm{tor}}$ from the determinant of the narrow Mordell–Weil lattice Q^0, the $E(K)^0$ in the list of Oguiso and Shioda [155, pp. 84–86]. The group structure of $\mathrm{MW}_{\mathrm{tor}}$ is uniquely determined by $\#(\mathrm{MW}_{\mathrm{tor}})$ except if $\#(\mathrm{MW}_{\mathrm{tor}}) = 4$, in which case one still has to decide between ${\mathbb{Z}_2}^2$ and $\mathbb{Z}_4$. It follows from Section 5.2.3 that if the Mordell–Weil group has an element of order four, then the configuration of singular fibers is equal to $2\,\mathrm{I}_4\ \mathrm{I}_2\ 2\,\mathrm{I}_1$, $2\,\mathrm{I}_4\ 2\,\mathrm{I}_2$, $\mathrm{I}_8\ \mathrm{I}_2\ 2\,\mathrm{I}_1$, or $\mathrm{I}_0^*\ \mathrm{I}_4\ \mathrm{I}_1$, which is an efficient criterion for ruling out $\mathrm{MW}_{\mathrm{tor}} \simeq {\mathbb{Z}_2}^2$. See Remark 5.2.5 for the case in question. It turns out

that in each case, $\#(\mathrm{MW}_{\mathrm{tor}}) = \#(H)$, hence $\mathrm{MW}_{\mathrm{tor}} \simeq H$, except in the cases $\mathrm{A}_1^{\oplus 4}$, $\mathrm{A}_3 \oplus \mathrm{A}_1^{\oplus 2}$, $\mathrm{A}_3^{\oplus 2}$, $\mathrm{A}_5 \oplus \mathrm{A}_1$, or A_7, when it can also happen that $\mathrm{MW}_{\mathrm{tor}}$ is trivial. □

The *list of Persson* [156, pp. 7–14] enumerates all configurations of singular fibers of rational elliptic surfaces, in combination with the group structure of the Mordell–Weil group in each case. Persson's list with 289 items is much longer than the list of Oguiso and Shioda with 74 items, because the Kodaira types I_2 and III have the same Dynkin diagram A_1, the Kodaira types I_3 and IV have the same Dynkin diagram A_2, and the irreducible singular fibers of Kodaira type I_1 and II do not show up in the list of Oguiso and Shioda. In Persson's list the configuration I_3 III I_2 2 I_1 is a typo, since the sum of the Euler numbers would be $3+3+2+1+1 = 10 \neq 12$. Lemma 9.2.7 below yields that this configuration should be replaced by I_3 III I_2 2 II.

Lemma 9.2.7 *Persson's list is characterized by the following conditions:*

(i) *The lattice* Q^{irr} *is isomorphic to a root sublattice of the root lattice of type* E_8.
(ii) *The sum of the topological Euler numbers of the fibers is equal to* 12.
(iii) *For each singular fiber* S_c *not of type* I_b, $b > 0$, *the torsion subgroup* $\mathrm{MW}_{\mathrm{tor}}$ *of the Mordell–Weil group is isomorphic to a subgroup of* $\mathfrak{F}_c/\mathfrak{F}_c^\circ$; *see the end of Section 6.3.6.*
(iv) *Write* $i_b, i_b^*, ii, iii, iv, ii^*, iii^*,$ *and* iv^* *for the number of singular fibers in the configuration of type* I_b, I_b^*, II, III, IV, II*, III*, *and* IV*, *respectively. Let* $j := \sum_{b>0} b\,(i_b + i_b^*)$. *If* $j = 0$ *then* $iii = iii^* = 0$ *or* $ii = iv = iv^* = ii^* = 0$. *If* $j > 0$ *then*

$$\begin{aligned} &j - ii - iv^* - 2\,iv - 2\,ii^* \in 3\,\mathbb{Z}_{\geq 0}, \\ &j - iii - iii^* \in 2\,\mathbb{Z}_{\geq 0}, \quad \text{and} \\ &\textstyle\sum_{b>0} (6-b)\,(i_b + i_b^*) \; + 2\,(iv + ii^*) + 3\,(iii + iii^*) + 4\,(ii + iv^*) - 12 \\ &\quad \in 6\,\mathbb{Z}_{\geq 0}. \end{aligned}$$

(v) *Not* 4 III *and* $\mathrm{MW}_{\mathrm{tor}} \simeq \{0\}$, 3 III I_2 I_1 *and* $\mathrm{MW}_{\mathrm{tor}} \simeq \mathbb{Z}/2\,\mathbb{Z}$, I_5 I_3 2 II, *or* 2 IV I_3 I_1.

Proof. Condition (i) has been discussed in the paragraphs preceding Lemma 9.2.6, which also contains a description of the root sublattices of E_8 modulo automorphisms of E_8. Condition (ii) follows from Lemma 6.2.29, and has already been used in Lemma 9.2.6.

It follows from Corollary 7.5.5(iii) that $\mathrm{MW}_{\mathrm{tor}}$ acts freely on the set of regular points $S_c \cap S^{\mathrm{reg}}$ of S_c, and the first paragraph in Theorem 6.3.29 implies that the action is by means of elements of $\mathfrak{F}_c$. According to the last part of Section 6.3.6, $\mathfrak{F}_c^\circ \simeq \mathbb{C}$ if S_c is a singular fiber not of type I_b, $b > 0$. Condition (iii) follows because the additive group of complex numbers has no nontrivial finite subgroups.

If the configuration of Kodaira types is equal to the configuration of the singular fibers of a rational elliptic surface $\kappa : S \to \mathbb{P}^1$, then (6.2.48) implies that the number j in condition (iv) is equal to the mapping degree of the modulus function $J : \mathbb{P}^1 \to \mathbb{P}^1$

of the fibration. If $j = 0$, then the fact that $\kappa : S \to \mathbb{P}^1$ has singular fibers implies that $J \equiv 0$ or $J \equiv 1$, when $iii = iii^* = 0$ or $ii = iv = iv^* = ii^* = 0$, respectively.

Let $j > 0$, when $J : \mathbb{P}^1 \to \mathbb{P}^1$ is a j-fold branched covering. Let $c \in \mathbb{P}^1$ and denote by $\deg_c J$ the multiplicity of the divisor $J^{-1}(\{0\})$ at c. Then Table 6.2.40 implies that there exists $k_c \in \mathbb{Z}_{\geq 0}$ such that $\deg_c J = 3\,k_c$ if $J(c) = 0$ and S_c is a fiber of type I_0 or I_0^*, $\deg_c J = 1+3\,k_c$ if S_c is of type II or IV*, and $\deg_c J = 2+3\,k_c$ if S_c is of type IV or II*. Since j is equal to the sum of the $\deg_c J$ over all $c \in C$ such that $J(c) = 0$, it follows that $(j - ii - iv^* - 2\,iv - 2\,ii^*)/3$ is equal to the sum over all the k_c, and is therefore a nonnegative integer. Similarly there exists $k_c \in \mathbb{Z}_{\geq 0}$ such that $\deg_c J = 3\,k_c$ if $J(c) = 1$ and S_c is a fiber of type I_0 or I_0^*, and $\deg_c J = 1+2\,k_c$ if S_c is of type III or III*, and therefore $(j - iii - iii^*)/2 =$ the sum over all the k_c, hence $\in \mathbb{Z}_{\geq 0}$.

For each ramification point $c \in \mathbb{P}^1$ the order of ramification at c is defined as the order of the zero of J' at c, which is equal to $k-1$ if k denotes the order of the zero of $J - J(c)$ or $1/J$ at c if $J(c) \neq \infty$ or $J(c) = \infty$, respectively. Since the number $d - n_b$ in the Riemann–Hurwitz formula (2.3.18) is equal to the sum of the orders of ramification at all points in the fiber over the branch point b, and for $\pi = J$ we have $d = j$ and $C = D = \mathbb{P}^1$ hence $\mathrm{g}(C) = \mathrm{g}(D) = 0$, it follows that the sum of all orders of ramification is equal to $2\,j - 2$. The sum of the orders of ramification over ∞ is equal to $j - \sum_{b>0} (i_b + i_b)^*$. The sum of the orders of ramification over 0 is equal to the sum of the numbers $3\,k_c - 1$, $3\,k_c$, and $3\,k_c + 1$ when S_c is of type I_0 or I_0^* with $J(c) = 0$, of type II or IV*, and of type IV or II*, respectively. Writing $3\,k_c = 2\,k_c + k_c$ and using that the sum of all k_c is equal to $(j - ii - iv^* - 2\,iv - 2\,ii^*)/3$, we obtain that the sum of the orders of ramification over 0 is equal to $2\,(j - ii - iv* - 2\,iv - 2\,ii^*)/3 + iv + ii^*$ plus the sum of the numbers $k_c - 1$, where c is a ramification point with $J(c) = 0$ such that S_c is a fiber of type I_0 or I_0^*, and the numbers k_c such that S_c is of Kodaira type II, IV*, IV, or II*. Similarly the sum of the orders of ramification over 1 is equal to $(j - iii - iii^*)/2$ plus the sum of the numbers $k_c - 1$ where c is a ramification point with $J(c) = 0$ such that S_c is a fiber of type I_0 or I_0^*, and the numbers k_c such that S_c is of Kodaira type III or III*. Because $j + 2\,j/3 + j/2 = 2\,j + j/6$, $2\,(ii + iv* + 2\,iv + 2\,ii^*)/3 - iv - ii^* + (iii + iii^*)/2 = (2\,(iv + ii^*) + 3\,(iii + iii^*) + 4\,(ii + iv^*))/6$, and $j/6 = \sum_{b>0} b\,(i_b + i_b^*)/6$, the equation that $2\,j - 2$ is equal to the sum of all orders of ramification takes the form that $1/6$ times the left-hand side in the last line in condition (iv) is equal to the sum of the numbers $k_c - 1$, where c is a ramification point with $J(c) \in \{0,\ 1\}$ such that S_c is a fiber of type I_0 or I_0^*, the numbers k_c such that S_c is of type II, IV*, IV, II*, III, or III*, and the orders of ramification over any $b \notin \{0,\ 1,\ \infty\}$. This completes the proof of condition (iv).

If there are at least three singular fibers of type III, then Table 6.3.2 implies that the polynomial g_2 of degree four has at least three simple zeros, at each of which the polynomial g_3 of degree six has a zero of order ≥ 2. The Mordell–Weil group has an element of order two if and only if the Weierstrass equation $Y^2 - 4\,X^3 + g_2\,X + g_3 = 0$ has a solution with $Y \equiv 0$, that is, if and only if there is a homogeneous polynomial X of degree two such that $-4\,X^3 + g_2\,X + g_3 = 0$. At each of the common zeros of g_2 and g_3 we have $X = 0$; hence the only possibility is that $X \equiv 0$, which *is* a

solution if and only if $g_3 \equiv 0$, when the configuration of the singular fibers is 4 III. This excludes the first two cases mentioned in v).

If the configuration is $\mathrm{I}_5\ \mathrm{I}_3\ 2\ \mathrm{II}$, then the left-hand side in the last line in condition (iv) is equal to zero, when the proof of condition (iv) implies that $J : \mathbb{P}^1 \to \mathbb{P}^1$ branches only over ∞, 0, and 1, in the following way. Over ∞ the local branches form a 5-cycle and a 3-cycle. Over 0 the two fibers of type II do not correspond to ramification points, and there are two 3-cycles corresponding to smooth fibers. Over 1 there are four 2-cycles corresponding to smooth fibers. The restriction of J to $Y := \mathbb{P}^1 \setminus J^{-1}(\{0, 1, \infty\})$ is an unramified 8-fold covering from Y onto $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For any base point $* \in X$, the lift of loops defines an action of $\pi_1(X, *)$ onto the fiber $F := J^{-1}(\{*\})$ of J over $*$, where $\#(F) = 8$ and the action is transitive because Y is connected. As mentioned in the proof of Lemma 6.4.2, $\pi_1(X, *)$ is generated by the homotopy classes δ_0, δ_1, and δ_∞ of loops around 0, 1, and ∞, respectively, subject to the single relation $\delta_0\,\delta_1\,\delta_\infty = 1$. If σ_j denotes the action of δ_j on F, then σ_0, σ_1, and σ_∞ generate a transitive group of permutations of F, $\sigma_0\,\sigma_1\,\sigma_\infty = 1$, and the respective cycle structures of σ_0, σ_1, and σ_∞ are (1, 1, 3, 3), (2, 2, 2, 2), and (5, 3). There is a numbering of F such that $\sigma_0 = (1)\,(2)\,(3\,4\,5)\,(6\,7\,8)$. Because σ_0 and σ_1 generate a transitive group, $(1\,2)$ is not one of the four 2-cycles of σ_1, and the numbering of F can be arranged such that $(1\,3)$ is one of the 2-cycles of σ_1. Because $(a\,b\,c)\,(a\,b) : b \mapsto b$ and $\sigma_0\,\sigma_1$ has no 1-cycles, none of the 2-cycles of σ_1 can be part of $\{3, 4\,5\}$ or $\{6, 7, 8\}$. If $(2\,4)$ or $(2, 5)$ is a 2-cycle of σ_1, then one of the remaining 2-cycles of σ_1 is contained in $\{6, 7, 8\}$, a contradiction. Therefore the numbering of F can be arranged such that $\sigma_1 = (1\,3)\,(2\,6)\,(4\,7)\,(5\,8)$ or $(1\,3)\,(2\,6)\,(4\,8)\,(5\,7)$, when $\sigma_0\,\sigma_1 = (1\,4\,8\,3)\,(2\,7\,5\,6)$ or $(1\,4\,6\,2\,7\,3)\,(5\,8)$. Since neither case yields the correct cycle structure for $\sigma_\infty = (\sigma_0\,\sigma_1)^{-1}$, the configuration $\mathrm{I}_5\ \mathrm{I}_3\ 2\ \mathrm{II}$ cannot occur.

If the configuration is $2\ \mathrm{IV}\ \mathrm{I}_3\ \mathrm{I}_1$, then the left-hand side in the last line in condition (iv) is equal to zero, when the proof of condition (iv) implies that $J : \mathbb{P}^1 \to \mathbb{P}^1$ branches over ∞, 0, and 1, only in the following way. Over ∞ the local branches form a 3-cycle and a 1-cycle. Over 0 the two fibers of type IV correspond to two 2-cycles. Over 1 there are two 2-cycles corresponding to smooth fibers. In the notation of the previous paragraph, there is a numbering of F such that $\sigma_0 = (1\,2)\,(3\,4)$ and $\sigma_1 = (1\,3)\,(2\,4)$, when $\sigma_0\,\sigma_1 = (1\,4)\,(2\,3)$ does not have the cycle structure of ${\sigma_\infty}^{-1}$. This completes the proof of condition (v).

There are 346 cases that satisfy condition (i) and (ii), where for each item in the list of Oguiso and Shioda one collects the configurations of singular fibers with the given intersection diagrams of the reducible fibers, and sum of the Euler numbers of the singular fibers equal to 12. Of these, 37 cases are ruled out by condition (iii). Of the remaining 309 cases, 16 are ruled out by condition (iv), and 4 more by condition (v), leaving the 289 cases of Persson's list. □

Conditions (i), (ii), and (iii) in Lemma 9.2.7 correspond to conditions (2'), (1), and (3) in Persson [156, pp. 1, 2], respectively. Condition (iv) in Lemma 9.2.7 corresponds to conditions (1.8)–(1.10) in Miranda [137]. I learned the proof of the impossibility of the configurations $\mathrm{I}_5\ \mathrm{I}_3\ 2\ \mathrm{II}$ and $2\ \mathrm{IV}\ \mathrm{I}_3\ \mathrm{I}_1$ from Miranda [137, Lemma 2.6, (2.6.3) and (2.6.1)].

The proof that all 289 items in Persson's list are realized by rational elliptic surfaces constitutes the larger part of the paper of Persson [156]. The constructions are geometric, although Persson [156, p. 2] says that "the decision which configurations occur will actually boil down to finding the Weierstrass model for the surface." Another proof has been given by Miranda [137, Section 3, 4], based on the strategy of first constructing the modulus function J. In Miranda [137] the cases that for the same configurations of singular fibers the torsion subgroups of the Mordell–Weil group can be different are not distinguished, which explains why Miranda mentioned 279 cases instead of Persson's 289 cases.

According to (a) in Theorem 9.1.3, every rational elliptic surface can be obtained from a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$. For any given item in Persson's list one may attempt to compute such a pencil $\mathfrak{C}$, using the classification of Wall [206, §3] of the singular fibers that can arise from singular members of the pencil after the blowing up. If α is an element of the Mordell–Weil group without fixed point fibers, then Theorem 4.3.2 implies that α is the QRT transformation defined by a pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Given the action of α on the set of all irreducible components of the reducible fibers, one may attempt to use Section 12.1 to compute such a pencil $\mathfrak{B}$. Section 5.2 contains such a computation when α has finite order. The examples in Chapter 11 start out with the pencils $\mathfrak{B}$, where we note that Corollary 2.4.7 and Table 6.3.2 enabled the determination of the configuration of the singular fibers. The examples in this book cover about one-tenth of Persson's list; see Section 12.2. I did not try to find pencils of biquadratic curves for each of the $289 - 2 = 287$ items in Persson's list with a nontrivial Mordel-Weil group.

In [156, Epilogue], Persson wrote that "Another thing to work out is …the stratification of the fibrations, which fibrations may degenerate to which?" where I understand that the stratification is the one defined by the configurations of the singular fibers. The degeneration of fibrations into others corresponds to confluences of singular fibers. See Remark 6.3.4. I fully agree with Persson that it would be worthwhile to make a closer study of the stratification in the moduli space of isomorphisms classes of rational elliptic surfaces defined by the configurations of the singular fibers. In particular, it would be interesting to have a better understanding of the dimensions of the strata, which fibrations may degenerate to which, and a geometric understanding of the coalescence of singular fibers.

Remark 9.2.8. For a rational elliptic surface $\kappa : S \to P \simeq \mathbb{P}^1$ all the smooth fibers are isomorphic if and only if the modulus function $J : P \to \mathbb{P}^1$ is constant if and only if the mapping degree j of J is equal to zero. According to (6.2.48) this happens if and only if S has no singular fibers of type I_k with $k > 0$ or I_l^* with $l > 0$.

The rational elliptic surfaces with constant modulus function are the elliptic fibrations in Lemma 6.4.12 with $g = 0$ and $k = 1$. This leads to the following possibilities for the configurations of the singular fibers, where the isomorphism class of the Mordell–Weil group in each case is added after the dots. For $J = 0$ we have $6\ \mathrm{II} \ldots \mathbb{Z}^8$, $4\ \mathrm{II}\ \mathrm{IV} \ldots \mathbb{Z}^6$, $3\ \mathrm{II}\ \mathrm{I}_0^* \ldots \mathbb{Z}^4$, $2\ \mathrm{II}\ 2\ \mathrm{IV} \ldots \mathbb{Z}^4$, $2\ \mathrm{II}\ \mathrm{IV}^* \ldots \mathbb{Z}^2$, $\mathrm{II}\ \mathrm{IV}\ \mathrm{I}_0^* \ldots \mathbb{Z}^2$, $\mathrm{II}\ \mathrm{II}^* \ldots 0$, $3\ \mathrm{IV} \ldots \mathbb{Z}^2 \times (\mathbb{Z}/3\mathbb{Z})$, $\mathrm{IV}\ \mathrm{IV}^* \ldots \mathbb{Z}/3\mathbb{Z}$, or $2\ \mathrm{I}_0^* \ldots (\mathbb{Z}/2\mathbb{Z})^2$. For $J = 1$ we have $4\ \mathrm{III} \ldots \mathbb{Z}^4$, $2\ \mathrm{III}\ \mathrm{I}_0^* \ldots \mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$, $\mathrm{III}\ \mathrm{III}^* \ldots \mathbb{Z}/2\mathbb{Z}$, or $2\ \mathrm{I}_0^* \ldots (\mathbb{Z}/2\mathbb{Z})^2$. Finally, if $J \neq 0$ and $J \neq 1$ then we have

$2\ \mathrm{I}_0^* \ldots\ (\mathbb{Z}/2\mathbb{Z})^2$. I obtained the Mordell–Weil groups from Lemma 9.2.6. It is quite remarkable that there are still so many configurations of singular fibers for which the modulus function is constant. Note that Lemma 6.4.12 implies that the positions of the points in $\mathbb{P}^1$ over which we have the singular fibers can be chosen aribtrarily.

This classification also follows from the Weierstrass model of the rational elliptic surfaces, in Example 5 with $N = 1$. For $J \equiv 0$, we have $g_2 \equiv 0$ and g_3 an arbitrary nonzero homogeneous polynomial of degree 6 in $(z_0,\ z_1)$. According to Table 6.3.2 we have a singular fiber of type II, IV, I_0^*, IV*, or II* over the point $[z_0 : z_1]$ if and only if g_3 has a zero at that point of order 1, 2, 3, 4, or 5, respectively. For $J \equiv 1$, we have $g_3 \equiv 0$ and g_2 an arbitrary nonzero homogeneous polynomial of degree 4 in $(z_0,\ z_1)$. According to Table 6.3.2 we have a singular fiber of type III, I_0^*, or III* over the point $[z_0 : z_1]$ if and only if g_2 has a zero at that point of order 1, 2, or 3, respectively. We have the configuration of singular fibers $2\ \mathrm{I}_0^*$ if and only if after a suitable linear substitution of variables in $(z_0,\ z_1)$, there are constants $a,\ b \in \mathbb{C}$, not both equal to zero, such that $g_2(z_0,\ z_1) = a\, {z_0}^2\, {z_1}^2$ and $g_3(z_0,\ z_1) = b\, {z_0}^3\, {z_1}^3$. Here $J = a^3/(a^3 - 27\, b^2)$. ⊘

Remark 9.2.9. Let $w_0,\ w_1$ be a basis of W, the space of all holomorphic two-vector fields on S, and let z be the affine coordinate on $P \setminus \{[w_1]\}$ such that $z([w_0 + c\, w_1]) = c$, $c \in \mathbb{C}$. For each $c \in \mathbb{C}$ we have the vector field v_c on the smooth part of the fiber over $[w_0 + c\, w_1]$, defined in Lemma 3.3.4, and this defines a meromorphic section v of the Lie algebra bundle $\mathfrak{f}$ that has a simple pole at $[w_1]$ and no other zeros or poles. This confirms the statement in Lemma 9.1.2(ii) that the number of zeros minus the number of poles, counted with multiplicities, of any meromorphic section v of the Lie algebra bundle $\mathfrak{f}$ is equal to -1.

With these choices of v and z, the complex two-form ω dual to the exterior two-vector field w_1 in Lemma 3.3.4 is holomorphic and nowhere vanishing on the complement of the single fiber $S_{[w_1]} = \kappa^{-1}(\{[w_1]\})$, which can be chosen at will. Because $\mathrm{d}z$ and $1/v$ have a double pole and simple zero at $[w_1]$, respectively, ω has a simple pole along $S_{[w_1]}$, and actually the holomorphic exterior two-vector field $1/\omega$ is equal to w_1.

Viewing the surface as a real four-dimensional manifold, it follows that the complement of any fiber in a rational elliptic surface can be viewed as a completely integrable bi-Hamiltonian system of two degrees as in Remark 6.2.22. Because Persson's list [156, pp. 7–14] shows a great variety of configurations of singular fibers of rational elliptic surfaces, the conclusion is that already the rational elliptic surfaces form a very rich source of examples of integrable bi-Hamiltonian systems, where the nonsingular fibers are Lagrangian tori. ⊘

In the following proposition, $\mathcal{M}$ is the subgroup of $\mathbb{Z}^2$ defined in Remark 6.4.9 in terms of the monodromy, $\mathrm{MW} := \mathrm{Aut}(S)^+_\kappa$ the Mordell–Weil group, $\varpi : \mathrm{MW} \to \Gamma := \prod_{r \in C^{\mathrm{red}}} \mathfrak{F}_r/\mathfrak{F}_r^{\mathrm{o}}$ is the action of the Mordell–Weil group on the multiplicity-one irreducible components of the reducible fibers, and $\mathrm{MW}_{\mathrm{tor}}$ is the torsion subgroup of the Mordell–Weil group.

Proposition 9.2.10 *Let $\kappa : S \to \mathbb{P}^1$ be a rational elliptic surface, and $\psi : R \to \mathbb{P}^1$ a relatively minimal elliptic fibration without multiple singular fibers. If ψ has the same*

modulus function J and monodromy representation M as κ, then ψ is isomorphic to κ. The finite groups $\mathbb{Z}^2/\mathcal{M}$, $\Gamma/\varpi(\mathrm{MW})$, and $\mathrm{MW}_{\mathrm{tor}}$ are isomorphic to each other.

Proof. The assumptions for $\psi : R \to \mathbb{P}^1$ mean that its isomorphism class belongs to the family $\mathcal{F}(J, M)$ in Section 6.4.3, or Section 6.4.4 if the modulus function J is constant, where $\kappa : S \to \mathbb{P}^1$ represents the basic member, the unique member of $\mathcal{F}(J, M)$ that admits a holomorphic section. Therefore the first statement amounts to the claim that the fibration $\psi : R \to \mathbb{P}^1$ admits a holomorphic section. It follows from the description in Section 6.4.3 of the members of $\mathcal{F}(J, M)$ in terms of the cohomology group $\mathrm{H}^1(C, \mathcal{O}(\mathfrak{F}))$ that the Lie algebra bundle of $\psi : R \to \mathbb{P}^1$ is equal to the Lie algebra bundle $\mathfrak{f}$ of the basic member $\kappa : S \to \mathbb{P}^1$. Therefore Theorem 6.2.18, Lemma 2.1.3, and Lemma 9.1.1 imply that we have a canonical isomorphism ι from $W_R := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(\mathrm{O}(1)))$ onto $\mathrm{H}^0(R, \mathcal{O}(\mathrm{K}_R^*))$ such that $\mathrm{Div}(w) = \psi^*(\mathrm{Div}(\alpha))$ if $\alpha \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(\mathrm{O}(1)))$ and $w = \iota(\alpha)$ is the corresponding holomorphic complex two-form on R. If α is nonzero, then it has a simple zero at a unique point $c \in \mathbb{P}^1$, and $\mathrm{Div}(\psi^*(\alpha))$ is equal to the fiber of ψ over c. It follows that the anticanonical system of R defines the fibration ψ. Because ψ has connected fibers, the implication (d) $\Rightarrow$ (b) in Theorem 9.1.3 yields that ψ admits a holomorphic section.

Because $\deg(\mathfrak{f}) = -1 < 0$, see (ii) in Lemma 9.1.2, we have $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(\mathfrak{f})) = 0$, and therefore the homomorphism $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(\mathfrak{F}^{\mathrm{o}})) \to \mathrm{H}^2(\mathbb{P}^1, P) \simeq \mathbb{Z}^2/\mathcal{M}$ in (6.4.8) is an isomorphism. On the other hand, the triviality of the family $\mathcal{F}(J, M)$ implies that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(\mathfrak{F})) = 0$, when the paragraph after (6.4.7) yields an isomorphism from $\Gamma/\varpi(\mathrm{MW})$ onto $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(\mathfrak{F}^{\mathrm{o}}))$. The composition of this isomorphism with the previous one yields a canonical isomorphism from $\Gamma/\varpi(\mathrm{MW})$ onto $\mathbb{Z}^2/\mathcal{M}$. Because the Néron–Severi lattice of S is unimodular, see Theorems 9.2.4 and 9.2.3, we have the identity (7.6.11), which states that the finite commutative groups $\Gamma/\varpi(\mathrm{MW}) \simeq \mathbb{Z}^2/\mathcal{M}$ and $\mathrm{MW}_{\mathrm{tor}}$ have the same number of elements. Since I do not see a natural homomorphism from $\Gamma/\varpi(\mathrm{MW}) \simeq \mathbb{Z}^2/\mathcal{M}$ to $\mathrm{MW}_{\mathrm{tor}}$ or vice versa, I resort to a case by case check for the proof that these groups are isomorphic.

There are nine cases in which the structure of $\Gamma/\varpi(\mathrm{MW}) \simeq \mathbb{Z}^2/\mathcal{M}$ is not uniquely determined by the number of elements of $\mathrm{MW}_{\mathrm{tor}}$. These are $\mathrm{I}_2^*\ 2\ \mathrm{I}_2 - (\mathbb{Z}/2\mathbb{Z})^2$, $\mathrm{I}_1\ \mathrm{I}_4\ \mathrm{I}_1 - \mathbb{Z}/4\mathbb{Z}$, $2\ \mathrm{I}_0^*\ -\ (\mathbb{Z}/2\mathbb{Z})^2$, $\mathrm{I}_0^*\ 3\ \mathrm{I}_2 - (\mathbb{Z}/2\mathbb{Z})^2$, $2\ \mathrm{I}_4\ 2\ \mathrm{I}_2 - (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $2\ \mathrm{I}_4\ \mathrm{I}_2\ 2\ \mathrm{I}_1 - \mathbb{Z}/4\mathbb{Z}$, $\mathrm{I}_4\ 4\ \mathrm{I}_2 - (\mathbb{Z}/2\mathbb{Z})^2$, $4\ \mathrm{I}_3 - (\mathbb{Z}/3\mathbb{Z})^2$, and $6\ \mathrm{I}_2 - (\mathbb{Z}/2\mathbb{Z})^2$. Here the configurations of the singular fibers and the structures of $\mathrm{MW}_{\mathrm{tor}}$ appear before and after the horizontal bars, respectively. If all the matrix coefficients of $M - 1$ are divisible by $n \in \mathbb{Z}_{>1}$, then the same is true for any $\mathrm{SL}(2, \mathbb{Z})$-conjugate of $M - 1$. Therefore, if this holds for all monodromy matrices M, it follows that $\mathcal{M} \subset (n\mathbb{Z})^2$, and therefore the canonical projection $\mathbb{Z}^2 \to \mathbb{Z}^2/\mathcal{M}$ induces a surjective homomorphism $\mathbb{Z}^2/\mathcal{M} \to \mathbb{Z}^2/(n\mathbb{Z})^2 = (\mathbb{Z}/n\mathbb{Z})^2$. Because $(\mathbb{Z}/n\mathbb{Z})^2$ is not generated by one element, $\mathbb{Z}^2/\mathcal{M}$ is not generated by one element. It follows from Table 6.2.40, that in the first, third, fourth, fifth, seventh, and ninth cases this applies with $n = 2$. In the fifth case it follows that $\mathbb{Z}^2/\mathcal{M}$ is not isomorphic to $\mathbb{Z}/8\mathbb{Z}$, and because $\mathbb{Z}^2$, hence $\mathbb{Z}^2/\mathcal{M}$, is generated by two elements, $\mathbb{Z}^2/\mathcal{M}$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$; hence $\mathbb{Z}^2/\mathcal{M} \simeq (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{MW}_{\mathrm{tor}}$. In the other cases $\mathbb{Z}^2/\mathcal{M} \simeq (\mathbb{Z}/2\mathbb{Z})^2 \simeq \mathrm{MW}_{\mathrm{tor}}$. In the eighth case this argument with $n = 3$ yields $\mathbb{Z}^2/\mathcal{M} \simeq (\mathbb{Z}/3\mathbb{Z})^2 \simeq \mathrm{MW}_{\mathrm{tor}}$. In the second and sixth cases there

is a fiber of type I_1 with monodromy matrix $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, see Table 6.2.40, when $(M-1)(\mathbb{Z}^2) = \mathbb{Z} \times \{0\}$, and it follows that $\mathbb{Z}^2/\mathcal{M}$ is isomorphic to a quotient of $\mathbb{Z}$, hence to $\mathbb{Z}/m\,\mathbb{Z}$, where m is equal to the order of the group. Since this also holds for $\mathrm{MW}_{\mathrm{tor}}$, the groups are isomorphic. □

9.2.1 The Full Automorphism Group

The group $\mathrm{Aut}(S)^+_\kappa$ is a normal subgroup of the somewhat larger group $\mathrm{Aut}(S)_\kappa$ of all automorphisms of S that leave each fiber of κ invariant. In turn, $\mathrm{Aut}(S)_\kappa$ is a normal subgroup of the group $\mathrm{Aut}(S)$ of all automorphisms of the surface S. See Definition 6.3.25. We conclude this section with a number of facts about the full automorphism group of a rational elliptic surface.

Lemma 9.2.11 *Let $\kappa : S \to P$ be rational elliptic surface in the sense of Definition* 9.1.4*, and let $\alpha \in \mathrm{Aut}(S)$. Then the pushforward α_* of holomorphic exterior two-vector fields defines an invertible linear transformation of W, whose induced action on $P = \mathrm{P}(W)$ is an automorphism α_P of P. We have $\kappa \circ \alpha = \alpha_P \circ \kappa$, meaning that for every $p \in P$, α maps the fiber of κ over $p \in P$ onto the fiber of κ over $\alpha_P(p)$. Furthermore, if $J : P \to \mathbb{P}^1$ denotes the modulus function, then α_P leaves J invariant, in the sense that $J \circ \alpha_P = J$.*

The automorphism group $\mathrm{Aut}(S)$ of S is a complex Lie group, with Lie algebra $\mathfrak{aut}(S)$ equal to the finite-dimensional complex vector space of all holomorphic vector fields on S. The mapping $\alpha \mapsto \alpha_P$ is a homomorphism of Lie groups from $\mathrm{Aut}(S)$ to the group of automorphisms of P that leave the modulus function J and the configuration of singular values of κ, provided with the information of the types of the singular fibers, invariant. The kernel of the homomorphism $\alpha \mapsto \alpha_P$ is equal to $\mathrm{Aut}(S)_\kappa$, and is a discrete closed normal subgroup of $\mathrm{Aut}(S)$.

For any $\alpha \in \mathrm{Aut}(S)$, let α^L denote the action on the lattice $L := \mathrm{H}_2(S, \mathbb{Z})$ of α. Let $f \in L := \mathrm{H}_2(S, \mathbb{Z})$ denote the homology class of any fiber of the fibration $\kappa : S \to P$. Then the mapping $\alpha \mapsto \alpha^L$ is a homomorphism of Lie groups from $\mathrm{Aut}(S)$ to the discrete group $\mathrm{Aut}(L)$ of the automorphisms of the lattice L, and we have $\alpha^L(f) = f$ for every $\alpha \in \mathrm{Aut}(S)$.

The identity component $\mathrm{Aut}(S)^o$ of $\mathrm{Aut}(S)$ is contained in the kernel of the homomorphism $\alpha \mapsto \alpha^L$.

Proof. Let $w \in W$, $w \neq 0$, and $s \in S$. Then $\alpha_* w \in W$, and $(\alpha_* w)(\alpha(s)) = 0$ if and only if $w_{(s)} = 0$, which means that $\alpha_P([w]) = [\alpha_* w] = \kappa(\alpha(s))$ if and only if $[w] = \kappa((s))$, or $\alpha_P(\kappa(s)) = \kappa(\alpha(s))$. This proves the first statement in the lemma.

For every regular value p of κ in P, the restriction of $\alpha \in \mathrm{Aut}(S)$ to $S_p := \kappa^{-1}(\{p\})$ is an isomorphism from the fiber S_p onto the fiber $S_{\alpha_P(p)}$, which implies that $J(p) = J(\alpha_P(p))$, where $J(p) \in \mathbb{C}$ is the modulus of the elliptic curve S_p defined in (2.3.23). The function $J : P^{\mathrm{reg}} \to \mathbb{C}$ extends to a rational function on P, that is, a holomorphic mapping $J : P \to \mathbb{P}^1$ and $J = J \circ \alpha_P$ on P^{reg} implies $J = J \circ \alpha_P$ on P by continuity.

According to the theorem of Bochner and Montgomery [18, Theorem 1], the automorphism group $\mathrm{Aut}(M)$ of any compact complex analytic manifold M is a Lie group, with Lie algebra equal to the finite-dimensional complex vector space $\mathfrak{aut}(M)$ of all holomorphic vector fields on M. The closed normal Lie subgroup $\mathrm{Aut}(S)_\kappa$ of $\mathrm{Aut}(S)$ has the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ as a closed normal subgroup, and Lemma 6.3.26 implies that the quotient group $\mathrm{Aut}(S)_\kappa / \mathrm{Aut}(S)^+_\kappa \simeq \mathbb{Z}/m\,\mathbb{Z}$ is finite, where $m = 6$, 4, or 2 if respectively $J \equiv 0$, $J \equiv 1$, or otherwise. Because $\mathrm{Aut}(S)^+_\kappa$ is countable, see for instance Lemma 9.2.6, it follows that $\mathrm{Aut}(S)_\kappa$ is a countable, hence discrete, subgroup of $\mathrm{Aut}(S)$.

The fibers of κ are the members of a pencil. If $\alpha \in \mathrm{Aut}(S)$, then α maps fibers of κ to fibers of κ, and therefore each element of the pencil to an element of the pencil. Because the elements of a pencil are algebraically equivalent, hence homologous to each other, see Griffiths and Harris [74, p. 462], it follows that $\alpha^L(f) = f$, where f is the homology class of a fiber. Because $\mathrm{Aut}(L)$ is discrete, the continuous homomorphism of $\alpha \mapsto \alpha^L$ maps $\mathrm{Aut}(S)^o$ to $\{1\}$. □

Remark 9.2.12. For many rational elliptic surfaces $\kappa : S \to P$, the invariance under α_P of the modulus function J and the configuration of singular values of κ, provided with the information of the types of the singular fibers, implies that $\alpha_P = 1$. That is, for such rational elliptic surfaces we have $\mathrm{Aut}(S) = \mathrm{Aut}(S)_\kappa$.

For instance, in the generic case that all singular fibers are of type I_1, the configuration of the singular fibers corresponds to the twelve zeros in $\mathbb{P}^1$ of $\Delta := {g_2}^3 - 27\,{g_3}^2 \in \mathrm{H}_{12}(\mathbb{C}^2)$. Because the space of $(g_2,\, g_3) \in \mathrm{H}_4(\mathbb{C}^2) \times \mathrm{H}_6(\mathbb{C}^2)$ modulo the action $(g_2,\, g_3) \mapsto (\lambda^4\, g_2,\, \lambda^6\, g_3)$ of $\lambda \in \mathbb{C}^\times$ has dimension $(4+1) + (6+1) - 1 = 11$, the variety of zero-sets of Δ's has dimension ≤ 11. In the other direction, for us the interesting one, it contains an open subset such that deleting one of the points defines a nonempty open subset of the variety of all 11-tuples in $\mathbb{P}^1$. That generically $\mathrm{Aut}(S) = \mathrm{Aut}(S)_\kappa$ therefore follows from the following fact.

Let $\mathfrak{F}_n$ denote, for any $n \in \mathbb{Z}_{>0}$, the algebraic variety of all subsets $\mathcal{F}$ of $\mathbb{P}^1$ with n elements. On $\mathfrak{F}_n$ we have the action of the projective linear group $\mathrm{Aut}(\mathbb{P}^1)$, where $T \in \mathrm{Aut}(\mathbb{P}^1)$ maps the set $\mathcal{F}$ to its image set $T(\mathcal{F})$. If $n > 4$, then there exists an $\mathrm{Aut}(\mathbb{P}^1)$-invariant proper algebraic subvariety $\mathfrak{B}_n$ of $\mathfrak{F}_n$ of codimension one such that every $\mathcal{G} \in \mathfrak{F}_n \setminus \mathfrak{B}_n$ has the property that if $T \in \mathrm{Aut}(\mathbb{P}^1)$ and $T(\mathcal{G}) = \mathcal{G}$, then T is the identity.

Proof If $\mathcal{F} \in \mathfrak{F}_4$, then there is homogeneous polynomial f of degree four in two variables, unique up to a nonzero constant factor, whose zero-set in $\mathbb{P}^1$ is equal to $\mathcal{F}$. If $D(f)$ and $E(f)$ denote the Eisenstein invariants of f as in Section 2.3.5, then $F(f) := D(f)^3 - 27\,E(f)^2 \neq 0$, and the number $I(\mathcal{F}) := D(f)^3/F(f) \in \mathbb{C}$ does not depend on the choice of f, and is an absolute invariant in the sense that $I(T(\mathcal{F})) = I(\mathcal{F})$ for every $T \in \mathrm{Aut}(\mathbb{P}^1)$ and $\mathcal{F} \in \mathfrak{F}_4$.

If $n > 4$, let $\mathfrak{B}_n$ denote the set of all $\mathcal{G} \in \mathfrak{F}_n$ such that there exist two distinct subsets $\mathcal{F}$ and $\mathcal{F}'$ with four elements such that $I(\mathcal{F}) = I(\mathcal{F}')$. A straightforward calculation for $5 \leq n \leq 8$ by means of a formula manipulation computer program shows that $\mathfrak{B}_n$ is a proper algebraic subvariety of codimension one in $\mathfrak{F}_n$. If $a \in \mathcal{G} \in \mathfrak{F}_n \setminus \mathfrak{B}_n$, then there exists $\{a,\, b,\, c,\, d,\, e\} \in \mathfrak{F}_5$ such that $\{a,\, b,\, c,\, d,\, e\} \subset \mathcal{G}$, the

values of I lead to a determination of $\mathcal{F}_1 = \{a, b, c, d\}$ and $\mathcal{F}_2 = \{b, c, d, e\}$, and therefore to a determination of the complement $\{a\}$ of $\mathcal{F}_2$ in $\mathcal{F}_1$. It follows that if $T \in \mathrm{Aut}(\mathbb{P}^1)$ and $T(\mathcal{G}) = \mathcal{G}$, then $T(a) = a$ for every $a \in \mathcal{G}$; hence T is the identity because $\#(\mathcal{G}) = n \geq 3$. ⊘

Remark 9.2.13. For any nonzero holomorphic exterior two-vector field w on S and affine coordinate function $z : P \setminus \{[w]\} \to \mathbb{C}$ on $P \setminus \{[w]\}$, we have the holomorphic vector field (6.2.21) on $S \setminus \kappa^{-1}(\{[w]\})$ obtained by contracting w with $\mathrm{d}(z \circ \kappa)$. This is also the Hamiltonian vector field of the function $z \circ \kappa$ with respect to the complex Poisson structure w; see Remark 3.4.5.

This vector field is nonzero at every point of $S^{\mathrm{reg}} \setminus \kappa^{-1}(\{[w]\})$, and has a pole along the fiber $\kappa^{-1}(\{[w]\})$ over the point $[w] \in P = \mathrm{P}(W)$ of the elliptic fibration $\kappa : S \to P$. The vector field v corresponds to a meromorphic section v of the holomorphic line bundle $\mathfrak{f}$ over $P \simeq \mathbb{P}^1$ that has one simple pole at $[w]$ and no other poles or zeros, see Theorem 6.2.18. This reconfirms that $\deg(\mathfrak{f}) = -1$ for a rational elliptic surface, see Lemma 9.1.2(ii). ⊘

Remark 9.2.14. Lönne [122] proved that for a rational elliptic surface S, every automorphism of $\mathrm{H}_2(S, \mathbb{Z})$ that preserves the intersection form is induced by a diffeomorphism Φ of S, where S is viewed as a smooth real four-dimensional manifold. On the other hand, it follows from Lemma 9.2.11 that every complex analytic diffeomorphism α of S maps fibers to fibers, and therefore its action on $\mathrm{H}_2(S, \mathbb{Z})$ leaves f fixed. Because most automorphisms of $\mathrm{H}_2(S, \mathbb{Z})$ that preserve the intersection form do not leave f fixed, we have that usually the action of the real diffeomorphism Φ on $\mathrm{H}_2(S, \mathbb{Z})$ is not equal to the action on $\mathrm{H}_2(S, \mathbb{Z})$ of a complex analytic diffeomorphism α of S. ⊘

The homomorphism $\alpha \mapsto \alpha^L$ from $\mathrm{Aut}(S)$ to $\mathrm{Aut}(L)$ is called *the representation of* $\mathrm{Aut}(S)$ *in the homology group* $L = \mathrm{H}_2(S, \mathbb{Z})$. This representation is called faithful if the homomorphism $\alpha \mapsto \alpha^L$ is injective, or equivalently the action of $\mathrm{Aut}(S)$ is free. In turn this is equivalent to the statement that if $\alpha \in \mathrm{Aut}(S)$ acts trivially on L, then α is the identity on S. That is, any automorphism of S is determined by its action on L. The following lemma, which was suggested to me by Eduard Looijenga, is as close as I could get to a classification of the automorphisms of S that act trivially on L.

Lemma 9.2.15 *Let S be a rational elliptic surface, obtained by successivly blowing up the base points of the anticanonical pencils, starting with a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with at least one smooth member. Then $\alpha \in \mathrm{Aut}(S)$ acts trivially on $L := \mathrm{H}_2(S, \mathbb{Z})$ if and only if α is induced by a projective linear transformation A of $\mathbb{P}^2$ that leaves each of the base points of $\mathfrak{C}$ fixed. In the case of multiple base points, the latter condition has to be interpreted in the sense that the automorphism induced by A on each intermediate blown-up surface leaves each of the base points of the corresponding anticanonical pencil fixed.*

Proof. Every projective linear transformation A of $\mathbb{P}^2$ maps any projective line in $\mathbb{P}^2$ to a projective line in $\mathbb{P}^2$, and therefore ${A^S}_L(l) = l$ in the notation of the proof of

Lemma 9.2.3. If furthermore A leaves each of the base points of $\mathfrak{C}$ fixed then A^S leaves the total transforms on the -1 curves that appear at the blowing-up transformations fixed; hence $A^S{}_L(\epsilon_i) = \epsilon_i$ for every $1 \le i \le 9$. Because $L = \mathrm{H}_2(S, \mathbb{Z})$ is generated by l and the ϵ_i, $1 \le i \le 9$, it follows that A^S acts trivially on L.

Now let $\alpha \in \mathrm{Aut}(S)$ and $\alpha_L = 1$. Let E denote the -1 curve that appeared at the last blowing up. Because $\mathrm{H}(\alpha(E)) = \alpha_L(\mathrm{H}(E) = \mathrm{H}(E)$ and $E \cdot E = -1 < 0$, we have $\alpha(E) = E$ in view of Lemma 2.1.9. Let $\pi : S \to T$ denote the surface obtained by blowing down E, where $\pi(E) = \{t\}$ for a point $t \in T$. Then π is a complex analytic diffeomorphism from $S \setminus E$ onto $T \setminus \{t\}$, and $S \setminus E$ is invariant under α; hence

$$\beta : (\pi|_{S\setminus E}) \circ (\alpha|_{S\setminus E}) \circ (\pi|_{S\setminus E})^{-1}$$

is a complex analytic diffeomorphism from $T \setminus \{t\}$ onto itself, whereas the compactness of E and the continuity of α imply that $\beta(x) \to t$ when $x \to t$. It follows from Riemann's theorem on removable singularities of analytic functions that the extension of β to T that leaves the point t fixed is a complex analytic mapping from T to T, and because we have a similar conclusion for the map β^{-1} obtained from $\alpha^{-1} \in \mathrm{Aut}(S)$, we have that this extension is a complex analytic diffeomorphism of T, which we also denote by β.

We have $\pi \circ \alpha = \beta \circ \pi$, hence $\alpha^* \circ \pi^* = \pi^* \circ \beta^*$ on $\mathrm{H}^2(S', \mathbb{Z})$. Therefore, if $c \in \mathrm{H}^2(S', \mathbb{Z})$, we have

$$\pi^*(\beta^*(c)) = \alpha^*(\pi^*(c)) = \pi^*(c),$$

which implies that $\beta^*(c) = c$ because π^* is injective; see the text preceding (3.2.4).

Continuing the blowdowns, we arrive at the conclusion that α is induced by an element A of $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(\mathbb{C}^3)$, see Lemma 2.2.1, which fixes each of the base points of $\mathfrak{C}$ in the sense of Lemma 9.2.15. □

Remark 9.2.16. In many cases the condition that the projective linear transformation A in Lemma 9.2.15 leaves each of the base points of $\mathfrak{C}$ fixed implies that $A = 1$, hence $\alpha = 1$, leading to the conclusion that the action of $\mathrm{Aut}(S)$ on $\mathrm{H}_2(S, \mathbb{Z})$ is free. This happens for instance if $\mathfrak{C}$ has at least four base points, no three of which are on a projective line in $\mathbb{P}^2$.

An example in which even the action of $\mathrm{Aut}(S)_\kappa$ on $\mathrm{H}_2(S, \mathbb{Z})$ is not free is provided by the rational elliptic surface defined by the pencil (4.5.1) in Lemma 4.5.1. Here the nontrivial projective linear transformation $A : [x_0 : x_1 : x_2] \mapsto [-x_0 : x_1 : x_2]$ leaves both q^0 and q^1 unchanged, hence $A^S \in \mathrm{Aut}(S)_\kappa$. It therefore leaves the base points of the pencil fixed in the sense of Lemma 9.2.15; hence A^S acts trivially on $\mathrm{H}_2(S, \mathbb{Z})$. In the case of the pencil (4.5.1) in Lemma 4.5.1, also each projective linear transformation $A : [x_0 : x_1 : x_2] \mapsto [x_0 : \omega x_1 : x_2]$, where ω is a third root of unity, leaves q^0 and q^1 unchanged, and therefore $A^S \in \mathrm{Aut}(S)_\kappa$ acts trivially on $\mathrm{H}_2(S, \mathbb{Z})$. ⊘

Example 7. In Willox, Grammaticos, and Ramani [211], an *antisymmetric QRT map* is defined as a QRT map for which A_0 is an antisymmetric matrix and A_1 is a symmetric matrix. Then the switch $\sigma : (x, y) \mapsto (y, x)$ defines an automorphism

σ_S of S such that $\sigma_P = \pi_P(\sigma_S)$ is equal to the automorphism of $P = \mathrm{P}(V)$, $V = \mathbb{C}\,A_0 + \mathbb{C}\,A_1$, that sends $[z_0\,A_0 + z_1\,A_1]$ to $[-z_0\,A_0 + z_1\,A_1]$. That is, the affine coordinate $c = -z_1/z_0$ is mapped to its opposite $-c$. Therefore in this case $\pi_P(\mathrm{Aut}(S))$ is a nontrivial subgroup of $\mathrm{Aut}(P)_J$, containing at least the switch $[z_0\,A_0 + z_1\,A_1] \mapsto [-z_0\,A_0 + z_1\,A_1]$. ⊘

9.2.2 Rational Elliptic Surfaces with nondiscrete Automorphism Groups

In the following proposition we classify the rational elliptic surfaces with a nondiscrete automorphism group, a complex Lie group of positive dimension.

Proposition 9.2.17 *Let $\kappa : S \to P$ be a rational elliptic surface, and let $\mathrm{Aut}(S)$ denote the group of all complex analytic diffeomorphisms from S onto S. Then the following conditions are equivalent;*

(a) *The quotient group $\mathrm{Aut}(S)/\mathrm{Aut}(S)^+_\kappa$ has more than $6 \cdot 12!$ elements.*
(b) *The fibration $\kappa : S \to P$ has at most two singular fibers.*
(c) *S is isomorphic to the successive blowing up, nine times, at the base points of the anticanonical pencils, starting with one of the following pencils of cubic curves in $\mathbb{P}^2$:*

$$z_0\,({x_1}^3 + x_2\,{x_0}^2) + z_1\,{x_2}^3 = 0, \tag{9.2.3}$$
$$z_0\,(x_0\,x_2 - {x_1}^2)\,x_0 + z_1\,{x_2}^3 = 0, \tag{9.2.4}$$
$$z_0\,x_1\,x_2\,(x_1 - x_2) + z_1\,{x_0}^3 = 0, \tag{9.2.5}$$
$$z_0\,x_1\,x_2\,(x_1 - x_2) + z_1\,(x_1 - a\,x_2)\,{x_0}^2 = 0, \tag{9.2.6}$$

where in (9.2.6) *the complex number a is a parameter with $a \neq 0$ and $a \neq 1$.*

For (9.2.3)*, the identity component $\mathrm{Aut}(S)^\circ$ of $\mathrm{Aut}(S)$ is the group of all automorphisms of S that are induced by the projective linear transformations $[x_0 : x_1 : x_2] \mapsto [c\,x_0 : x_1 : c^{-2}\,x_2]$, with $c \in \mathbb{C}^\times$; hence $\mathrm{Aut}(S)^\circ \simeq \mathbb{C}^\times$. The configuration of the singular fibers is* II* II*, the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is trivial, and $J \equiv 0$.*

For (9.2.4)*, $\mathrm{Aut}(S)^\circ \simeq \mathbb{C}^\times$ is the group of all automorphisms of S that are induced by the projective linear transformations $[x_0 : x_1 : x_2] \mapsto [c\,x_0 : x_1 : c^{-1}\,x_2]$, with $c \in \mathbb{C}^\times$. The configuration of the singular fibers is* III* III*, the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and $J \equiv 1$.*

For (9.2.5)*, $\mathrm{Aut}(S)^\circ \simeq \mathbb{C}^\times$ is the group of all automorphisms of S that are induced by the projective linear transformations $[x_0 : x_1 : x_2] \mapsto [c\,x_0 : x_1 : x_2]$, with $c \in \mathbb{C}^\times$. The configuration of the singular fibers is* IV* IV*, the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and $J \equiv 0$.*

For (9.2.6), $\mathrm{Aut}(S)^\circ \simeq \mathbb{C}^\times$ is the group of all automorphisms of S that are induced by the projective linear transformations $[x_0 : x_1 : x_2] \mapsto [c\,x_0 : x_1 : x_2]$, with

$c \in \mathbb{C}^\times$. The configuration of the singular fibers is $2\ \mathrm{I}_0^*$*, the Mordell–Weil group* $\simeq \mathrm{Aut}(S)_\kappa^+$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$*, and* $J \equiv 4\,(1-a+a^2)^3/27\,(-1+a)^2\,a^2$*. Two elliptic surfaces defined by pencils (9.2.6) are isomorphic if and only if they have the same constant modulus* J*.*

Proof. (a) $\Rightarrow$ (b). Let F denote the set of singular values of κ, where $0 < \#(F) \leq 12$ in view of Lemma 9.2.2 and the fact that the Euler number of each singular fiber is strictly positive. Assume that $\#(F) \geq 3$. If $\alpha \in \mathrm{Aut}(S)$, then it follows from Lemma 9.2.11 that $\alpha_P(F) = F$. Furthermore, if the automorphism α_P of P leaves F pointwise fixed, then $\alpha_P = 1$, that is, $\alpha \in \mathrm{Aut}(S)_\kappa$. Therefore $\alpha \mapsto \alpha_P|_F$ is a homomorphism from $\mathrm{Aut}(S)$ to the permutation group $\mathrm{S}(F)$ of F with kernel equal to $\mathrm{Aut}(S)_\kappa$, and therefore $\#(\mathrm{Aut}(S)/\mathrm{Aut}(S)_\kappa) \leq \#(\mathrm{S}(F)) \leq 12!$. On the other hand, Lemma 6.3.26 implies that $\mathrm{Aut}(S)_\kappa/\mathrm{Aut}(S)_\kappa^+ \simeq \mathbb{Z}/m\,\mathbb{Z}$, where $m = 6$, 4, or 2, and therefore $\#(\mathrm{Aut}(S)/\mathrm{Aut}(S)_\kappa^+) = \#(\mathrm{Aut}(S)_\kappa/\mathrm{Aut}(S)_\kappa^+) \cdot \#(\mathrm{Aut}(S)/\mathrm{Aut}(S)_\kappa) \leq 6 \cdot 12!$.

(b) $\Rightarrow$ (c) It follows from the proof of Lemma 6.4.2 that the product of the monodromy matrices $M(\delta_j)$ around the singular values of κ is equal to the identity matrix, where for this fact the assumption that the modulus function J is not constant does not play a role. Table 6.2.40 shows that the monodromy matrix around every singular value is not the identity matrix, which implies that there are at least two, hence exactly two, singular values. Furthermore, no element in $\mathrm{SL}(2,\ \mathbb{Z})$ is conjugate to the inverse of the monodromy matrix of a singular fiber of type I_b, $b > 0$, or I_b^*, $b > 0$. This leaves the configurations of the singular fibers II* II, III* III, IV* IV, and $2\ \mathrm{I}_0^*$ as the only possibilities. If we apply Aronhold's recipes in Section 4.4 to compute the coefficients $g_2 = 27\,S$ and $g_3 = -27\,T$ of the Weierstrass normal form in Proposition 4.4.3, then we arrive at

$g_2 = 0$, $g_3 = 4\,{z_0}^5\,z_1$, $\Delta = -432\,{z_0}^{10}\,{z_1}^2$, and $J = 0$ for the pencil (9.2.3),

$g_2 = -4\,{z_0}^3\,z_1$, $g_3 = 0$, $\Delta = -64\,{z_0}^9\,{z_1}^3$, and $J = 1$ for the pencil (9.2.4),

$g_2 = 0$, $g_3 = -{z_0}^4\,{z_1}^2$, $\Delta = -27\,{z_0}^8\,{z_1}^4$, and $J = 0$ for the pencil (9.2.5), and finally

$g_2 = (4/3)\,(1-a+a^2)\,{z_0}^2\,{z_1}^2$, $g_3 = -(4/27)\,(-2+a)\,(1+a)\,(-1+2\,a)\,{z_0}^3\,{z_1}^3$, $\Delta = 16\,(-1+a)^2\,a^2\,{z_0}^6\,{z_1}^6$, and $J = 4\,(1-a+a^2)^3/27\,(-1+a)^2\,a^2$ for the pencil (9.2.6).

In view of Table 6.3.2 and Lemma 9.2.6 this leads to the configurations of the singular fibers, the Mordell–Weil groups, and the values of the modulus as mentioned in (9.2.3)–(9.2.6). Under the homomomorphism $\alpha \to \alpha_P$ the symmetry $c \in \mathbb{C}^\times$ sends $[z_0 : z_1]$ to $[z_0 : c^{-6}\,z_1]$ for (9.2.3), $[c\,z_0 : c^{-3}\,z_1]$ for (9.2.4), $[z_0 : c^3\,z_1]$ for (9.2.5), and $[z_0 : c^2\,z_1]$ for (9.2.6). Since the kernel of $\alpha \mapsto \alpha_P$ is discrete, see Lemma 9.2.11, it follows that $\mathrm{Aut}(S)$ is a complex one-dimensional complex Lie group, with identity component isomorphic to $\mathbb{C}^\times$.

That any rational elliptic surface with at most two singular fibers is isomorphic to the elliptic surface defined by one of the pencils (9.2.3)–(9.2.6) follows from the uniqueness statement in Theorem 6.4.11. Finally, (c) $\Rightarrow$ (a) follows because in (c) we have $\mathrm{Aut}(S)^\circ \simeq \mathbb{C}^\times$, whereas the Mordell–Weil group $\mathrm{Aut}(S)_\kappa^+$ is discrete. □

We have met (9.2.3) before in (4.5.2), the second "Weierstrass pencil." The II^* fiber corresponds to a triple projective line in $\mathbb{P}^2$, and the II fiber to a cubic curve with a contact of order three with the line and a cusp not on the line. I found the pencils of cubic curves in (9.2.4), (9.2.5), and (9.2.6) by applying the blowing down procedure in the proof of (c) $\Rightarrow$ (a) in Theorem 9.1.3. The III^* fiber became a triple line and the III fiber became the union of a smooth quadric and a line tangent to it, where the triple line is tangent to the quadric at another point. The IV^* fiber became a triple line and the IV fiber became the union of three distinct lines passing through a common point not on the triple line. One of the I_0^* fibers became the union of a double line and another line and the second I_0^* fiber became the union of three lines, where all multiplicity one lines pass through a common point not on the double line. Using projective linear transformations, these configurations then have been put into the standard forms (9.2.4)–(9.2.6).

Remark 9.2.18. In all cases in Proposition 9.2.17 the modulus function J is constant. The converse, that $\mathrm{Aut}(S)$ is not discrete if J is constant, is not true, since there are nine configurations of singular fibers for which the modulus function is constant and that are not equal to the four configurations of the singular fibers mentioned in Proposition 9.2.17. See Remark 9.2.8.

If m denotes the maximum of $\#(\mathrm{Aut}(S)/\mathrm{Aut}(S)_\kappa^+)$ over all rational elliptic surfaces S with at least three singular fibers, then Proposition 9.2.17 implies that $m \leq 6 \cdot 12!$. I have not tried to determine m, but it should be a lot smaller than $6 \cdot 12!$. ⊘

9.2.3 Rational Elliptic Surfaces with Finite Mordell–Weil Groups

I learned the following proposition from Miranda and Persson [138].

Proposition 9.2.19 *For a rational elliptic surface $\kappa : S \to \mathbb{P}^1$, the following conditions are equivalent:*

(i) *The rank of the Mordell–Weil group $\mathrm{Aut}(S)_\kappa^+$ is equal to zero.*
(ii) *The Mordell–Weil group is finite.*
(iii) *The group $\mathrm{Aut}(S)_\kappa$ of all automorphisms of S that preserve the fibers of κ is finite.*
(iv) *There are only finitely many -1 curves in S.*
(v) *There are only finitely many irreducible curves C in S such that $C \cdot C < 0$.*
(vi) *There are only finitely many curves C in S for which there exists a modification $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ in a finite subset B of $\mathbb{P}^2$ such that $C = \pi^{-1}(B)$.*
(vii) *The configuration of the singular fibers is one of those in Table 9.2.20, where the second and third columns after the type contains the structure of the Mordell–Weil group and the (sub)section in which the configuration is discussed elsewhere in this book.*

In each case the rational elliptic surface is uniquely determined up to isomorphism by the configuration of singular fibers, except for $2\ \mathrm{I}_0^*$, where for each $J \in \mathbb{C}$ there is

a unique isomorphism class in which all the nonsingular fibers have modulus equal to J.

Table 9.2.20

Type	Group structure	Section	Type	Group structure	Section
II^* II	$\{0\}$	9.2.2	IV^* I_3 I_1	$\mathbb{Z}/3\,\mathbb{Z}$	
II^* 2 I_1	$\{0\}$	4.5	I_2^* 2 I_2	$(\mathbb{Z}/2\,\mathbb{Z})^2$	
I_4^* 2 I_1	$\mathbb{Z}/2\,\mathbb{Z}$		I_1^* I_4 I_1	$\mathbb{Z}/4\,\mathbb{Z}$	5.2.3
I_9 3 I_1	$\mathbb{Z}/3\,\mathbb{Z}$	4.5	I_6 I_3 I_2 I_1	$\mathbb{Z}/6\,\mathbb{Z}$	5.2.5
III^* III	$\mathbb{Z}/2\,\mathbb{Z}$	9.2.2	2 I_0^*	$(\mathbb{Z}/2\,\mathbb{Z})^2$	9.2.2
III^* I_2 I_1	$\mathbb{Z}/2\,\mathbb{Z}$		2 I_5 2 I_1	$\mathbb{Z}/5\,\mathbb{Z}$	5.2.4
I_8 I_2 2 I_1	$\mathbb{Z}/4\,\mathbb{Z}$	5.2.3	2 I_4 2 I_2	$(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$	5.2.3
IV^* IV	$\mathbb{Z}/3\,\mathbb{Z}$	9.2.2	4 I_3	$(\mathbb{Z}/3\,\mathbb{Z})^2$	11.1.3

Proof. Condition (i) holds if and only if rank $Q^{\mathrm{irr}} = 8$ if and only if Q^{irr} is one of the thirteen root sublattices of Q of type E_8, A_8, D_8, $\mathrm{A}_7 + \mathrm{A}_1$, $\mathrm{A}_5 + \mathrm{A}_2 + \mathrm{A}_1$, 2 A_4, 4 A_2, $\mathrm{E}_6 + \mathrm{A}_2$, $\mathrm{A}_7 + \mathrm{A}_1$, $\mathrm{D}_6 + 2\ \mathrm{A}_1$, $\mathrm{D}_5 + \mathrm{A}_3$, 2 D_4, or 2 $\mathrm{A}_3 + 2\ \mathrm{A}_1$, obtained from E_8 by elementary transformations as mentioned in one of the paragraphs preceding Lemma 9.2.6, where the cases $\mathrm{D}_4 + 4\ \mathrm{A}_1$ and 8 A_1 have been deleted because of condition (ii) in Lemma 9.2.7. There are 23 corresponding configurations of singular fibers that satisfy condition (ii) in Lemma 9.2.7, seven of which do not satisfy condition (iii) in Lemma 9.2.7. The remaining 16 satisfy all conditions in Lemma 9.2.7. These are the ones in Table 9.2.20, where the structure of the Mordell–Weil group follows from Lemma 9.2.6.

The statement about the isomorphism class of the rational elliptic surfaces with the prescribed configuration of singular fibers follows for II^* II, III^* III, IV^* IV, and 2 I_0^* from Proposition 9.2.17. For II^* 2 I_1, I_9 3 I_1, I_6 I_3 I_2 I_1, 2 I_5 2 I_1, and 4 I_3 it follows from Corollary 4.5.6, the proof of Lemma 4.5.3, Section 5.2.5, Section 5.2.4, and Proposition 11.1.6, respectively. For I_8 I_2 2 I_1, I_1^* I_4 I_1, and 2 I_4 2 I_2 the Mordell–Weil group has an element of order four, when the description of the Weierstrass models in Section 5.2.3 implies that the rational elliptic surface exists and is unique up to isomorphism. The Weierstrass model is given by $g_2 = (D^4 - 12\,D^2\,X + 24\,X^2)/2$, $g_3 = (D^2 - 8\,X)\,(-D^4 + 8\,D^2\,X + 16\,X^2)/16$, and $\Delta = (D^2 - 12\,X)^4\,D^2\,(5\,D^2 - 48\,X)/256$, where D and X are homogeneous polynomials in $(z_0,\ z_1)$ of the respective degrees one and two. The configuration of the singular fibers is I_8 I_2 2 I_1 if and only if the discriminant of $D^2 - 12\,X$ is equal to zero, where the resultant of D and X and the discriminant of $5\,D^2 - 48\,X$ are nonzero. By means of a projective linear transformation it can be arranged that $D = z_1$ and $D^2 - 12\,X = {z_0}^2$, hence $X = ({z_1}^2 - z_0)/12$, when indeed the resultant of D and X and the discriminant of $5\,D^2 - 48\,X$ are nonzero. The proof for the configurations I_1^* I_4 I_1 and 2 I_4 2 I_2 is analogous.

In the remaining cases I_4^* 2 I_1, III^* I_2 I_1, IV^* I_3 I_1, and I_2^* 2 I_2 there are three singular fibers. For every relatively minimal elliptic fibration $\varphi : S \to \mathbb{P}^1$ with a section and at most three singular fibers, Schmickler-Hirzebruch [176, §5] proved that given the configuration of the singular fibers, S is uniquely determined up to

isomorphisms, except when the configuration of singular fibers is empty or $2\ \mathrm{I}_0^*$. Putting the three subsequent singular fibers over $[z_0 : z_1] = [0 : 1]$, $[1 : 0]$, and $[1 : 1]$, we have the Weierstrass model $g_2 = 3\,{z_0}^2\,({z_0}^2 - 16\,z_0\,z_1 + 16\,{z_1}^2)$, $g_3 = {z_0}^3\,(2\,z_1 - z_0)\,({z_0}^2 + 32\,z_0\,z_1 - 32\,{z_1}^2)$, and $\Delta = 2^2\,3^6\,{z_0}^{10}\,z_1\,(z_1 - z_0)$ for $\mathrm{I}_4^*\ 2\,\mathrm{I}_1$, $g_2 = 3\,{z_0}^3\,(4\,z_0 - 3\,z_1)/4$, $g_3 = {z_0}^5\,(8\,z_0 - 9\,z_1)/8$, and $\Delta = 3^6\,{z_0}^9\,{z_1}^2\,(z_0 - z_1)/2^6$ for $\mathrm{III}^*\ \mathrm{I}_2\ \mathrm{I}_1$, $g_2 = {z_0}^3\,(9\,z_0 - 8\,z_1)/3$, $g_3 = {z_0}^4\,(27\,{z_0}^2 - 36\,z_0\,z_1 + 8\,{z_1}^2)/27$, and $\Delta = 64\,{z_0}^8\,{z_1}^3\,(z_0 - z_1)/27$ for $\mathrm{IV}^*\ \mathrm{I}_3\ \mathrm{I}_1$, and $g_2 = 3\,{z_0}^2\,({z_0}^2 - z_0\,z_1 + {z_1}^2)$, $g_3 = {z_0}^3\,(z_0 - 2\,z_1)\,(2\,z_0 - z_1)\,(z_0 + z_1)/2$, and $\Delta = 3^6\,{z_0}^8\,{z_1}^2\,(z_0 - z_1)^2/4$ for $\mathrm{I}_2^*\ 2\ \mathrm{I}_2$. I obtained these from Schmickler-Hirzebruch [176, pp. 121, 122]. It is also not too hard to deduce these from Table 6.3.2, in combination with the special form of the Weierstrass model in Section 5.2.1 and 5.2.2 for which the Mordell–Weil group has an element of order two and three, respectively. This completes the proof of (i) $\Rightarrow$ (vii) $\Rightarrow$ (ii) $\Rightarrow$ (i), together with the uniqueness statement.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 6.3.26. The equivalence (ii) $\Leftrightarrow$ (iv) follows from Lemma 7.1.1 in combination with Lemma 9.2.1. It follows from Theorem 9.1.3(d) and Corollary 3.3.9 that the curve C in (v) is either a -1 curve or an irreducible component of a reducible fiber. Because there are only finitely many irreducible components of reducible fibers, this proves (iv) $\Leftrightarrow$ (v). The implication (v) $\Rightarrow$ (vi) follows from Lemma 6.2.49, whereas (vi) $\Rightarrow$ (iv) follows from the fact that in the blowing down from S to $\mathbb{P}^2$ we can start by blowing down any of the -1 curves in S; see the proof of (c) $\Rightarrow$ (a) in Theorem 9.1.3. □

Miranda and Persson [138] called a rational elliptic surface *extremal* if it satisfies one of the equivalent conditions in Proposition 9.2.19. Note that regarding the rank of the Mordell–Weil group, the extremal rational elliptic surfaces are at the other extreme of the Manin elliptic surfaces; see (i) in Proposition 9.2.22. Table 9.2.20 functioned as a predecessor of Perrson's list of all 289 configurations of singular fibers of rational elliptic surfaces.

Naruki [149] called a rational elliptic surface *maximal* if the Mordell–Weil group is finite, it has no singular fibers of type II, III, or IV, and the modulus function is not constant. This amounts to deleting the rational elliptic surfaces with the configurations of singular fibers $\mathrm{II}^*\ \mathrm{II}$, $\mathrm{III}^*\ \mathrm{III}$, $\mathrm{IV}^*\ \mathrm{IV}$, and $2\ \mathrm{I}_0^*$ from Table 9.2.20, the rational elliptic surfaces with a nondiscrete automorphism group as described in Proposition 9.2.17. For each of the remaining twelve cases, Naruki gave a pencil of cubic curves in $\mathbb{P}^2$ that defines an elliptic surface with a corresponding configuration of singular fibers, without telling how he found these pencils.

In [149, Introduction], Naruki wrote "…while the moduli space of rational elliptic surfaces has eight parameters. Thus we have to impose our maximality assumption on our elliptic surface to make it rigid; this roughly amounts to assuming that the surface has only finitely many global sections." In [149] I could not find the explicit statement that the moduli space of isomorphism classes of maximal rational elliptic surfaces is discrete, as suggested in [149, Introduction]. Note that the uniqueness statement in Morrison and Persson's Proposition 9.2.19 is stronger.

9.2.4 Manin Elliptic Surfaces

For any elliptic fibration $\varphi : S \to C$ with singular fibers, a quantity of information has been collected in Chapter 7 about the Mordell–Weil group of S, isomorphic to the group $\mathrm{Aut}(S)^+_\varphi$ of all complex analytic diffeomorphisms of S that act on each smooth, hence elliptic, fiber as a translation. All these results apply to a rational elliptic surface $\kappa : S \to P$, where $\varphi = \kappa$, $C = P \simeq \mathbb{P}^1$, $\mathrm{NS}(S) \simeq \mathrm{H}_2(S, \mathbb{Z})$, and $\chi(S, \mathcal{O}) = 1$; see Theorem 9.2.4. For characterizations of the Manin QRT automorphisms in $\mathrm{Aut}(S)^+_\kappa$, see Sections 4.3 and 4.5.

Definition 9.2.21. Let L be a holomorphic line bundle over a compact complex analytic manifold M and let E denote the space of all holomorphic sections of L over M, where E is a finite-dimensional vector space; see Section 6.2.8. The line bundle L is called *very ample* if $\kappa : x \mapsto \{e \in E \mid e(x) = 0\}$ is an embedding from X into the space of all codimension-one linear subspaces of E, which is isomorphic to $\mathbb{P}^n$ if $\dim E = n + 1$. The line bundle L is called *ample* if there exists a strictly positive integer k such that L^k is very ample. The *criterion of Grauert* states that L is ample if and only if for every irreducible compact complex analytic subset A of X with $\dim A > 0$ there exist $k \in \mathbb{Z}_{>0}$ and a nonzero holomorphic section of $L^k|_A$ with at least one zero on A. See Barth, Hulek, Peters, and van de Ven [11, pp. 56, 57] and Grauert [70, p. 347].

A (complex) *del Pezzo surface* is a compact complex analytic surface S such that the anticanonical line bundle K_S^* of S is ample. The number $d = \mathrm{K}_S \cdot \mathrm{K}_S = \mathrm{K}_S^* \cdot \mathrm{K}_S^*$ is called the *degree* of the del Pezzo surface. Because a suitable power of K_S^* is very ample, S is a complex projective algebraic surface, and therefore the definition of del Pezzo surfaces is equivalent to the one in Iskovskikh and Shafarevich [94, p. 232]. In [94, pp. 232, 233] it is established that S is rational, and therefore the definition in turn agrees with Manin's original definition in [130, p. 105], which generalizes several classes of projective algebraic surfaces studied by del Pezzo. The definitions in [94, p. 232] and [130, p. 105] are for general fields, not only $\mathbb{C}$.

Manin [131, Theorem 24.3–5] proved the following facts. We have $1 \le d \le 9$. If $d = 9$ then $S \simeq \mathbb{P}^2$. If $d = 8$ then S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup Σ_1 of $\mathbb{P}^2$ at one point. If $1 \le d \le 7$ then S is isomorphic to the blowup of $\mathbb{P}^2$ at $9 - d$ distinct points, with no three points on a projective line and no six points on a conic. If $3 \le d \le 7$ then every such blowup of $\mathbb{P}^2$ is a del Pezzo surface. The dimension of the space of all holomorphic sections of K_S^*, the holomorphic two-vector fields on S, is equal to $d + 1$. It follows that if $d = 1$, then the anticanonical system of S defines a pencil of curves in S with a unique base point, of multiplicity one.

Manin [131, Theorem 26.2] proved that the number of -1 curves on a del Pezzo surface S of degree d is equal to 240, 56, 27, 16, 10, or 6, as d runs from 1 up to 6. The computation is based on the determination of the lattice structure of the Néron–Severi groups of the del Pezzo surfaces in [131, Section 25]. If $d = 7$ we have $S \simeq \Sigma_{01}$, when there are three -1 curves in S; see Remark 9.1.5. If $d = 8$ there are no -1 curves if $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and there is exactly one if $S \simeq \Sigma_1$. If $d = 9$ then $S \simeq \mathbb{P}^2$ and there are no -1 curves. ⊘

Proposition 9.2.22 *Let $\kappa : S \to P$ be a rational elliptic surface, obtained by successively blowing up $\mathbb{P}^2$ at base points of the anticanonical pencils, starting with a pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ with at least one smooth member. Then the following conditions are equivalent:*

(i) *The rank of the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is equal to* 8.
(ii) *All fibers of κ are irreducible.*
(iii) *Every, and equivalently some, pencil $\mathfrak{C}$ leading to S has nine distinct base points and all members of $\mathfrak{C}$ are irreducible.*
(iv) *Every, and equivalently some, pencil $\mathfrak{C}$ leading to S has nine distinct base points and no three base points are on a projective line in $\mathbb{P}^2$.*
(v) *There exist a del Pezzo surface T of degree one and a blowing up $\pi : S \to T$ of T at the unique base point of the pencil of anticanonical curves on T.*
(vi) *For any modification $\pi : S \to T$ of a smooth surface T that is not an isomorphism, T is a del Pezzo surface, of degree equal to the number of blowups.*
(vii) *There are* 240 *Manin QRT automorphisms of S.*
(viii) *For each -1 curve E_0 in S, there are* 240 -1 *curves E in S such that $E \cap E_0 = \emptyset$.*

Proof. The equivalence between (i) and (ii) follows from the formula for the rank of the Mordell–Weil group in Lemma 9.2.6.

The first sentence in the proof of Theorem 4.3.4 shows that (iii) $\Rightarrow$ (ii). The converse follows from the fact that both a reducible member of $\mathfrak{C}$ and a member of $\mathfrak{C}$ that is singular at a multiple base point of $\mathfrak{C}$ give rise to a reducible fiber of κ.

For the proof of (iii) $\Leftrightarrow$ (iv) we may assume that $\mathfrak{C}$ has nine distinct base points.

(iv) $\Rightarrow$ (iii). If the member $D = \mathrm{Div}(q)$ of $\mathfrak{C}$ is reducible, then the cubic polynomial q has a linear factor; hence D contains a projective line L. The pencil $\mathfrak{C}$ has a smooth, hence irreducible, member C, which therefore does not contain L. Because C is a cubic curve in $\mathbb{P}^2$, it intersects L in points p_i with multiplicities m_i, where $\sum_i m_i = 3$. These points p_i are intersection points of C with D of multiplicities $\mu_i \geq m_i$, hence are base points of $\mathfrak{C}$ with multiplicities μ_i. Because all base points of $\mathfrak{C}$ have multiplicity one, we have $1 \leq m_i \leq \mu_i = 1$, hence $m_i = 1$, which in combination with $\sum_i m_i = 3$ means that the p_i are three distinct base points of $\mathfrak{C}$, lying on the projective line L.

(iii) $\Rightarrow$ (iv). Assume that p_i are three distinct base points of $\mathfrak{C}$, lying on a projective line L. In an affine coordinate chart containing the points p_i, we have the linear map $q \mapsto q|_L$, from the two-dimensional vector space Q of cubic forms defining the pencil $\mathfrak{C}$ to the space of polynomials of degree ≤ 3 on L. Because the p_i are base points of $\mathfrak{C}$, we have $q(p_i) = 0$ for every $q \in Q$. Because the space of all polynomials of degree ≤ 3 in one variable that vanish at three given points is one-dimensional, it follows that the image of Q under the linear mapping $q \mapsto q|_L$ is at most one-dimensional, which implies that there exists $q \in Q$ such that $q \neq 0$ and $q|_L = 0$. This means that L is contained in $D = \mathrm{Div}(q)$. The curve D is a reducible member of $\mathfrak{C}$.

(v) $\Rightarrow$ (ii). Assume that (v) holds, and denote by W the two-dimensional space of all holomorphic two-vector fields on T. Because K_T^* is ample, Grauert's criterion

yields a positive integer k such that $0 < (\mathrm{K}_T^*)^k \cdot A = k\, \mathrm{K}_T^* \cdot A$; hence $Z_{[w]} \cdot A = \mathrm{K}_T^* \cdot A > 0$ for every zero-set $Z_{[w]}$ of $w \in W \setminus \{0\}$. If $Z_{[w]}$ has the irreducible components Θ_j, then $1 = Z_{[w]} \cdot Z_{[w]} = \sum_j Z_{[w]} \cdot \Theta_j$, where each of the summands is stricitly positive, and it follows that $Z_{[w]}$ is irreducible. If $Z_{[w]} = Z_{[w']}$, then $w' = 0$ on the zero-set of w, and it follows that w'/w is a holomorphic function on T, hence a constant in view of the maximum principle. Therefore $Z_{[w]} \neq Z_{[w']}$ if $[w] \neq [w']$, when $Z_{[w]} \cdot Z_{[w']} = Z_{[w]} \cdot Z_{[w]} = 1$ imply that $Z_{[w]}$ and $Z_{[w']}$ have a single point of intersection, at which both curves are smooth and the intersection is transversal. We recover that the anticanonical pencil on T has a single base point b that has multiplicity one. Blowing up T at b we obtain a surface S that satisfies (d) in Theorem 9.1.3, and therefore is a rational elliptic surface. Because the anticanonical curves in S are the proper images of the anticanonical curves in T, hence isomorphic to these, see Lemma 3.2.6, and the anticanonical curves in T are irreducible, we have proved (ii).

(ii) $\Rightarrow$ (vi). Assume that (ii) holds and that $\pi : S \to T$ is a blowing up of T at a point $b \in T$. According to Proposition 3.3.7, the pushforward of holomorphic two-vector fields defines a linear isomorphism π_* from the space W of all holomorphic two-vector fields on S onto $\{w_\sim \in W_\sim \mid w_\sim(b) = 0\}$, where $W_\sim$ denotes the space of all holomorphic two-vector fields on T. It follows that $\dim \pi_*(W) \geq \dim W_\sim - 1$, that is, $W_\sim = \pi_*(W)$ or $W_\sim$ is a three-dimensional vector space containing $\pi_*(W)$ as a two-dimensional linear subspace. For each $w \in W \setminus \{0\}$ the fiber $Z_{[w]}$ of the elliptic fibration $\kappa : S \to \mathrm{P}(W)$ is equal to the proper transform of the zero-set $Z_{[\pi_* w]}$ of $\pi_* w$ in T. Because $Z_{[w]}$ is irreducible and $Z_{[\pi_* w]} \simeq Z_{[w]}$, see Lemma 3.2.6, the curve $Z_{[\pi_* w]}$ in T is irreducible. It follows that the pencil of curves in T defined by $\pi_*(W)$ has b as its single base point. Because the -1 curve $\pi^{-1}(\{b\})$ is a section of $\kappa : S \to \mathrm{P}(W)$, see Lemma 9.2.1, the multiplicity of the base point is equal to one; hence $\mathrm{K}_T^* \cdot \mathrm{K}_T^* = 1$.

Assume that $\dim W_\sim = 3$. Then for each $t \in T$ the space $\kappa(t) = \{w_\sim \mid w_\sim(t) = 0\}$ is a linear subspace of codimension ≤ 1 of $W_\sim$. If $\kappa(t) = W_\sim$ then $(\pi_* w)(t) = 0$ for every $w \in W \setminus \{0\}$; hence $t = b$, in contradiction to $\kappa(b) = \pi_*(W)$. Therefore κ defines a complex analytic mapping from T to the space of all two-dimensional linear subspaces of $W_\sim$, which is isomorphic to $\mathbb{P}^2$. If H is a two-dimensional linear subspace of $W_\sim$ not equal to $\pi_*(W)$, then $H \cap \pi_*(W) \neq 0$, and H is spanned by an element $w_\sim \notin \pi_*(W)$ and an element $\pi_* w$ such that $w \in W \setminus \{0\}$. We have $Z_{[w_\sim]} \cdot Z_{[\pi_* w]} = \mathrm{K}_T^* \cdot \mathrm{K}_T^* = 1$. If $\dim(Z_{[w_\sim]} \cap Z_{[\pi_* w]}) > 0$, then $Z_{[\pi_* w]} \subset Z_{[w_\sim]}$ because $Z_{[\pi_* w]}$ is irreducible; hence $w_\sim / \pi_* w$ is a holomorphic function on T, therefore a constant, in contradiction to $[w_\sim] \neq [\pi_* w]$. It follows that $Z_{[w_\sim]}$ and $Z_{[\pi_* w]}$ intersect each other at a single point $\iota(H)$, where both curves are smooth and the intersection is transversal. It follows that the mapping ι is a two-sided inverse of κ, and T is isomorphic to $\mathbb{P}^2$. However, because $\mathrm{rank}\, \mathrm{H}_2(S, \mathbb{Z}) = 10$, see Lemma 9.2.3, hence $\mathrm{rank}\, \mathrm{H}_2(T, \mathbb{Z}) = 9$, see (3.2.4), we arrive at a contradiction to $\mathrm{rank}\, \mathrm{H}_2(\mathbb{P}^2, \mathbb{Z}) = 1$, see Griffiths and Harris [74, p. 60], and we conclude that $W_\sim = \pi_*(W)$.

If A is an irreducible complex analytic curve in T, then for any $a \in A$ there exists a $w \in W \setminus \{0\}$ such that $a \in Z_{[\pi_* w]}$. If A is not contained in $Z_{[\pi_* w]}$, then the irreducibility of A implies that $Z_{[\pi_* w]} \cdot A > 0$. If $A \subset Z_{[\pi_* w]}$, then the irreducibility

of A and $Z_{[\pi_* w]}$ implies that $A = Z_{[\pi_* w]}$, and therefore $Z_{[\pi_* w]} \cdot A = Z_{[\pi_* w]} \cdot Z_{[\pi_* w]} = 1 > 0$. It follows that K_T^* satisfies Grauert's criterion for being ample with $k = 1$. Hence T is a del Pezzo surface, of degree one because $\mathrm{K}_T^* \cdot \mathrm{K}_T^* = 1$.

If $\pi : S \to T$ is any modification of a smooth surface T, not an isomorphism, then Lemma 6.2.50 implies that $\pi = \pi_n \circ \cdots \circ \pi_1$, where $n \in \mathbb{Z}_{>0}$ and $\pi_i : S_i \to S_{i-1}$ is a blowup for each $1 \leq i \leq n$, $S_n = S$, and $S_0 = T$. The previous arguments yield that S_{n-1} is a del Pezzo surface of degree one. According to Manin [131, Corollary 24.5.2], V is a del Pezzo surface if V' is a del Pezzo surface and $f : V' \to V$ is a birational morphism, and therefore $T = S - 0$ is a del Pezzo surface. If we define inductively $W_{i-1} = (\pi_i)_*(W_i)$, $W_n = W$, then W_i is a two-dimensional vector space of holomorphic sections of $\mathrm{K}_{S_i}^*$, and $Z_{[w_i]} = \pi_i'(Z_{[(\pi_i)_* w_i]})$ for every $w_i \in W_i \setminus \{0\}$, see Lemma 3.2.6, when (3.2.8) yields that $Z_{[w_i]} \cdot Z_{[w_i]} = Z_{[(\pi_i)_* w_i]} \cdot Z_{[(\pi_i)_* w_i]} - 1$. It follows that $\mathrm{K}_{S_i}^* \cdot \mathrm{K}\, {S_i}^* = \mathrm{K}_{S_{i-1}}^* \cdot \mathrm{K}_{S_{i-1}}^* - 1$, and therefore $\mathrm{K}_T^* \cdot \mathrm{K}_T^* = \mathrm{K}_{S_0}^* \cdot \mathrm{K}_{S_0}^* = n$.

(vi) $\Rightarrow$ (v). Because S is a rational surface, it has at least one holomorphic section E, which according to Lemma 9.2.1 is a -1 curve in S. The Castelnuovo–Enriques criterion, Theorem 3.2.4, yields a smooth surface T and a blowing up $\pi : S \to T$ of T at a point b such that $E = \pi^{-1}(\{b\})$. Then (vi) implies that T is a del Pezzo surface of degree one, and (v) holds.

(ii) $\Leftrightarrow$ (vii) follows from Theorem 4.3.3.

(vii) $\Leftrightarrow$ (viii). Lemma 9.2.1 implies that E is a -1 curve in S if and only if E is a holomorphic section of κ; Lemma 7.1.1 implies that $\alpha \mapsto \alpha(E_0)$ is a bijective mapping from $\mathrm{Aut}(S)_\kappa^+$ onto the set of all holomorphic sections of κ; and Theorem 4.3.2(iii) implies that $\alpha \in \mathrm{Aut}(S)_\kappa^+$ is a Manin QRT automorphism of S if and only if $\alpha(E_0) \cap E_0 = \emptyset$. □

The following corollary of Lemma 7.3.2, Corollary 7.3.3, and Corollary 7.4.7, which shows that the description of the Mordell–Weil group is particularly simple for a rational elliptic surface without reducible fibers, goes back to Manin [129, §4], see Theorem 4.3.4. Therefore I propose to call a rational elliptic surface without reducible fibers a *Manin elliptic surface*.

Corollary 9.2.23 *Let $\kappa : S \to P$ be a rational elliptic surface without reducible fibers, and let $\alpha \in \mathrm{Aut}(S)_\kappa^+$, an automorphism of S that acts as translations on the elliptic fibers. Then the action A of α on $L = \mathrm{H}_2(S, \mathbb{Z})$ is an Eichler–Siegel transformation of the form E_q for $q = A(e) - e + \mathbb{Z} f$, where e is the homology class of any section E of $\kappa : S \to P$. For any $k \in \mathbb{Z}$, the number $\nu(\alpha^k)$ of k-periodic fibers of α, counted with multiplicity, is equal to $-1 + (1 + \nu(\alpha))\, k^2$.*

Remark 9.2.24. Because an irreducible singular fiber is of type I_1 or II, with respective Euler number 1 or 2, condition (ii) in Lemma 9.2.7 implies that the configuration of the singular fibers of a Manin elliptic surface can only be k II $+(12 - 2k)$ I_1 with $0 \leq k \leq 6$. If $k = 5$ then $j = 12 - 2k > 0$ and $j - ii - iv^* - 2\,iv - 2\,ii^* = (12 - 2k) - k < 0$, in contradiction to condition (iv) in Lemma 9.2.7. The other cases $0 \leq k \leq 4$ and $k = 6$ satisfy all the conditions of Lemma 9.2.7, and therefore belong to Persson's list. For 12 I_1 ($k = 0$), see Section 9.2.5. The Weierstrass model of 6 II ($k = 6$) is given by $g_2 \equiv 0$ and g_3 a polynomial of degree 6 with 6 simple zeros. For $0 \leq k \leq 4$, the Weierstrass model is given by $g_2 = a\, h_2$ and $g_3 = a\, h_3$,

where a is a polynomial of degree k with k simple zeros, h_2 and h_3 are polynomials of the respective degrees $4-k$ and $6-k$, and $a\,{h_2}^3 - 27\,{h_3}^2$ has $12-2k$ simple zeros not equal to one of the zeros of a. See Table 6.3.2. The number of moduli is $8-k$ if $0 \le k \le 4$ and 3 if $k=6$. According to Lemma 9.2.6, the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ is isomorphic to $Q/Q^{\mathrm{irr}} = Q \simeq \mathbb{Z}^8$. ⊘

Remark 9.2.25. The blowup of a del Pezzo surface of rank one at the base point of its anticanonical pencil defines a bijection from the moduli space of isomorphism classes of del Pezzo surfaces of degree one onto the moduli space of isomorphism classes of Manin elliptic surfaces. This statement is known; see, for instance, Naruki [148, p. 328]. The surjectivity follows from (v) in Proposition 9.2.22. Let $\pi : S \to T$ and $\pi' : S \to T'$ be blowups of the respective del Pezzo surfaces T and T' of degree one in the base points b and b' of their anticanonical pencils. Then $E = \pi^{-1}(\{b\})$ and $E' = (\pi')^{-1}(\{b'\})$ are -1 curves in the Manin elliptic surface S. According to Lemma 9.2.1, E and E' are sections of $\kappa : S \to \mathrm{P}(W)$; hence Lemma 7.1.1 implies that $E' = \alpha(E)$ for some element α of the Mordell–Weil group of S. Because α is an automorphism of S, it follows that T is isomorphic to T'. This proves the injectivity. ⊘

Remark 9.2.26. Let T be a del Pezzo surface of rank one, $\mathfrak{P}$ the anticanonical pencil of T, b the base point of $\mathfrak{P}$, and $\pi : S \to T$ the blowing up of T at b. If E is a -1 curve in the del Pezzo surface T, then (6.2.5) implies that $F \cdot E = -\mathrm{K}_T \cdot E = E \cdot E - 2\,\mathrm{g}(E) + 2 = -1 - 2 \cdot 0 + 2 = 1$ for every $F \in \mathfrak{P}$, which in turn implies that E intersects F at one point, which is a nonsingular point of F where the intersection is transversal. Because every one-dimensional linear subspace of $\mathrm{T}_b\, T$ is equal to $\mathrm{T}_b\, F$ for some $F \in \mathfrak{P}$, it follows that $b \notin E$. Therefore $E \mapsto \pi^{-1}(E)$ is a bijective mapping from the set of all -1 curves in T onto the set of all -1 curves in S which are disjoint from the -1 curve $\pi^{-1}(\{b\})$ in S, and (viii) in Proposition 9.2.22 confirms that the number of -1 curves on a del Pezzo surface of degree one is equal to 240. ⊘

Remark 9.2.27. Let $\mathfrak{C}$ be a pencil of cubic curves in $\mathbb{P}^2$ as in Proposition 9.2.22(iv), with base points P_i, $1 \le i \le 9$, and let $\pi : S \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the base points. Let $T_i = \iota_{\mathfrak{C},\, P_i}$ denote the birational involution of $\mathbb{P}^2$ that on each smooth member of $\mathfrak{C}$ acts as the inversion in P_i, as in the paragraph preceding (4.2.2). Let R_i be the automorphism of S such that $\pi \circ R_i = T_i \circ \pi$, see Lemma 3.4.1, and r_i the action on $\mathrm{NS}(S) = \mathrm{H}_2(S, \mathbb{Z})$ induced by R_i, where $r_i \circ r_i$ is equal to the identity. As in Lemma 9.2.3, we denote the homology class of the -1 curve $E_i = \pi^{-1}(\{P_i\})$ by e_i, and the common homology class of the total transform of any projective line in $\mathbb{P}^2$ by l.

If $j \ne i$, then $R_i(E_j)$ contains the proper transform $\pi'(P_i\, P_j)$ of the projective line $P_i\, P_j$ containing P_i and P_j, and because $R_i(E_j)$ is irreducible as the image of the irreducible curve E_j under the automorphism R_i, it follows that $R_i(E_j) = \pi'(P_i\, P_j)$. Because (3.2.5) implies that $\pi'(P_i\, P_i) = \pi^*(P_i\, P_j) - E_i - E_j$, it follows that $r_i(e_j) = l - e_i - e_j$. Because $e_j = r_i(r_i(e_j)) = r_i(l) - r_i(e_i) - r_i(e_j) = r_i(l) - r_i(e_i) - (l - e_i - e_j)$, we have $r_i(l) - r_i(e_i) = l - e_i$. Because R_i preserves the fibers, we have $3\,l - \sum_i e_k = f = r_i(f) = 3\,r_i(l) - r_i(e_i) - \sum_{j \ne i} r_i(e_j)$. The

two inhomogeneous linear equations for $r_i(l)$ and $r_i(e_i)$ have the unique solutions $r_i(l) = 5\,l - 4\,e_i - \sum_{j\neq i}\, e_j$ and $r_i(e_i) = 4\,l - 3\,e_i - \sum_{j\neq i}\, e_j$. The latter equation confirms the statement "The proper image of P_i with respect to T_i is a curve F_i in $\mathbb{P}^2$ of degree four, having P_i as a triple point and passing simply through the P_j, $j \neq i$". in Manin [129, p. 95]. In the same vein, the image under T_i of a projective line in $\mathbb{P}^2$ not passing through any of the base points is a quintic curve in $\mathbb{P}^2$, having P_i as a quadruple point and passing simply through the P_j, $j \neq i$.

If $j \neq i$, then $r_i(e_j - e_i) = -(e_j - e_i) + f$. On the other hand, $(e_j - e_i)\cdot(e_j - e_i) = -2$ if $j \neq i$ and $(e_j - e_i)\cdot(e_k - e_i) = -1$ if $j \neq i$, $k \neq i$, and $k \neq j$, the intersection matrix $(e_j - e_i)\cdot(e_k - e_i)|_{j\neq i,\,k\neq i}$ has determinant equal to $9 \neq 0$; hence the $e_j - e_i + \mathbb{Z}\,f$ form a $\mathbb{Q}$-basis of $Q \otimes \mathbb{Q}$. It follows that r_i acts as minus the identity on $Q := f^{\perp}/\mathbb{Z}\,f$. ⊘

Remark 9.2.28. The relation in Proposition 9.2.22 between the pencil $\mathfrak{C}$ of cubic curves in $\mathbb{P}^2$ and the rational elliptic fibration $\kappa : S \to \mathbb{P}^1$ defines a bijective mapping from the category of isomorphism classes of pencils onto the category of isomorphism classes of rational elliptic surfaces, provided with an unordered set of nine mutually disjoint sections. For a given rational elliptic fibration $\kappa : S \to \mathbb{P}^1$, two sets $\mathcal{E}$, $\mathcal{E}'$ of nine unordered mutually disjoint sections are isomorphic if and only if there is an automorphism α of S that maps $\mathcal{E}$ onto $\mathcal{E}'$. In the sequel we restrict ourselves to Manin elliptic fibrations $\kappa : S \to \mathbb{P}^1$ such that $\mathrm{Aut}(S) = \mathrm{Aut}(S)_\kappa$, which exist; see Remark 9.2.12. Note that this excludes the example with twelve singular fibers in Section 9.2.5. If we also exclude the configuration of singular fibers 6 II discussed in Remark 9.2.24, when $J \equiv 0$, then the isomorphism classes can be determined as follows.

We use the notation of Lemma 9.2.3. If $e \in \mathrm{NS}(S)$, $e \cdot f = 1$, and $e \cdot e = -1$, then Lemma 7.2.1 with $\varphi = \kappa$, in combination with the assumption that all fibers of κ are irreducible, implies that there exist a holomorphic section E of κ and $k \in \mathbb{Z}$ such that $e = \mathrm{H}(E) + k\,f$. Then $-1 = e\cdot e = \mathrm{H}(E)\cdot\mathrm{H}(E) + 2\,k\;\mathrm{H}(E)\cdot f + k^2\,f\cdot f = -1 + 2\,k$ implies that $k = 0$; hence e is the homology class of a holomorphic section of κ. Therefore $e_i = \mathrm{H}(E_i)$ defines a bijection from the collection of all ordered sets E_i, $1 \leq i \leq 9$, of mutually disjoint holomorphic sections of κ, onto the collection of all ordered sets $e_i \in \mathrm{NS}(S)$, $1 \leq i \leq 9$, such that for all i and j, $e_i \cdot f = 1$, $e_i \cdot e_i = -1$, and $e_i \cdot e_j = 0$ if $i \neq j$. If E_0 is a given holomorphic section of κ with homology class e_0, then it follows from Lemma 7.1.1 that some element of $\mathrm{Aut}(S)^+_\kappa \subset \mathrm{Aut}(S)$ maps e_9 to e_0, and therefore we can restrict ourselves to the nine-tuples e_i such that $e_9 = e_0$. If we write $d_i := e_i - e_0 + \mathbb{Z}\,f \in Q$, then we obtain a bijective mapping onto the collection of sequences $d_i \in Q$, $1 \leq i \leq 8$, such that $d_i \cdot d_i = -2$ and $d_i \cdot d_j = -1$ when $i \neq j$. Two such sequences are isomorphic if and only if they are mapped to each other by means of the action on Q of an automorphism of S. Because we excluded the configuration of singular fibers 6 II, there is at least one singular fiber of type I_1 where the modulus function J has a pole. It follows from Lemma 6.3.26 with nonconstant J that the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ is a subgroup of index two in $\mathrm{Aut}(S) = \mathrm{Aut}(S)_\kappa$, where each involution $R_i \in \mathrm{Aut}(S)_\kappa$ of Remark 9.2.27 represents the nontrivial element of $\mathrm{Aut}(S)/\,\mathrm{Aut}(S)^+_\kappa$. Because there are no reducible fibers, it follows from Lemma 7.3.2(vii) that the action of $\mathrm{Aut}(S)^+_\kappa$

on Q is trivial, whereas Remark 9.2.27 yields that R_i acts as minus the identity on Q. Therefore two sequences d_i are isomorphic if and only if they are equal or each other's opposite.

The description of the simple root system α_i, $0 \le i \le 7$, in the proof of Lemma 9.2.3 defines a bijective mapping from the collection of sequences d_i onto the collection of simple root systems in the root lattice Q of type E_8, where the numbering of the simple roots is unique because the Dynkin diagram of type E_8 has no symmetries. Because the Weyl group $\mathrm{W}(E_8)$ acts freely and transitively on the collection of simple root systems, see the first remark on p. 154 in Bourbaki [23], we arrive at the statement that the number of sequences d_i is equal to $\#(\mathrm{W}(\mathrm{E}_8)) = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. See Bourbaki [23, p. 270] for the number of elements in the Weyl group. As pointed out to me by Eduard Looijenga, this count follows from Demazure [45, Proposition 4 on p. 31] for $r = 8$, when $P_8 = {e_0}^{\perp}$, and the number of sequences $e_j \in P_8$, $1 \le j \le 8$, is counted such that $e_i \cdot e_i = -1$ and $e_i \cdot e_j = 0$ when $i \ne j$. Since our isomorphism classes are unordered sequences in which opposite sequences are isomorphic, it follows that the number of isomorphism classes of nine unordered mutually disjoint sections is equal to $(\#(\mathrm{W}(\mathrm{E}_8))/8!)/2 = 2^6 \cdot 3^3 \cdot 5 = 8640$. Therefore the generic fiber of the Aronhold morphism AM : MCP $\to$ MRES in Section 6.3.3 has $8640 = 2^6 \cdot 3^3 \cdot 5$ elements. ⊘

Remark 9.2.29. If we start out from a pencil of biquadratic curves, then we arrive at a rational elliptic fibration $\kappa : S \to \mathbb{P}^1$ together with a Manin QRT transformation, an element τ of the Mordell–Weil group that maps each holomorphic section of φ to a disjoint one. The assumption that all fibers of φ are irreducible implies the assumptions of Corollary 5.1.9, and therefore τ acts on $\mathrm{NS}(S) = \mathrm{H}_2(S, \mathbb{Z})$ as E_q with $q = l_2 - l_1 + \mathbb{Z}\, f$. If the e_i are the homology classes of the -1 curves E_i that appear at the blowing up of the base points, then $e_i \cdot l_j = 0$ for each $1 \le i \le 8$ and $j = 1, 2$; hence $d_i \cdot q = 0$ for each $1 \le i \le 7$ if we write $d_i = e_i - e_8 + \mathbb{Z}\, f \in Q$. We have $q \cdot q = -2$, hence q is a root in the E_8 root lattice Q. The orthogonal complement $q^{\perp}$ of q in Q is a root lattice of type E_7, because $d_i \cdot d_i = -2$, the d_i forming a sequence of seven roots in $q^{\perp}$, where $e_i \cdot e_j = 0$ implies that $d_i \cdot d_j = -1$ if $i \ne j$.

Assume conversely that $q \in Q, q \cdot q = -2$, and $d_i \in q^{\perp}, 1 \le i \le 7, d_i \cdot d_i = -2$, and $d_i \cdot d_j = -1$ if $i \ne j$. Let E_8 be a given holomorphic section of κ. Because all fibers of κ are irreducible, Lemma 7.2.1 with $\varphi = \kappa$ implies that for each $1 \le i \le 7$ there is a holomorphic section E_i of κ such that $d_i = [E_i] - [E_8] + \mathbb{Z}\, f$, where $d_i \cdot d_i = -2$ implies that $E_i \cdot E_8 = 0$ and $d_i \cdot d_j = -1$ implies that $E_i \cdot E_j = 0$ if $i \ne j$, $i, j \le 7$. That is, the holomorhic sections E_i, $1 \le i \le 8$, are mutually disjoint. Blowing down the E_i, we arrive at a Hirzebruch surface Σ_n with $n \le 2$, where $n = 2$ is excluded because κ has no reducible fibers.

If $n = 1$, then the preimage in S of the -1 curve T_1 in Σ_1 would be a ninth holomorphic section E_9 of κ, disjoint of the E_i, $1 \le i \le 8$, whereas blowing down T_1, we arrive at $\mathbb{P}^2$, and we have the description of $\mathrm{H}_2(S, \mathbb{Z})$ as in Lemma 9.2.3. There exist $\lambda, \epsilon_i \in \mathbb{Z}$ such that $q = \lambda\, f + \sum_i \epsilon_i\, e_i + \mathbb{Z}\, f$. For each $1 \le j \le 7$ we have $0 = q \cdot d_i = (\lambda\, f + \sum_i \epsilon_i\, e_i) \cdot (e_i - e_8) = -\epsilon j + \epsilon_8$, whereas $q \in f^{\perp}$ implies that $0 = (\lambda\, f + \sum_i \epsilon_i\, e_i) \cdot (3\, l - \sum e_i) = 3\,\lambda + \sum \epsilon_i = 3\,\lambda + 8\,\epsilon_8 + \epsilon_9$, and it

follows that

$$q = \lambda\, l + \epsilon_8 \sum_{i=1}^{8} e_i - (3\,\lambda + 8\,\epsilon_8)\, e_9 + \mathbb{Z}\, f.$$

Working modulo $\mathbb{Z}\, f$, we may assume that $\lambda = 0$, 1, or 2, and in either case we have a contradiction to $q \cdot q = -2$.

The conclusion is that $n = 0$, that is, blowing down the E_i, $1 \le i \le 8$, we arrive at $\mathbb{P}^1 \times \mathbb{P}^1$ with a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, where the E_i, $1 \le i \le 8$, are the -1 curves that appear at the blowing up of the base points of this pencil, and $q = l_2 - l_1 + \mathbb{Z}\, f$, where the sign of q can be arranged by applying the symmetry switch $(x,\ y) \mapsto (y,\ x)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, which interchanges l_1 and l_2. The birational transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ in Section 4.1 at the base point in $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to E_8 yields $l = e_8 + l_1 + l_2$. The simple root system α_i, $0 \le i \le 7$, in Q in the proof of Lemma 9.2.3 satisfies $\alpha_0 = l - e_1 - e_2 - e_3 = l_1 + l_2 + e_8 - e_1 - e_2 - e_3 \in (l_2 - l_1)^\perp = q^\perp$, $\alpha_j = e_j - e_{j+1} = d_j - d_{j+1} \in q^\perp$ for $1 \le j \le 6$. Because the Dynkin diagram of α_i, $0 \le i \le 6$, is equal to the Dynkin diagram of α_i, $0 \le i \le 7$, with the vertex α_7 removed, the α_i, $0 \le i \le 6$ form a simple root system in the root lattice $q^\perp$ of type E_7. Since conversely, $d_1 = 2\,\alpha_0 + 2\,\alpha_1 + 3\,\alpha - 2 + 4\,\alpha_3 + 3\,\alpha_4 + 2\,\alpha_5 + \alpha_6$, and $d_{j+1} = d_j - \alpha_j$ for $1 \le j \le 6$ determine the d_j in terms of α_k, this leads to a bijective mapping from the set of all ordered sequences d_j in $q^\perp$ as above to the set of all simple root systems $q^\perp$, where the ordering of the roots is unique because the Dynkin diagram of type E_7 has no symmetries. Since the Weyl group $\mathrm{W}(\mathrm{E}_7)$ of E_7 acts freely and transitively on the set of simple root systems, see the first remark on p. 154 in Bourbaki [23], the number of ordered 7-tuples d_i as above in $q^\perp$ is equal to $\#(\mathrm{W}(\mathrm{E}_7)) = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$. See Bourbaki [23, p. 265] for the number of elements in the Weyl group. Therefore the number of unordered such sequences d_i, where moreover opposite sequences are identified with each other, is equal to $(\#(\mathrm{W}(\mathrm{E}_8))/7!)/2 = 2^5 \cdot 3^2 = 288$. If $\pi, \pi' : S \to \mathbb{P}^1 \times \mathbb{P}^1$ are blowdowns of the E_i such that $\pi(E_i) = \{b_i\}, \pi'(E_i') = \{b_i'\}$, then $\pi' \circ \pi^{-1}$ is a complex analytic diffeomorphism from the complement of the b_i in $\mathbb{P}^1 \to \mathbb{P}^1$ onto the complement of the b_i' in $\mathbb{P}^1 \times \mathbb{P}^1$ that has a continuous extension $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $\psi(b_i) = b_i'$ for every i, when the theorem on removable singularities implies that ψ is a complex analytic diffeomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. Lemma 9.2.30 below implies that ψ induces an isomorphism from the pencil of biquadratic curves with the base points b_i to the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with the base points b_i'. It follows that generically the number of isomorphism classes of pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ that give rise to a given isomorphism class of rational elliptic surfaces with a given pair $\{\tau,\ \tau^{-1}\}$ of Manin QRT transformations is equal to $2^5 \cdot 3^2 = 288$. Therefore the generic fiber of the Frobenius morphism FM in Section 6.3.3 has $288 = 2^5 \cdot 3^2$ elements.

Lemma 9.2.30 *Let A be a complex analytic diffeomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. Then there are projective linear transformations A_1 and A_2 in $\mathbb{P}^1$ such that $A : (x,\ y) \mapsto (A_1(x),\ A_2(y))$ or $A : (x,\ y) \mapsto (A_2(y),\ A_1(x))$.*

Proof. If l_1 and l_2 denote the common homology classes of the horizontal and vertical axes, respectively. Then (2.4.4) yields that $H_2(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} l_1 \oplus \mathbb{Z} l_2$, where the intersection form is determined by $l_i \cdot l_i = 0$ and $l_1 \cdot l_2 = 1$. If a denotes the action of A on $H_2(\mathbb{P}^1 \times \mathbb{P}^1)$, then $a(l_i) = \sum_j a_i^j l_j$ for $a_i^j \in \mathbb{Z}$, and $0 = l_i \cdot l_i = a(l_i) \cdot a(l_i) = a_i^1 a_i^2$, $1 = l_1 \cdot l_2 = a_1^1 a_2^2 - a_1^2 a_2^1$. It follows that either $a(l_i) \pm l_i$ or $a(l_1) = \pm l_2$ and $a(l_2) = \pm l_1$. Composing A with the symmetry switch $(x, y) \mapsto (y, x)$ if necessary, we can arrange that $a(l_i) = \pm l_i$.

For any pair H, H' of horizontal axes, we have $A(H) \cdot H' = a(l_1) \cdot l_1 = \pm l_1 \cdot l_1 = 0$. Because both $A(H)$ and H' are irreducible curves, it follows that $A(H) \cap H' = \emptyset$ or $A(H) = H'$. Since this holds for every horizontal axis H', it follows that $A(H)$ is a horizontal axis. A similar argument yields that A maps each vertical axis to a vertical axis, and it follows that there are mappings $A_i : \mathbb{P}^1 \to \mathbb{P}^1$ such that $A(x, y) = (A_1(x), A_2(y))$ for every $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$. The assumption that A is a complex analytic diffeomorphism implies that A_1 and A_2 are complex analytic diffeomorphsims of $\mathbb{P}^1$, when Lemma 2.2.1 with $V = \mathbb{C}^2$ yields that $A_i \in \mathrm{PGL}(\mathbb{C}^2)$. □

Because on the other hand there are $240/2 = 120 = 2^3 \cdot 3 \cdot 5$ pairs $\{\tau, \tau^{-1}\}$ of Manin QRT automorphism on a given Manin elliptic surface, see Proposition 9.2.22, it follows that generically there are $(2^3 \cdot 3 \cdot 5) \cdot (2^5 \cdot 3^2) = 2^8 \cdot 3^3 \cdot 5 = 34560$ isomorphism classes of pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ that give rise to a given isomorphism class of rational elliptic surfaces. Therefore the generic fiber of the morphism WM ∘ FM in Section 6.3.3 has $34560 = 2^8 \cdot 3^3 \cdot 5$ elements. I have no explanation, in terms of a morphism from the moduli space of isomorphism classes of pencils of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ to the moduli space of isomorphism classes of pencils of cubic curves in $\mathbb{P}^2$, of the fact that 34560 is four times the number 8640 in Remark 9.2.28. ⊘

9.2.5 An Example with Twelve Singular Fibers

9.2.5.1 Generic Configuration of the Singular Fibers

For a rational elliptic surface, the discriminant Δ is a homogeneous polynomial of degree 12 in (z_0, z_1). The configuration of the singular fibers is 12 I_1 if and only if Δ has 12 distinct zeros, if and only if the discriminant of Δ is not equal to zero. It follows from Example 2 after Corollary 3.3.14 that this occurs, and therefore the set of coefficients of pairs of biquadratic polynomials such that the configuration of the singular fibers in the corresponding rational elliptic surface is 12 I_1 is equal to the complement of a proper complex codimension-one algebraic set. The rational elliptic surface will be called *generic* if it has 12 singular fibers of type I_1, even if with respect to other properties the rational elliptic surface might be special. Corollary 6.2.42 implies that the modulus function J of a generic rational elliptic surface has the maximal degree 12. For the monodromy representation of a generic rational elliptic

surface, see Example 4 after Lemma 6.2.36. The moduli spaces in Heckman and Looijenga [81] are compactifications of the moduli space $\mathcal{M}$ of rational elliptic surfaces with 12 singular fibers of type I_1, one of which is Miranda's space QW discussed in Section 6.3.3.

Remark 9.2.31. Let $U = \mathrm{H}_4(\mathbb{C}^2) \times \mathrm{H}_6(\mathbb{C}^2)$ and the weighted projective space $M = (U \setminus \{0\})/\mathbb{C}^\times$ be as in Miranda's construction of the moduli space QC in Section 6.3.3. Write $D = \mathrm{H}_{12}(\mathbb{C}^2)$. The mapping $\Delta : U \to D : (g_2,\ g_3) \mapsto {g_2}^3 - 27\,{g_3}^2$ induces an algebraic morphism $\Delta_M : M' \to \mathrm{P}(D)$, where $M' = (U \setminus \Delta^{-1}(\{0\}))/\mathbb{C}^\times$ is a Zariski-open subset of M. If $\delta(\Delta)$ denotes the discriminant of any $\Delta \in D$, then the complement of the zero-set of δ in D defines a Zariski-open subset of $\mathrm{P}(D)$, denoted by $\mathrm{P}(D)_{\delta\neq 0}$. The set $\mathcal{B} := {\Delta_M}^{-1}(\mathrm{P}(D)_{\delta\neq 0})$, corresponding to the Weierstrass models of rational elliptic surfaces with 12 singular fibers of type I_1, is a nonempty Zariski-open subset of M'. The restriction π of Δ_M to $\mathcal{B}$ is an algebraic morphism from $\mathcal{B}$ to $\mathrm{P}(D)$. Chevalley's theorem implies that $\mathcal{Z} := \pi(\mathcal{B})$ is a constructible subset of $\mathrm{P}(D)$, where $\mathcal{Z} \subset \mathrm{P}(D)_{\delta\neq 0}$. According to Vakil [198, Theorem 6.1], the morphism $\pi : \mathcal{B} \to \mathcal{Z}$ is unramified and proper, birational but not an isomorphism, and a normalization of $\mathcal{Z}$. Because $\dim \mathcal{B} = \dim M = (4+1) + (6+1) - 1 = 11$ and $\dim \mathrm{P}(D) = (12+1) - 1 = 12$, it follows that $\mathcal{Z}$ has codimension one in $\mathrm{P}(D)$. According to [198, Theorem 6.2], the hypersurface $\mathcal{Z}$ in $\mathrm{P}(D)$ has degree 3762, which according to [198, Abstract] means that there are 3762 rational elliptic fibrations with singular fibers of type I_1 over 11 given general points in $\mathbb{P}^1$. ⊘

It is somewhat surprising that none of the examples in Chapter 11 from the literature have the configuration of singular fibers $12\ \mathrm{I}_1$. In this subsection we present an explicit example of a pencil of biquadratic curves with the configuration of singular fibers $12\ \mathrm{I}_1$.

9.2.5.2 The Pencil

Consider the pencil of biquadrics (2.5.3) with

$$A^0 := \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{9.2.7}$$

where the coefficients a and b are nonzero. The pencil where all the nonzero entries in (9.2.7) are replaced by arbitrary nonzero numbers can be brought into the form (9.2.7) by means of a scaling of the variables $x_0,\ x_1,\ y_0,\ y_1,\ z_0,\ z_1$, where the scaling factors can be taken to be real if all the matrix entries we started out with are real. Note that the pencil has $([1:0],\ [0:1])$ and $([0:1],\ [1:0])$ as base points.

9.2.5.3 The Weierstrass Invariants

The Weierstrass invariants of Corollary 2.4.7 and Proposition 2.5.6 are given by

$$
\begin{aligned}
g_2 &= (4/3)\, l\, {z_0}^2\, {z_1}^2,\\
g_3 &= -a^2\, b^2\, {z_0}^6 + (2/27)\, m\, {z_0}^3\, {z_1}^3 - {z_1}^6,\\
X &= (1/3)\, (2 - a - b)\, z_0\, z_1,\\
Y &= a\, b\, {z_0}^3 + {z_1}^3,
\end{aligned}
$$

where we have used the abbreviations $l := a^2 + b^2 + 1 - a\, b - b - a$ and $m := (a - 2\,(b + 1))\,(b - 2\,(1 + a))\,(1 - 2\,(a + b))$.

The discriminant is equal to

$$\Delta := {g_2}^3 - 27\, {g_3}^2 = a_0\, {z_0}^{12} + 4\, a_1\, {z_0}^9\, {z_1}^3 + 6\, a_2\, {z_0}^6\, {z_1}^6 + 4\, a_3\, {z_0}^3\, {z_1}^9 + a_4\, {z_1}^{12},$$

where $a_0 = -27\, a^4\, b^4$, $a_1 = a^2\, b^2\, m$, $3\, a_2 = 8\,((a^2\, b^2 + a^2 + a\, b + b^2)\,(a - b)^2 - a\, b\,(a\, b + 1)\,(a + b) - 2\,(a^3 + b^3) + a^2 + b^2) - 33\, a^2\, b^2$, $a_3 = m$, and $a_4 = -27$. We view Δ as a homogeneous polynomial of degree four in $({z_0}^3,\ {z_1}^3)$, and as such, its discriminant, see the text following (2.3.21), is equal to 2^{24} times

$$\delta := a^4\, b^4\, (a - b)^2\, (a - 1)^2\, (b - 1)^2\, (a\, b + a + b)^2\, ((a - b)^2 - 2\,(a + b) + 1)\, l^6.$$

Therefore we have four different solutions $[{z_0}^3 : {z_1}^3]$ if and only if $\delta \neq 0$, that is, $a \neq 0$, $b \neq 0$, $a \neq b$, $a \neq 1$, $b \neq 1$, $a\, b + a + b \neq 0$, $(a - b)^2 - 2\,(a + b) + 1 \neq 0$, and $l := a^2 + b^2 + 1 - a\, b - b - a \neq 0$.

If $a\, b = 0$ then $a_0 = a_1 = 0$, and $[1 : 0]$ is a zero of order ≥ 6 of Δ. Assume that $a\, b \neq 0$. Then the coefficients of ${z_0}^{12}$ and of ${z_1}^{12}$ in Δ are nonzero, and we have $z_0 \neq 0$ and $z_1 \neq 0$ for the zeros of Δ. For each nonzero $\zeta = (-z_1/z_0)^3$, we have three solutions $z = -z_1/z_0$, which lie in an equilateral triangle in the complex plane with center at the origin. It follows that $\delta \neq 0$ if and only if we have four different solutions $[{z_0}^3 : {z_1}^3]$ of $\Delta = 0$ if and only if we have 12 different solutions $[z_0 : z_1]$ of $\Delta = 0$ if and only if the configuration of the singular fibers is equal to the generic one 12 I_1. The equation $\delta = 0$, describing when the configuration of the singular fibers is not equal to 12 I_1, defines a bifurcation diagram in the $(a,\ b)$-plane.

These z form a regular 12-gon in the complex plane with center at the origin if and only if $m = 0$, $(4\, l/3)^3 - 54\, a^2\, b^2 = 0$, and $a\, b \neq 0$, which happens for instance when $a + b = 1/2$ and $(1 - 4\, a\, b)^3 - 54\, (a\, b)^2 = 0$. In Example 2 the existence of a QRT surface with singular fibers over a regular 12-gon in $\mathbb{P}^1$ was proved using a Weierstrass model, and almost no computations.

9.2.5.4 Simplicity of the Base Points

Let $\kappa : S \to \mathbb{P}^1$ be the elliptic surface defined by the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, according to Corollary 3.3.10, where $\pi : S \to \mathbb{P}^1 \to \mathbb{P}^1$ is the blowing-up map. If a base point b of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ is multiple, then there is a unique member C of the pencil that is singular at b, and the fiber F in the elliptic surface $\kappa : S \to \mathbb{P}^1$ such that $\pi(F) = C$ is reducible. If $\delta \neq 0$, then all singular fibers are of type I_1, hence irreducible, and it follows that all base

points are simple. Because, counted with multiplicities, there are eight base-points, see Lemma 3.1.1, we conclude that we have eight distinct bases points, all simple. Moreover, S can be identified with the surface S in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$, defined in Section 3.1, with π equal to the projection from S onto the second factor $\mathbb{P}^1 \times \mathbb{P}^1$. In contrast with rarely having the configuration of singular fibers 12 I_1, it happens quite often in the examples in the literature that all eight base points are simple.

9.2.5.5 The Action on the Homology

If a horizontal or vertical axis is contained in a biquadratic curve, then this curve is reducible, and the same holds for the corresponding fiber in the elliptic surface $\kappa : S \to \mathbb{P}^1$ in Corollary 3.3.10. Therefore, if $\delta \neq 0$, no member of the pencil of biquadratic curves contains a horizontal or vertical axis, the assumptions in Corollary 5.1.9 hold, and the action of the QRT map on the homology is as described in Corollary 5.1.9. In particular for each $k \in \mathbb{Z}$ the number of k-periodic fibers, counted with multiplicities, is equal to $k^2 - 1$. This too happens in quite a number of the examples in Chapter 11 from the literature.

9.2.5.6 A Symmetry

Although our example has the above generic properties, it also has the highly non-generic property that the surface in $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ defined by (2.5.3) is invariant under the $\mathbb{Z}/3\,\mathbb{Z}$-action

$$([z_0 : z_1],\, ([x_0 : x_1],\, [y_0 : y_1])) \mapsto ([z_0 : \omega^2 z_1],\, ([x_0 : \omega x_1],\, [y_0 : \omega^2 y_1])),$$

where ω runs over the three third roots of unity. This defines a $\mathbb{Z}/3\,\mathbb{Z}$-action by automorphism on the rational elliptic surface $\kappa : S \to P$ for which the two nontrivial automorphisms permute the fibers, as discussed in Section 9.2.1. A slight perturbation of the coefficients will yield a rational elliptic surface without such automorphisms, retaining the other generic properties.

9.2.5.7 Real Singular Fibers

Assume that the coefficients $a,\ b$ in (9.2.7) are real. The base points ($[1 : 0]$, $[0 : 1]$) and ($[0 : 1]$, $[1 : 0]$) are real, and the existence of a real base point implies that for each $[z_0 : z_1] \in \mathbb{P}^1$ the corresponding biquadratic curve has a nonempty real part, which implies that the elliptic fibration $\kappa : S \to \mathbb{P}^1$ maps $S(\mathbb{R})$ onto the whole real projective line $\mathbb{P}(\mathbb{R})$.

The number of real zeros $[{z_0}^3 : {z_1}^3]$ of Δ, which is equal to the number of real zeros $[z_0 : z_1]$ of Δ, is equal to zero or four when $\delta > 0$, and equal to two when $\delta < 0$; see the proof of Lemma 2.6.3. In combination with Proposition 8.5.1 this

leads to the conclusion that there are four or two real singular fibers when $\delta > 0$ or $\delta < 0$, respectively.

The configuration of the real singular fibers does not change if the parameters (a, b) vary in a connected component of the complement R^{reg} of the zero-set R^{sing} of δ in the real (a, b)-plane R. The *bifurcation diagram* R^{sing} is the union of the straight lines $a = 0$, $b = 0$, $a = b$, $a = 1$, $b = 1$, the hyperbola $a\,b + a + b = 0$, and the parabola $u := (a - b)^2 - 2\,(a + b) + 1 = 0$. Note that the factor $l = (a - 1)^2 + (b - 1)^2 - (a - 1)\,(b - 1)$, which occurs in δ with multiplicity six, is strictly positive unless $a = b = 1$, which is the common intersection point of the three straight lines $a = b$, $a = 1$, and $b = 1$ in the bifurcation diagram. For $(a, b) \in R$, the sign of δ is equal to the sign of $u := (a - b)^2 - 2\,(a + b) + 1$. Because the solutions of the latter equation are $a = ((v + 1)^2 - u)/4$, $b = ((v - 1)^2 - u)/4$, where $v = a - b$ is any real number, both signs for δ can occur.

The real singular fiber of type I_1 is elliptic or hyperbolic if and only if $g_3 > 0$ or $g_3 < 0$, respectively. See Remark 8.3.1. On the real variety where $\Delta = 0$, we have $g_3 = 0$ if and only if $g_2 = g_3 = 0$. Therefore, if we stay away from $a = b = 1$, we have $z_0 = 1$, which is in contradiction to $0 = g_3 = -{z_1}^6$, or $z_1 = 0$, when $g_3 = 0$ if and only if $a = 0$ or $b = 0$. Therefore the type with respect to the real structure can change only if we pass from one connected component of R^{reg} to another via one of the straight lines $a = 0$ and $b = 0$. On the other hand, when $m = 0$ and $a\,b \neq 0$, then $g_3 < 0$. Because each of the connected components of the set where $a\,b \neq 0$ meets at least one of three straight lines of $m := (a - 2\,(b + 1))\,(b - 2\,(1 + a))\,(1 - 2\,(a + b)) = 0$, it follows that when $(a, b) \in R^{\mathrm{reg}}$ each of the real singular fibers is hyperbolic of type I_1, with respect to the real structure.

9.2.5.8 A Bifurcation

The bifurcation when u changes sign from positive, when there are four singular fibers, to negative, when there are two singular fibers, is quite interesting.

If we cross the parabola $u = 0$ at a point in the (a, b)-plane that is not on one of the straight lines in the bifurcation diagram, two of the real singular fibers of hyperbolic type I_1 merge to a hyperbolic singular fiber of type I_2, and then disappear, in the sense that the two reappearing complex singular fibers of type I_1 no longer have any real points. This can happen only if the two hyperbolic singular points s_1, s_2 of the real biquadratic curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ for $0 < u \ll 1$ merge at a singular point s when $u \downarrow 0$ and become nonreal singular points when $-1 \ll u < 0$. For $u = 0$ the singular point s of the singular real biquadratic curve must be a base point, because otherwise the real surfaces $S(\mathbb{R})$ over a neighborhood of s in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ would be diffeomorphic to each other when passing through the bifurcation, the real fibers in $S(\mathbb{R})$ would be level curves of a Morse function, and there would be no change in the nature of the singular points. At the bifurcation the real base point is double, when in the real blowing up the single singular point becomes the two singular points on the singular fiber of type I_2, where one of the irreducible components of the singular

fiber is the proper image after the second blowing up of the -1 curve that appears at the first blowing up.

The resultant of the polynomials $x \mapsto p^0(x,\, y) = a\,x^2\,y + b\,x\,y^2$ and $x \mapsto p^1(x,\, y) = x^2\,y^2 + x + y$ is $b^2\,y^6 + (1 - 2\,a + a^2 - b - a\,b)\,y^3 + a$, whose discriminant as a quadratic polynomial in y^3 is equal to $(a-1)^2\,u$. This proves that when $0 < u \ll 1$ there are two real base points near s, which merge at s when $u \to 0$, and become nonreal when $-1 \ll u < 0$. The two base points near s for $0 < s \ll 1$ actually are the intersection points of the two nearby singular curves, which have normal crossings at the respective singular points s_1 and s_2. When $-1 \ll u < 0$, the biquadratic curves in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ near s form a smooth fibration, where the curves look pinched together near s, as a reminder of the vanished base points.

Every real blowing up replaces a point by a circle = real projective line, of which the normal bundle in the blown-up surface is a Möbius strip; see Section 8.5 and the explanation of Figure 3.2.1. Therefore the real part $S(\mathbb{R})$ of the elliptic surface S for $-1 \ll u < 0$ is even topologically different from the real part of the elliptic surface for $0 < u \ll 1$, whereas the complex elliptic surfaces are diffeomorphic; see Remark 6.2.35. This explains why the real fibrations before and after the bifurcation cannot be viewed as fibrations on one and the same smooth surface.

Figures 9.2.1, 9.2.2, and 9.2.3 illustrate the bifurcation when crossing $u = 0$. The pictures show the pencil of real curves for $(a,\, b) = (0.538,\, 0.038)$, $(0.562,\, 0.062)$, and $(0.588,\, 0.0388)$, when $u := (a-b)^2 - 2\,(a+b) + 1 = 0.1$, 0, and -0.1, respectively.

For the choice of the members of the real pencils that appear in the pictures, see the text under the heading "Figure 2.3.2," starting with "In all pictures in this book of real pencils ...."

9.2.5.9 The Real Period Function

Because $l = 1$ in Lemma 8.4.1, and because $0 < T(z) < p(z)$, hence $0 < \rho(z) < 1$, for a QRT map, and because in Lemma 8.1.5 the rotation function $\rho(z)$ was discussed modulo integers, it follows that the limit value of the rotation function at the singular values of $z = -z_1/z_0$ is 0 or 1 when we approach it from an interval I_j where $\Delta > 0$ and the real fiber has two connected components, whereas the limit value is 0, $1/2$, or 1 when coming from an interval I_j where $\Delta < 0$ and the real fiber is connected.

It follows from Remark 8.4.2 that the approach of the rotation function of the real QRT map to its limit value is infinitely steep, unless the QRT map switches two connected components of the real fiber and its square acts as the identity on the limit singular fiber. If the QRT map preserves the two connected components over the interval where $\Delta > 0$, then the discussion in Remark 8.4.3 yields that the limit value of the rotation function from the other side, over which $\Delta < 0$ and the real fiber is connected, is equal to $1/2$. If the QRT map switches the two connected components, then we have to pass to the square of the QRT map and the limit value is always 0 or 1.

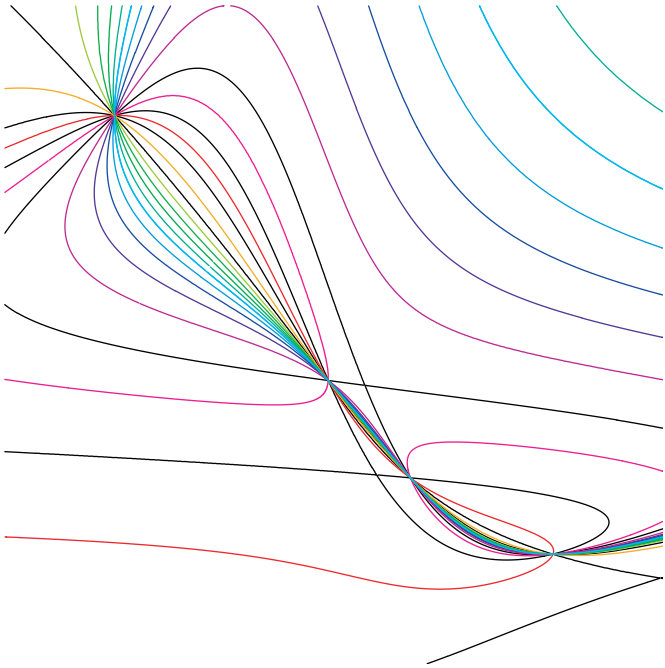


Fig. 9.2.1 The pencil for $u = 0.1$.

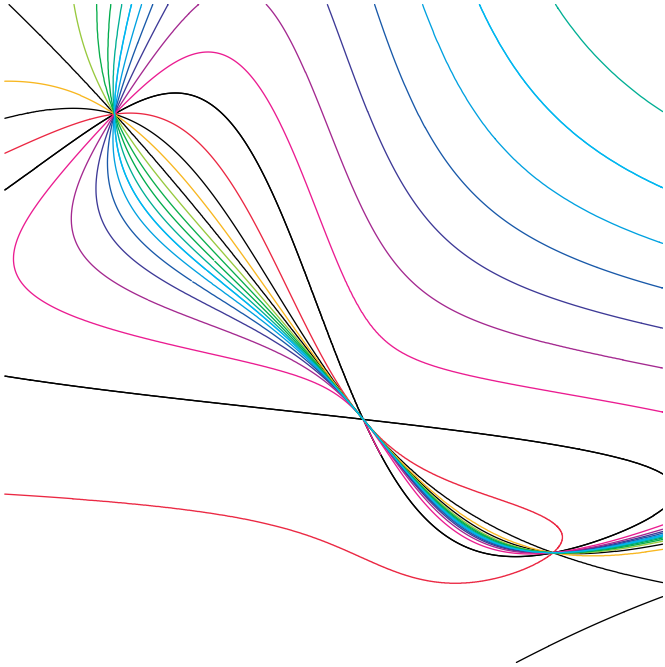


Fig. 9.2.2 The pencil for $u = 0$.

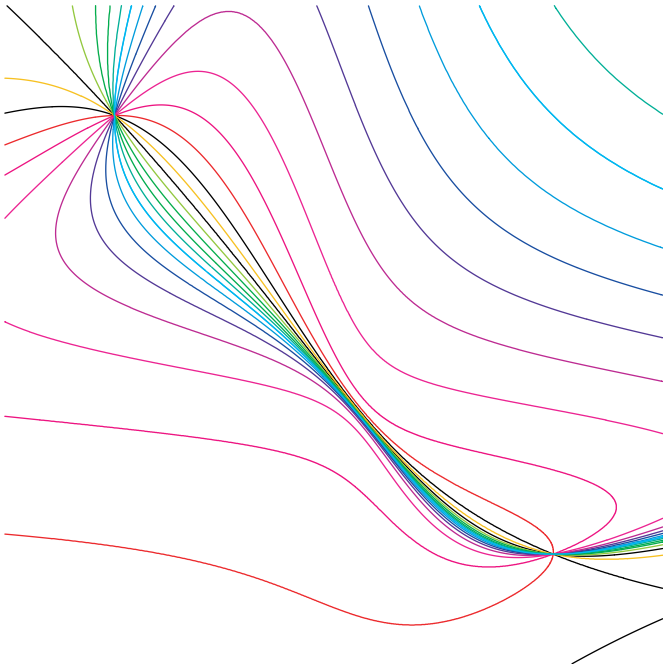


Fig. 9.2.3 The pencil for $u = -0.1$.

9.2.5.10 The Inhomogeneous Picard–Fuchs Equation

For arbitrary values of the coefficients a and b, the polynomial $\Gamma(z)$ and the rational function $\mu(z)$, Manin's function of the QRT map, which appear in Sections 2.5.3 and 2.6.3, are given by

$$\Gamma(z) = -8\,(a^2 - a\,b + b^2 - a - b + 1)\,z\,(z^3 - a\,b)\,(z^3 + a\,b),$$
$$\mu(z) = -9\,(z^3 + a\,b)^2\,(\alpha\,z^6 + \beta\,z^3 + \gamma)/z\,(z^3 - a\,b)\,\Delta(z),$$

where $\alpha = a^2\,b^2\,(a + b - 2)$,

$$\beta/2 = 2\,a^2\,b^2 + a^3 - 2\,a^2\,b - 2\,a\,b^2 + b^3 - a^2 + 4\,a\,b - b^2,$$

and $\gamma = a + b - 2$. The relatively simple form of μ and $P(z) = 3\,\Gamma(z)/\Delta(z)$ might make it feasible to prove the monotonicity properties of the period functions, using (2.6.8).

Figures 9.2.4 and 9.2.5 Figure 9.2.4 shows the rotation function of the QRT map for $a = 0.538$, $b = 0.038$, corresponding to the pencil of biquadratic curves shown in Figure 9.2.1. The horizontal coordinate in each computer plot is $(2/\pi)\,\arctan z$, $z = -z - 1/z_0$, and therefore the endpoints ± 1 correspond to $z = \pm\infty$, that is, $[z_0 : z_1] = [0 : 1]$. Because the orientation of the Hamiltonian vector field on the biquadratic curve is reversed if z goes from $z \gg 0$ to $z \ll 0$, the natural continuation of the rotation function ρ is such that $\rho(1+\epsilon) = 1 - \rho(-1+\epsilon)$ for $0 < \epsilon \ll 1$. This

is related to the fact that the real Lie algebra bundle $\mathfrak{f}(\mathbb{R})$ over $\mathbb{P}^1(\mathbb{R})$ is a Möbius strip; see Remark 8.1.3.

There are four real singular fibers, each of hyperbolic Kodaira type I_1, over $z = 0.486622\ldots$, $-0.153645\ldots$, $0.123443\ldots$, and $0.605682\ldots$. For $0.123443\ldots < z < 0.605682\ldots$ the rotation number is not defined, because for these z the real biquadratic curve has two connected components that are interchanged by the QRT map. The limit value of $\rho_\tau(z)$ for $z \uparrow 0.123443\ldots$ and for $z \downarrow 0.605682\ldots$ is equal to $1/2$. At the other two singular values of z the limit value of the rotation number is equal to 1. There is one point where the derivative of the rotation function is equal to zero.

In order also to get a rotation function on the interval $0.123443\ldots < z < 0.605682\ldots$, we have added, in Figure 9.2.5, the analogous computer plot of the rotation function ρ_{τ^2} of the square of the QRT map, which preserves the connected components of each real fiber. Note that $\rho_{\tau^2}(z) = 2\,\rho_\tau(z)$ modulo 1, whenever $\rho_\tau(z)$ is defined. Two more zeros of $\rho'_{\tau^2}(z)$ appear in the interval $0.123443\ldots < z < 0.605682\ldots$, and in total $\rho'_{\tau^2}(z)$ has three zeros. A similar situation occurs for the rotation function of Jogia's QRT map; see Figures 11.8.1 and 11.8.2.

It appears that $\rho_{\tau^2}(z)$, for increasing z, decreases to the value 0 at some point $z = \zeta$ in the regular interval $0.123443\ldots < z < 0.605682\ldots$, where it jumps from 0 to 1, and after which it continues to decrease. The real fiber over $z = \zeta$ is a fixed-point fiber for τ^2, that is, a 2-periodic fiber for τ. For regular values of z the fiber over z is 2-periodic if and only if z satisfies the third-order equation $Y = a\,b - z^3 = 0$, corresponding to the fact that there are $2^2 - 1 = 3$ complex

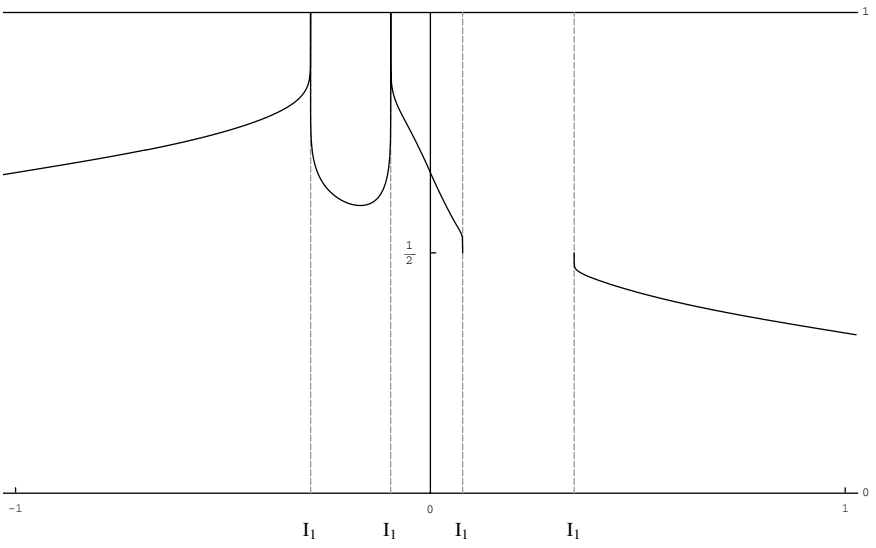


Fig. 9.2.4 Rotation function for the QRT map, $a = 0.538$, $b = 0.038$.

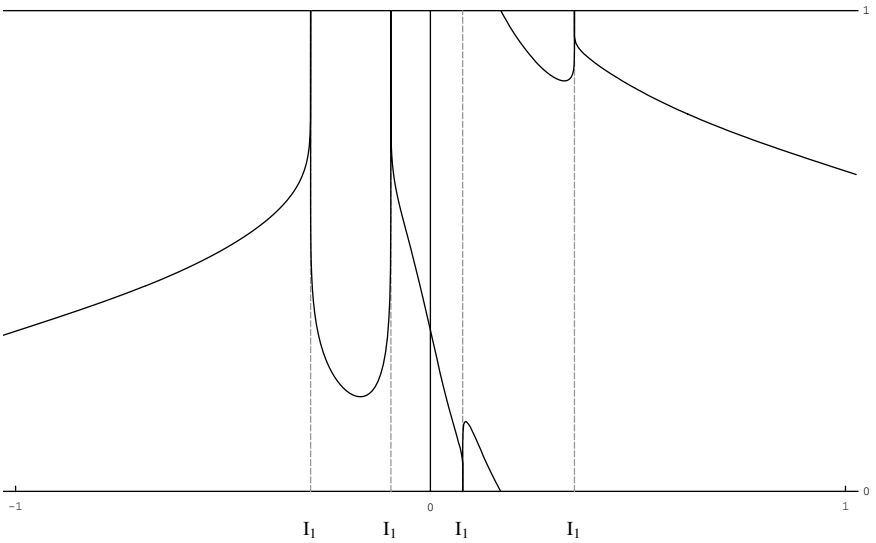


Fig. 9.2.5 Rotation function for the square of the QRT map shown in Figure 9.2.4.

2-periodic fibers for τ. The real solution of $Y = 0$ is $z = 0.273436\ldots$, which indeed lies in the interval $0.123443\ldots < z < 0.605682\ldots$. The other two solutions of $Y = 0$ are $z = -0.136718 \pm 0.236802\,\mathrm{i}\ldots$, where the corresponding 2-periodic curves for τ have no real points and are complex conjugate to each other.

Chapter 10
Symmetric QRT Maps

All the biquadratic polynomials p_z in (2.5.3) are symmetric in the sense that $p_z(x, y) = p_z(y, x)$ for all $x, y \in \mathbb{C}^2$ if and only if both coefficient matrices A^0 and A^1 in (2.5.4) are symmetric 3×3 matrices. In this case the corresponding QRT map will be called a *symmetric QRT map*.

In [169, Section 3], Quispel, Roberts, and Thompson observed that all examples of QRT mappings in their paper are symmetric, although [168] contains a nonsymmetric example; see Section 11.5.4. Also the classical examples of the elliptic billiard and the Lyness map are symmetric; see Sections 11.2 and 11.4, respectively. The planar four-bar link is symmetric if two opposite bars have the same length; see Section 11.3. Because of their prominent place in the examples in the literature, we collect in this chapter a number of basic facts about symmetric QRT mappings.

We start in Section 10.1 with the simple observation that the symmetry condition implies that the QRT map is equal to the square of the QRT root, the composition of the horizontal switch and the symmetry switch. It follows that the Mordell–Weil group contains a Manin QRT automorphism, an element that maps a section to a disjoint one, whose square is also a Manin QRT automorphism. This leads to quite strong restrictions on the configuration of the singular fibers. In Section 10.1.3 we describe the QRT root on the Weierstrass curve, and at the end of Section 10.1.3 we define the generic QRT roots.

In Section 10.2 we show that pencils of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are in bijective correspondence with pencils of *planar quadrics*, quadrics in $\mathbb{P}^2$, also called *conics*. This leads to an identification of the QRT root, defined by a pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, with the Poncelet mapping, defined by a pair of planar quadrics, where one of the quadrics varies in a pencil. See Section 10.3. When the other quadric belongs to the same pencil, then the Poncelet mapping corresponds to the elliptic billiard, which is discussed in more detail in Section 11.2. A lovely book on the subject, close in spirit to Section 2.3 and Chapter 10, is Flatto [62].

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_10,
© Springer Science+Business Media, LLC 2010

10.1 The QRT Root

The condition that the biquadratic polynomial $p(x,\ y)$ is symmetric means that p is invariant under the *symmetry switch*

$$\sigma : (x,\ y) \mapsto (y,\ x) : \mathbb{C}^2 \to \mathbb{C}^2. \tag{10.1.1}$$

If we denote the switch in $\mathbb{P}^1 \times \mathbb{P}^1$ induced by (10.1.1) by the same letter σ, then σ is an involution, that is, $\sigma \circ \sigma$ is equal to the identity. The biquadratic curve $C_{[z]}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation $p_z = 0$ is *symmetric*, that is, invariant under the symmetry switch σ. When $C_{[z]}$ is smooth, hence an elliptic curve, the restriction to $C_{[z]}$ of the involution σ has fixed points at the intersection points of $C_{[z]}$ with the diagonal

$$D := \{(x,\ y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x = y\}. \tag{10.1.2}$$

Because $C_{[z]}$ and D are a $(1,\ 1)$ curve and a $(2,\ 2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, the intersection number of $C_{[z]}$ and D is equal to 4. It follows that $\sigma|_{C_{[z]}}$ has at least one fixed point, and therefore $\sigma|_{C_{[z]}}$ is an inversion on the elliptic curve $C_{[z]}$.

Recall the inversions ι_1 and ι_2 on $C_{[z]}$, the horizontal and vertical switches, respectively; see (1.1.4), (1.1.5), and Section 2.5. Because the symmetry switch σ interchanges horizontal with vertical axes, we have that

$$\rho := \sigma \circ \iota_1 = \iota_2 \circ \sigma. \tag{10.1.3}$$

Note that ρ, as a composition of two inversions on the elliptic curve $C_{[z]}$, is a translation on the elliptic curve $C_{[z]}$. It follows that $\iota_2 = \sigma \circ \iota_1 \circ \sigma^{-1} = \sigma \circ \iota_1 \circ \sigma$, and therefore

$$\tau = \iota_2 \circ \iota_1 = \sigma \circ \iota_1 \circ \sigma \circ \iota_1 = \rho \circ \rho.$$

In other words, $\tau = \rho^2$: the QRT mapping is equal to two times the translation ρ on each elliptic curve $C_{[z]}$. For this reason the mapping ρ in (10.1.3) will be called the *QRT root of the symmetric QRT mapping* τ.

Figure 10.1.1 shows the QRT root and its square, on the same symmetric biquadratic curve as in Figures 1 and 2.

Remark 10.1.1. In affine coordinates, we have $\rho : (x,\ y) \mapsto (x',\ y) \mapsto (y,\ x')$, where $x' = F(x,\ y)$ is equal to the right-hand side in (1.1.4). The equations $\rho^n(x,\ y) = (x_n,\ y_n)$, $n \in \mathbb{Z}$, then are equivalent to $y_n = x_{n+1}$, $x_{n+2} = y_{n+1} = F(x_n,\ y_n) = F(x_n,\ x_{n+1})$. That is, the x_n, $n \in \mathbb{Z}$, satisfy the *second-order recurrence equation*

$$x_{n+2} = F(x_n,\ x_{n+1}), \tag{10.1.4}$$

whereas conversely, the QRT root ρ is defined by $\rho(x_n,\ x_{n+1}) = (x_{n+1},\ x_{n+2})$ for any solution $(x_n)_{n\in\mathbb{Z}}$ of (10.1.4). In the literature, QRT roots of pencils of symmetric biquadratic curves are often presented in the form of second-order recurrence equations. ⊘

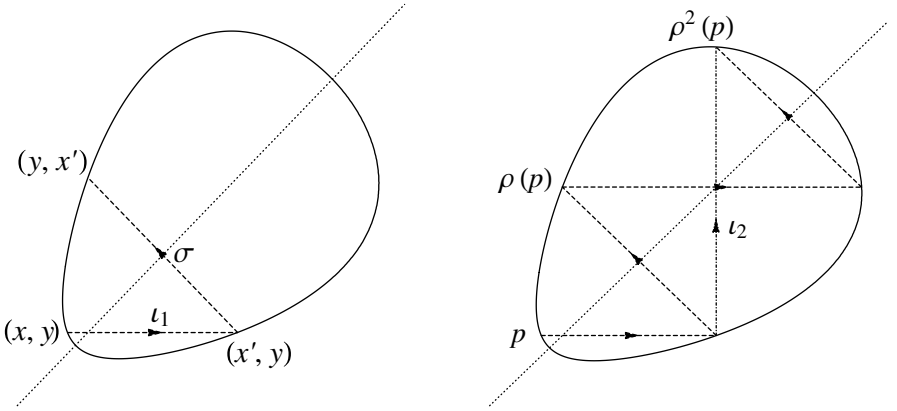


Fig. 10.1.1 The QRT root (left) and its square (right) = the QRT map.

Let $\kappa : S \to P$ be the rational elliptic surface obtained by successively blowing up, eight times, in base points of the anticanonical pencils starting with the given pencil $\mathfrak{B}$ of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The symmetry switch σ induces a birational transformation σ^S of S that preserves the fibers of κ, and therefore it follows from Lemma 3.4.1 that $\sigma^S \in \mathrm{Aut}(S)_\kappa$, that is, σ^S is a complex analytic diffeomorphism of S that leaves each fiber of κ invariant. It follows that ρ induces the automorphism

$$\rho^S := \sigma^S \circ \iota_1^S = \iota_2^S \circ \sigma^S \in \mathrm{Aut}(S)_\kappa^+ \tag{10.1.5}$$

of S, which acts as a translation on each smooth, hence elliptic, fiber of κ. We have $\tau^S = \rho^S \circ \rho^S$, that is, the QRT automorphism of S is equal to the square of ρ^S. The automorphism ρ^S in (10.1.5) will be called the *QRT root of the QRT automorphism* τ^S.

10.1.1 Reconstruction of the Symmetric Pencil

There is an explicit test whether a rational transformation $(x,\ y) \mapsto (y,\ \xi(x,\ y))$ is equal to the QRT root defined by a pencil of symmetric biquadratic curves, and if this is the case, there is a way to compute the pencil of symmetric biquadratic curves explicitly.

The first condition is that $\xi(x,\ y)$ be of the form (1.1.4) with $f(y) = (f_0(y),\ f_1(y),\ f_2(y)$ a vector-valued polynomial in y of degree ≤ 4, with coefficients f_{kl} as in (1.3.1).

We use the notation of Section 1.3, where this time E is the space of all symmetric 3×3 matrices A_{ij}, on which we use the coordinates $s_1 = A_{00}$, $s_2 = A_{01} = A_{10}$, $s_3 = A_{02} = A_{20}$, $s_4 = A_{11}$, $s_5 = A_{12} = A_{21}$, and $s_6 = A_{22}$. Let d_j, $1 \leq j \leq 6$, denote the corresponding basis of E. Then the equation $(A^0\, Y) \times (A^1\, Y) = f(y)$, see (1.1.3), is equivalent to the equation

$$
\begin{aligned}
w = {} & f_{20}\, d_1 \wedge d_2 - f_{10}\, d_1 \wedge d_3 + f_{21}\, d_1 \wedge d_4 - (f_{00} + f_{11})\, d_1 \wedge d_5 \\
& - (f_{12} + (f_{01} + f_{23})/2)\, d_1 \wedge d_6 + f_{00}\, d_2 \wedge d_3 + (2f_{00} + f_{11} + f_{22})\, d_2 \wedge d_4 \\
& + ((f_{01} + f_{23})/2)\, d_2 \wedge d_5 - (f_{13} + f_{24})\, d_2 \wedge d_6 - ((f_{01} - f_{23})/2)\, d_3 \wedge d_4 \\
& + f_{24}\, d_3 \wedge d_5 - f_{14}\, d_3 \wedge d_6 + (f_{02} + f_{13} + 2f_{24})\, d_4 \wedge d_5 + f_{03}\, d_4 \wedge d_6 \\
& + f_{04}\, d_5 \wedge d_6
\end{aligned}
$$

for $w = A^0 \wedge A^1$. In other words, the mapping $A^0 \wedge A^1 \to f$ is a linear isomorphism from $\Lambda^2 E$ onto the f-space, with inverse given by the above equations.

The second and last condition is that w have rank two, or equivalently $w \wedge w = 0$, which are $\binom{6}{4} = 15$ quadratic equations for the $3 \cdot 5 = 15$ coefficients f_{kl}. Note that the solution set in the complement of the origin in the f-space is a 9-dimensional smooth conic manifold, since it is isomorphic to the cone over the Grassmann manifold $\mathrm{G}_2(E)$, which is a smooth manifold of dimension $2 \cdot (6-2) = 8$. If the test is passed, then $w(E^*)$ is the two-dimensional space of symmetric matrices corresponding to the pencil of symmetric biquadratic curves, of which $(x,\ y) \mapsto (y,\ \xi(x,\ y)$ is the QRT root.

If the vector-valued function $f(y)$ has degree $d < 4$, then one can try to find a scalar factor $\varphi(y)$ of degree $4 - d$ such that the vector-valued function $\varphi(y)\, f(y)$ passes the test. For an illustration of this procedure, see the paragraph with the heading "Reconstruction" in Section 11.4.

If the points of indeterminacy of ι_1 in $\mathbb{P}^1 \times \mathbb{P}^1$ are explicitly known, then one can determine the space P of all symmetric biquadratic polynomials that vanish at these points of indeterminacy and their images under $\sigma : (x,\ y) \mapsto (y,\ x)$, and the test would be that $\dim P = 2$ and the given mapping is the QRT root of the pencil defined by P. For an example in which this works, see Section 11.7.

10.1.2 The Sction of the QRT Root on the Elliptic Surface

Let $\kappa : S \to P$ be a rational elliptic surface. Recall that the Manin QRT automorphisms of S are the elements α of the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ that have no fixed points in S^{reg}, that is, $\nu(\alpha) = 0$; see (ii) $\Rightarrow$ (iv) in Theorem 4.3.2. The circumstance that such an automorphism has a root has strong consequences. The following proposition and its proof give a quite detailed description of any elements ρ of the Mordell–Weil group whose square is a Manin QRT automorphism. For the "contributions," see Lemma 7.5.3. Such ρ could be called an *abstract QRT root*, where "abstract" means that ρ need not have been introduced by means of a pencil of symmetric biquadratic curves. For a discussion of the question whether every abstract QRT root is equal to the QRT root defined by a pencil of symmetric biquadratic curves, see the end of Section 10.1.3.

Proposition 10.1.2 *Let $\kappa : S \to P$ be a rational elliptic surface and $\rho \in \mathrm{Aut}(S)^+_\kappa$. If $\tau := \rho \circ \rho$ is a Manin QRT automorphism, then ρ is a Manin QRT automorphism. Conversely, if ρ is a Manin QRT automorphism, then τ is a Manin QRT automorphism*

if and only if the sum of the contributions to ρ of the reducible fibers is equal to $3/2$ *plus* $1/4$ *times the sum of the contributions to* τ. *It follows that* κ *has at least one reducible fiber whose irreducible components are permuted in a nontrivial way by* ρ.

If τ *is a Manin QRT automorphism, then the sum of the contributions to* ρ *is equal to its minimal value* $3/2$ *if and only if* τ *leaves every irreducible component of every reducible fiber invariant if and only if* τ *acts on* $\mathrm{H}_2(S, \mathbb{Z})$ *as an Eichler–Siegel transformation, if and only if the number of* k*-periodic fibers for* ρ, *counted with multiplicity, is given by*

$$\nu(\rho^k) = \begin{cases} -1 + n^2 \text{ when } & k = 2n, \quad n \in \mathbb{Z}, \\ n^2 + n \text{ when } & k = 2n + 1, \quad n \in \mathbb{Z}, \end{cases} \tag{10.1.6}$$

if and only if the configurations of reducible fibers whose irreducible components are permuted by ρ *in a nontrivial way are as in a), b), or c) in the proof of the proposition.*

The rank of the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ *of* S *is* ≤ 5, *with equality if and only if there are exactly three reducible fibers, each of Kodaira type* I_2 *or* III, *where* α *switches the two irreducible components of each of the three reducible fibers. We have* $\mathrm{Aut}(S)^+_\kappa \simeq \mathbb{Z}^5$ *if there are three reducible fibers, each of Kodaira type* I_2 *or* III.

Proof. Because every fixed point of ρ is a fixed point of τ, $\nu(\tau) = 0$ implies $\nu(\rho) = 0$. Now assume that $\nu(\rho) = 0$. Equation (4.3.2) with $k = 2$ and $\alpha = \rho$ yields that $\nu(\tau) = 3 + (1/2) \sum_r \mathrm{contr}_r(\tau) - 2 \sum_r \mathrm{contr}_r(\rho)$, where the sums are over all $r \in P^{\mathrm{red}}$. This completes the proof of the first paragraph in the proposition.

Because all contributions are nonnegative, the sum of the contributions of τ is equal to zero if and only if each contribution of τ is equal to zero if and only if τ leaves each irreducible component of each reducible fiber invariant. In view of Lemma 7.3.2(x) this happens if and only if τ^L is an Eichler–Siegel transformation. If ρ^2 leaves all irreducible components invariant, then ρ^{2l} leaves all irreducible components invariant, and $\rho^{2l+1} = \rho \circ \rho^{2l}$ permutes these in the same way as ρ. Therefore (10.1.6) follows from (4.3.2) with $\alpha = \rho$, and the sum of the contributions of α equal to $3/2$. Conversely, (10.1.6) and (4.3.2) with $\alpha = \rho$ imply that

$$\frac{1}{4} = \lim_{k \to \infty} \frac{\nu(\rho^k)}{k^2} = 1 - \frac{1}{2} \sum_{r \in P^{\mathrm{red}}} \mathrm{contr}_r(\rho);$$

hence the sum of the contributions to ρ of the reducible fibers is equal to $3/2$.

According to Lemma 6.3.31, ρ permutes the irreducible components of a reducible fiber S_c by means of an element of $\mathfrak{F}_c/\mathfrak{F}_c{}^\circ$. The description of the $\mathfrak{F}_c$ in Section 6.3.6 shows that ρ permutes the irreducible components of S_c in a nontrivial way and ρ^2 leaves each irreducible component of S_c invariant if and only if we are in one of the following cases (i)–(iv). For the contribution to ρ of a reducible fiber S_c, see Lemma 7.5.3.

(i) S_c is of Kodaira type I_{2k}, $k \in \mathbb{Z}_{>0}$, ρ translates the cycle over k units, $\mathrm{contr}_c(\rho) = k/2$.

(ii) S_c is of Kodaira type I_b^*, $b \in \mathbb{Z}_{\geq 0}$, and ρ leaves each multiplicity-two component invariant if b is odd. We have $\mathrm{contr}_c(\rho) = 1$ if ρ leaves each multiplicity-two component invariant and $\mathrm{contr}_c(\rho) = 1 + (b/4)$ otherwise.
(iii) S_c is of Kodaira type III, where ρ switches the two irreducible components of S_c and $\mathrm{contr}_c(\rho) = 1/2$.
(iv) S_c is of Kodaira type III*, and $\mathrm{contr}_c(\rho) = 3/2$.

In addition, the sum of the contributions of ρ is equal to $3/2$ if and only if we have the following configurations of reducible fibers whose irreducible components are permuted by ρ in a nontrivial way:

(a) There are three fibers of type I_2 or III, where ρ switches the two irreducible components of each of these.
(b) There is one fiber of type I_4 or I_b^*, where in the first case ρ shifts the cycle over two units and in the second case leaves each multiplicity-two component invariant while switching the multiplicity-one components, combined with another fiber of type I_2 or III where ρ switches its two irreducible components.
(c) There is one fiber of type I_6 where ρ shifts the cycle over three units, or one fiber of type I_2^* where ρ inverts the chain of multiplicity-two components, or one fiber of type III* where ρ switches its two multiplicity-one components.

This completes the proof of the second paragraph in the proposition. According to Lemma 9.2.6, the rank of the Mordell–Weil group is equal to $8 - \sum_r (n_r - 1)$, where n_r is the number of irreducible components of the reducible fiber S_r. It follows from Lemma 7.5.3 that the contribution of S_r to ρ is $\leq 1/2$ if $n_r = 2$, $\leq 2/3$ if $n_r = 3$, and ≤ 1 if $n_r = 4$. Since the sum of the contributions to ρ has to be $\geq 3/2$, it follows that the rank of the Mordell–Weil group is ≤ 5, with equality if and only if there are three reducible fibers, each with two irreducible components that are switched by ρ. Lemma 9.2.6 also implies that the Mordell–Weil group is isomorphic to $\mathbb{Z}^5$ as soon as there are three reducible fibers, each of type I_2 or III. □

Remark 10.1.3. For an arbitrary elliptic fibration $\varphi : S \to C$ the Mordell–Weil group $\simeq \mathrm{Aut}(S)_\varphi^+$ is commutative. Therefore, if $\beta = \alpha \circ \alpha$ is the square of an element $\alpha \in \mathrm{Aut}(S)_\varphi^+$, then $\beta := \alpha' \circ \alpha'$ for $\alpha' \in \mathrm{Aut}(S)_\varphi^+$ if and only $\alpha' = \eta \circ \alpha$, where η is an element of order two in $\mathrm{Aut}(S)_\varphi^+$. It follows that a root in $\mathrm{Aut}(S)_\varphi^+$ is not unique if and only if $\mathrm{Aut}(S)_\varphi^+$ contains nontrivial elements of order two. Lemma 9.2.6 shows that there are quite a few configurations of singular fibers of rational elliptic surfaces for which the Mordell–Weil group contains nontrivial elements of order two. ⊘

Remark 10.1.4. Because QRT automorphisms have no fixed points, see (i) ⇔ (iv) in Theorem 4.3.2, Proposition 10.1.2 can be applied to $\rho = \rho^S =$ the QRT root and $\tau = \tau^S =$ the symmetric QRT automorphism. Regarding the conclusion in the second paragraph of Proposition 10.1.2, we observe that (ii) ⇔ (iii) in Corollary 5.1.12 implies that τ leaves every irreducible component of every reducible fiber invariant if and only if no member C of the pencil of biquadratic curves contains a horizontal or a vertical axis. Note that in the symmetric case the invariance of C under the switch σ implies that if C contains the horizontal (vertical) axis L, then it also contains

the vertical (horizontal) axis $\sigma(L)$. For a more explicit description of the irreducible fiber components that are nontrivially permuted by the QRT root, and the condition in that the rank of the Mordell–Weil group is equal to 5, see Corollary 10.2.6.

The third paragraph in Proposition 10.1.2, which holds for each symmetric QRT surface, should be compared with Proposition 9.2.22, which implies that for a general rational elliptic surface the rank of the Mordell–Weil group is ≤ 8, and that the rank 8 occurs if and only if S is a Manin elliptic surface, that is, all the fibers of κ are irreducible. In particular, a symmetric QRT surface cannot be a Manin elliptic surface. ⊘

10.1.3 The QRT Root on the Weierstrass Curve

Let $\kappa : S \to \mathbb{P}^1$ be a rational elliptic surface, $\rho \in \mathrm{Aut}(S)^+_\kappa$, and $\tau = \rho \circ \rho$. Let g_2 and g_3 be the homogeneous polynomials in $(z_0,\ z_1)$ of the respective degrees 4 and 6 that define the Weierstrass model of S as described in Example 5 after Remark 6.3.11 with $N = 1$. If $(X,\ Y)$ and $(X_\rho,\ Y_\rho)$ are the affine coordinates $(x,\ y)$ of the point on the Weierstrass curve $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$ to which τ and ρ send the point at infinity, respectively, then the formulas (7.8.4) and (7.8.5) with $\alpha = \rho$ imply that

$$X = D^2/4 - 2\,X_\rho \quad \text{and} \quad Y = -D^3/4 + 3\,X_\rho\,D - Y_\rho. \tag{10.1.7}$$

Here we have substituted $D := (12\,{X_\rho}^2 - g_2)/2\,Y_\rho$, the derivative $\mathrm{d}y/\,\mathrm{d}x$ at $x = X_\rho$ and $y = Y_\rho$ on the Weierstrass curve $y^2 - 4\,x^3 + g_2\,x + g_3 = 0$. As observed in Remark 7.8.2, the respective elements ρ and τ of the Mordell–Weil group are Manin QRT automorphisms if and only if $(X_\rho,\ Y_\rho)$ and $(X,\ Y)$ are polynomials in $(z_0,\ z_1)$. Therefore if ρ (τ) is a Manin QRT automorphism, then τ (ρ) is a Manin QRT automorphism if and only if D^2 is a polynomial if and only if D is a polynomial, which then is homogeneous of degree one in $(z_0,\ z_1)$.

Conversely, for any homogeneous polynomials D, X_ρ, and Y_ρ in $(z_0,\ z_1)$ of respective degrees 1, 2, and 3, the formulas

$$g_2 = 12\,{X_\rho}^2 - 2\,D\,Y_\rho \quad \text{and} \tag{10.1.8}$$

$$g_3 = -{Y_\rho}^2 + 4\,{X_\rho}^3 - g_2\,X_\rho = -{Y_\rho}^2 - 8\,{X_\rho}^3 + 2\,D\,X_\rho\,Y_\rho \tag{10.1.9}$$

define the Weierstrass model of a rational elliptic surface such that $(X_\rho,\ Y_\rho)$ is the image of the point at inifinity under the action of an abstract QRT root ρ, where $(X,\ Y)$ defined by (10.1.7) is the image of the point at inifinity under the action of the Manin QRT automorphism $\tau = \rho \circ \rho$. If H_k denotes the space of all homogeneous polynomials of degree k in $(z_0,\ z_1)$, then $(D,\ X_\rho,\ Y_\rho) \in \mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$ will be called the *Weierstrass data of the abstract QRT root* ρ.

The conditions in Example 5 after Remark 6.3.11 with $N = 1$ are that the discriminant

$$\Delta = {Y_\rho}^2 \left[36\,(D^2 - 12\,X_\rho)\,{X_\rho}^2 - (8\,D^3 - 108\,D\,X_\rho + 27\,Y_\rho)\,Y_\rho)\right] \tag{10.1.10}$$

is not identically equal to zero and g_2 and g_3 do not have a common zero of respective orders ≥ 4 and ≥ 6 at some point z. If the latter happens, then Δ has a zero of order 12 at z, when the fact that Y_ρ^2 is a factor of Δ implies that Y_ρ has a zero of order ≥ 3 at z. Then (10.1.8) and (10.1.9) imply that X_ρ and D have a zero of the respective orders ≥ 2 and ≥ 1 at z. Conversely, if D, X_ρ, and Y_ρ have a zero of respective orders 1, 2, and 3 at z, then g_2 and g_3 have a zero of respective orders ≥ 4 and ≥ 6 at z. Therefore the conditions for the Weierstrass model are that there is no $z = (z_0, z_1) \neq (0, 0)$ such that D, X_ρ, and Y_ρ have a zero at z of respective order 1, 2, and 3, that Y_ρ is not identically equal to zero, and that $R := 36\,(D^2 - 12\,X_\rho)\,{X_\rho}^2 - (8\,D^3 - 108\,D\,X_\rho\,Y_\rho + 27\,Y_\rho)\,Y_\rho$ is not identically equal to zero. Because the discriminant of the last expression with respect to Y_ρ is equal to $64\,(D^2 - 9\,X_\rho)^3$, the last condition holds if and only if there does not exist a homogeneous polynomial L of degree one in (z_0, z_1) such that $X_\rho = (D^2 - L^2)/9$ and $Y_\rho = 2\,(D - L)^2\,(D + 2\,L)/27$. It follows that the set of all (D, X_ρ, Y_ρ) such that g_2 and g_3 in (10.1.8), (10.1.9) do not satisfy the conditions in Example 5 after Remark 6.3.11 with $N = 1$ form a subset of codimension four in $\mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$. Working modulo linear substitutions of variables in (z_0, z_1), we obtain a moduli space of QRT roots of dimension $(1 + 1) + (2 + 1) + (3 + 1) - 4 = 5$, as compared to the dimension 8 of the moduli space of all QRT maps; see Section 6.3.3.

According to Table 6.3.2, the configuration of the singular fibers is $3\ \mathrm{I}_2\ 6\ \mathrm{I}_1$ if and only if the discriminants of the factors Y_ρ and R of Δ are not equal to zero, and the resultants of Y_ρ and X_ρ and of Y_ρ and $D^2 - 12\,X_\rho$ are not equal to zero. In this case the QRT root with the Weierstrass data (D, X_ρ, Y_ρ) will be called the *generic QRT root*. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}^5$ if we have the configuration of singular fibers $3\ \mathrm{I}_2\ 6\ \mathrm{I}_1$. None of the symmetric examples in Chapter 11 is generic.

According to Proposition 10.1.2, the rank of the Mordell–Weil group is is ≤ 5, with equality if and only the Mordell–Weil group is isomorphic to $\mathbb{Z}^5$ if and only if there are exactly three reducible fibers, each of type I_2 or III. If, for example, $D = z_0 - z_1$, $X_\rho = z_0\,z_1$, and $Y_\rho = 2\,z_0\,z_1\,(z_0 - z_1)/3$, then $g_2 = -4\,z_0\,z_1\,({z_0}^2 - 11\,z_0\,z_1 + {z_1}^2)/3$, $g_3 = 8\,{z_0}^2\,{z_1}^2\,({z_0}^2 - 11\,z_0\,z_1 + {z_1}^2)/9$, $\Delta = -64\,{z_0}^3\,{z_1}^3\,(z_0 - z_1)^2\,({z_0}^2 - 11\,z_0\,z_1 + {z_1}^2)^2/27$, and the configuration of the singular fibers is $2\ \mathrm{III}\ \mathrm{I}_2\ 2\ \mathrm{II}$. These are Weierstrass data of a QRT root where the Mordell–Weil group is still isomorphic to $\mathbb{Z}^5$, but the configuration of the singular fibers is quite a bit more degenerate than $3\ \mathrm{I}_2\ 6\ \mathrm{I}_1$, the generic configuration of singular fibers for a QRT root.

Remark 10.1.5. Because Y_ρ is homogeneous of degree three, its zero-set in $\mathbb{P}^1$ has three elements, when counted with multiplicities. Let $z = (z_0, z_1) \neq (0, 0)$ and $Y_\rho(z) = 0$. If $12\,{X_\rho}^2(z) - g_2(z) \neq 0$. Then Remark 7.8.5 with $\alpha = \rho$ implies that $S_{[z]}$ is a 2-periodic fiber of ρ, hence a fixed-point fiber of $\tau = \rho \circ \rho$, in contradiction to the assumption that τ is a Manin QRT automorphism. Therefore Remark 7.8.5 implies that $Y_\rho(z) = 0$ if and only if $S_{[z]}$ is one of the reducible fibers of κ whose irreducible components are permuted in a nontrivial way by ρ. These are the reducible fibers with a positive contribution to ρ, mentioned in Proposition 10.1.2. It follows that there is at least one and at most three such fibers. This proof of the existence of such fibers in quite different from the proof of Proposition 10.1.2. The above

argument also shows that $Y_\rho(z) = 0 \Rightarrow S_{[z]}$ is singular $\Rightarrow \Delta(z) = 0$, in agreement with the fact that Δ has Y_ρ as a factor; see (10.1.10). ⊘

For the QRT root we have the following analogue of Proposition 2.5.6.

Proposition 10.1.6 *For a symmetric biquadratic polynomial with the symmetric coefficient matrix A_{ij}, the Weierstrass data of the QRT root are given by*

$$D = A_{11} - 4\,A_{02}, \tag{10.1.11}$$

$$X_\rho = ({A_{11}}^2 + 4\,A_{01}\,A_{12} - 4\,A_{00}\,A_{22} - 12\,A_{02}\,A_{11} + 20\,{A_{02}}^2)/12, \tag{10.1.12}$$

$$\begin{aligned} Y_\rho = {} & -4\,{A_{02}}^3 + 4\,{A_{02}}^2\,A_{11} - A_{02}\,{A_{11}}^2 - 2\,A_{01}\,A_{02}\,A_{12} \\ & + A_{01}\,A_{11}\,A_{12} - A_{00}\,{A_{12}}^2 - {A_{01}}^2\,A_{22} + 4\,A_{00}\,A_{02}\,A_{22}. \end{aligned} \tag{10.1.13}$$

Proof. Assume that $A_{22} = 0$, that is, the point $([1 : 0], [1 : 0])$ lies on the biquadratic curve C. Let $([1 : x(t)], [1 : y(t)])$ be the solution curve of the Hamiltonian system in Lemma 2.4.5 such that $([1 : x(0)], [1 : y(0)]) = \rho([1 : 0], [1 : 0])$. Note that $([1 : x(t)], [1 : y(t)]) = \rho([1 : \xi(t)], [1 : \eta(t)])$, if $([1 : \xi(t)], [1 : \eta(t)])$ is the solution curve of the Hamiltonian system in Lemma 2.4.5 such that $([1 : \xi(0)], [1 : \eta(0)]) = ([1 : 0], [1 : 0])$. Then (10.1.12) and (10.1.13) follow from taking the limit for $t \to 0$ in the formulas in Lemma 2.4.13 for $\wp(x(t),\ y(t))$ and $wp'(x(t),\ y(t))$, respectively. In order to obtain a well-defined limit, we need the Taylor expansion of the functions $(\xi(t),\ \eta(t))$ or $(x(t),\ y(t))$ at $t = 0$ up to the order two.

The formula (10.1.11) follows from the substitution of (10.1.12), (10.1.13), and g_2, see Corollary 2.4.7, in $D = g_2 = (12\,{X_\rho}^2 - g_2)/2\,Y_\rho$; see (10.1.8), (10.1.9). □

The formula (10.2.4) expresses Y_ρ, see (10.1.13), as the determinant of a suitable symmetric 3×3 matrix B.

Remark 10.1.7. Let $\mathfrak{S}$ denote the six-dimensional vector space of all *symmetric* biquadratic polynomials p, parametrized as in Proposition 10.1.6 by the coefficients A_{ij} with $0 \le i \le j \le 2$. For $L \in \mathrm{GL}(2, \mathbb{C})$ we write $(L,\ L)(x,\ y) = (L\,x,\ L\,y)$. Then $(L, p) \mapsto p \circ (L,\ L)$ defines an action of $\mathrm{GL}(2, C)$ on $\mathfrak{S}$. The homogeneous polynomials D, X_ρ, and Y_ρ of degrees 1, 2, and 3 on $\mathfrak{S}$ are projective invariants of orders 1, 2, and 3 for this action of $\mathrm{GL}(2, C)$ on $\mathfrak{S}$, in the sense that $D(p \circ (L,\ L)) = (\det L)\,D(p)$, $X_\rho(p \circ (L,\ L)) = (\det L)^2\,X_\rho(p)$, and $Y_\rho(p \circ (L,\ L)) = (\det L)^3\,Y_\rho(p)$, respectively.

The restrictions to $\mathfrak{S}$ of the projective invariants X, Y, g_2, and g_3 of the action of G on $\mathfrak{B}$, see Remark 2.5.7, are projective invariants of the action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathfrak{S}$, of orders 2, 3, 4, and 6, respectively. The equations (10.1.7) and (10.1.8), (10.1.9) show that these projective invariants are polynomials in D, X_ρ, and Y_ρ.

Frobenius [64, (5) on p. 177] defined the projective invariants n, p, and q of the respective degrees 1, 2, and 3, and his [64, Theorem VI on p. 181] says that every polynomial projective invariant for symmetric biquadratic polynomials is a polynomial in n, p, and q. A direct comparison shows that $n = D/4$, $p = X_\rho/4$, and $q = -Y_\rho/8$, and therefore D, X_ρ, and Y_ρ are basic polynomial projective invariants

for symmetric biquadratic polynomials too. Because of this, I propose to call D, X_ρ, and Y_ρ *the Frobenius invariants of symmetric biquadratic polynomials*. ⊘

Every abstract QRT root is the QRT root of a pencil of symmetric biquadratic curves if and only if every Weierstrass data $(D,\ X_\rho,\ Y_\rho)$ of an abstract QRT root is given by (10.1.11), (10.1.12), (10.1.13), where $A_{ij} = z_0\, A^0_{ij} + z^1\, A^1_{ij}$ and A^0, A^1 are suitable symmetric 3×3 matrices of coefficients of symmetric biquadratic polynomials.

For the admittedly somewhat arbirary example

$$A^0 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

we have $D = 3\, z_0$, $X_\rho = (5\, {z_0}^2 - 4\, {z_1}^2)/12$, and $Y_\rho = -2\, z_0\, z_1\, (z_0 + 3\, z_1)$. These are the Weierstrass data of a generic QRT root. The derivative at this special $(A^0,\ A^1)$, of the coefficients of D, X_ρ, Y_ρ with respect to the coefficients of arbitrary pairs of symmetric matrices, is surjective. In view of the implicit function theorem it follows that the set R of $(D,\ X_\rho,\ Y_\rho)$ for such pairs of matrices has a nonempty interior. Chevalley's theorem on constructible sets implies that the set R is constructible, and therefore contains the complement of an algebraic subset of positive codimension in $\mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3$. This shows that in a quite strong sense "most" abstract QRT roots are defined by a symmetric pencil. I do not know whether every abstract QRT root can be defined by a symmetric pencil.

10.2 Pencils of Planar Quadrics

The algebra of all polynomials in two variables $(x,\ y) \in \mathbb{C} \times \mathbb{C}$ that are invariant under the symmetry switch $(x,\ y) \mapsto (y,\ x)$ is freely generated by the polynomials $x + y$ and $x\, y$. We have $(1,\ x + y,\ x\, y) = \Psi((1,\ x),\ (1,\ y))$, where we use the bihomogeneous mapping

$$\Psi : ((x_0,\ x_1),\ (y_0,\ y_1)) \mapsto (x_0\, y_0,\ x_0\, y_1 + x_1\, y_0,\ x_1\, y_1) \tag{10.2.1}$$

of bidegree $(1,\ 1)$ from $\mathbb{C}^2 \times \mathbb{C}^2$ to $\mathbb{C}^3$. The mapping Ψ induces a linear isomorphism $\Psi^* : f \mapsto f \circ \Psi$ from the space of all homogeneous polynomials of degree 2 on $\mathbb{C}^3$ onto the space of all biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$. More explicitly, if

$$f(u) = u\, B\, u = \sum_{i,\, j=0}^{2} u_i\, B_{ij}\, u_j, \tag{10.2.2}$$

in which B_{ij} is a symmetric 3×3 matrix, then $p = f \circ \Psi$ is given by (2.4.1), where $A_{00} = B_{22}$, $A_{01} = A_{10} = B_{12} + B_{21} = 2\, B_{12}$, $A_{02} = A_{20} = B_{11}$, $A_{11} = B_{02} + 2\, B_{11} + B_{20} = 2\, (B_{02} + B_{11})$, $A_{12} = A_{21} = B_{01} + B_{10} = 2\, B_{01}$, $A_{22} = B_{00}$.

For a symmetric matrix A, these equations are equivalent to $B_{00} = A_{22}$, $B_{01} = B_{10} = A_{12}/2$, $B_{02} = B_{20} = A_{11}/2 - A_{02}$, $B_{11} = A_{02}$, $B_{12} = B_{21} = A_{01}/2$, $B_{22} = A_{00}$.

Lemma 10.2.1 *The mapping* Ψ *in* (10.2.1) *induces an algebraic morphism = a complex analytic mapping*

$$\psi : ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0\, y_0 : x_0\, y_1 + x_1\, y_0 : x_1\, y_1] \tag{10.2.3}$$

from $\mathbb{P}^1 \times \mathbb{P}^1$ *to* $\mathbb{P}^2$.

The mapping $C \mapsto \psi(C)$ *defines a bijective mapping from the set of all symmetric biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$ *onto the set of all quadrics in* $\mathbb{P}^2$*, thereby identifying pencils of symmetric biquadratic curves in* $\mathbb{P}^1 \times \mathbb{P}^1$ *with pencils of quadrics in* $\mathbb{P}^2$.

Proof. If $x_0\, y_0 = x_0\, y_1 + x_1\, y_0 = x_1\, y_1 = 0$ and $(x_0, x_1) \neq (0, 0)$, $(y_0, y_1) \neq (0, 0)$, then $x_0 = 0$ implies that $x_1 \neq 0$, $x_1\, y_0 = 0$, $x_1\, y_1 = 0$, hence $y_0 = y_1 = 0$, a contradiction. Similarly $y_0 = 0$ implies $y_1 \neq 1$, $x_0\, y_1 = 0$, $x_1\, y_1 = 0$, hence $x_0 = x_1 = 0$, and we have a contradiction again. It follows that the homogeneous mapping Ψ maps $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ to $\mathbb{C}^3 \setminus \{0\}$, hence it induces an algebraic morphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. □

Motivated by Lemma 10.2.1, we collect some classical facts about planar quadrics.

Lemma 10.2.2 *Let* Q *be a planar quadric defined by an equation* $f = 0$ *with* f *as in* (10.2.2)*, where* B *is a symmetric* 3×3 *matrix. Then* $[u]$ *is a smooth point of* Q *if and only if* $B\, u \neq 0$*, and if this is the case, then the complex projective line in* $\mathbb{P}^2$ *that is tangent to* Q *at* u *is equal to*

$$\ker(B\, u) = \{[v] \in \mathbb{P}^2 \mid v\, B\, u = 0\}.$$

It follows that Q *is smooth if and only if* $\det B \neq 0$*. With* B *expressed in terms of* A *by means of the formulas preceding Lemma* 10.2.1*, and* Y_ρ *given by* (10.1.13)*, we have*

$$\det B = Y_\rho/4. \tag{10.2.4}$$

If Q *is smooth, then every other smooth planar quadric can be transformed into* Q *by means of a projective linear transformation. Furthermore,* Q *is isomorphic to* $\mathbb{P}^1$.

If Q *is not smooth, then either* Q *is equal to the union of two distinct complex projective lines, with the intersection point as the only nonsmooth point of* Q*, or* Q *is equal to a double complex projective line and* Q *has no smooth points.*

Proof. Because of the symmetry of B, the total derivative of f at u is equal to $v \mapsto 2\, v\, B\, u$. This proves the first statement. We have $\det B = 0$ if and only if there exists $u \neq 0$ such that $B\, u = 0$, which implies that $u\, B\, u = 0$, and therefore $[u] \in Q$ and Q is not smooth at $[u]$. A direct computation yields (10.2.4).

If $[u]$ is a smooth point of Q then we can arrange by means of a projective linear transformation that $[u] = [0 : 0 : 1]$, and that the tangent line of Q at u is given by the equation $u_0 = 0$. This means that $B_{22} = 0$, $B_{12} = B_{21} = 0$, and $B_{02} \neq 0$.

Replacing u_2 by $-(B_{00}/2B_{02})\,u_0 - (B_{01}/B_{02})\,u_1 + (1/B_{02})\,u_2$, we can arrange that $B_{00} = 0$, $B_{01} = B_{10} = 0$, and $B_{02} = B_{20} = 1$. If Q is not smooth, then $0 = \det B = -B_{02}^2\,B_{11}$, hence $B_{11} = 0$, and Q is equal to the union of the two distinct projective lines defined by the equations $u_0 = 0$ and $u_2 = 0$. If Q is smooth, then $B_{11} \neq 0$, and by a scaling of u_1 we can arrange that $B_{11} = 1$, and we have arrived at the quadric $2u_0\,u_2 + {u_1}^2 = 0$. This normal form shows that all smooth planar quadrics are projectively equivalent.

Let Q be a smooth quadric, and let $q \in Q$. Let P be the space of all complex projective lines through q. For each $L \in P$, let $\iota(L)$ be the second intersection point of L with Q, where $\iota(L) = q$ if L is tangent to Q at q. Then ι is an isomorphism from $P \simeq \mathbb{P}^1$ onto Q.

If Q has no smooth points, then $u\,B\,u = 0$ implies that $B\,u = 0$. Because Q contains at least two points, it follows that $\dim\ker B \geq 2$, and because $B \neq 0$, we have $\dim\ker B = 2$. By means of a linear transformation we can arrange that $\ker B = \{u_2 = 0\}$, which means that the first two columns, and therefore also the first two rows, of B are equal to zero. This leads to $u\,B\,u = B_{22}\,{u_2}^2$, and Q is equal to the line $u_2 = 0$ with multiplicity 2. □

The following lemma implies that there are only five isomorphism classes of pencils of planar quadrics with at least one smooth member.

Lemma 10.2.3 *Let Ω be a pencil of quadrics in $\mathbb{P}^2$ with at least one smooth member. Then, counted with multiplicities, Ω has four base points and three singular members. We have the following cases.*

(i) *There are four distinct base points, no three of them on a complex projective line. Any two distinct smooth members intersect each other at the four distinct base points. There are three singular members, and for each partition of the set B of base points into two pairs $B = \{b_1,\ b_2\} \cup \{b_3,\ b_4\}$ we take the union of the line through b_1 and b_2 with the line through b_3 and b_4. By means of a projective linear transformation in $\mathbb{P}^2$ and of $[z_0 : z_1] \in \mathbb{P}^1$ the pencil can be brought into the form $z_0\,u_0\,(u_0 - u_1 - u_2) + z_1\,u_1\,u_2 = 0$.*

(ii) *One base point b has multiplicity two and the two other base points b_1 and b_2 are simple, where b, b_1, and b_2 do not lie on a line. There are two singular members, one equal to the union of the line through b and b_1 with the line through b and b_2. The other singular member is the union of the line through b_1 and b_2 and a line L through b and not through b_1 or b_2. For each smooth member Q, L is tangent to Q at b. By means of a projective linear transformation in $\mathbb{P}^2$ and of $[z_0 : z_1] \in \mathbb{P}^1$ the pencil can be brought into the form $z_0\,u_0\,(u_1 - u_2) + z_1\,u_1\,u_2 = 0$.*

(iii) *There are two base points b and b', of multiplicity three and one, respectively. There is only one singular member, equal to the union of the line $b\,b'$ though b and b' and another line L through b. For each smooth member Q of Ω, L is tangent to Q at b. By means of a projective linear transformation in $\mathbb{P}^2$ and of $[z_0 : z_1] \in \mathbb{P}^1$ the pencil can be brought into the form $z_0\,({u_1}^2 - u_0\,u_2) + z_1\,u_1\,u_2 = 0$.*

(iv) *There are two base points b_1 and b_2, each of multiplicity two. There are two singular members, one of which is equal to the double line L through b_1 and*

b_2, whereas the other is the union of two distinct lines L_1 and L_2, $L_i \neq L$, with $b_i \in L_i$. The double line through b_1 and b_2 is the singular member of multiplicity 2. For each smooth member Q of $\mathfrak{Q}$, L_i is tangent to Q at b_i. By means of a projective linear transformation in $\mathbb{P}^2$ and of $[z_0 : z_1] \in \mathbb{P}^1$ the pencil can be brought into the form $z_0\, u_1\, u_2 + z_1\, {u_0}^2 = 0$.

(v) *There is only one base point b of multiplicity 4, and only one singular member, equal to a double line L through the point b. For each smooth member Q of $\mathfrak{Q}$, L is tangent to Q at b. By means of a projective linear transformation in $\mathbb{P}^2$ and of $[z_0 : z_1] \in \mathbb{P}^1$ the pencil can be brought into the form $z_0\, ({u_1}^2 - u_0\, u_2) + z_1\, {u_0}^2 = 0$.*

Proof. The pencil of quadrics is given by $u\, B\, u = 0$ where $B = z_0\, B^0 + z_1\, B^1$ where B^0 and B^1 are linearly independent symmetric 3×3 matrices.

Let one of the singular members be a double line. By means of a projective linear transformation we can arrange that this singular member corresponds to $[z_0 : z_1] = [0 : 1]$, and is given by the equation $u\, B^1\, u = {u_0}^2 = 0$. Then

$$\det(z_0\, B^0 + z_1\, B^1) = z_1\, {z_0}^2\, (B^0_{11}\, B^0_{22} - B^0_{12}\, B^0_{21}) + {z_0}^3\, \det B^0.$$

We have only one singular member if and only if $d := B^0_{11}\, B^0_{22} - B^0_{12}\, B^0_{21} = 0$, and by means of a linear substitution in the variables (u_1, u_2) we can arrange that $B^0_{12} = B^0_{21} = B^0_{22} = 0$. If $B^0_{11} = 0$, then $\det B^0 = 0$ and $\mathfrak{Q}$ has no smooth members. Therefore $B^0_{11} \neq 0$. Replacing u_1 by $u_1 - u_0\, (B^0_{01} + B^0_{10})/B^0_{11}$ we can arrange that in addition $B^0_{01} = B^0_{10} = 0$, and by adding a suitable multiple of B^1 to B^0 we can arrange that in addition $B^0_{00} = 0$. If $B^0_{02} = B^0_{20} = 0$ then $\det B \equiv 0$, and $\mathfrak{Q}$ has no smooth members. After a suitable scaling we arrive at the normal form in (v). The line $u_0 = 0$ is tangent to the smooth member ${u_1}^2 - u_0\, u_2 = 0$ at the point $[0 : 0 : 1]$, and therefore $[0 : 0 : 1]$ is a base point of multiplicity four.

If $d \neq 0$, when the double line is a singular member of multiplicity two, we can replace z_1 by $z_1 - z_0\, \det B^0/d$, after which we have $\det B^0 = 0$, hence $\operatorname{rank} B^0 \leq 2$. Because $d \neq 0$ implies $\operatorname{rank} B^0 \geq 2$, we have $\operatorname{rank} B^0 = 2$, and $u\, B^0\, u$ is the union of two distinct complex projective lines, different from $u_0 = 0$. Because their intersection point $\ker B^0$ is not contained in $u_0 = 0$, we can arrage by means of a projective linear transformation that the second singular member is given by the equation $u_1\, u_2 = 0$, which means that we are in case (iv).

In the remaining cases every singular member is the union of two distinct lines, and by means of a projective linear transformation we can arrange that one of these, corresponding to $[z_0 : z_1] = [0 : 1]$, is given by the equation $u\, B^1\, u = u_1\, u_2 = 0$. In this case

$$\det(z_0\, B^0 + z_1\, B^1) = -{z_1}^2\, z_0\, B^0_{00} - 2\, z_1\, {z_0}^2\, (B^0_{00}\, B^0_{12} - B^0_{10}\, B^0_{02}) + {z_0}^3\, \det B^0.$$

There is only one singular member if and only if $B^0_{00} = 0$ and $B^0_{10}\, B^0_{02} = 0$, where we can arrange that $B^0_{10} = B^0_{01} = 0$ by interchanging u_1 and u_2 if necessary. Because $\det B^0 = 0$ would imply that $\det B \equiv 0$, we then have $B^0_{02} = B^0_{20} \neq 0$ and $B^0_{11} \neq 0$. replacing u_0 by $u_0 - u_1\, B^0_{12}/B^0_{02} - u_2\, B^0_{22}/2\, B_{02}$, we can arrange that

$B^0_{12} = B^0_{21} = 0$ and $B^0_{22} = 0$, and by scaling u_0, u_1, u_2 we arrive at the normal form in (iii). Note that at $[u] = [1 : 0 : 0]$ the line $u_2 = 0$ is tangent to the quadric ${u_1}^2 - u_0\, u_2 = 0$, and therefore this is a base point of multiplicity three, whereas $[0 : 0 : 1]$ is the other base point of multiplicity one.

The singular member is of multiplicity two if and only if $B^0_{00} = 0$, $B^0_{01} = B^0_{10} \neq 0$ and $B^0_{02} = B^0_{20} \neq 0$. In this case there is a second singular member C, of multiplicity one, which we can arrange to correspond to $[z_0 : z_1] = [1 : 0]$, which means that $\det B^0 = 0$. Because we already have seen that the multiplicity of the singular member is ≥ 2 if it is a double line, C is equal to the union of two distinct lines L_1 and L_2, neither of which is equal to $u_1 = 0$ or $u_2 = 0$. Because $B^0_{00} = 0$ means that $[1 : 0 : 0] \in C$, one of these lines, say L_1, passes through $[1 : 0 : 0]$. Then L_2 does not pass through $[1 : 0 : 0]$, because otherwise $[1 : 0 : 0]$ would be a singular point of each member of $\mathcal{Q}$. By means of a projective linear transformation that leaves $u_1 = 0$ and $u_2 = 0$ invariant, we can arrange that L_2 and L_1 are defined by the equations $u_0 = 0$ and $u_1 - u_2 = 0$, respectively. After a rescaling we arrive at the normal form in (ii). The point $[1 : 0 : 0]$ is a base point of multiplicity two, whereas $[0 : 0 : 1]$ and $[0 : 1 : 0]$ are base points of multiplicity one.

Finally we have the generic situation that there are three singular members, each of multiplicity one, which implies that each of these singular members is equal to the union of two distinct lines, and actually all lines are distinct. If one of the lines passes through one of the singular points of another singular member, then we are in case (ii). By means of a projective linear transformation we can arrange that one of the singular members is given by $u_1\, u_2 = 0$ and a second is given by $u_0\,(u_0 - u_1 - u_2) = 0$, which leads to the normal form of (i). □

Lemma 10.2.4 *The mapping $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ in* (10.2.3) *is a twofold branched covering, with noninvertible derivative at the diagonal D in $\mathbb{P}^1 \times \mathbb{P}^1$ as defined in* (10.1.2)*, and branching over the curve*

$$P = \psi(D) = \{[u_0 : u_1 : u_2] \in \mathbb{P}^2 \mid 4\, u_0\, u_2 = {u_1}^2\} \tag{10.2.5}$$

in $\mathbb{P}^2$. The curve P is a smooth quadric in $\mathbb{P}^2$. The symmetry switch $\sigma : (x,\ y) \mapsto (y,\ x)$ permutes the two points in each fiber over a point in the complement of P in $\mathbb{P}^2$, whereas $\psi^{-1}(P) = D$ and $\psi|_D$ is an isomorphism from D onto P. For each $(x,\ y) \notin D$, the tangent mapping $\mathrm{T}_{(x,\,y)}\psi$ of ψ at $(x,\ y)$ is a linear isomorphism from $\mathrm{T}_{(x,\,y)}(\mathbb{P}^1 \times \mathbb{P}^1)$ onto $\mathrm{T}_{\psi(x,\,y)}\mathbb{P}^2$. For each $x \in \mathbb{P}^1$, the kernel of $\mathrm{T}_{(x,\,x)}\psi$ is equal to $\{(u,\ -u) \mid u \in \mathrm{T}_x\mathbb{P}^1\}$, the eigenspace of $\mathrm{T}_{(x,\,x)}\sigma$ for the eigenvalue -1.

Proof. Let $\psi([x_0 : x_1],\ [y_0 : y_1]) = \psi([x'_0 : x'_1],\ [y'_0 : y'_1])$. If $x_0\, y_0 \neq 0$, then $x'_0\, y'_0 \neq 0$, and we can use the affine coordinates $[x_0 : x_1] = [1 : x]$, $[y_0 : y_1] = [1 : y]$, $[x'_0 : x'_1] = [1 : x']$, $[y'_0 : y'_1] = [1,\ y']$, for which we have the equations $x + y = x' + y'$ and $x\, y = x'\, y'$, hence $x\, y = x'\,(x + y - x')$, and therefore $(x - x')\, y = x'\,(x - x')$. It follows that either $x' = x$, and therefore $y' = x$, or $x' = y$, and therefore $y' = x$. Similarly, if $x_1\, y_1 \neq 0$, then $x'_1\, y'_1 \neq 0$, and in the affine coordinates $[x_0 : x_1] = [\xi : 1]$, $[y_0 : y_1] = [\eta : 1]$, $[x'_0 : x'_1] = [\xi' : 1]$, $[y'_0 : y'_1] = [\eta',\ 1]$, we arrive at $(\xi',\ \eta') = (\xi,\ \eta)$ or $(\xi',\ \eta') = (\eta,\ \xi)$.

If $x_0\,y_0 = 0$ and $x_1\,y_1 = 0$, then $x_0'\,y_0' = 0$ and $x_1'\,y_1' = 0$. There are four cases, $([x_0 : x_1], [y_0 : y_1]) = ([0 : 1], [1 : 0])$ and $([x_0' : x_1'], [y_0 : y_1']) = ([0 : 1], [1 : 0])$; $([x_0 : x_1], [y_0 : y_1]) = ([0 : 1], [1 : 0])$ and $([x_0' : x_1'], [y_0 : y_1']) = ([1 : 0], [0 : 1])$; $([x_0 : x_1], [y_0 : y_1]) = ([1 : 0], [0 : 1])$ and $([x_0' : x_1'], [y_0 : y_1']) = ([1 : 0], [0 : 1])$; $([x_0 : x_1], [y_0 : y_1]) = ([1 : 0], [0 : 1])$ and $([x_0' : x_1'], [y_0 : y_1']) = ([0 : 0], [1 : 0])$.

It follows that in all cases $\psi(x,\ y) = \psi(x',\ y')$, for $(x,\ y), (x',\ y') \in \mathbb{P}^1 \times \mathbb{P}^1$, implies that $(x',\ y') = (x,\ y)$ or $(x',\ y') = (y,\ x')$, where $(x,\ y) = (y,\ x)$ if and only if $(x,\ y) \in D$.

That $\psi(D)$ is as in (10.2.5) follows from

$$\psi([x_0 : x_1],\ [x_0 : x_1]) = [{x_0}^2 : 2\,x_0\,x_1 : {x_1}^2]$$

for all $[x_0 : x_1] \in \mathbb{P}^1$. In the affine coordinates $[1 : u_1 : u_2]$, the curve P is equal to the parabola $u_2 = {u_1}^2/4$, which is smooth, and the smoothness of P at the point $[0 : 0 : 1]$ at infinity is also easily verified.

It follows from $\psi \circ \sigma = \psi$ and the chain rule that $\mathrm{T}_a\psi \circ \mathrm{T}_a\sigma = \mathrm{T}_a\psi$ for each fixed point a of σ, which implies that $(\mathrm{T}_a\psi)(v) = 0$ if v is a tangent vector at a such that $(\mathrm{T}_a\sigma)(v) = -v$. □

This leads to the following classification of the symmetric biquadratic curves C in $\mathbb{P}^1 \times \mathbb{P}^1$, in terms of the quadric $Q = \psi(C)$ and the intersection behavior of Q with the smooth quadric P.

Lemma 10.2.5 *Let C be a biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and $Q = \psi(C)$ the corresponding quadric in $\mathbb{P}^2$ as in Lemma* 10.2.1. *If the quadric Q is smooth, we have the following cases:*

(i) *Q intersects P in four distinct points. The biquadratic curve $C = \psi^{-1}(Q)$ is smooth and intersects the diagonal D in four distinct points, with tangent space equal to the -1 eigenspace of the symmetry switch. The restriction to C of ψ is a twofold branched covering from C to Q, ramifying at $C \cap D$ and branching over $Q \cap P$.*

(ii) *Q intersects P with multiplicity two at p and with multplicity one at two other points. The biquadratic curve $C = \psi^{-1}(Q)$ is irreducible and has one singular point, equal to $\psi^{-1}(p) \in D$, which is an ordinary double point. The intersection of C with D at $d = \psi^{-1}(p)$ has multiplicity two.*

(iii) *Q intersects P with multiplicity three at p and with multiplicity one at one other point. The biquadratic curve $C = \psi^{-1}(Q)$ is irreducible and has one singular point, equal to $d = \psi^{-1}(p)$, which is an ordinary cusp point of C, where the cusp points in the direction of D. That is, the intersection of C with D has multiplicity three at d.*

(iv) *Q intersects P at two points p_1 and p_2, each with multiplicity two. The biquadratic curve $C = \psi^{-1}(Q)$ is the union of two smooth $(1,\ 1)$ curves, which intersect each other at the two points $\psi^{-1}(p_1)$ and $\psi^{-1}(p_2)$, transversally and also transversally to D. The two irreducible components are switched by the mapping $\sigma : (x,\ y) \mapsto (y,\ x)$.*

(v) *Q intersects P with multiplicity 4 at p. The biquadratic curve $C = \psi^{-1}(Q)$ is the union of two smooth $(1,\ 1)$ curves, which intersect each other at the point $\psi^{-1}(p)$, with multiplicity 2. Both $(1,\ 1)$ curves are tangent to D at $\psi^{-1}(p)$. The two irreducible components are switched by the mapping $\sigma : (x,\ y) \mapsto (y,\ x)$.*

(vi) *$Q = P$. The biquadratic curve $C = \psi^{-1}(Q)$ is equal to the diagonal D, counted twice.*

If Q is not smooth then Q is equal to either the union of two complex projective lines or a double complex projective line. For a complex projective line $L \subset Q$ we have the following alternatives:

(a) *L intersects P in two distinct points. Then $\psi^{-1}(L)$ is a smooth and irreducible $(1,\ 1)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, invariant under the symmetry switch and intersecting the diagonal D in two points, with tangent space equal to the -1 eigenspace of the symmetry switch. $\psi^{-1}(L)$ is an irreducible component of $C = \psi^{-1}(Q)$.*

(b) *L intersects P at one point p, where it is tangent to P. Then $\psi^{-1}(L)$ is equal to the union of the horizontal axis $L_1 = \mathbb{P}^1 \times \{x\}$ and the vertical axis $L_2 = \{x\} \times \mathbb{P}^1$ through the point $(x,\ x) = \psi^{-1}(p) \in D$. The axes L_1 and L_2 are two irreducible components of the biquadratic curve $C = \psi^{-1}(Q)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. The descriptions in (i)–(v) at the intersection points of $C = \psi^{-1}(Q)$ with D follow from the fact that $\psi|_D$ is an isomorphism from D onto P, whereas in the direction transversal to D the mapping ψ contracts in a quadratic fashion toward P.

If the line $l_0\,u_0 + l_1\,u_1 + l_2\,u_2 = 0$ is tangent to $P = \{{u_1}^2 - 4\,u_0\,u_2 = 0\}$ at the point $[u_0^0 : u_1^0 : u_2^0]$, then $l_0\,u_0^0 + l_1\,u_1^0 + l_2^0 = 0$, $(u_1^0)^2 - 4\,u_0^0\,u_2^0 = 0$, and $[l_0 : l_1 : l_2] = [-4\,u_2^0 : 2\,u_1^0 : -4\,u_0^0]$. Let $u_a^0 = a^2$ and $u_2^0 = b^2$, hence $(u_1^0)^2 = 4\,b^2\,a^2$, and therefore $u_1^0 = 2\,b\,a$, for a suitable choice of the sign in b and/or a. In view of (10.2.3) the curve $\psi^{-1}(L)$ is given by the equation

$$\begin{aligned} 0 &= u_2^0\,x_0\,y_0 - u_1^0\,(x_0\,y_1 + x_1\,y_0)/2 + u_0^0\,x_1\,y_1 \\ &= b^2\,x_0\,y_0 - b\,a\,(x_0\,y_1 + x_1\,y_0) + a^2\,x_1\,y_1 \\ &= (b\,x_0 - a\,x_1)\,(b\,y_0 - a\,y_1). \end{aligned}$$

This proves (b) with $x = [a : b]$. □

Corollary 10.2.6 *Let $\mathfrak{B}$ be a pencil of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, corresponding to the pencil $\mathfrak{Q}$ of quadrics in $\mathbb{P}^2$ as in Lemma* 10.2.1. *Let $\kappa : S \to \mathbb{P}^1$ be the rational elliptic surface defined by $\mathfrak{B}$, $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ the blowing-up map, and $\rho \in \mathrm{Aut}(S)^+_\kappa$ the QRT root of the QRT automorphism τ defined by $\mathfrak{B}$.*

If S_r is a reducible fiber of κ, then ρ leaves each irreducible component of S_r invariant if and only if the quadric $Q = \psi \circ \pi(S_r)$ in $\mathbb{P}^2$ is smooth. The sum of the contributions to ρ of the fibers S_r for which Q is not smooth, equal to a union of complex projective lines in $\mathbb{P}^2$, is $\geq 3/2$, with equality if and only if none of these lines is tangent to the quadric P in (10.2.5).

The rank of the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is ≤ 5, with equality if and only if (a) the pencil $\mathfrak{Q}$ is as in (i) in Lemma 10.2.3, (b) none of the complex projective

lines contained in a singular member of $\mathfrak{Q}$ is tangent to the quadric P, and (c) no smooth member of $\mathfrak{Q}$ has two intersection points of multiplicity two with P or one intersection point of multiplicity four with P.

Proof. With $r = [z_0 : z_1]$, it follows from Lemma 10.2.2 and Remark 10.1.5 that Q is smooth if and only if $\det(z_0 B^0 + z_1 B^1) = 0$ if and only if $Y_\rho(z_0, z_1) = 0$ if and only if ρ^S leaves each irreducible component of S_r invariant. This can also be verified directly for each case in Lemma 10.2.5. It follows from Proposition 10.1.2 that the sum of the contributions to ρ^S of the reducible fibers is $\geq 3/2$, with equality if and only if τ leaves every irreducible component of every fiber invariant. According to (iii) $\Leftrightarrow$ (ii) in Corollary 5.1.12, the latter condition is equivalent to the condition that no member of $\mathfrak{B}$ contains a horizontal or vertical axis, which in view of Lemma 10.2.5 is equivalent to the condition that no line in a singular member of $\mathfrak{Q}$ is tangent to P. Finally, Proposition 10.1.2 implies that the rank of the Mordell–Weil group is ≤ 5, with equality if and only there are exactly three reducible fibers, each of which has exactly two irreducible components that are switched by ρ^S. It follows from Lemmas 10.2.3 and 10.2.5 that the latter condition is equivalent to the conditions (a)–(c). □

Remark 10.2.7. The identification in Lemma 10.2.1 allows a classification of pencils of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of pencils of quadrics in $\mathbb{P}^2$, in which, however, the behavior with respect to the quadric P in Lemma 10.2.4 has to be taken into account. Two pencils of quadrics correspond to equivalent pencils of symmetric biquadratic curves if they can be mapped to each other by means of a projective linear transformation in $\mathbb{P}^2$ that leaves P invariant. These transformations form a three-dimensional projective orthogonal group. The classification of the symmetric QRT maps and roots is considerably simpler than the classification of the arbitrary QRT maps. For instance, in the list in Section 12.1 of the singular biquadratic curves and the action of the QRT map on these, only a minority are symmetric. ⊘

10.3 Poncelet

10.3.1 Dual Quadrics

Let E be a three-dimensional complex vector space with dual space E^* = the space of all complex linear forms on E. The elements of the projective plane $\mathrm{P}(E)$ are the one-dimensional linear subspaces $[e]$ of E, where $[e] = \mathbb{C}\, e$, $e \in E$, $e \neq 0$. A complex projective line in $\mathrm{P}(E)$ is defined by the zero-set ε^0 of a nonzero linear form ε on E, that is, $\varepsilon \in E^*$ and $\varepsilon \neq 0$. Because $\varepsilon^0 = \eta^0$ if and only if $[\varepsilon] = \mathbb{C}\,\varepsilon = \mathbb{C}\,\eta = [\eta]$, this induces a bijective mapping from the *dual projective plane* $\mathrm{P}(E^*)$ onto the set of all complex projective lines in $\mathrm{P}(E)$. This mapping will be used in order to identify $\mathrm{P}(E^*)$ with the set of all complex projective lines in $\mathrm{P}(E)$.

If B is a symmetric bilinear form on E, then the function $f_B : e \mapsto B(e, e)/2$ is a homogeneous polynomial of degree two on E, and the mapping $B \mapsto f_B$ is a linear isomorphism from the space $\mathrm{Symm}^2(E)$ of all symmetric bilinear forms on E, onto the space $\mathrm{Pol}^2(E)$ of all homogeneous polynomials on E, where $B(e, e') = f(e+e') - f(e) - f(e')$ if $f = f_B$. Furthermore, if B is a bilinear form on E, then $L_B := e \mapsto (e' \mapsto B(e, e'))$ is a linear mapping from E to E^*, and the mapping $B \mapsto L_B$ is a linear isomorphism from the space $\mathrm{Bilin}(E)$ of all bilinear forms on E onto the space $\mathrm{Lin}(E, E^*)$ of all linear mappings from E to E^*. We will actually identify B with L_B and write $B(e, e') = (B\,e)(e')$, where on the left- and right-hand sides B is viewed as a bilinear form on E and a linear mapping from E to E^*, respectively. The bilinear form B on E is symmetric if and only if the linear mapping $B : E \to E^*$ is equal to its transpose $B^* : E \simeq (E^*)^* \to E^*$, where we identify $e \in E$ with the linear form $\varepsilon \mapsto \varepsilon(e)$ on E^* in order to identify E with $(E^*)^*$.

If $B \in \mathrm{Symm}^2(E)$ and $B \neq 0$, then we have the corresponding quadric $Q = Q_B$ in $\mathrm{P}(E)$, which corresponds to the zero-set of f_B. According to Lemma 10.2.2, the complex projective line in $\mathrm{P}(E)$ that is tangent to Q at the point $e \in Q$ corresponds to the zero-set of the linear form $B\,e$ on E^*. We have that Q is smooth if and only if the linear mapping $B : E \to E^*$ is bijective, in which case B induces the projective linear mapping $[B] : [e] \mapsto [B\,e]$ from $\mathrm{P}(E)$ onto $\mathrm{P}(E^*)$. With the aforementioned identification of $\mathrm{P}(E^*)$ with the set of all complex projective lines in $\mathrm{P}(E)$, we have that $[B](Q)$ is equal to the set of all complex projective lines in $\mathrm{P}(E)$ that are tangent to Q. If we write $B\,e = \varepsilon$, then $B(e, e) = B^{-1}(\varepsilon, \varepsilon)$. It follows that $[e] \in Q_B$ if and only if $[\varepsilon] = [B\,e] \in Q_{B^{-1}}$. In this way the set of all complex projective lines is equal to the quadric $Q_{B^{-1}}$ in $\mathrm{P}(E^*)$ defined by the symmetric bilinear form B^{-1} on E^*. The smooth quadric $Q_{B^{-1}}$ in $\mathrm{P}(E^*)$ is called the *dual quadric* of the smooth quadric Q_B in $\mathrm{P}(E)$, and denoted by Q^*.

The following lemma is an application of dual quadrics.

Lemma 10.3.1 *Let $Q = Q_B$ be a smooth quadric in a complex projective plane $\mathrm{P}(E)$. If $[e] \in \mathrm{P}(E)$, $[e] \notin Q$, then there are exactly two complex projective lines through $[e]$ that are tangent to Q, mapped by $[B]$ to the two intersection points of the complex projective line in $\mathrm{P}(E^*)$ defined by the zero-set of e, with the dual quadric $Q_{B^{-1}}$. If $[e] \in Q$, then there is exactly one complex projective line through $[e]$ that is tangent to Q, mapped by $[B]$ to the multiplicity-two intersection point of the e^0 line with $Q_{B^{-1}}$.*

Proof. We have that $[e]$ lies on the ε^0 line if and only if $\varepsilon(e) = 0$ if and only if $[\varepsilon]$ lies on the e^0 line, and $[B]$ is a bijective map from the set of all lines tangent to Q_B onto the quadric $Q_{B^{-1}}$. □

Remark 10.3.2. If the quadric $Q = Q_B$ is not smooth then it is a union of two distinct complex projective lines L_1, L_2, or it is a double complex projective line L; see Lemma 10.2.2. In both cases the symmetric linear mapping $B : E \to E^*$ is nonzero and not bijective.

In the first case the rank of B is equal to 2, the kernel of B corresponds to the intersection point s of L_1 and L_2, the singular point of Q, whereas the range $B(E) = (\ker B)^0$ of B corresponds to the complex projective line S in $\mathrm{P}(E^*)$ that

consists of all complex projective lines in $\mathrm{P}(E)$ that pass through s. If π denotes the linear projection from E onto the two-dimensional vector space $F = E/\ker B$, then there is a unique homogeneous polynomial g of degree two on F such that $f_B = g \circ \pi$, and the zero-set of g in F corresponds to two points l_1 and l_2 in the projective line $\mathrm{P}(F)$, which in turn correspond to the projective lines L_1 and L_2. The linear mapping $B : E \to E^*$ induces a linear isomorphism from $F = E/\ker B$ onto $B(E)$, which in turn induces a projective linear isomorphism from $\mathrm{P}(F)$ onto S, thereby mapping l_1 and l_2 to two distinct points s_1 and s_2 on S. The configuration of the complex projective line S in $\mathrm{P}(E^*)$ together with the two points s_1 and s_2 on S is viewed as playing the role of the dual quadric of Q, which does not exist when the quadric Q is not smooth.

If $Q = Q_B$ is a double complex projective line L, then the rank of B is equal to 1. The two-dimensional kernel of B corresponds to L, and the one-dimensional range $B(E) = (\ker B)^0$ corresponds to L, viewed as an element of $\mathrm{P}(E^*)$. This point in $\mathrm{P}(E^*)$ is the object in the dual projective plane that replaces the dual quadric. ⊘

10.3.2 The Theorem of Poncelet

We continue the notation of Section 10.3.1. Let P and Q be two smooth quadrics in the complex projective plane $\mathrm{P}(E)$. The *Poncelet curve* is the set $R = R_{Q,\,P^*}$ of all pairs $(q,\ L)$ such that $q \in Q$, L is a complex projective line in $\mathrm{P}(E)$ that is tangent to P, and $q \in L$. This set can be viewed as the incidence relation

$$R_{Q,\,P^*} = \{(q,\ L) \in Q \times P^* \mid q \in L\} \tag{10.3.1}$$

in $Q \times P^*$, and therefore defines an algebraic curve in $Q \times P^* \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

If $(q,\ L) \in R$, then the complex projective line L intersects Q in the point q and one other point q', where $q' = q$ if and only if L is tangent to Q at q. This defines a mapping $\iota_Q : (q,\ L) \mapsto (q',\ L)$ from R onto itself, which is an involution of R, that is, ι_Q is bijective with inverse equal to ι_Q.

Similarly, if $(q,\ L) \in R$, then it follows from Lemma 10.3.1 that we have a second complex projective line L' through q that is tangent to P, where $L' = L$ if and only if $q \in P$. This defines a mapping $\iota_{P^*} : (q,\ L) \mapsto (q,\ L')$ from R onto itself that is also an involution of R. The bijective mapping

$$\rho = \rho_{Q,\,P^*} := \iota_{P^*} \circ \iota_Q \tag{10.3.2}$$

from R to itself is called the *Poncelet mapping* of R. Note that $\rho(q,\ L) = (q',\ L')$ if and only if q' is the second intersection point of L with Q, and L' is the second line through q' that is tangent to P.

Figure 10.3.1 illustrates the Poncelet mapping for quadrics in the real affine plane. The left picture is for the ellipses $P : x^2 + 4\,y^2 - x\,y - (2/3)\,x = 1/2$ and $Q : x^2 + 3\,y^2 = 2$ in the real affine $(x,\ y)$-plane, whereas in the right picture we have

taken the hyperbola $P : (1/2)\,x^2 + x\,y - x/2 = 1/50$ and the ellipse $Q : x^2 + 2\,y^2 = 2$.

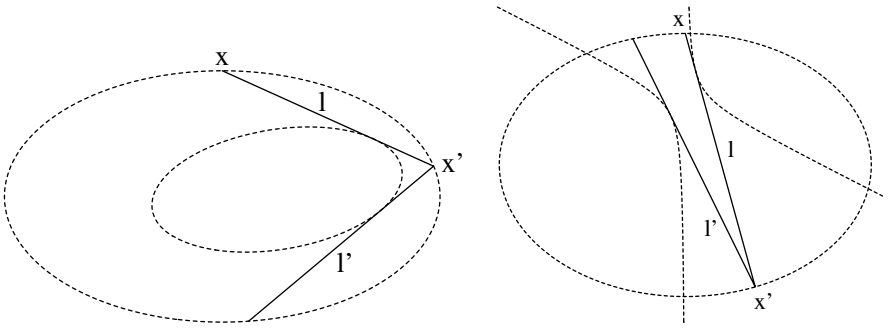


Fig. 10.3.1 The Poncelet mapping.

If $k \in \mathbb{Z}_{>0}$ and $\rho^k(q,\ L) = (q,\ L)$, that is, $(q,\ L)$ is a k-periodic point of the Poncelet mapping ρ, then we have the k-cycle $(q_j,\ L_j) := \rho^j(q,\ L)$, $j \in \mathbb{Z}/k\,\mathbb{Z}$, of elements of R. The points q_j, $j \in \mathbb{Z}/k\,\mathbb{Z}$, form a k-cycle on the quadric Q, the lines L_j, $j \in \mathbb{Z}/k\,\mathbb{Z}$, form a k-cycle of complex projective lines that are tangent to P, and we have $q_j \in L_j$ and $q_{j+1} \in L_j$ for every $j \in \mathbb{Z}/k\,\mathbb{Z}$. In other words, the L_j are the sides of the polygon such that the q_j are the vertices, and this polygon is both an *inscribed polygon* to the quadric Q and a *circumscribed polygon* to the quadric P. Such a polygon is called a *Poncelet polygon in Q and around P*.

We may now stat the *Poncelet closure theorem*:

Theorem 10.3.3 *If there is a Poncelet k-gon in Q and around P, then every point on Q and every tangent line to P is a vertex and side, respectively, of a Poncelet k-gon in Q and around P. In other words, if $\rho^k(q,\ L) = (q,\ L)$ for some $(q,\ L) \in R$, then $\rho^k(q,\ L) = (q,\ L)$ for every $(q,\ L) \in R$.*

According to Bos, Kers, Oort, and Raven [20, p. 297], Poncelet found and proved his closure theorem in 1813–1814, and published it in 1822; see Poncelet [165, Section 566]. Bos, Kers, Oort, and Raven [20, Section 4.1] also pointed out that Poncelet actually presented a much more general theorem, of which the closure theorem is a corollary.

Under the assumption that Q and P are in general position in the sense that they are nowhere tangent, that is, they intersect each other at four distinct points, Bos, Kers, Oort, and Raven [20, pp. 322–329], inspired by Griffiths [73, pp. 345–348], gave the following proof of Poncelet's closure theorem.

Proof. The mapping $R \ni (q,\ L) \mapsto q \in Q$ from R to Q is a twofold branched covering, where $q \in Q$ is a branch point if and only if there is only one line through q that is tangent to P if and only if $q \in P$. Because $Q \simeq \mathbb{P}^1$, see Lemma 10.2.2, and because there are four branch points, it follows from the Riemann–Hurwitz formula (2.3.18) that R is an elliptic curve. Moreover, the mapping ι_{P*} that interchanges the

elements in the fibers is an involution of R with the ramification points of the branched covering as fixed points, and therefore ι_{P^*} is an inversion on R; see Lemma 2.3.6 and Remark 2.3.8.

On the other hand, the mapping $R \ni (q,\ L) \mapsto L \in P^*$ from R to P^* is a twofold branched covering. Because $\mathrm{g}(R) = 1$ and $P^* \simeq \mathbb{P}^1$, another application of (2.3.18) yields that there are four branch branch points, which correspond to the fixed points of the involution ι_Q of R that interchanges the elements in the fibers. Therefore also ι_Q is an inversion on R.

But then the Poncelet mapping ρ in (10.3.2), which is the composition of two inversions on R, is a translation on the elliptic curve R. It follows that for each $k \in \mathbb{Z}$, the kth iterate ρ^k of ρ is a translation on R. Because a translation is equal to the identity as soon as it leaves one point fixed, it follows that ρ^k is equal to the identity on R as soon as it leaves one point of R fixed. □

Proposition 10.3.5 in the next subsection implies that the mapping ψ in Lemma 10.2.1 induces an isomorphism from the symmetric biquadratic curve C onto R that conjugates the QRT root on C with the Poncelet mapping on R. In this way Poncelet mappings are identified with QRT roots of symmetric biquadratic curves.

Remark 10.3.4. Cayley [31] found the following criterion for the Poncelet transformation $\rho_{Q,\,P^*}$ to have order k. Let A and B be the bilinear forms in three variables such that $P = Q_A$ and $Q = Q_B$, respectively. Let c_j, $j \in \mathbb{Z}_{\geq 0}$, be the coefficients in the power series expansion $(\det(A + t\,B))^{1/2} = \sum_{j\geq 0} c_j\, t^j$. If $k = 2\,m + 1$ is odd, let C_k denote the $m \times m$ matrix such that $(C_k)_{p\,q} = c_{p+q}$, $1 \leq p,\ q \leq m$. If $k = 2\,m$ is even, then C_k is the $(m-1) \times (m-1)$ matrix such that $(C_k)_{p\,q} = c_{p+q+1}$, $1 \leq p,\ q \leq m-1$. Then the Poncelet transformation has order k if and only if $\det C_k = 0$.

For a modern algebraic geometric proof, see Griffiths and Harris [75]. More references on this subject can be obtained by typing "cayley AND poncelet" in "Anywhere" in MathSciNet. ⊘

10.3.3 The QRT Root and the Poncelet Mapping

Let Q and P be smooth quadrics in $\mathbb{P}^2$, and let $R = R_{Q,\,P^*}$ and $\rho = \rho_{Q,\,P^*}$ denote the Poncelet curve and the Poncelet automorphism of it, defined in (10.3.1) and (10.3.2), respectively.

According to Lemma 10.2.2, we can transform the quadric P by means of a projective linear transformation to the quadric $\psi(D)$ in (10.2.5). We will assume in the remainder of this section that $P = \psi(D)$.

Proposition 10.3.5 *Let C be a smooth symmetric biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $Q = \psi(C)$ be the corresponding smooth quadric in $\mathbb{P}^2$. Let ρ_C denote the QRT root on C defined in* (10.1.3)*. That is, ρ_C is the automorphism of C that is equal to the horizontal switch on C followed by the symmetry switch on C = the symmetry switch on C followed by the vertical switch on C. Then the mapping*

$$\psi' : (x,\ y) \mapsto (\psi(x,\ y),\ \psi(\mathbb{P}^1 \times \{y\}))$$

defines an isomorphism from C onto $R_{Q,\,P^*}$, which conjugates the QRT root ρ_C on C with the Poncelet mapping $\rho_{Q,\,P^*}$ on $R_{Q,\,P^*}$, in the sense that $\psi' \circ \rho_C = \rho_{Q,\,P^*} \circ \psi'$.

Proof. It follows from (b) in Lemma 10.2.5 that for each $(x,\ y) \in C$, ψ maps both the horizontal axis $\mathbb{P}^1 \times \{y\}$ and the vertical axis $\{x\} \times \mathbb{P}^1$ through $(x,\ y)$ to the complex projective line L in $\mathbb{P}^2$ that is tangent to Q at the point $q = \psi(x,\ y) \in Q$. Because ψ is surjective, it follows that ψ' is a surjective morphism , that is, a complex analytic mapping from C onto $R_{Q,\,P^*}$.

Let $(x,\ y),\ (x',\ y') \in C$, and $\psi'(x,\ y) = \psi'(x',\ y')$. Then $\psi(x,\ y) = \psi(x',\ y')$, which in view of Lemma 10.2.4 implies that $(x',\ y') = (x,\ y)$, or $x \neq y$ and $(x',\ y') = (y,\ x)$. In the second case the horizontal axis $\mathbb{P}^1 \times \{y\}$ through $(x,\ y)$ is disjoint from the horizontal axis $\mathbb{P}^1 \times \{x\}$ through $(x',\ y') = (y,\ x)$. It follows from $\psi'(x,\ y) = \psi'(x',\ y')$ that ψ maps both horizontal axes to the same line L through $q = \psi(x,\ y) = \psi(x',\ y')$ that is tangent to P. Because Lemma 10.2.4 implies that $\psi^{-1}(L)$ is equal to the union of only one horizontal axis and one vertical axis, we arrive at a contradiction. This shows that $\psi' : C \to R_{Q,\,P^*}$ is injective, and therefore an isomorphism from C onto $R_{Q,\,P^*}$.

Let $(x,\ y) \in C$ and $\psi'(x,\ y) = (q,\ L)$. Then $\iota_{1,\,C}(x,\ y)$ and $(x,\ y)$ lie on the same horizontal axis $\mathbb{P}^1 \times \{y\}$, and therefore $\psi'(\iota_{1,\,C}(x,\ y)) = (q',\ L)$ for some $q' \in Q$, which must be the other point of intersection of L with Q. Therefore $(q',\ L) = \iota_Q(q,\ L)$, and we have proved that $\psi' \circ \iota_{1,\,C} = \iota_Q \circ \psi'$. On the other hand, $\psi(x,\ y) = \psi(x,\ y)$, and therefore $\psi'(y,\ x) = (q,\ L')$ for some L', which must be the other line through q that is tangent to P. This shows that $\psi' \circ \sigma|_C = \iota_{P^*} \circ \psi'$. Combining the two indentities, we arrive at

$$\psi' \circ \rho_C = \psi' \circ \sigma \circ \iota_{1,\,C} = \iota_{P^*} \circ \psi' \circ \iota_{1,\,C} = \iota_{P^*} \circ \iota_Q \circ \psi' = \rho_{Q,\,P^*} \circ \psi'.$$

□

In Proposition 10.3.5 we can take the quadrics Q in the Poncelet mapping $\rho_{Q,\,P^*}$, with P equal to the fixed quadric $P = \psi(D)$, in a pencil $\mathfrak{Q}$ of quadrics in $\mathbb{P}^2$. Then the biquadratic curves $C = \psi^{-1}(Q)$ are the members of a corresponding pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, and the QRT roots on the members of $\mathfrak{B}$ range together to the QRT root $\rho_{\mathfrak{B}}$, which is a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$. After successively blowing up, eight times, in the base points of the anticanonical pencils, we obtain the rational elliptic surface $\kappa : S \to P$ on which $\rho_{\mathfrak{B}}$ corresponds to the automorphism $\rho^S_{\mathfrak{B}} \in \mathrm{Aut}(S)^+_\kappa$, the QRT root of the QRT automorphism $\tau^S_{\mathfrak{B}}$.

Corollary 10.3.6 *Let P be a given smooth quadric in $\mathbb{P}^2$, and let $\mathfrak{Q}$ be a pencil of quadrics in $\mathbb{P}^2$ with at least one smooth member such that none of the lines in any of its singular members is tangent to P. Then the number of k-periodic fibers, counted with multiplicities, is equal to (10.1.6). Here the smooth k-periodic fibers correspond to the smooth members Q of $\mathfrak{Q}$ for which we have a k-gon in Q and around P.*

Proof. This follows from combining Proposition 10.3.5 with Corollary 10.2.6 and Proposition 10.1.2. □

For the asymptotics for $k \to \infty$ of the values of the parameter of the members Q of $\mathfrak{Q}$ for which we have a k-gon in Q and around P, see Section 7.4. For the asymptotics in the real case, see Chapter 8.

Remark 10.3.7. Regarding the relation between Poncelet and symmetric QRT maps, I found it exciting to read the following at the beginning of Cayley [32]: "The porism of the in-and-circumscribed polygon has its foundation in the theory of the symmetrical $(2,\ 2)$ correspondence of points on a conic; viz. a $(2,\ 2)$ correspondence is such that to any given position of either point there correspond two positions of the other point. ... or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form $p((x_0,\ x_1),\ (y_0,\ y_1)) = 0$, bihomogeneous of bidegree $(2,\ 2)$, which is symmetrical in regard to the parameters $(x,\ y)$. For the relation may be expressed as a second oreder equation in $(x_0\, y_0,\ x_1\, y_0 + x_0\, y_1,\ x_1\, y_1)$. ..." The only change I have made in the text is a translation of Cayley's notation of homogeneous coordinates into ours. In this way Chapter 10 can be viewed as working out Cayley [32]. ⊘

Chapter 11
Examples from the Literature

In this chapter we collect a number of examples from the literature, apart from the exceptional QRT surfaces and the periodic QRT mappings discussed previously in Sections 4.5 and 5.2, respectively. In all the examples, we freely use the theory of the previous chapters, illustrating how effectively the theory can be used to answer questions about a given special QRT map.

Historically one of the oldest pencils of elliptic curves that has been studied in detail is Hesse's pencil, generated by an arbitrary smooth cubic curve and the zero-set of its Hessian determinant. This pencil has very many striking properties, a good part of which we present in Section 11.1. The next classical example, the elliptic billiard, is treated extensively in Section 11.2.

In Roberts and Quispel [175, p. 166], the authors wrote that "one of the first people to study integrable mappings was McMillan" [133] a reference that also occurred in [168] and [169]. McMillan's map is the QRT root for which the invariant rational function p^0/p^1 is a polynomial; see Section 11.5. Special cases are the KdV, modified KdV, and nonlinear Schrödinger equations, whereas the isotropic Heisenberg spin chain map is not a McMillan map. For all these examples we only make a few observations, mainly concerning the configurations of singular fibers.

The example that was shown to me by Theo Tuwankotta, and which led to this book, is the sine–Gordon map. We give a quite complete discussion of it in Section 11.7. Because of its prominent place in the pure mathematical literature, we also treat the Lyness map in detail, in Section 11.4. The classical example of the planar four-bar link, which is one of the very few examples in the literature of a nonsymmetric QRT map, is discussed in Section 11.3. In Section 11.8 we discuss Jogia's QRT map τ and its QRT-like companion τ_H.

11.1 Hesse

Hesse [82, Aufgabe 2 on p. 90] formulated the problem to bring any cubic curve in $\mathbb{P}^2$ into the normal form

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*,
Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_11,
© Springer Science+Business Media, LLC 2010

$$q = z_0\,({x_0}^3 + {x_1}^3 + {x_2}^3) - 3\,z_1\,x_0\,x_1\,x_2 = 0 \tag{11.1.1}$$

by means of a projective linear transformation. Hesse took $z_0 = 1$ and $z_1 = -2\,c$, and his discussion of the problem indicates that he was convinced that it has a positive solution. Hesse's normal form is older than the Weierstrass normal form (2.3.6) obtained in Lemma 4.4.1. The second conclusion in the following lemma is the positive answer to Hesse's question.

Lemma 11.1.1 *Let $\mathfrak{C}$ be a pencil of cubic curves in $\mathbb{P}^2$ such that the modulus function $J : \mathfrak{C} \to \mathbb{P}^1$ is not constant, and let C be any smooth cubic curve in $\mathbb{P}^2$. Then there exists a projective linear transformation L of $\mathbb{P}^2$ such that $L(C) \in \mathfrak{C}$.*

The modulus function of the Hesse pencil (11.1.1) *has degree* 12*, hence is not constant, and it follows that every smooth cubic curve can be brought into the Hesse normal form* (11.1.1) *by means of a projective linear transformation.*

Proof. The modulus function is a rational function, that is, a complex analytic mapping from $\mathfrak{C} \simeq \mathbb{P}^1$ to $\mathbb{P}^1$. Because it is not constant, its mapping degree is strictly positive, and therefore the modulus function is surjective. Let J be the modulus of our given smooth cubic curve C, and let D be a member of $\mathfrak{C}$ with modulus equal to J. Because $J \neq \infty$, D is smooth.

According to Lemma 4.4.1, there are projective linear transformations that map C and D to a Weierstrass normal form F and G with coefficients g_2, g_3 and h_2, h_3, respectively. Because the modulus J does not change under isomorphisms of curves, we have in view of (2.3.10) that

$${g_2}^3/({g_3}^3 - 27\,{g_3}^2) = J = {h_2}^3/({h_2}^3 - 27\,{h_3}^2),$$

or equivalently ${g_2}^3\,{h_3}^2 = {h_2}^3\,{g_3}^2$, or equivalently $(g_0/f_0)^3 = (g_1/f_1)^2$, or equivalently there exists $\mu \in \mathbb{C} \setminus \{0\}$ such that $h_2 = \mu^2\,g_2$ and $h_3 = \mu^3\,h_3$. Note that if $g_2 = 0$ then $g_3 \neq 0$, $h_2 = 0$, $h_3 \neq 0$ and the same conclusion holds.

Replacing $(x_0,\, x_1,\, x_2)$ by $(\lambda^2\,x_0,\, x_1,\, x_2/\lambda)$ in the Weierstrass normal form (2.3.6), the left-hand side is replaced by a Weierstrass polynomial in which g_2 and g_3 are replaced by $\lambda^4\,g_2$ and $\lambda^6\,g_3$, respectively. Taking λ such that $\lambda^2 = \mu$ as above, we obtain that there is a projective linear transformation that maps F to G. The projective linear transformation that maps C to F, followed by the projective linear transformation that maps F to G, and concluded by the inverse of the projective linear transformation that maps D to G is a projective linear transformation that maps C to D. This proves the first statement in the lemma.

The Weierstrass invariants, given in terms of the Aronhold invariants as in Proposition 4.4.3, of the cubic polynomial q in (11.1.1) are given by

$$\begin{aligned}
2^2\,g_2 &= 3^3\,z_1\,({z_1}^3 + 8\,{z_1}^3),\\
2^3\,g_3 &= 3^3\,(-{z_1}^6 + 20\,{z_1}^3\,{z_0}^3 + 8\,{z_0}^6),\\
\Delta &= 3^9\,({z_1}^3 - {z_0}^3)^3\,{z_0}^3, &(11.1.2)\\
J &= 2^{-6}\,{z_1}^3\,({z_1}^3 + 8\,{z_0}^3)^3/({z_1}^3 - {z_0}^3)^3\,{z_0}^3. &(11.1.3)
\end{aligned}$$

Because the numerator and denominator in (11.1.3) have no common factors, the degree of the rational mapping $J : \mathbb{P}^1 \to \mathbb{P}^1$ has degree 12. □

Figure 11.1.1 shows the real curves of the pencil (11.1.1), where the affine coordinates in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ have been chosen in such a way that all the real base points are shown, together with the real singular fiber over $[z_0 : z_1] = [0 : 1]$.

For the choice of the members of the real pencils that appear in the pictures, see the text under the heading "Figure 2.3.2," starting with "In all pictures in this book of real pencils …."

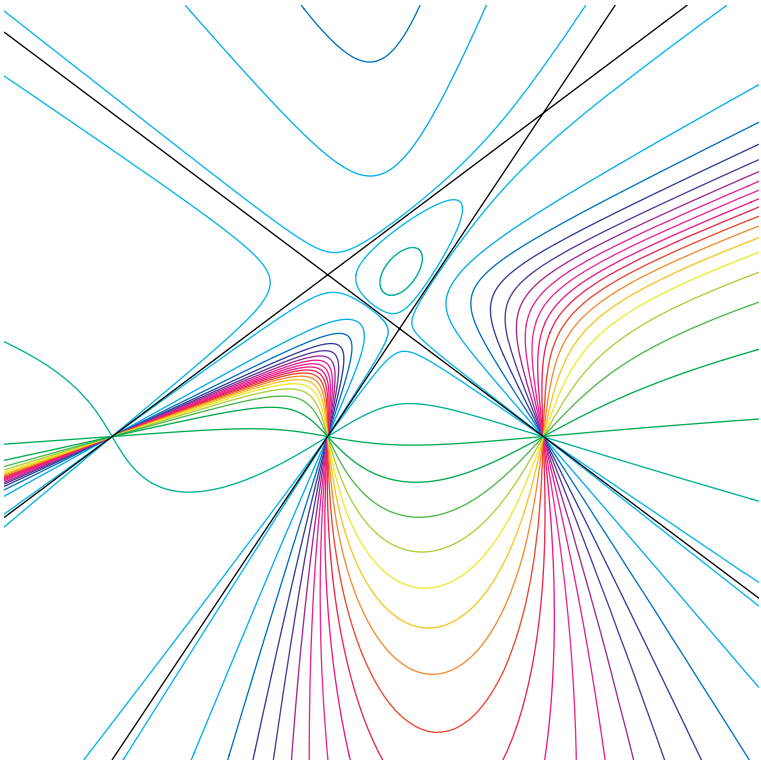


Fig. 11.1.1 The Hesse pencil.

11.1.1 Properties of the Hesse Curves

The curves $C_{[z]}$ defined by (11.1.1), parametrized by $[z] \in \mathbb{P}^1$, form a pencil $\mathfrak{H}$ of cubic curves in $\mathbb{P}^2$ as in (4.1.9), with $q^0 = {x_0}^3 + {x_1}^3 + {x_2}^3$ and $q^1 = -3\,x_0\,x_1\,x_2$.

Let $\omega \in \mathbb{C}$, $\omega \neq 1$, $\omega^3 = 1$ denote one of the nontrivial third roots of unity. With this notation, the pencil has the nine distinct base points

$$[0 : 1 : -\omega^i],\ [-\omega^j : 0 : 1],\ [1 : -\omega^k : 0], \quad i,\ j,\ k \in \mathbb{Z}/3\,\mathbb{Z}. \tag{11.1.4}$$

The Hesse determinant $r = \mathrm{H}(q)$ of q, the determinant of the symmetric 3×3 matrix of all second-order partial derivatives of q, is equal to

$$r = -2\,3^3\,z_0\,{z_1}^2\,({x_0}^3 + {x_1}^3 + {x_2}^3) + 2\,3^3\,(4\,{z_0}^3 - {z_1}^3)\,x_0\,x_1\,x_2, \tag{11.1.5}$$

which is in Hesse's normal form (11.1.1), but with $(z_0,\ z_1)$ replaced by $(-2\,3^3\,z_0\,{z_1}^2,\ 2\,3^2\,({z_1}^3 - 4\,{z_0}^3))$. This defines a rational mapping

$$H : [z_0 : z_1] \mapsto [3\,z_0\,{z_1}^2 : 4\,{z_0}^3 - {z_1}^3], \tag{11.1.6}$$

from $\mathbb{P}^1$ to $\mathbb{P}^1$, called the *Hesse map*, which has mapping degree equal to 3.

As Hesse [82, p. 104] observed, if $q = 0$ is smooth, then $[x]$ is a flex point of $q = 0$ if and only if $q(x) = 0$ and $r(x) = 0$. Because $q(x) = 0$ implies that $z_0\,q^0(x) = -z_1\,q^1(x)$, we have $0 = r(x)$ if and only if

$$0 = 3\,z_0\,{z_1}^2\,q^0(x) + (4\,{z_0}^3 - {z_1}^3)\,q^1(x) = 4\,({z_0}^3 - {z_1}^3)\,q^1(x).$$

Because $q = 0$ is smooth if and only if $z_0 \neq 0$ and ${z_0}^3 - {z_1}^3 \neq 0$, it follows that $q(x) = r(x) = 0$ if and only if $q(x) = 0$ and $q^1(x) = 0$ if and only if $q^0(x) = 0$ and $q^1(x) = 0$ if and only if $[x]$ is a base point of the pencil $\mathfrak{H}$ defined by (11.1.1). In other words, for each smooth member C of the pencil $\mathfrak{H}$, the flex points of C are the nine base points of the pencil. In still other words, if C_1 and C_2 are smooth members of the pencil, then $C_1 \cap C_2$ is equal to the set of flex points of C_1 and equal to the set of flex points of C_2.

It follows from (11.1.2) that the pencil $\mathfrak{H}$ has four singular members, one for $[z_0 : z_1] = [0 : 1]$ and the other three for $[z_0 : z_1] = [1 : \omega^i]$, $i = 0,\ 1,\ 2$. For $[z_0 : z_1] = [0 : 1]$ the fiber is equal to the triangle of complex projective lines in $\mathbb{P}^2$ given by the equation $t_\infty := x_0\,x_1\,x_2 = 0$, whereas for $[z_0 : z_1] = [1 : \omega^i]$ we have the triangle of complex projective lines given by

$$t_i := (x_0 + x_1 + \omega^i\,x_2)\,(x_0 + \omega\,x_1 + \omega^{i+2}\,x_1)\,(x_0 + \omega^2\,x_1 + \omega^{i+1}\,x_1) = 0, \tag{11.1.7}$$

for the verification of which we use the identities $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. It follows that each of the four singular members is a triangle of complex projective lines.

The singular points of the Hesse map (11.1.6) from $\mathbb{P}^1$ to $\mathbb{P}^1$ are the four points determined by the equation $z_1\,(z_1 + 8\,{z_0}^3) = 0$, which correspond to the members for which $J = 0$, the anharmonic members of the pencil. The singular values of the Hesse map (11.1.6) from $\mathbb{P}^1$ to $\mathbb{P}^1$ are the four points $[0 : 1]$ and $[1 : \omega^i]$, $i \in \mathbb{Z}/3\,\mathbb{Z}$, which correspond to the four singular members of the pencil $\mathfrak{H}$. These points are also precisely the fixed points of H. At each of the fixed points the derivative of H is equal to -3, which implies that all the fixed points are repellers for the map H.

The preimage of each of these fixed points consists of the fixed point itself, which is a regular point, and a singular point s, where the tangent map of H is equal to

zero, $H'(s) = 0$ in local coordinates. Because the degree of the mapping is equal to zero, there are no other points in the preimage, and $H''(s) \neq 0$. It follows that if p is a point at distance of order ϵ to S, then the distance of $H(p)$ to the fixed point $H(s)$ is of order ϵ, and the distance of $H^{n+1}(p) = H^n(H(p))$ to $H(s)$ is of order $3^n \epsilon^2$, which is small of order δ if n is of order $^3 \log(\delta/\epsilon^2)$. Simulations of orbits with a large number of iterations of H show a strong clustering near each of the fixed points of H.

Figure 11.1.2 shows the image points of the initial point $(1.02,\ 0.02)$ under 20,000 iterates of the Hesse map, except for the first 100 ones. The picture illustrates that for most initial values the high-order iterates pass close to the fixed points. The following proposition implies that the Hesse map H is totally chaotic.

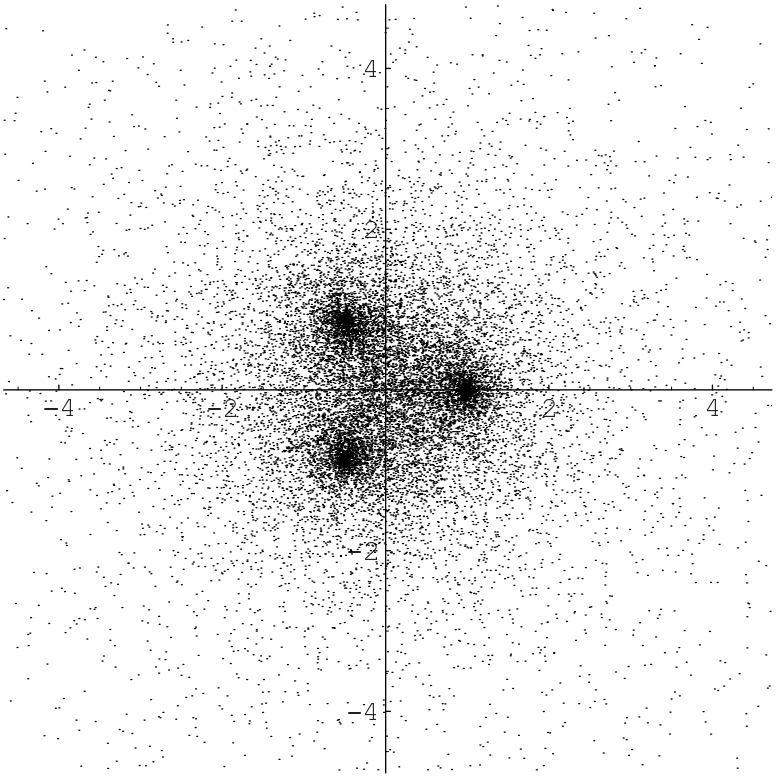


Fig. 11.1.2 Many iterates of the Hesse map applied to one initial point.

Proposition 11.1.2 *For every nonempty open subset U of $\mathbb{P}^1$ there exists an $n \in \mathbb{Z}_{>0}$ such that $H^n(U) = \mathbb{P}^1$. Furthermore, the mapping H is ergodic.*

Proof. Let $z \in \mathbb{P}^1$, $p \in \mathbb{Z}_{>0}$, $H^p(z) = z$, and $\mathrm{T}_z(H^p) = 0$. Because of the chain rule for tangent maps, we have

$$\mathrm{T}_z(H^p) = \mathrm{T}_{H^{p-1}(z)}H \circ \cdots \circ \mathrm{T}_z H,$$

and therefore there exists $0 \le n \le p-1$ such that $H^j(z)$ is a critical point of H. But then $v = H^{j+1}(z) = H(H^j(z))$ is a critical value of H, hence a fixed point of H, and therefore $z = H^p(z) = H^{p-(j+1)}(v)$ is a fixed point of H, and $H^j(z) = z$ for all $j \in \mathbb{Z}_{\ge 0}$, which in turn implies that $\mathrm{T}_z H = 0$. Because $H'(z) = -3 \ne 0$, we have arrived at a contradiction.

This proves that f has no superattracting cycle, and therefore it follows from McMullen [134, Corollary 3.10] that the Julia set $\mathrm{J}(H)$ of H is equal to $\mathbb{P}^1$ and that H is ergodic. The first statement in the proposition now follows from McMullen [134, Lemma 3.12] with $\mathrm{J}(H) = \mathbb{P}^1$. □

Because all nine base points are simple, every member of the pencil is smooth at every base point, and two distinct members intersect transversally at each base point. It follows that if S denotes the set of all $([x],\ [z]) \in \mathbb{P}^2 \times \mathbb{P}^1$ such that x and z satisfy (11.1.1), then S is a smooth surface in $\mathbb{P}^2 \times \mathbb{P}^1$. The restriction $\pi : S \to \mathbb{P}^2$ of the projection $([x],\ [z]) \mapsto [x]$ exhibits S as the blowing up of $\mathbb{P}^2$ at the nine base points of the pencil (11.1.1), whereas the restriction $\kappa : S \to \mathbb{P}^1$ of the projection $([x],\ [z]) \mapsto [z]$ exhibits S as a rational elliptic surface with fibers equal to the proper transforms of the members of the pencil (11.1.1). For any fiber C' of κ, the restriction to C' of π is an isomorphism from C' onto the member C of the pencil (11.1.1) of which C' is the proper transform.

The holomorphic exterior two-vector fields on S, the sections of the anticanonical bundle of S, correspond to the cubic polynomials that vanish at the base points. Because the space of all holomorphic exterior two-vector fields on any rational elliptic surface S is two-dimensional, see d) in Theorem 9.1.3 and the definition in the beginning of Section 9.1, it follows that the space of all cubic polynomials that vanish at the base points is two-dimensional. Because the latter space contains the polynomials in $(x_0,\ x_1,\ x_2)$ on the left-hand side of (11.1.1), it follows that the space of all cubic polynomials that vanish at (11.1.4) is equal to the space of polynomials in $(x_0,\ x_1,\ x_2)$ on the left-hand side of (11.1.4). See also Proposition 4.1.4.

It follows from the description of the singular members of the Hesse pencil (11.1.1) that the configuration of singular fibers in S consists of four singular fibers of Kodaira type I_3, with intersection diagram $\mathrm{A}_2^{(1)}$. This can also be checked by means of Table 6.3.2. According to Lemma 9.2.6, the Mordell–Weil group in this case is isomorphic to $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, which has nine elements. Because the Mordell–Weil group is identified with the set of all sections in S, it follows that there are no other sections in S other than the nine -1 curves that appeared at the blowing up of the nine distinct base points. Because the Mordell–Weil group is also isomorphic to the group $\mathrm{Aut}(S)_\kappa^+$ of all automorphisms of S that act as translations on the smooth fibers of κ, it also follows that for each base point o of the pencil we have for each such automorphism α a unique base point o' such that $\alpha = \tau^S_{\mathfrak{H},\, o,\, o'}$, the Manin automorphism of S defined by the pencil $\mathfrak{H}$ and the base points o, o' as in (4.2.3), that is, the automorphism of S that is induced by the translations on the smooth members of (11.1.1) that send o to o'.

Let α be an automorphism of S. Because α permutes the -1 curves in S, which are equal to the -1 curves in S that are mapped by π to the base points of the pencil (11.1.1) in $\mathbb{P}^2$, and because the restriction of π to the complement of these -1 curves is a complex analytic diffeomorphism onto the complement in $\mathbb{P}^2$ of the base points, there is a unique complex analytic diffeomorphism $L = L_\alpha : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\pi \circ \alpha = L \circ \pi$ that permutes the base points. Because every complex analytic diffeomorphism of $\mathbb{P}^2$ is a projective linear tansformation, see Griffiths and Harris [74, p. 64], and conversely every projective linear transformation L that permutes the base points induces a unique automorphism α of S such that $\pi \circ \alpha = L \circ \pi$, the mapping $\alpha \mapsto \alpha_L$ is an isomorphism from the group $\mathrm{Aut}(S)$ of all automorphisms of S onto the group G of all projective linear transformations of $\mathbb{P}^2$ that permute the base points (11.1.4).

For the linear transformation $L : (x_0,\, x_1,\, x_2) \mapsto (x_0,\, x_1,\, \omega^j\, x_2)$, which obviously preserves the pencil (11.1.1), we have $[t_\infty \circ L] = [t_\infty]$ and $t_i \circ L = t_{i+j}$ for all $i \in \mathbb{Z}/3\,\mathbb{Z}$, and therefore L induces an automorphism of S that cyclically permutes the singular fibers over $[z_0 : z_1] = [1 : -\omega^i/2]$. For the linear transformation

$$L : (x_0,\, x_1,\, x_2) \mapsto (x_0 + x_1 + \omega\, x_2,\; x_0 + \omega\, x_1 + x_1\, x_0 + \omega^2\, x_1 + \omega^2\, x_1)$$

we have $[t_\infty \circ L] = [t_1]$, $[t_0 \circ L] = [t_\infty]$, $[t_1 \circ L] = [t_0]$, and $[t_2 \circ L] = [t_2]$. Because two of the t_i already generate the pencil (11.1.1), it follows that L leaves the pencil (11.1.1) invariant and induces an automorphism that cyclically permutes the singular fibers over $[1 : 1]$, $[0 : 1]$, and $[1 : \omega]$. It follows that the image of the homomorphism p that assigns to each automorphism α of S the permutation of the four singular fibers contains the group A_4 of all even permutations of the four singular fibers. For each $\alpha \in \mathrm{Aut}(S)$ the induced automorphism β of $\mathbb{P}^1$, the projective linear transformation β of $\mathbb{P}^1$ such that $\kappa \circ \alpha = \beta \circ \kappa$, permutes the four points $[0 : 1]$, $1 : \omega^i]$, $i \in \mathbb{Z}/3\,\mathbb{Z}$. Because each permutation of four given points in $\mathbb{P}^1$ by means of a projective linear transformation is even, we conclude that $p(\mathrm{Aut}(S)) \simeq \mathrm{A}_4$. Because each projective linear transformation of $\mathbb{P}^1$ that leaves four points of $\mathbb{P}^1$ fixed is equal to the identity, the kernel of p is equal to the group $\mathrm{Aut}(S)_\kappa$ of all automorphisms of S that leave each fiber of κ invariant. In other words, we have an exact sequence $1 \to \mathrm{Aut}(S)_\kappa \to \mathrm{Aut}(S) \to \mathrm{A}_4 \to 1$, which in turn implies that

$$\#(\mathrm{Aut}(S)) = \#(\mathrm{A}_4) \cdot \#(\mathrm{Aut}(S)_\kappa) = 12\,\#(\mathrm{Aut}(S)_\kappa).$$

For each base point o of the pencil $\mathfrak{H}$ we have the automorphism $\iota^S_{\mathfrak{H},\,o}$ of S that is induced by the inversion on each smooth member of $\mathfrak{H}$ about the point o; see the paragraph preceding (4.2.3), which refers back to (4.2.2) and (4.2.1). On the other hand, if C is a smooth member of $\mathfrak{H}$ with modulus not equal to 0 or 1, then the only automorphisms of C are the inversion and the translations. It follows that the group $\mathrm{Aut}(S)^+_\kappa$ of automorphisms of S that act as a translation on each smooth fiber of κ is a subgroup of index two in $\mathrm{Aut}(S)_\kappa$, that is, $\mathrm{Aut}(S)_\kappa / \mathrm{Aut}(S)^+_\kappa \simeq \mathbb{Z}/2\,\mathbb{Z}$. This implies in turn that

$$\#(\mathrm{Aut}(S)_\kappa) = 2\,\#(\mathrm{Aut}(S)^+_\kappa) = 2 \cdot 9 = 18.$$

In combination with the previously obtained equation $\#(\mathrm{Aut}(S)) = 12\,\#(\mathrm{Aut}(S)_\kappa)$ this yields that the full automorphism group of S has 216 elements.

11.1.2 Conclusions for Arbitrary Smooth Cubic Curves

If L is a linear transformation and q'' denotes the second-order derivative matrix of the function q, then $(q \circ L)''(x) = L^* (q''(L(x)))\, L$, where L^* denotes the transpose of L. Using the well-known facts, due to Jacobi [95, p. 310], that the determinant of a product is equal to the product of the determinants and that $\det L^* = \det L$, Hesse [82, p. 89] concluded that

$$\mathrm{H}(q \circ L) = (\det L)^2\, (\mathrm{H}(q) \circ L). \tag{11.1.8}$$

Let q be a homogeneous polynomial of degree 3 on $\mathbb{C}^3$ such that the curve C in $\mathbb{P}^2$ defined by the equation $q = 0$ is smooth. In view of Lemma 11.1.1, (11.1.8), and the fact that $H([z_0 : z_1]) \neq [z_0 : z_1]$ for any smooth member of the pencil defined by (11.1.1), where H we denotes the Hesse map (11.1.6), we have that the Hesse determinant $\mathrm{H}(q)$ of q is linearly independent of q. We define the *Hesse pencil* $\mathfrak{H}_C$ *of* C as the family of curves, parametrized by $[z_0 : z_1] \in \mathbb{P}^1$, defined by the equation

$$z_0\, q(x) + z_1\, (\mathrm{H}(q))(x) = 0. \tag{11.1.9}$$

We define the *Hesse surface* S_C *of* C as the surface in $\mathbb{P}^2 \times \mathbb{P}^1$ defined by the equation (11.1.9) for $(x,\ (z_0,\ z_1)) \in \mathbb{C}^3 \times \mathbb{C}^2$. It follows from (11.1.8) that a projective linear transformation that maps a smooth cubic curve C to a curve C' maps the Hesse pencil $\mathfrak{H}_C$ of C isomorphically onto the Hesse pencil $\mathfrak{H}_{C'}$ of C', and induces an automorphism of $\mathbb{P}^2 \times \mathbb{P}^1$ that maps the Hesse surface S_C onto the Hesse surface $S_{C'}$. In view of Lemma 11.1.1, all the properties of the special Hesse pencil (11.1.1) and the corresponding rational elliptic surface S therefore carry over to the Hesse pencil and Hesse surface of any smooth cubic curve C in $\mathbb{P}^2$.

In the conclusions (i)–(x) below, C is an arbitrary smooth cubic curve in $\mathbb{P}^2$ defined by the zero-set of a cubic polynomial q on $\mathbb{C}^3$, with Hesse pencil $\mathfrak{H} = \mathfrak{H}_C$ and Hesse surface $S = S_C$.

(i) The Hesse pencil $\mathfrak{H}$ has nine base points and the Hesse surface S is a smooth surface in $\mathbb{P}^2 \times \mathbb{P}^1$, where the projection $\pi : S \to \mathbb{P}^2$ to the first factor exhibits S as the blowing up of $\mathbb{P}^2$ in the base points of $\mathfrak{H}$, and the projection $\kappa : S \to \mathbb{P}^1$ exhibits S as a rational elliptic surface. For each fiber F of κ, the restriction to F of π is an isomorphism from F onto the corresponding member of $\mathfrak{H}$.

(ii) For each smooth member C' of $\mathfrak{H}_C$, the set of all flex points of C' is equal to the set of all base points of $\mathfrak{H}_C$.

(iii) The space of all cubic polynomials that vanish at the base points of $\mathfrak{H}$ is two-dimensional, hence equal to the vector space spanned by q and $\mathrm{H}(q)$. As

a consequence, each cubic curve that passes through the base points of the pencil $\mathfrak{H}$ belongs to $\mathfrak{H}$.

(iv) For any smooth cubic curve C' in $\mathbb{P}^2$, the following statements are equivalent:

(a) $C' \in \mathfrak{H}_C$.
(b) $\mathfrak{H}_{C'} = \mathfrak{H}_C$.
(c) $C \cap C'$ is equal to the set of flex points of C' is equal to the set of flex points of C.

(v) $\mathfrak{H}$ has four singular members, each of which is a triangle of complex projective lines in $\mathbb{P}^2$. Each side S of each triangle contains three of the base points that lie on an equilateral triangle centered at the origin in the complex z-plane, if z is an affine coordinate on $S \simeq \mathbb{P}^1$ such that $z = 0$ and $z = \infty$ correspond to the intersection points of S with the two other sides of the triangle.

(vi) The mapping that assigns to each cubic polynomial its Hesse determinant induces a rational map H from $\mathfrak{H} \simeq \mathbb{P}^1$ to itself of mapping degree equal to 3. The singular values of H are the four singular members of $\mathfrak{H}$, see (v), which are also the fixed points of H. At each fixed point of H, the derivative of H is equal to -3, and the Hesse map H is totally chaotic in the sense of Proposition 11.1.2.

(vii) The configuration of singular fibers in S is 4 I_3, and the Mordell–Weil group of S, isomorphic to the group $\mathrm{Aut}(S)^+_\kappa$ of automorphisms of S that act on each smooth, hence elliptic, fiber of κ as a translation, is isomorphic to $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$.

(viii) The surface S has no other -1 curves = sections, other then the nine -1 curves that appeared at the blowing up of the base points. If G denotes the group of all projective linear transformations that permute the base points of $\mathfrak{H}$, then the relation $\pi \circ \alpha = L \circ \pi$ between $\alpha \in \mathrm{Aut}(S)$ and $L \in G$ defines an isomorphism between $\mathrm{Aut}(S)$ and G.

(ix) Let G be the group of all projective linear transformations in $\mathbb{P}^2$ that permute the base points of the Hesse pencil. The mapping that assigns to each $g \in G$ the automorphism g^S of S such that $\pi \circ g^S = g \circ \pi$ is an isomorphism from G onto $\mathrm{Aut}(S)$. The mapping that assigns to each $\alpha \in \mathrm{Aut}(S)$ the corresponding permutation of the four singular fibers is a surjective homomorphism from $\mathrm{Aut}(S)$ onto the group of all even permutations of the singular fibers, with kernel equal to the group $\mathrm{Aut}(S)_\kappa$ of all $\alpha \in \mathrm{Aut}(S)$ such that $\kappa \circ \alpha = \kappa$. Furthermore, $\mathrm{Aut}(S)_\kappa / \mathrm{Aut}(S)^+_\kappa \simeq \mathbb{Z}/2\,\mathbb{Z}$, where each element of $\mathrm{Aut}(S)_\kappa \setminus \mathrm{Aut}(S)^+_\kappa$ acts on each smooth fiber as an inversion. It follows that the group $G \simeq \mathrm{Aut}(S)$ has 216 elements. Jordan [99, p. 209] called G the *Hesse group*.

(x) All Hesse pencils and Hesse surfaces are isomorphic by means of projective linear transformations to the pencil and surface defined by (11.1.1), respectively.

For the following corollary one can also give a more direct proof. See for instance Griffiths and Harris [74, pp. 280, 281].

Corollary 11.1.3 *Let C be a smooth cubic curve in $\mathbb{P}^2$. Let F denote the set of flex points of C. Then $\#(F) = 9$. The transformations $\iota_{C,o}$, with $o \in F$ and $\tau_{C,o,o'}$, with $o,\ o' \in F$, see* (4.2.1)*, leave F invariant. That is, the third intersection point with C of a complex projective line through each pair of distinct flex points of C is again a flex point of C. The $\tau_{C,o,o'}$, for $o,\ o' \in F$ form a group of translations on C that is isomorphic to $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ and that acts freely and transitively on F. If $o \in F$ and $o' \in C$, then $\tau_{C,o,o'}$ has order 3 if and only if $o' \in F$.*

Proof. The "if" part of the last statement follows from the fact that each element of the group $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ is of order 3. For the converse, let $o \in F$. The group $\mathrm{Aut}(C)^{\mathrm{o}}$ of translations on C is isomorphic to the additive group $\mathbb{C}/P$, where P is the period lattice, and acts freely and transitively on C, which means that the mapping $\mathbb{C}/P\ \mathrm{Aut}(C)^{\mathrm{o}} \ni \tau \to \tau(o)$ is an isomorphism; see Section 2.3.1. Furthermore, $\tau_{C,o,o'}(o) = o'$ for each $o' \in C$. The element $z + P \in \mathbb{C}/P$ has order 3 if and only if $3\,z \in P$ if and only if $z \in ((1/3)\,P)/P$. Because $((1/3)\,P)/P$ has nine elements, it follows that the set G of all $o' \in C$ such that $\tau_{C,o,o'}$ has order three has nine elements. because $F \subset G$ and F has nine elements, it follows that $F = G$. □

Many of the aforementioned facts are due to Hesse [82]. His Lehrsatz 5 on p. 85 states that the Hesse determinant of a cubic polynomial q is a linear combination of q and its Hesse determinant; this is related to the implication (a) ⇒ (b) in (iv). [82, Lehrsatz 6 on pp. 88, 89] corresponds to the first statement in (vi). [82, Lehrsatz 7 on p. 91] corresponds to the statement that the Hesse pencil of a smooth cubic curve contains a triangle of complex projective lines; [82, Lehrsatz 8 on p. 92 and Lehrsatz 12 on p. 106] correspond to the stronger statement that this pencil contains exactly four triangles of complex projective lines, for which Hesse refers to Plücker [162, p. 284], see also [161, p. 589]. The first statement in (v) adds to this that there are no other singular members in this pencil. [82, Aufgabe 3 on pp. 95, 96] corresponds to the determination of the group G in (viii), where our statement (viii) adds to this that G is canonically isomorphic to the automorphism group of the surface S. [82, Lehrsatz 9 on p. 104] corresponds to our statement (ii). [82, Lehrsatz 13 on p. 107] corresponds to our (iv), (c). Finally [82, Lehrsatz 14 on p. 107] states that if the flex points of the two distinct cubic curves C and C' lie on the same triangle T of complex projective lines, and C'' is a cubic curve that passes through all points of $C \cap C'$, then the flex points of C'' also lie on T. Indeed in suitable projective linear coordinates, T is given by the equation $x_0\,x_1\,x_2 = 0$, and the flex points of C and C' on each side of T form an equilateral triangle with center at 0 if on this side we use the affine coordinate that is 0 and ∞ at the intersection point with one and the second other side of T, respectively. It follows that the equation for C and C' is of the form

$$q := \sum_{i=0}^{2} a_i\,{x_i}^3 + b\,x_0\,x_1\,x_2 = 0 \quad \text{and} \quad q' := \sum_{i=1}^{2} a'_i\,{x_i}^3 + b'\,x_0\,x_1\,x_2 = 0,$$

for suitable coefficients a_i, b and a'_i, b', respectively. Because C'' is in the pencil containing C and C', it is defined by an equation $c\,q + c'\,q' = 0$ for suitable constants c

and c', which is an equation of the same form, and therefore C'' has the same property that its flex points lie on T.

Remark 11.1.4. Plücker [161, p. 100] stated that every (smooth) cubic curve in $\mathbb{P}^2$ can be obtained as a member of a pencil that contains the union of three complex projective lines (not passing through a common point) and a triple projective line (not passing through any of the vertices of the triangle). If we add the conditions within the parentheses then there are projective coordinates in which the pencil has the form

$$z_0\,(x_0 + x_1 + x_2)^3 + z_1\,x_0\,x_1\,x_2 = 0.$$

The modulus function $J : \mathbb{P}^1 \to \mathbb{P}^1$ of this pencil is given by

$$-2^6\,3^3\,J = z_1\,(24\,z_0 + z_1)^3/{z_0}^3\,(27\,z_0 + z_1),$$

which has mapping degree 4 and therefore is surjective. With a proof similar to the proof of Lemma 11.1.1, we therefore obtain that every smooth cubic curve in $\mathbb{P}^2$ can be transformed by means of a projective linear transformation to a member of the Plücker pencil. Plücker continued by saying that "this immediately leads to a multitude of remarkable properties of cubic curves in $\mathbb{P}^2$." However, the Plücker pencil does not have the property of the Hesse pencil, the property that its base points are equal to the flex points of its members.

In any case, Plücker's normal form of cubic curves is more than ten years older than Hesse's. Schoenflies [177, p. 597] observed that pencils of planar curves of a given degree were already used systematically by Lamé (1818) and even earlier by Gergonne, references that I have not consulted. ⊘

Remark 11.1.5. Many more properties of the Hesse pencil can be found in the recent survey article by Artebani and Dolgachev [7]. ⊘

11.1.3 Characterizations of the Hesse Surface

Proposition 11.1.6 *Let $\kappa : S \to P$ be a rational elliptic surface. Then the following conditions are equivalent:*

(a) *There is a complex analytic diffeomorphism from S onto the Hesse surface = the set of all $([x],\ [z]) \in \mathbb{P}^2 \times \mathbb{P}^1$ such that $(x,\ z)$ satisfies* (11.1.1).
(b) *The Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is finite, but not isomorphic to one of the groups $\mathbb{Z}/k\,\mathbb{Z}$ for $1 \le k \le 6$, $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$, or $(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.*
(c) *The Mordell–Weil group is isomorphic to $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$.*

Proof. (a) ⇒ (b) follows from the second statement in (vii) above.

(b) ⇒ (c) follows from Lemma 9.2.6.

(c) ⇒ (a). Assume that c) holds. In view of Lemmas 7.1.1 and 9.2.1 it follows from the fact that that $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ has nine elements, that there are exactly

nine -1 curves in S. If E and E' are -1 curves in S and $E \neq E'$, then there is a unique $\alpha \in \mathrm{Aut}(S)^+_\kappa$ such that $\alpha(E) = E'$, α has order 3, hence is a torsion element of the Mordell–Weil group, and therefore it follows from Oguiso and Shioda [155, Proposition 3.5(iv)] that $E \cap E' = \emptyset$. Blowing down all the mutually disjoint -1 curves, we obtain that S is the blowing up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ at nine points, which are the base points of a pencil $\mathfrak{C}$ of cubic curves that defines the elliptic fibration $\kappa : S \to P$.

If $A_i, i \in \mathbb{Z}/3\,\mathbb{Z}$, are the irreducible components of one of the four singular fibers S_r of type I_3, then the nine -1 curves = sections of κ intersect S_r at nine distinct points, not equal to any of the singular points of S_r = the intersection points of A_i and A_j for $i \neq j$. If A_i contains m_i intersection points then, because $A_i \cdot A_i = -2$, see I_3 in Kodaira's list of singular fibers in Section 6.2.6, we have $\pi(A_i) \cdot \pi(A_i) = -2 + m_i$ in view of (3.2.8), where we use that A_i is the proper transform of $\pi(A_i)$ and each blowing-up point has degree one. Because any curve in $\mathbb{P}^2$ has self-intersection number ≥ 1, it follows that $m_i \geq 3$ for every i. Because, on the other hand, the sum of the m_i over $i \in \mathbb{Z}/3\,\mathbb{Z}$ is equal to 9, we conclude that $m_i = 3$, hence $\pi(A_i) \cdot \pi(A_i) = 1$, hence $\pi(A_i)$ is a complex projective line in $\mathbb{P}^2$, for every $i \in \mathbb{Z}/3\,\mathbb{Z}$. In other words, each of the singular fibers S_r is mapped by π onto a triangle T of complex projective lines in $\mathbb{P}^2$, where each of the sides of each triangle contains three base points, each of which is not equal to a vertex of the triangle.

Each $\alpha \in \mathrm{Aut}(S)^+_\kappa$ acts on the smooth part of a singular fiber S_r as an element of the group $\mathfrak{F}_r$ described in Corollary 6.3.31 with $c = r$, which for a singular fiber S_r of type I_3 is isomorphic to $\mathbb{C}^\times \times (\mathbb{Z}/3\,\mathbb{Z})$; see the last part of Section 6.3.6. Here the action on the set of the three connected components of the smooth part of S_r is a cyclic permutation that corresponds to a translation in the $\mathbb{Z}/3\,\mathbb{Z}$-component. Because $\mathrm{Aut}(S)^+_\kappa$ acts freely and transitively on the set of sections and leaves S_r invariant, it acts freely and transitively on the set of the nine intersection points of the nine sections with S_r. Because each of the connected components of the smooth part of S_r is intersected by three sections, it follows that the action of $\mathrm{Aut}(S)^+_\kappa \simeq (\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ on the set of these components defines a surjective homomorphism from $(\mathbb{Z}/3\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ onto the group $\mathrm{AS}_3 \simeq \mathbb{Z}/3\,\mathbb{Z}$ of all cyclic permutations of these three components. Therefore the kernel of this homomorphism, which leaves each of the A_i invariant and acts freely on the set of the three intersection points on A_i, is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$. Each of the nontrivial elements α of this kernel acts on $A_i \simeq \mathbb{P}^1$ as an automorphism, a projective linear transformation of $\mathbb{P}^1$, which moreover leaves the two intersection points $A_{i-1} \cap A_i$ and $A_i \cap A_{i+1}$ fixed. Using an affine coordinate z on A_i such that $A_{i-1} \cap A_i$ and $A_i \cap A_{i+1}$ correspond to $z = 0$ and $z = \infty$, respectively, the action of α on A_i is a multiplication by a constant $c_\alpha \in \mathbb{C}^\times$. Because α is of order 3, we have ${c_\alpha}^3 = 1$, and because α acts as a nontrivial permutation of the three intersection points $p_k, k \in \mathbb{Z}/3\,\mathbb{Z}$, of A_i with the sections, we have that $c_\alpha \neq 1$ and the $p_k, k \in \mathbb{Z}/3\,\mathbb{Z}$, form an equilateral triangle in $\mathbb{C}$ with center at the origin.

Because π maps A_i isomorphically onto the complex projective line $T_i = \pi(A_i)$ in $\mathbb{P}^2$, one of the sides of the triangle T, and maps the sections to the base points of $\mathfrak{C}$, it follows that the three base points of $\mathfrak{C}$ on T_i also form an equilateral triangle

with center at the origin, in an affine coordinate on T_i that is equal to 0 and ∞ at the intersection points of T_i with T_{i-1} and T_{i+1}.

Using a suitable projective linear change of variables, we can arrange that $T = T_0 \cup T_1 \cup T_2$ is equal to the triangle $x_0\,x_1\,x_2 = 0$ in $\mathbb{P}^2$. The projective linear transformation

$$[x_0 : x_1 : x_1] \mapsto [\lambda_0\,x_0 : \lambda_1\,x_1 : \lambda_2\,x_2]$$

leaves each of the sides $x_i = 0$ of the triangle invariant, and acts on it as the projective linear transformation that multiplies the affine coordinate x_{i+2}/x_i by $\mu_i = \lambda_{i+2}/\lambda_{i+1} \in \mathbb{C}^\times$. For arbitrary $\mu_i \in \mathbb{C}^\times$, $i \in \mathbb{Z}/3\,\mathbb{Z}$, these equations have solutions $\lambda_j \in \mathbb{Z}/3\,\mathbb{Z}$; for instance we can take for λ_1 any of the two solutions of ${\lambda_1}^2 = \mu_2\,\mu_0/\mu_1$, and then $\lambda_2 = \mu_0/\lambda_1$ and $\lambda_0 = \lambda_1\,\mu_1/\mu_0$ complete the solution. Because by multiplication by a suitable nonzero complex number μ any equilateral triangle in $\mathbb{C}$ with center at the origin can be brought into the position -1, $-\omega$, $-\omega^2$, where $\omega \in \mathbb{C}$, $\omega^3 = 1$, $\omega \neq 1$, we can choose the projective linear change of variables in $\mathbb{P}^2$ such that not only T is equal to the triangle $x_0\,x_1\,x_2 = 0$, but also the base points of $\mathfrak{C}$ on it are given by (11.1.4), that is, are equal to the base points of the Hesse pencil (11.1.1). It follows from (iii) above that $\mathfrak{C}$ is equal to the Hesse pencil (11.1.1), which completes the proof of (a). □

Remark 11.1.7. Let $\kappa : S \to P \simeq \mathbb{P}^1$ be a Hesse surface, and let $\alpha \in \mathrm{Aut}(S)^+_\kappa$, $\alpha \neq 1$. It then follows from Proposition 11.1.6 that α has order 3, which in view Theorem 4.3.3 implies that α is a Manin QRT automorphism. However, because $\alpha^2 \neq 1$ and α^2 also has finite order 3, α^2 is a Manin QRT automorphism as well, and therefore α is an abstract QRT root as defined in Chapter 10.1. In turn, $(\alpha^2)^2 = \alpha^4 = \alpha$ shows that α^2 is an abstract QRT root of α. In other words, for every nontrivial element α of the Mordell–Weil group of S, α and α^2 are each other's QRT roots. ⊘

11.2 The Elliptic Billiard

11.2.1 Confocal Quadrics

According to the *gardener's construction of an ellipse*, an ellipse in the Euclidean plane with the two focal points p_+ and p_- can be defined as the set of points p in the plane such that the sum of the distances from p to p_+ and to p_- is equal to a given constant l, with l strictly larger than the distance between p_+ and p_-. By a rotation and a scaling we can arrange that $p_\pm = (\pm 1,\ 0) \in \mathbb{R}^2$, and then a straightforward computation shows that the ellipse in the $(x_1,\ x_2)$-plane is given by the equation

$$\frac{4}{l^2}\,{x_1}^2 + \frac{4}{l^2 - 4}\,{x_2}^2 = 1. \tag{11.2.1}$$

In complex projective coordinates this would be the real part of the quadric

$$-{x_0}^2 + \frac{4}{l^2}\,{x_1}^2 + \frac{4}{l^2-4}\,{x_2}^2 = 0, \tag{11.2.2}$$

whose dual quadric in the dual projective $[\xi_0 : \xi_1 : \xi_2]$-plane is given by the equation

$$-{\xi_0}^2 + \lambda\,{\xi_1}^2 + (\lambda - 1)\,{\xi_2}^2 = 0, \quad \lambda := l^2/4. \tag{11.2.3}$$

This shows that the dual quadrics of the ellipses with the given focal points $(\pm 1,\ 0)$ form members of the pencil of quadrics in the dual projective plane, where for each $[z_0 : z_1] \in \mathbb{P}^1$ we take the quadric in the projective $[\xi_0 : \xi_1 : \xi_2]$-plane defined by the equation $z_0\, f^0(\xi) + z_1\, f^1(\xi) = 0$, where

$$f^0(\xi) = {\xi_0}^2 + {\xi_2}^2, \quad f^1(\xi) = {\xi_1}^2 + {\xi_2}^2. \tag{11.2.4}$$

Note that in (11.2.3) we have $z_0 = -1$ and $z_1 = \lambda = l^2/4$.

Actually, in the discussion in Section 11.2.2 of the elliptic billiard, all the confocal quadrics will appear, including the hyperbolas in (11.2.1) that occur for $l < 2$. The projective geometric definition of a *confocal family of quadrics* in a projective space $\mathrm{P}(E^*)$ is that there is a *pencil* $\mathfrak{Q}$ of quadrics in $\mathrm{P}(E)$, and the corresponding confocal family of quadrics in $\mathrm{P}(E^*)$ is the family of all dual quadrics Q^* in $\mathrm{P}(E^*)$, of the smooth members Q of the pencil $\mathfrak{Q}$. Recall that for each smooth quadric Q in $\mathrm{P}(E)$, the dual quadric Q^* is canonically identified with the set of all complex projective lines in $\mathrm{P}(E)$ that are tangent to Q. Conversely, Q is canonically identified with the set of all complex projective lines in E^* that are tangent to Q^*. See Section 10.3.1.

Before we continue the discussion of general confocal families of quadrics, let us determine the singular members and the base points of the pencil of quadrics defined by (11.2.4). The member for $[z_0 : z_1] = [1 : 0]$ is defined by the equation

$${\xi_0}^2 + {\xi_2}^2 = (\xi_0 + \mathrm{i}\,\xi_2)\,(\xi_0 - \mathrm{i}\,\xi_2) = 0, \tag{11.2.5}$$

and is therefore equal to the union of the two complex projective lines $L^0_\pm$ defined by the equations $\xi_0 \pm \mathrm{i}\,\xi_2 = 0$. The member for $[z_0 : z_1] = [0 : 1]$ is defined by the equation

$${\xi_1}^2 + {\xi_2}^2 = (\xi_1 + \mathrm{i}\,\xi_2)\,(\xi_1 - \mathrm{i}\,\xi_2) = 0, \tag{11.2.6}$$

and is therefore equal to the union of the two complex projective lines $L^1_\pm$ defined by the equations $\xi_1 \pm \mathrm{i}\,\xi_2 = 0$. The member for $[z_0 : z_1] = [1 : -1]$ is defined by the equation

$${\xi_0}^2 - {\xi_1}^2 = (\xi_0 + \xi_1)\,(\xi_0 - \xi_1) = 0, \tag{11.2.7}$$

and is therefore equal to the union of the two complex projective lines $L^2_\pm$ defined by the equations $\xi_0 \pm \xi_1 = 0$. It follows that the pencil $\mathfrak{Q}$ is of the type (i) in Lemma 10.2.3. The four base points, obtained for instance by determining the intersections $L^1_+ \cap L^2_+$, $L^1_+ \cap L^2_-$, $L^1_- \cap L^2_+$, $L^1_- \cap L^2_-$, are $[1 : -1 : -\mathrm{i}]$, $[1 : 1 : \mathrm{i}]$, $[1 : -1 : \mathrm{i}]$, $[1 : 1 : -\mathrm{i}]$.

Only one of the singular members of $\mathfrak{Q}$ is real, namely (11.2.7). The points in the $[x_0 : x_1 : x_2]$-plane that are dual to the two lines $L^2_\pm$ in (11.2.7) are $[1 : \pm 1 : 0]$, which correspond to the two focal points $(\pm 1,\ 0)$ in the affine $(x_1,\ x_2)$-plane with

which we started out. However, when working projectively and over $\mathbb{C}$, there are actually three pairs of focal points, the pair $[1 : 0 : \pm\mathrm{i}]$, the pair $[0 : 1 : \pm\mathrm{i}]$, and the real pair $[1 : \pm 1 : 0]$. Of these, the second pair lies on the complex line at infinity of the complex affine plane.

The base points of $\mathfrak{Q}$ are the points that lie on each member Q of $\mathfrak{Q}$, and therefore the lines in $\mathrm{P}(E^*)$ that are dual to the base points are the four complex projective lines in $\mathrm{P}(E^*)$ that are tangent to each smooth member Q^* of the family of confocal quadrics. These complex projective lines in $\mathrm{P}(E^*)$ could be called the *base lines* of the family of confocal quadrics. Because in our example none of the base points of $\mathfrak{Q}$ is real, none of the base lines is real. That is, in the real projective $[x_0 : x_1 : x_2]$-plane we don't see any real projective lines that are tangent to each of the real confocal quadrics.

Lemma 11.2.1 *Let $\mathfrak{Q}$ be a pencil of quadrics in the projective plane $\mathrm{P}(E)$, and let L be a complex projective line in $\mathrm{P}(E)$ that does not contain any of the base points of $\mathfrak{Q}$ and is not tangent to each $Q \in \mathfrak{Q}$. Furthermore, we assume that L is not a component of any of the singular members of $\mathfrak{Q}$.*

Then there are exactly two pairs $(l, P) = (l^{\pm}, P_{\pm}) \in L \times \mathfrak{Q}$ such that L is tangent to P at the point l. We have $P_+ \neq P_-$ and $l^+ \neq l^-$.

Let $l^{\pm} = [\xi^{\pm}]$, where ξ^+ and ξ^- are nonzero vectors in the three-dimensional complex vector space E. Let $\mathfrak{B}$ be the two-dimensional space of symmetric bilinear forms on E such that the members of $\mathfrak{Q}$ are the quadrics Q_B with $B \in \mathfrak{B}$, $B \neq 0$. Then $B(\xi^-, \xi^+) = 0$ for every $B \in \mathfrak{B}$.

Let ι be the mapping that assigns to each $l \in L$ the other point of intersection l' with L of the unique $Q \in \mathfrak{Q}$ such that $l \in Q$. Then ι is the unique involution of $L \simeq \mathbb{P}^1$ with l^+ and l^- as its fixed points. See Remark 2.3.7 *for the nontrivial involutions of complex projective lines.*

Proof. If $l \in L$ then l is not a base point of $\mathfrak{Q}$; hence there is a unique $Q = Q(l) \in \mathfrak{Q}$ such that $l \in Q$. Because L is not contained in Q, Q has one other point l' of intersection with L, where $l' = l$ if and only if Q is tangent to L at l. This defines the involution $\iota : L \to L$. Because L is not tangent to every $Q \in \mathfrak{Q}$, there exist $Q \in \mathfrak{Q}$ and l, $l' \in L$ such that $Q(l) = Q(l')$ and $l \neq l'$. That is, the mapping $l \mapsto Q(l)$ is a twofold branched covering mapping from $L \simeq \mathbb{P}^1$ onto $\mathfrak{Q} \simeq \mathrm{P}(\mathfrak{B}) \simeq \mathbb{P}^1$. It therefore follows from the Riemann–Hurwitz formula (2.3.18) with $\mathrm{g}(C) = \mathrm{g}(D) = 0, d = 2$, and therefore $n_b = 1$ for every $b \in B$, that there are exactly two points $l^{\pm}$ in L, mapped to the two branch points $P_{\pm} = Q(l^{\pm}) \in \mathfrak{Q}$.

Finally, if $P_{\pm} = Q_{B_{\pm}}$, then $[B_+] \neq [B_-]$, hence B_+ and B_- form a basis of $\mathfrak{B}$. If we write $L = x^0$, where x is a nonzero element of E^*, then $l^{\pm} \in L$ means that $x(\xi^{\pm}) = 0$. Furthermore, $l^+ \neq l^-$ implies that ξ^+ and ξ^- are linearly independent; hence x^0 is equal to the span of ξ^+ and ξ^-. The fact that L is tangent to $P_{\pm}$ at $l^{\pm}$ means that $B_{\pm}(\xi^{\pm}, \eta) = 0$ for all $\eta \in x^0$, or equivalently $B_{\pm}(\xi^{\pm}, \xi^+) = 0$ and $B_{\pm}(\xi^{\pm}, \xi^-) = 0$. In particular, $B_+(\xi^+, \xi^-) = 0$ and $B_-(\xi^+, \xi^-) = B_-(\xi^-, \xi^+) = 0$. Because B_+ and B_- form a basis of $\mathfrak{B}$, it follows by linearity that $B(\xi^+, \xi^-) = 0$ for every $B \in \mathfrak{B}$. □

11.2.2 The Billiard Map, the Poncelet Map, and QRT

Through each point x in the real affine plane there pass two confocal quadrics P_+^* and P_-^*, one an ellipse and the other a hyperbola, which moreover intersect each other orthogonally.

Furthermore, each line l through x is tangent to a unique confocal quadric Q^*, and the second line $\iota(l)$ through x that is tangent to Q^* is the reflected line with respect to the quadric P_+^*, in the sense that the angle between $\iota(l)$ and the tangent line l^+ of P_+^* at x is equal to the angle between l and l^+. The same holds with the confocal quadric P_+^* through x replaced by the other confocal quadric P_-^* through x.

Let Q_0^* be a given member of the family of confocal quadrics. The *billiard map* for Q_0^* is the mapping that assigns to each pair $(x,\ l)$ such that $x \in Q_0^*$ and l is a line through x the pair $(x',\ l')$ such that x' is the second intersection point of l with Q_0^*, and l' is the line through x' that is reflected with respect to Q_0^*. We will show that the above facts make the billiard map isomorphic to the QRT root defined by the pencil of symmetric biquadratic curves, which in turn is defined by the pencil of quadrics dual to our family of confocal quadrics.

The aforementioned facts about confocal quadrics follow from Corollary 11.2.2 below, which is dual to Lemma 11.2.1. Corollary 11.2.2 is formulated for the family of confocal quadrics Q^* in $\mathbb{P}^2$ defined by (11.2.2). As we have seen in Section 11.2.1, this family of confocal quadrics is dual to a pencil of quadrics with four base points, no three of which are on a projective line. Since all such pencils of quadrics are equivalent to each other by a projective linear transformation, the theorem can also be formulated for any family of confocal quadrics that is dual to a pencil of quadrics with four base points in general position.

Corollary 11.2.2 *Let $x = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ be not equal to one of the focal points $[1 : 0 : \pm\mathrm{i}]$, $[0 : 1 : \pm\mathrm{i}]$, $[1 : \pm 1 : 0]$. Also assume that $[x_0 : x_1 : x_2]$ does not lie on any of the four base lines of the family of confocal quadrics, that is, $x_0 + \epsilon_1 x_1 + \epsilon_2\ \mathrm{i}\ x_2 \neq 0$ for $\epsilon_j \in \{-1, +1\}$, $j = 1, 2$.*

Then there are exactly two confocal quadrics $P_\pm^*$ such that $x \in P_\pm^*$, and there are exactly two lines $l^\pm$ through x such that $l^\pm$ is tangent to $P_\pm^*$ at x. We have $l^+ \neq l^-$, and therefore also $P_+^* \neq P_-^*$. Furthermore, If $l_\pm = (\xi^\pm)^0$ for a nonzero dual vector $\xi^\pm \in \mathbb{C}^3$, then we have $\xi_0^+\,\xi_0^- + \xi_2^+\,\xi_2^- = 0$ and $\xi_1^+\,\xi_1^- + \xi_2^+\,\xi_2^- = 0$.

For each line l through x there is a unique confocal quadric Q^* such that l is tangent to Q^*. Let $\iota(l)$ denote the other line through x that is tangent to Q^*. Let L denote the set of all complex projective lines in $\mathbb{P}^2$ that pass through x, which is a complex projective line in the dual complex projective plane. Then ι is equal to the unique involution in L that leaves l^+ and l^- fixed.

In the real affine plane the conditions for $[x_0 : x_1 : x_2]$ in Corollary 11.2.2 are equivalent to avoiding the real focal points $[1 : \pm 1 : 0]$. In the affine coordinates $[1 : x_1 : x_2]$ and dual coordinates $[1 : \xi_1 : \xi_2]$, the equation $\xi_1^+\,\xi_1^- + \xi_2^+\,\xi_2^- = 0$ means that the lines $l^\pm$, defined by the equations $1 + \xi_1^\pm\,x_1 + \xi_2^\pm\,x_2 = 0$, are orthogonal to each other.

The unique involution ι in the space of all lines through x that leaves l^+ and l^- fixed has the property that the angle between l and l^+ is equal to the angle between $\iota(l)$ and l^+, and also the angle between l and l^- is equal to the angle between $\iota(l)$ and l^-. That is, $\iota(l)$ is the line through x which is obtained by reflecting l with respect to the confocal quadric P_+^* according to the law of reflection that l and $\iota(l)$ have equal angles with the tangent line of P_+^* at x, and the same holds with P_+^* replaced by P_-^*. In other words, the image (x', l') of (x, l) under the billiard map can also be described by the conditions that x and x' are the two intersection points of l with P_+^*, whereas l and l' are tangent to the same confocal quadric Q^*. Letting one of the two confocal quadrics $P_\pm^*$ play the role of the confocal quadric Q_0^* for which we have defined the billiard map, then we have proved the following:

Lemma 11.2.3 *If we fix Q^*, then the billiard map $(x, l) \mapsto (x', l')$ is equal to the Poncelet map in Section* 10.3.2*, with Q and P replaced by Q_0^* and Q^*, respectively. This implies that the set of all (x, l) such that x is a point on Q_0^*, the line l is tangent to Q^*, and x lies on l, is an elliptic curve, and that the billiard map acts a translation on it.*

Corollary 11.2.4 *Assume that by a projective linear transformation in $\mathbb{P}^2$ we have arranged that the member Q_0 of the pencil $\mathfrak{Q}$ of quadrics such that the duals of its members form our family of confocal quadrics, is equal to the quadric P in* (10.2.5).

Then the dualizing transformation $(x, l) \mapsto (l^0, x^0)$ conjugates the billiard map to the Poncelet map on the curves $R_{Q,\, P^*}$, with $Q \in \mathfrak{Q}$.

In turn, the inverse of the mapping ψ' in Proposition 10.3.5 conjugates this Poncelet map with the QRT root defined by the pencil $\mathfrak{B}$ of symmetric biquadratic curves $\psi^{-1}(Q)$, $Q \in \mathfrak{Q}$, in $\mathbb{P}^1 \times \mathbb{P}^1$.

Figure 11.2.1 shows how, for the billiard map, the reflected lines are tangent to a common confocal quadric. In the left- and right-hand pictures the confocal quadrics are an ellipse and a hyperbola, respectively. In the right-hand picture both l and l' happen to be tangent to the same branch of the confocal hyperbola, where l and l' touch it outside and inside the elliptic mirror, respectively. Figure 11.2.1 is the same as Figure 10.3.1 for a Poncelet map, where this time the circumscribed quadrics are confocal to the given inscribed one.

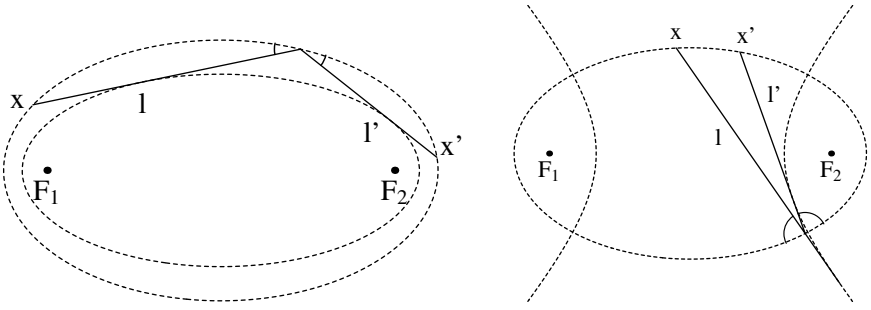


Fig. 11.2.1 Reflected lines tangent to a common confocal quadric

The fact that $P \in \mathfrak{Q}$ implies that the diagonal $D = \psi^{-1}(P)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, counted with multiplicity two, is a member of $\mathfrak{B}$. The four distinct base points of $\mathfrak{Q}$, which determine the pencil $\mathfrak{Q}$ of quadrics in $\mathbb{P}^2$, lie on P, and therefore correspond to four distinct multiplicity base points for $\mathfrak{B}$ on D, and these are all the base points of $\mathfrak{B}$. If follows that the singular member $2\,D$ of $\mathfrak{B}$ corresponds to a singular fiber of the rational elliptic surface $\kappa : S \to \mathbb{P}^1$ defined by $\mathfrak{B}$ of Kodaira type I_0^*, which according to Section 6.2.10 has Euler number 6. For each of the three partitions of the set of four base points of $\mathfrak{Q}$ into two pairs, we have the corresponding singular member of $\mathfrak{B}$ equal to the union of the two lines, each of which going through the pair of base points. These singular members of $\mathfrak{Q}$ correspond to the three remaining singular members of $\mathfrak{B}$, which in turn correspond to the three remaining singular fibers of $\kappa : S \to \mathbb{P}^1$, each of which is of Kodaira type I_2. It follows that the configuration of the singular fibers of $\kappa : S \to \mathbb{P}^1$ is $\mathrm{I}_0^*\ 3\ \mathrm{I}_2$, a fact that also can be inferred from Table 6.3.2 if we use the formulas for the Weierstrass invariants g_2, g_3, and Δ given in Section 11.2.3. According to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. The degree of the modulus function J is equal to $j = 6$; see (6.2.48).

Note that all these properties just follow, without further computations, from the fact that $\mathfrak{B}$ is a pencil $\mathfrak{B}$ of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with $2\,D$ as a member and having four distinct base points. This pencil is uniquely determined by the choice of the four points on the diagonal D that are declared to be the multiplicity-two base points of $\mathfrak{B}$.

The QRT root ρ leaves all the base points fixed and therefore leaves each of the multiplicity-one irreducible components of the singular fiber of type I_0^* invariant; hence the contribution of this singular fiber to $\rho^S \in \mathrm{Aut}(S)_\kappa^+$ is equal to zero. On the other hand, ρ^S switches the irreducible components of each of the singular fibers of type I_2, with contribution 1/2, and the sum of the contributions of all the reducible fibers to ρ^S is equal to 3/2. Corollary 7.5.4 with $\chi(S, \mathcal{O}) = 1$ and $\nu(\rho^S) = 0$ implies that

$$\begin{aligned} \nu(\rho^S)^k &= \frac{1}{4}\,k^2 + 3\,\left\{\frac{k}{2}\right\}\left(1 - \left\{\frac{k}{2}\right\}\right) - 1 \\ &= \begin{cases} n^2 - 1 \text{ when } \quad k = 2\,n, \\ n^2 + n \text{ when } k = 2\,n + 1, \end{cases} \end{aligned} \tag{11.2.8}$$

for every integer k. Here $\{x\}$ denotes the *fractional part of* x, that is, $0 \le \{x\} < 1$ and $x - \{x\} \in \mathbb{Z}$. The QRT mapping $\tau^S = (\rho^S)^2$ acts on $\mathrm{H}_2(S, \mathbb{Z})$ as an Eichler–Siegel transformation, and $\nu((\tau^S)^n) = n^2 - 1$ for every $n \in \mathbb{Z}$.

11.2.2.1 The Mordell–Weil group

Let α be an element of the Mordell–Weil group $\mathrm{Aut}(S)_\kappa^+$ that generates $\mathrm{Aut}(S)_\kappa^+$ modulo torsion, that is, modulo its elements of order two. Then $\rho = \alpha^m\,\beta$, where

$m \in \mathbb{Z}_{\neq 0}$ and $\beta^2 = 1$. By replacing α by α^{-1} if necessary, we can arrange that $m > 0$. Then $\tau = \rho^2 = \alpha^{2m}\beta^2 = \alpha^{2m} = (\alpha^2)^m$. Because τ has no fixed points in S^{reg}, it follows that α^2 has no fixed points in S^{reg}, which in turn implies that $\nu(\alpha^2) = 0$. Because each singular fiber of type I_2 has two irreducible components, and α acts on the set of multiplicity-one components of the singular fiber of type I_0^* as an element of $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$, see the end of Section 6.3.6, it follows that α^2 leaves all irreducible components of all singular fibers invariant. That is, all contributions to α^2 of all reducible fibers are equal to zero. Therefore (4.3.2) implies that $\nu((\alpha^2)^n) = n^2 - 1$, in particular, $0 = \nu(\tau) = \nu((\alpha^2)^m) = m^2 - 1$, hence $m = 1$. We conclude that $\rho = \alpha\,\beta$, that is, *the billiard map ρ generates the Mordell–Weil group modulo torsion*, and $\tau = \rho^2$ generates the subgroup of all squares in the Mordell–Weil group.

If β is a nontrivial torsion element of the Mordell–Weil group, then it follows from (7.5.4) with $\chi(S,\,\mathcal{O}) = 1$ that the sum of the contributions to β of the reducible fibers is equal to 2. Because each of the three singular fibers of type I_2 contributes 0 or $1/2$, and the singular fiber of type I_0^* contributes 0 or 1, we have that there is one singular fiber F_β of type I_2 whoe irreducible components are invariant under β, whereas β switches the two irreducible components of each of the two other singular fibers of type I_2, and also permutes the multiplicity-one irreducible components of the I_0^* fiber in a nontrivial way. The latter are the proper images of the -1 curves that appear at the first blowing up at the four multiplicity-two base points on the diagonal D, which are the intersection points of the biquadratic curve $C = \pi(F_\beta)$ with Δ, where $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ denotes the blowing-up map. C is the union of two $(1,\,1)$-curves C_1 and C_2, and the set of all four base points is partitioned into the two subsets $C_1 \cap D$, $C_2 \cap D$, each consisting of two base points. Because $\pi^{-1}(C_1)$ and $\pi^{-1}(C_2)$ are invariant under β, it follows that $\pi^{-1}(C_1 \cap D)$ and $\pi^{-1}(C_2 \cap D)$ are invariant under β. Each of these subsets of the singular fiber $\pi^{-1}(D)$ of type I_0^* is the union of two multiplicity-one irreducible components of $\pi^{-1}(D)$, which are switched by β, according to the description at the end of Section 6.3.6 of the action of the group $\mathfrak{F}_c$ on the set of multiplicity-one components of a fiber of type I_0^*. In this way the action of β on the set of multiplicity-one components of the I_0^* fiber is determined by the fiber F_β of type I_2, the irreducible components of which are preserved by β.

The product, or sum, of two distinct nontrivial elements of $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$ is again nontrivial. Therefore, if β_1 and β_2 are two distinct nontrivial torsion elements of the Mordell–Weil group, then $\beta_3 = \beta_2 \circ \beta_1$ is the third nontrivial torsion element. It follows that the map $\beta \mapsto F_\beta$ is bijective from the set of nontrivial torsion elements of the Mordell–Weil group onto the set of the three singular fibers of type I_2. The billiard map ρ, together with any two of the three nontrivial torsion elements, generates the Mordell–Weil group.

For any $\alpha \in \mathrm{Aut}(S)^+_\kappa$ we have $\alpha = \rho^m \circ \beta$, with $m \in \mathbb{Z}$ and $\beta = 1$ or $\beta = \beta_l$, a torsion element of the Mordell–Weil group. If we apply (7.5.1) with $\chi(S,\,\mathcal{O}) = 1$ to α, where $\pi^0 \circ j$ is a homomorphism with the torsion element β in its kernel, and use that $\nu(\rho) = 0$ and the sum of the contributions to ρ is equal to $3/2$, we obtain

$$\nu(\alpha) = -1 + \frac{m^2}{4} + \frac{1}{2} \sum_{r \in C^{\mathrm{red}}} \mathrm{contr}_r(\alpha). \tag{11.2.9}$$

The sum of the contributions to α is equal to 0 if m is even and $\beta = 1$, equal to 2 if m is even and $\beta = \beta_l$, and equal to 3/2 if m is odd. Therefore $\nu(\tau^n \circ \beta_l) = n^2$ and $\nu(\rho^{2n+1} \circ \beta_l) = n^2 + n$, whereas we already had obtained before that $\nu(\tau^n) = n^2 - 1$ and $\nu(\rho^{2n+1}) = n^2 + n$.

According to Theorem 4.3.2, $\alpha \in \mathrm{Aut}(S)^+_\kappa$ is a Manin QRT automorphism if and only if α maps a section to a disjoint one, that is, $\nu(\alpha) = 0$. For the elliptic surface of the billiard map, these α are β_l, ρ, $\rho \circ \beta_l$, ρ^{-1}, $\rho^{-1} \circ \beta_l$, τ, and τ^{-1}, where $l = 1, 2, 3$. That is, the Mordell–Weil group contains 13 Manin QRT automorphisms, as compared to the maximum of 240 Manin QRT automorphisms when there are no reducible fibers; see Theorem 4.3.3.

The action of β_l on the Weierstrass curve maps the point $[0 : 0 : 1]$ at infinity to a point $[1 : X_{\beta_l} : Y_{\beta_l}]$. We have $X_{\beta_l} = X_l$ with X_l given by (11.2.12), whereas $Y_{\beta_l} = 0$ because β_l has order two. This leads to explicit formulas for the action of the generators of $\mathrm{Aut}(S)^+_\kappa$ in the Weierstrass model, and therefore in principle of any element of the Mordell–Weil group.

11.2.3 Further Computations for the Billiard Map

Let $l = l_0$ be the parameter of the quadric Q_0^* in the family of confocal quadrics (11.2.2), which is the boundary of the billiard. That is, the dual quadric Q_0 is defined by (11.2.3) with $\lambda = \lambda_0 := {l_0}^2/4$. The linear substitution of variables

$$\xi_0 = u_2 + u_0, \ \xi_1 = {\lambda_0}^{-1/2} u_1, \ \xi_2 = (\lambda_0 - 1)^{-1/2} (u_2 - u_0) \tag{11.2.10}$$

brings Q_0 into the form (10.2.5). The pencil $\mathfrak{Q}$ is defined by the equation $z_0 f^0(u) + z_1 f^1(u) = 0$, where

$$f^0(u) = (u_2+u_0)^2+(\lambda_0-1)^{-1} (u_2-u_0)^2, \ f^1(u) = {\lambda_0}^{-1} {u_1}^2+(\lambda_0-1)^{-1} (u_2-u_0)^2;$$

see (11.2.4). We multiply both functions by the common factor $\lambda_0 (\lambda_0 - 1)$, in order to remove the denominators. The substitution $u = \Psi(x, y)$ in (10.2.1) leads to the pencil $\mathfrak{B}$ of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by (2.5.3), where

$$A^0 = \begin{pmatrix} {\lambda_0}^2 & 0 & 0 \\ 0 & 2\lambda_0(\lambda_0 - 2) & 0 \\ 0 & 0 & {\lambda_0}^2 \end{pmatrix}, \quad A^1 = \begin{pmatrix} \lambda_0 & 0 & \lambda_0 - 1 \\ 0 & -2 & 0 \\ \lambda_0 - 1 & 0 & \lambda_0 \end{pmatrix}. \tag{11.2.11}$$

Figure 11.2.2 shows the real pencil of curves for $\lambda_0 = 2$, where the affine coordinates are chosen in such a way that the elliptic singular points on the axes at infinity have become visible. Note that the unbounded curves, including the hyperbolic singular

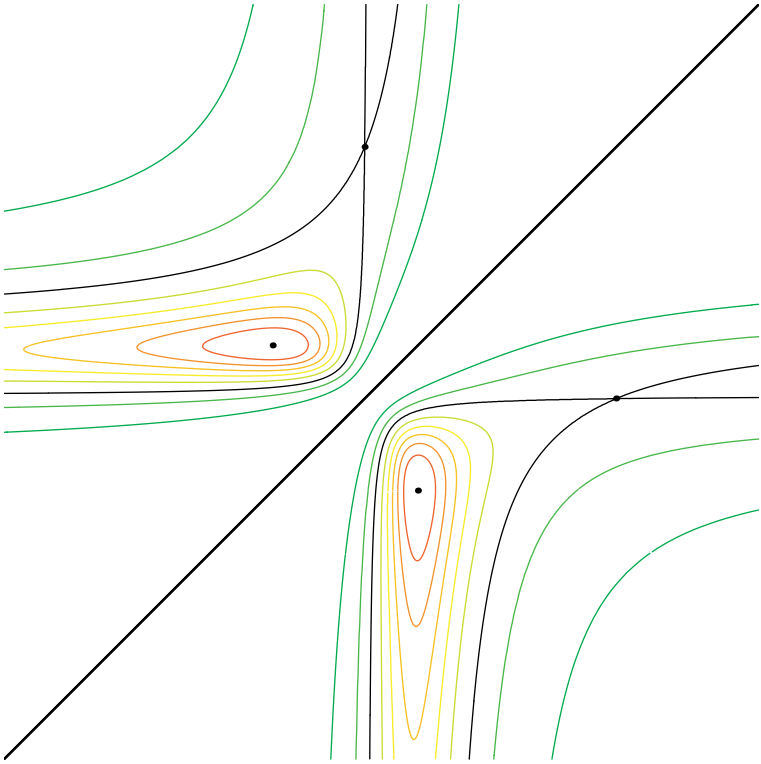


Fig. 11.2.2 The pencil of real curves for the elliptic billiard.

curves, that run to infinity horizontally and vertically are connected in the projective space $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. Each branch running horizontally (vertically) to $+\infty$ comes back horizontally (vertically) at the opposite side from $-\infty$. In this way Figure 11.2.2 can be viewed as a numerical version of the picture of Birkhoff [17, p. 249]. The elliptic singular points correspond to the billiard ball bouncing vertically up and down between the foci. The hyperbolic singular curves correspond to the billiard ball passing alternately through the foci. The diagonal is the (1, 1)-curve of the pencil of multiplicity two, and corresponds to the billiard ball gliding along the rim.

For the choice of the members of the real pencil that appear in Figure 11.2.2, see the text under the heading "Figure 2.3.2," starting with "In all pictures in this book of real pencils …."

11.2.3.1 The Weierstrass invariants

The Weierstrass invariants of the pencil of biquadratic polynomials, the QRT map τ, and the QRT root = the billiard map ρ, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$
\begin{aligned}
3\,g_2 &= 2^6\,(\lambda_0-1)^2\,{\lambda_0}^2\,(\lambda_0\,z_0+z_1)^2\,({z_0}^2+z_0\,z_1+{z_1}^2),\\
3^3\,g_3 &= 2^8\,(\lambda_0-1)^3\,{\lambda_0}^3\,(\lambda_0\,z_0+z_1)^3\,(z_0-z_1)\,(2\,z_0+z_1)\,(z_0+2\,z_1),\\
\Delta &= 2^{16}\,(\lambda_0-1)^6\,{\lambda_0}^6\,(\lambda_0\,z_0+z_1)^6\,{z_0}^2\,{z_1}^2\,(z_0+z_1)^2,\\
3\,X &= {\lambda_0}^2\,(4-4\,\lambda_0+3\,{\lambda_0}^2)\,{z_0}^2+2\,\lambda_0\,(2-\lambda_0+2\,{\lambda_0}^2)\,z_0\,z_1+(3-4\,\lambda_0+4\,{\lambda_0}^2)\,{z_1}^2,\\
Y &= 2\,((2\,\lambda_0-{\lambda_0}^2)\,z_0+z_1)\,({\lambda_0}^2\,z_0+z_1)\,({\lambda_0}^2\,z_0+(2\,\lambda_0-1)\,z_1),\\
3\,X_\rho &= -2^2\,\lambda_0\,(\lambda_0-1)\,(\lambda_0\,{z_0}^2+2\,(\lambda_0-1)\,z_0\,z_1-{z_1}^2),\\
Y_\rho &= 2^4\,{\lambda_0}^2\,(\lambda_0-1)^2\,z_0\,z_1\,(z_0+z_1).
\end{aligned}
$$

The confocal quadric with the parameter λ corresponds to $z_1/z_0=-\lambda$, where $z_0=0$ corresponds to $\lambda=\infty$. The modulus of the elliptic curves is equal to

$$J={g_2}^3/\Delta=\frac{4}{27}\,\frac{({z_0}^2+z_0\,z_1+{z_1}^2)^3}{{z_0}^2\,{z_1}^2\,(z_0+z_1)^2};$$

see (2.3.23). Remarkably, it does not depend on the value of the parameter λ_0. We also recover the previously observed fact that J has degree $j=6$.

The discriminant equation $\Delta=0$ corresponds to the singular biquadratic curves and the singular fibers. The factor $(\lambda_0\,z_0+z_1)^6=(\lambda-\lambda_0)^6$ in Δ corresponds to the singular fiber of Kodaira type I_0^*, which occurs for $\lambda=\lambda_0$. This fiber has intersection diagram $\mathrm{D}_4^{(1)}$, and Euler number equal to 6; see Kodaira [109, III, p. 14, Table II]. Note that the value of J at this singular fiber is equal to $4\,({\lambda_0}^2-\lambda_0+1)^3/27\,{\lambda_0}^2\,(\lambda_0-1)^2$. This attains arbitrary values, depending on the value of the parameter λ_0.

The factors ${z_0}^2$, ${z_1}^2$, and $(z_0+z_1)^2$ in Δ correspond to the singular fibers of Kodaira type I_2, which occur at $\lambda=\infty$, $\lambda=0$, and $\lambda=1$, respectively. If $\lambda_0\in\mathbb{R}$, then at these values for λ we have $g_3<0$, $g_3>0$, $g_3<0$, and therefore these singular fibers are of hyperbolic, elliptic, and hyperbolic type with respect to the real structure, respectively.

Note that the zeros of Y_ρ correspond to the singular fibers of type I_2. Each of these singular fibers comes from a biquadratic curve C that is the union of two $(1,\,1)$-curves C_1 and C_2, where C_1 and C_2 are invariant under the symmetry switch, and therefore are permuted by ρ. It is also easily verified that $X_\rho=-3\,g_3/2\,g_2$ if $Y_\rho=0$. In this way the formulas for the Weierstrass invariants X_ρ and Y_ρ of the billiard map ρ illustrate Remark 10.1.5.

The three nontrivial torsion elements β_l, $l=1,\,2,\,3$, of the Mordell–Weil group are of order two, and therefore $Y_{\beta_l}(z_0,\,z_1)\equiv 0$; see Corollary 2.5.9. Because $[1:X_{\beta_l}:Y_{\beta_l}]$ lies on the Weierstrass curve with the coefficients g_2 and g_3, the X_{β_l} are the three zeros of the polynomial $x\mapsto -4\,x^3+g_2\,x+g_3$. It follows from the formulas for $g_2=g_2(z_0,\,z_1)$ and $g_3=g_3(z_0,\,z_1)$ that $x^3-(g_2\,x+g_3)/4=(x-X_1)\,(x-X_2)\,(x-X_3)$, where

$$X_1=w\,(2\,z_0+z_1),\quad X_2=w\,(-z_0+z_1),\quad\text{and}\quad X_3=w\,(-z_0-2\,z_1),\tag{11.2.12}$$

with the common factor $w=(4/3)\,\lambda_0\,(\lambda_0-1)\,(\lambda_0\,z_0+z_1)$. Because X_l is a homogeneous polynomial of degree two in $(z_0,\,z_1)$, and $X=X_l$, $Y=0$ satisfies the

Weierstrass equation $Y^2 - 4\,X^3 + g_2\,X + g_3 = 0$, it follows from Proposition 7.8.1 that $X = X_l$, $Y = 0$ defines an element of the Mordell–Weil group, which has order two in view of Corollary 2.5.9. Because the X_l are distinct, the elements of the Mordell–Weil group are distinct, and we can arrange the enumeration of the β_l such that $X_{\beta_l} = X_l, l = 1,\ 2,\ 3$.

The common factor $\lambda_0\,z_0 + z_1$, over the zero of which we have the singular fiber of type I_0^*, where $g_2 = g_3 = 0$ and the singular point of the Weierstrass curve is $[1 : 0 : 0]$, corresponds to the fact that β_j switches the multiplicity-one components of the I_0^* fiber; see Remark 10.1.5. The I_2 fibers occur over $[z_0 : z_1] = [0 : 1]$, $[1 : 0], [1 : -1]$, where $-3\,g_3/2\,g_2$ is equal to $(4/3)\,\lambda_0\,(\lambda_0 - 1)$ times $1,\ -\lambda_0, \lambda_0 - 1$, respectively. At these points X_1 is equal to $(4/3)\,\lambda_0\,(\lambda_0 - 1)$ times $1,\ 2\,\lambda_0,\ \lambda_0 - 1$, respectively, and it follows from Remark 10.1.5 that β_1 switches the components of the fibers over $[0 : 1]$ and $[1 : -1]$, whereas it leaves the components of the fiber over $[1 : 0]$ invariant. That is, $F_{\beta_1} = S_{[1:0]}$. In the same way we obtain that $F_{\beta_2} = S_{[1:-1]}$ and $F_{\beta_3} = S_{[0:1]}$.

11.2.3.2 The Real Billiard

The substitution (11.2.10) has been chosen in such a way that it is real if the confocal quadric Q_0^* for $\lambda = \lambda_0$ is an ellipse, that is, if $\lambda_0 \in \mathbb{R}_{>1}$. Equation (11.2.3) has solutions in the real projective plane $\mathbb{P}^2(\mathbb{R})$ if and only if $\lambda \geq 0$, where in the real affine plane the solutions form an ellipse or hyperbola if $1 < \lambda < \infty$ and $0 < \lambda < 1$, respectively. On the other hand, for $[u_0 : u_1 : u_2] \in \mathbb{P}^2(\mathbb{R})$ the equation $\psi(x,\ y) = u$, see (10.2.3), has real solutions $(x,\ y) \in \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ if and only if ${u_1}^2 - 4\,u_0\,u_2 \geq 0$, or equivalently

$$-{\xi_0}^2 + \lambda_0\,{\xi_1}^2 + (\lambda_0 - 1)\,{\xi_1}^2 \geq 0. \tag{11.2.13}$$

If (11.2.3) holds then this inequality is equivalent to $\lambda \leq \lambda_0$. In other words, the member of the pencil $\mathfrak{B}$ of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with the parameter $\lambda = -z_1/z_0$ has real points, in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, if and only if $0 \leq \lambda \leq \lambda_0$. In particular, also the real locus of the singular fiber at $\lambda = \infty$ is empty. The singular values of κ in $[0,\ \lambda_0]$ are 0, 1, and λ_0, over which the singular fiber has Kodaira type I_2, I_2, and I_0^*, respectively.

11.2.3.3 $1 < \lambda < \lambda_0$

If $1 < \lambda < \lambda_0$, then the confocal quadric $Q^*(\mathbb{R})$ with parameter λ is an ellipse that lies entirely inside the confocal ellipse $Q_0^*(\mathbb{R})$ with parameter λ_0. This corresponds to reflections as in the left hand picture in Figure 11.2.1. In view of Corollary 11.2.4, the real locus $\psi^{-1}(Q)(\mathbb{R})$ of the member $\psi^{-1}(Q)$ of $\mathfrak{B}$ corresponds to the set R of all pairs $(x,\ l)$ such that x is a point on $Q_0^*(\mathbb{R})$, l is a straight line through x, and l is tangent to Q^*. If x' denotes the second point of intersection of l with Q^*, the direction

from x to x' defines an orientation of l, and Q^* lies either to the right or to the left of the line $x\,x'$. It follows that R has two connected components, each of which is diffeomorphic to a circle. The involution $(x,\, l) \mapsto (x',\, l)$ interchanges the two connected components of R, and the same is true for the involution $(x', l) \mapsto (x',\, l')$, where l' is the other straight line through x' that is tangent to Q^*. Therefore the billiard map leaves each of the two connected components of R invariant. In terms of the elliptic surface $\kappa : S \to \mathbb{P}^1$ with the real structure defined by the real pencil $\mathfrak{B}$, see Definition 8.1.1 and Section 8.5, this means that if $1 < \lambda < \lambda_0$, then the fiber of $\kappa(\mathbb{R}) : S(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ over λ has two connected components, each of which is invariant under the QRT root ρ^S.

The billiard ball, after repeated reflections, goes around the two focal points of the elliptic billiard, where the trajectory, when crossing the long axis, intersects the long axis in the complement of the interval on the long axis between the two focal points. The two connected components of the curve correspond to the two directions in which the billiard ball can travel around the focal points.

11.2.3.4 $0 < \lambda < 1$

If $0 < \lambda < 1$, then the confocal quadric $Q^*(\mathbb{R})$ with parameter λ is a hyperbola in the affine plane, each branch of which intersects the long axis of the ellipse $Q_0^*(\mathbb{R})$ between the two foci. This corresponds to reflections as in the right-hand picture in Figure 11.2.1. All tangent lines to $Q^*(\mathbb{R})$, including the asymptotes, which are the tangent lines to $Q^*(\mathbb{R})$ at infinity, lie in the domain B between the two branches of $Q^*(\mathbb{R})$. Therefore, if $(x,\, l) \in R$, then $x \in B$. The intersection $Q_0^*(\mathbb{R}) \cap B$ has two connected components, whose preimages in R under the projection $(x,\, l) \mapsto x$ are the two connected components of R. Note that because R is the real locus of an elliptic curve, it cannot have more than two connected components. This time the involution $(x,\, l) \mapsto (x',\, l)$ interchanges the two connected components, whereas the involution $(x',\, l) \mapsto (x',\, l')$ leaves each of the connected components of R invariant. The conclusion is that the billiard map interchanges the two connected components of R. In terms of the elliptic surface $\kappa : S \to \mathbb{P}^1$, this means that if $0 < \lambda < 1$, then the fiber of $\kappa(\mathbb{R}) : S(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ over λ has two connected components, which are interchanged by the QRT root ρ^S. It follows that if k is odd, then $(\rho^S)^k$ interchanges the two connected components, which implies that when $0 < \lambda < 1$, the fiber over λ cannot be periodic with period k.

The billiard ball bounces up and down, every time crossing the long axis of the billiard on the interval between the two focal points of the elliptic billiard. The two connected components of the real fiber correspond to the two connected components of the hyperbola to which the trajectories between the bounces are tangent.

11.2.3.5 $\lambda \downarrow 0$

When $\lambda \downarrow 0$, the hyperbola $Q^*(\mathbb{R})$ approaches the vertical axis $x_1 = 0$ with multiplicity two, and each of the connected components of R shrinks to a point. The points

on $Q^*(\mathbb{R})$ at which the straight lines through billiard trajectories are tangent to $Q^*(\mathbb{R})$ run to infinity, corresponding to the fact that the only solutions in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ of the equation $f^0 = 0$ are the points ([1 : 0], [0 : 1]) and ([0 : 1], [1 : 0]), which lie on the horizontal and vertical axes at infinity, respectively. These points correspond to *the billiard ball bouncing vertically up and down precisely between the two focal points*.

The fact that the connected components of R shrink to points means that over $\lambda = 0$ the singular fiber of Kodaira type I_2 is elliptic with respect to the real structure. The formula (8.4.1) implies that the rotation number $\rho(\lambda)$ converges to

$$\rho(+0) = \frac{1}{\pi} \arccos(1 - 2/\lambda_0) \tag{11.2.14}$$

when $\lambda \downarrow 0$. Here we have used that when $z_0 = 1$, $z_1 = 0$, we have $g_2 = 2^6 (\lambda_0 - 1)^2 {\lambda_0}^4/3$, $X = {\lambda_0}^2 (3{\lambda_0}^2 - 4\lambda_0 + 4)$, hence $X + (g_2/12)^{1/2} = {\lambda_0}^4$, and $Y = 2(2 - \lambda_0){\lambda_0}^5$. As a function of λ_0, $\rho(+0)$ decreases montonically from 1 to 0 as λ_0 increases from 1 to ∞. It follows that $\rho(+0)$ can take any real value in $]0, 1[$, and $\rho(+0)$ is in bijective correspondence with the parameter λ_0 of the elliptic billiard.

11.2.3.6 $\lambda = 1$

When $\lambda = 1$, R is equal to the set of all (x, l) such that $x \in P_+^*(\mathbb{R})$ and l is a line through x and one of the two foci. The billiard map applied to (x, l) leads to the element (x', l') of R where the reflected line l' passes through the other focal point. That is, for the iterates under the billiard map the lines pass alternately through the two focal points. In other words, the fiber over $\lambda = 1$ corresponds to *the perfect focusing of the elliptic mirror*. The name "focal point" of an ellipse historically comes from this phenomenon.

The real fiber of $\kappa(\mathbb{R})$ over $\lambda = 1$ is the union of two smooth closed real one-dimensional curves that intersect each other at two points, where the intersection is transversal. That is, over $\lambda = 1$ the singular fiber is hyperbolic with respect to the real structure. The singular points, that is, the intersection points, correspond to the $(x, l) \in R$ where x is one of the two intersection points of $P_+(\mathbb{R})$ with the long axis A, and $l = A$. The billiard map, isomorphic to the QRT root, interchanges the two singular points. For the QRT map $\tau^S = (\rho^S)^2$, both singular points are hyperbolic fixed points, where along each connected component of the smooth part of the singular fiber the points $(\tau^S)^k(s)$, $k \in \mathbb{Z}$, run from one of the fixed points to the other. At each fixed point s, the derivative of the QRT map in the contracting direction is equal to

$$\tau'(s) = ({\lambda_0}^{1/2} - 1)^2/({\lambda_0}^{1/2} + 1)^2 = (((l_0/2) - 1)/((l_0/2) + 1))^2 < 1, \tag{11.2.15}$$

where l_0 is the length of the long axis of the elliptic billiard.

11.2.3.7 $\lambda \uparrow \lambda_0$

When $\lambda \uparrow \lambda_0$, the confocal ellipse $Q^*(\mathbb{R})$ inside $Q_0^*(\mathbb{R})$ converges to $Q_0(\mathbb{R})$, and *the iterates of the billiard map approach a gliding motion along the ellipse* $Q_0^*(\mathbb{R})$. The complex singular fiber over $\lambda = \lambda_0$ is of Kodaira type I_0^*, with a multiplicity-two irreducible component that is equal to the proper transform of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, isomophic to a complex projective line. The multiplicity-one irreducible components of the singular fiber lie over the base points, none of which is real. As a consequence, the real fibers over λ converge to a multiplicity-two real projective line, the real part of the multiplicity two component of the complex singular fiber.

11.2.3.8 The Real Rotation Function

In Chapter 8, the asymptotics for $k \to \infty$ of the real k-periodic fibers for any element of the Mordell–Weil group are related to the properties of the rotation function $\rho(z)$. In this subsection we collect a number of qualitative properties of the rotation function of the QRT map equal to the square of the billiard map.

Assume that $\lambda_0 \in \mathbb{R}, \lambda_0 > 1$. At real values of z where the real fiber is smooth but has two connected components, the elliptic billiard mapping, i.e., the QRT root, might interchange these, and then the rotation number is not defined. However, the QRT map τ always preserves the connected components of the real fibers, and according to Lemma 8.1.5 its rotation function $\rho(z)$ is a real analytic function on every open interval I between values of z over which the real fiber is singular, and such that over I the real fibers are nonempty, that is, for $I =]0,\ 1[$ and for $I =]1,\ \lambda_0[$.

For the asymptotics for $k \to \infty$ of the real k-periodic fibers for the billiard map, we can apply Chapter 8 to the QRT root $\rho^S \in \mathrm{Aut}(S)^+_\kappa$. For instance, because $\lambda = 0$ corresponds to an elliptic singular fiber of Kodaira type I_2, both the rotation number and the density are analytic up to the boundary value $\lambda = 0$. On the other hand, because $\lambda = 1$ corresponds to a hyperbolic singular fiber of Kodaira type I_2 and the even iterates of the billiard map move the points on the regular part from one of the two singular points to the other, it follows from Remarks 8.4.3 and-8.4.2 that for $\lambda \uparrow 1$ and for $\lambda \downarrow 1$ the rotation number converges to an integer, where the graph of the rotation number function has an infinite order of contact with the vertical axis. Finally, it can be proved that the billiard map induces a nontrivial motion on the multiplicity-one connected components of the singular fiber of Kodaira type I_0^*, which implies that for $\lambda \uparrow \lambda_0$ the rotation number converges to an integer in a square root type of way. It follows that the density of the periodic real fibers is infinite when $\lambda \to 1$ and when $\lambda \uparrow \lambda_0$, that is, when the rays are close to passing through the foci and close to gliding along the boundary, respectively.

With our definition of the rotation number lying strictly between 0 and 1, we have $\rho(\lambda) \uparrow 1$ as $\lambda \uparrow \lambda_0$. Because $\lambda \uparrow \lambda_0$ corresponds to the gliding motion along the rim of the elliptic billiard, where the distance between the bouncing points converges to zero, it would have been more natural to say that $\rho(\lambda) \uparrow 0$ as $\lambda \uparrow \lambda_0$.

The first picture that I have seen of any rotation function of a QRT map was Figure 2(b) in Waalkens, Wiersig, and Dullin [202]. Our parameters $\lambda_0 = {l_0}^2/4$ and $\lambda = l^2/4$ correspond to their a and κ via $\lambda_0 = 1/a^2$ and $\lambda/\lambda_0 = \kappa^2$, where κ^2 on their horizontal axis runs from 0 to 1. Note that $a = 2/l_0 \in]0,\ 1[$ and $\kappa = l/l_0 \in]0,\ 1[$, where l_0 was the length of the string in the gardener's construction of the ellipse with focal points at distance 2 from each other. Their "winding number $\widetilde{w}$ for the desymmetrized billiard" on the vertical axis is equal to $1 - \rho$, where ρ is our rotation number of the QRT map. Their Figure 2(b), for $a = 0.7$, prominently shows an infinitely steep behavior at $\kappa^2 = a^2 = 0.49$, corresponding to our $\lambda = 1$, over which we have the hyperbolic singular fiber of Kodaira type I_2. For $\kappa^2 \uparrow 1$, corresponding to $\lambda \uparrow \lambda_0$, over which we have the I_0^* fiber, there is a square root behavior. Regarding the limit value for $\kappa^2 \downarrow 0$, corresponding to $\lambda \downarrow 0$, over which we have the elliptic I_2 fiber, our formula (11.2.14) agrees with the formula $\widetilde{w} = 2\ \arccos(a)/\pi$ of Waalkens, Wiersig, and Dullin [202, p. 55], because $(2/\lambda_0) - 1 = -\cos(\rho(+0)\,\pi) = \cos((1 - \rho(+0))\,\pi) = \cos(2\,\widetilde{w}\,\pi/2) = 2\cos^2(\widetilde{w}\,\pi/2) - 1 = 2\,a^2 - 1$ is equivalent to $\lambda_0 = 1/a^2$.

Figures 11.2.3 and 11.2.4 At the conference in honor of Richard Cushman's 65th birthday at Utrecht, 26–30 March 2007, Holger Dullin showed me Figure 11.2.3, of the rotation number as a function of the two variables $\kappa^2 = \lambda/\lambda_0$ and $a^2 = 1/\lambda_0$, that is, over the square $[0,\ 1] \times [0,\ 1]$. Figure 11.2.4, which shows the superimposed graphs of Figure 11.2.3 for $1/\lambda_0 = n/20, 0 \le n \le 20$, convincingly suggests that for every $\lambda_0 > 1$, the rotation function $\rho(\lambda)$ is monotonically decreasing and increasing for $0 < \lambda < 1$ and $1 < \lambda < \lambda_0$, respectively.

Staring at Figure 11.2.3, we were puzzled by the apparent symmetry

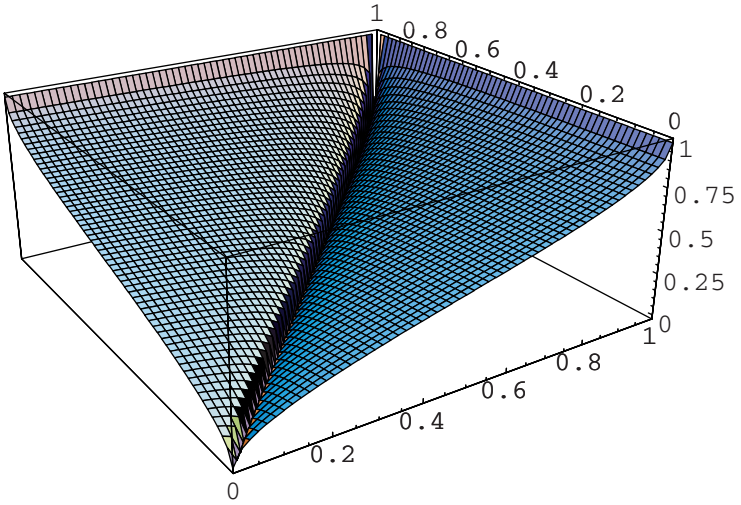


Fig. 11.2.3 The rotation number as a function of $(\lambda/\lambda_0,\ 1/\lambda_0)$.

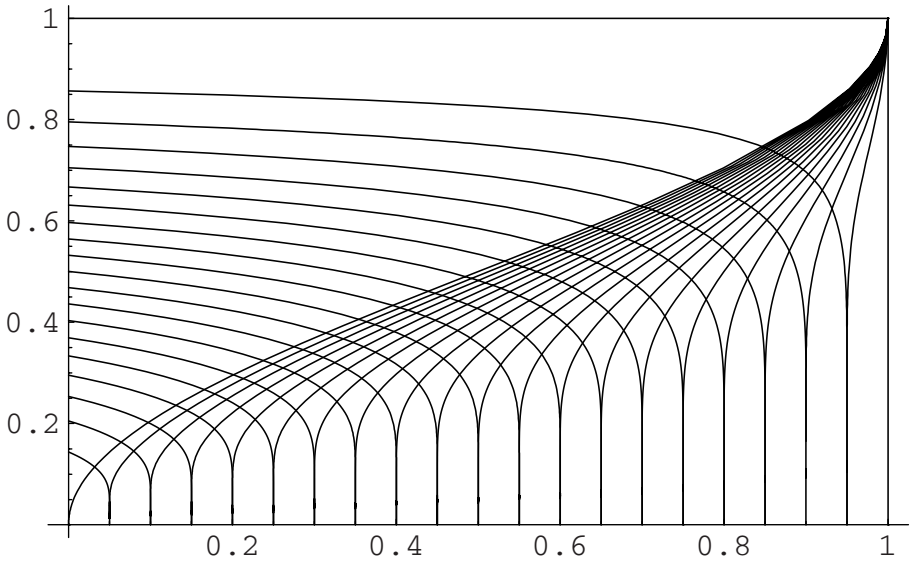


Fig. 11.2.4 The rotation functions $\lambda/\lambda_0 \mapsto \rho(\lambda)$ for $1/\lambda_0 = n/20$, $n = 0, \ldots, 20$.

$$(\lambda/\lambda_0,\ 1/\lambda_0) \mapsto (1/\lambda_0,\ \lambda/\lambda_0).$$

Note that keeping $1/\lambda_0$ constant corresponds to the elliptic surface for an elliptic billiard, but keeping λ/λ_0 constant does not have such an interpretation. Also note that this symmetry switches the domain $\lambda/\lambda_0 > 1/\lambda_0 \Leftrightarrow \lambda > 1$, where the confocal quadric is an ellipse, with the domain $\lambda/\lambda_0 < 1/\lambda_0 \Leftrightarrow \lambda < 1$, where the confocal quadric is a hyperbola. Also note that the symmetry implies that the rotation number for $\lambda/\lambda_0 = c$, $1/\lambda_0 \to 0$, that is, $\lambda_0 \to \infty$, is equal to $\rho(+0)$ as in (11.2.14) with $1/\lambda_0 = c$. The arc cosine function $c \mapsto (1/\pi)\arccos(1 - 2c)$ appears in Figure 11.2.4 for $n = 0$.

We now turn the suggestions of the computer pictures into mathematical theorems.

Proposition 11.2.5 *The rotation function $\rho(\lambda)$ of the billiard QRT map satisfies $\rho'(\lambda) < 0$ when $0 < \lambda < 1$ and $\rho'(\lambda) > 0$ when $\lambda > 1$. The rotation function $\rho(\lambda)$ decreases infinitely steeply to 0 when $\lambda \uparrow 1$ and when $\lambda \downarrow 1$.*

Proof. As a function of $z = \lambda = -z_1/z_0$, Manin's function $\mu = \mathcal{L}\,T$ of the QRT map, see Proposition 2.5.20, is equal to

$$\mu(\lambda) = 1/4\,(\lambda_0 - \lambda)^2\,\lambda\,(\lambda - 1),$$

which is < 0 and > 0 for $0 < \lambda < 1$ and $1 < \lambda < \lambda_0$, respectively. Because also

$$3\,\Gamma = 2^{14}\,(\lambda_0 - 1)^5\,{\lambda_0}^5\,(\lambda_0 - \lambda)^5\,(\lambda - 1)\,\lambda$$

has no zeros for $0 < \lambda < 1$ or $1 < \lambda < \lambda_0$, we can apply the Beukers–Cushman monotonicity criterion of Section 2.6.3. Since the singular fiber at $\lambda = 0$ is of *elliptic* Kodaira type I_2, it follows that the derivative $\rho'(\lambda)$ of the rotation function $\rho(\lambda)$ is strictly negative for $0 < \lambda < 1$, which implies that $\rho(\lambda)$ is strictly decreasing in the interval $]0,\ 1[$.

Over $\lambda = 1$ we have a *hyperbolic* fiber of type I_2, and it follows from Remark 8.4.2 and (11.2.15) that the rotation function approaches its common limit value 0 or 1 in an inifinitely steep way. Because $0 < \rho(\lambda) < 1$ and $\rho'(\lambda) < 0$ for $0 < \lambda < 1$, it follows that $\rho(1-0) = \rho(1+0) = 0$. Because $0 < \rho(\lambda) < 1$ for $1 < \lambda < \lambda_0$, we have $\rho(\lambda) \downarrow 0$ in an inifinitely steep way as $\lambda \downarrow 1$, and it follows that $\rho'(\lambda) > 0$, and therefore $\sigma(\lambda) > 0$, when $\lambda > 1$ and λ is close to 1. On the other hand (2.6.8), in combination with the fact that $P = 3\,\Gamma/2\,\Delta$ and μ have no zeros of poles for $1 < \lambda < \lambda_0$, implies that $(\sigma/|P|)'$ has the same sign as μ on $]1,\ \lambda_0[$, which is positive, and we conclude that $\sigma(\lambda)/|P(\lambda)|$, hence $\sigma(\lambda)$, hence $\rho'(\lambda)$ is strictly positive for $1 < \lambda < \lambda_0$. This proves the monotonicity and infinite steepness suggested by Figure 11.2.4. □

Proposition 11.2.6 *If λ_0' and λ' are defined by $\lambda/\lambda_0 = 1/\lambda_0'$ and $1/\lambda_0 = \lambda'/\lambda_0'$, then the rotation number of the QRT map on the real fiber over λ' for the elliptic billiard with the parameter λ_0' is equal to the rotation number of the QRT map on the real fiber over λ for the elliptic billiard with the parameter λ_0.*

Proof. In the formulas for the Weierstrass invariants g_2, g_3, X, and Y we first substitute $z_0 = 1$, $z_1 = -\lambda$, when the Weierstrass invariants are polynomials in $(\lambda,\ \lambda_0)$, and then $\lambda = 1/\lambda'$, $\lambda_0 = \lambda_0'/\lambda'$. This leads to $g_2(\lambda,\ \lambda_0) = c^4\, g_2(\lambda',\ \lambda_0')$ $g_3(\lambda,\ \lambda_0) = c^6\, g_3(\lambda',\ \lambda_0')$, $X(\lambda,\ \lambda_0) = c^2\, X(\lambda',\ \lambda_0')$, and $Y(\lambda,\ \lambda_0) = c^3\, Y(\lambda',\ \lambda_0')$, where $c = 1/\lambda'^2$. It follows that the isomorphism $[1 : x : y] \mapsto [1 : x/c^2 : y/c^3]$ maps the Weierstrass curve with the parameters $g_2 = g_2(\lambda,\ \lambda_0)$ onto the Weierstrass curve with the parameters $g_2' = g_2(\lambda',\ \lambda_0')$, $g_3' = g_3(\lambda',\ \lambda_0')$, leaving the point at infinity fixed and mapping the point $[1 : X(\lambda,\ \lambda_0) : Y(\lambda,\ \lambda_0)]$ to the point $[1 : X(\lambda',\ \lambda_0') : Y(\lambda',\ \lambda_0')]$. Therefore this isomorphism induces an isomorphism from the biquadratic curve C with the parameters $(\lambda,\ \lambda_0)$ onto the biquadratic curve C' with the parameters $(\lambda',\ \lambda_0')$, which conjugates the QRT map on C with the QRT map on C'. It follows that $\rho(\lambda',\ \lambda_0') = \rho(\lambda_0,\ \lambda)$, which proves the symmetry $(\kappa,\ a) \mapsto (a,\ \kappa)$ suggested by Figure 11.2.3. □

The isomorphism in the proof of Proposition 11.2.6 does not conjugate the QRT root of C with the QRT root of C', and the proof does not explain its geometric meaning, if it has one.

11.2.3.9 The Number of k-Periodic Real Fibers

As $\rho(\lambda)$ decreases from $\rho(+0)$ to 0 for $0 < \lambda < 1$, and then increases from 0 to 1 for $1 < \lambda < \lambda_0$, in a strictly monotonic and continuous fashion, this leads to the following conclusions for the number of k-periodic real fibers for the billiard map, of which the QRT map τ is the square.

If $k = 2n + 1$ is an odd positive integer, then the real fiber over λ is k-periodic for the billiard map if and only if $1 < \lambda < \lambda_0$ and $\rho(\lambda) = 2m/k$ for an integer m such that $0 < m < k$. There are n such λ's, where we note that the corresponding billiard k-gons go around the foci.

If $k = 2n$ is an even positive integer, then the real fiber over λ is k-periodic for the billiard map if and only if it is n-periodic for the QRT map τ if and only if $\rho(\lambda) = m/n$ for an integer m such that $0 < m < n$. There are $n - 1$ such λ's for which $1 < \lambda < \lambda_0$, that is, the corresponding billiard k-gons go around the foci. For $0 < \lambda < 1$, when the legs of the billiard k-gon intersect the long axis in between the foci, the condition is that $\rho(\lambda) = m/n$ for an integer m such that $0 < m/n < \rho(+0)$, where $\rho(+0)$ is given by (11.2.14). There are $\lceil n\,\rho(+0)\rceil - 1$ such λ's, where $\lceil x\rceil$ denotes the smallest integer m' such that $m' \geq x$. It follows that if $k = 2n$ is an even positive integer, then the total number of k-periodic real fibers for the billiard map is equal to $n + \lceil n\,\rho(+0)\rceil - 2$.

Using (7.8.2), (7.8.3), one can successively compute the coordinates $x = X_k$, $y = Y_k$ of the image point $[1 : x : y]$ of the point $[0 : 0 : 1]$ at infinity under the kth power of the billiard map in the Weierstrass model. For small k, we can use the formulas for g_2, g_3, X_ρ, and Y_ρ in order to compute X_k and Y_k as explicit rational expressions, where the zeros of the denominators correspond to the λ over which we have a k-periodic fiber for the billiard as map. Note that one could also use Cayley's criterion, as described in Remark 10.3.4, where I must admit that I have not tried out how efficient that would be.

For $k = 3$ this leads to a second-order equation for λ, with the solutions

$$\lambda_\pm = \lambda_0\,(2\,{\lambda_0}^2 - 3\,\lambda_0 + 2 \pm 2\,(\lambda_0 - 1)\,({\lambda_0}^2 - \lambda_0 + 1)^{1/2}).$$

We have $\lambda_+ > \lambda_0$ and $1 < \lambda_- < \lambda_0$, and therefore the real fiber over λ_- is the unique one on which the billiard map has period 3, that is, each billiard trajectory that is tangent to the confocal ellipse with parameter $\lambda = \lambda_-$ is a triangle.

For $k = 4$ the denominators in X_k and Y_k have the three zeros

$$\lambda = \lambda_0\,(2 - \lambda_0), \quad \lambda = {\lambda_0}^2/(2\,\lambda_0 - 1), \quad \lambda = {\lambda_0}^2.$$

Of these, the last one $\lambda = {\lambda_0}^2$ is $> \lambda_0$, and therefore the real fiber over it is empty. The second one $\lambda = {\lambda_0}^2/(2\,\lambda_0 - 1)$ lies between 1 and λ_0, corresponding to the unique confocal ellipse such that the billiard trajectories tangent to it are quadrangles going around the two foci. Over $\lambda = \lambda_0\,(2 - \lambda_0)$ we have a nonempty real fiber if and only if $1 < \lambda_0 < 2$, when $0 < \lambda < 1$, corresponding to the unique confocal hyperbola such that the billiard trajectories tangent to it are quadrangles passing between the foci. There is no such billiard quadrangle if $\lambda_0 \geq 2$.

These facts about the billiard triangles and quadrangles can be proved in an elementary geometric way by bringing one of the sides of the triangle or quadrangle into a vertical position. For $k = 5$, the equation for the denominators of X_k and Y_k to be equal to zero is a polynomial equation of degree 6 in λ, which I have not reproduced, because Mathematica did not factor it. For $k = 6$, the equation is of degree 8, with the following four factors of degree 2:

$$\lambda^2 - 2\,\lambda_0\,(2\,{\lambda_0}^2 - 3\,\lambda + 2)\,\lambda + {\lambda_0}^4 = 0, \tag{11.2.16}$$

$$\lambda^2 + 2\,{\lambda_0}^3\,(2\,\lambda_0 - 3)\,\lambda + {\lambda_0}^3\,(4 - 3\,\lambda_0) = 0, \tag{11.2.17}$$

$$(4\,\lambda_0 - 1)\,\lambda^2 - 6\,{\lambda_0}^2\,\lambda + {\lambda_0}^3\,(4 - \lambda_0) = 0, \tag{11.2.18}$$

$$(4\,\lambda_0 - 3)\,\lambda^2 - 2\,\lambda_0\,(3\,\lambda_0 - 2)\,\lambda + {\lambda_0}^4 = 0. \tag{11.2.19}$$

We recognize (11.2.16) as corresponding to the period 3 for the billiard map, where the solution $\lambda = \lambda_-$ corresponded to the nonempty real 3-periodic fiber. We had $1 < \lambda_- < \lambda_0$, corresponding to triangles around the foci, tangent to the confocal ellipse with the parameter $\lambda = \lambda_-$. For (11.2.17) there is one solution $\lambda = \lambda_+$ such that $1 < \lambda < \lambda_0$, corresponding to hexagons around the foci. The other solution $\lambda = \lambda_-$ satisfies $0 < \lambda < \lambda_0$ if and only if $1 < \lambda_0 < 4/3$, and then $0 < \lambda_- < 1$, corresponding to hexagons in between the foci. The equation (11.2.18) has no solutions λ such that $0 < \lambda < \lambda_0$ if $\lambda_0 > 4$, and one if $\lambda_0 < 4$, and then this solution satisfies $0 < \lambda < 1$, corresponding to hexagons between the foci. Finally (11.2.19) has no real solutions λ, because its discriminant is equal to $-16\,{\lambda_0}^2\,(\lambda_0 - 1)^3$. These findings agree with the previously made observation that of the eight $2\,n = 6$-periodic fibers only $m = n + \lceil n\,\rho(+0)\rceil - 2 = 1 + \lceil 3\,\rho(+0)\rceil$ are real. Here $\rho(+0)$ is given by (11.2.14); hence $m = 2$, 3, or 4 if $\lambda_0 > 4$, $4/3 < \lambda_0 < 4$, or $1 < \lambda_0 < 4/3$, respectively. With increasing k, the equations become rapidly more unwieldy.

The total variation of the rotation function of the QRT map is equal to

$$\int_0^{\lambda_0} |\rho'(\lambda)|\,\mathrm{d}\lambda = \rho(+0) + 1,$$

where $\rho(+0)$ is given by (11.2.14). Note that the total variation of the rotation function is equal to the limit for $k \to \infty$ of $1/k$ times the number of the real k-periodic curves for the QRT map, see (8.4.2). Since $\rho(+0)$ decreases analytically from 1 to 0 as λ_0 increases from 1 to ∞, the total variation of the rotation function decreases from 2 to 1, where any real number in $]1,\ 2[$ is attained for a unique corresponding value of the parameter λ_0 of the elliptic billiard. As a consequence, the parameter $\lambda_0 = {l_0}^2/4$ of the elliptic billiard is uniquely determined by the limit for $k \to \infty$ of $1/k$ times the number of the real k-periodic curves.

11.2.3.10 The Set of the Complex k-Periodic Fibers

Figure 11.2.5 shows the points in the complex λ-plane such that the billiard QRT map τ on the corresponding curve is periodic of period k, where $k = 67$ and $\lambda_0 = 2$. The gray horizontal segment in the middle is the real segment $[0,\ \lambda_0]$ over which we have nonempty real fibers. Note that $\tau = \rho^2$ for the billiard map ρ, and therefore this corresponds to the period 134 for the billiard map. The figure suggests in a prominent way that the density goes to zero as $\lambda \to 0$, the point in the λ-plane over which we have an elliptic singular fiber. See Remark 8.4.6 for a proof that indeed the density converges to zero here.

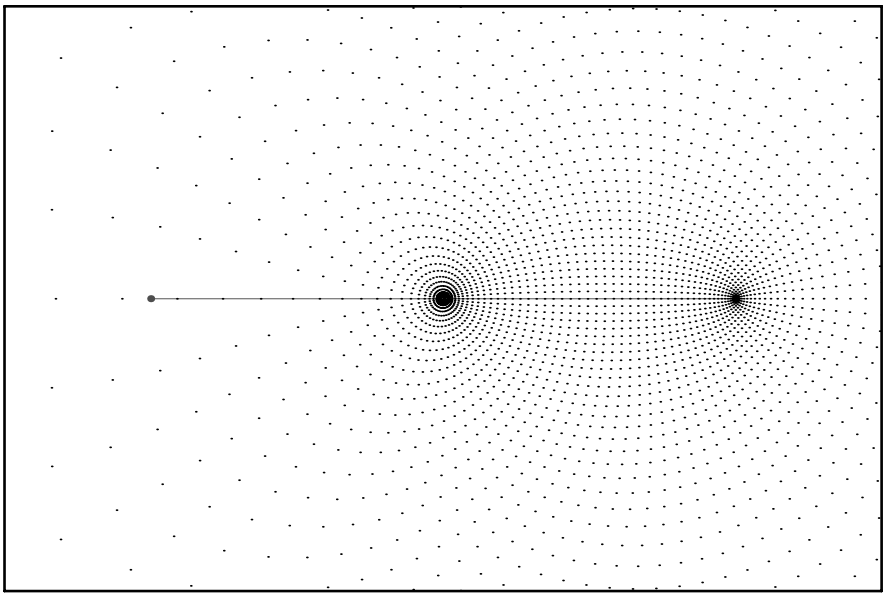


Fig. 11.2.5 Points in $\mathbb{P}^1(\mathbb{C})$ over which the billiard map has period 134.

The density converges to ∞ as $\lambda \to 1$, over which we have a hyperbolic singular fiber of type I_2, and the action of the billiard map on its regular part does not belong to the maximal compact subgroup. For the asymptotic behavior of the set $\mathbb{P}^1_{\tau^k}$ near such points, see Remark 7.7.11 with $C = \mathbb{P}^1$ equal to the complex projective line. Substituting (11.2.15) and $b = 2$ into (7.7.23), we obtain that $|z| \sim C\, a^{k/k_2}$, where $a = ({\lambda_0}^1/2 - 1)/({\lambda_0}^{1/2} + 1) = ((l_0/2) - 1)/((l_0/2) + 1)$. For $\lambda_0 = 2$ we have $a^2 \approx 0.17$, whose 67th power is $\approx 5.1 \times 10^{-52}$, which explains why the first 15 or so approximate circles around $\lambda = 1$ are too small to be visible in Figure 11.2.5. Figure 11.2.5 suggests the rough estimate $C \approx 8$, hence $|z| \approx C\, a^{67/15} \approx 0.003$ for the radius of the 15th circle. However, all the other features described in Remark 7.7.11, including the count of the points on the approximate circles and the monodromy around $\lambda = 1$, are clearly visible in Figure 11.2.5.

The density also converges to ∞ as $\lambda \to \lambda_0$, over which we have a singular fiber of type I_0^*. Figure 11.2.5 indicates that the high density is less pronounced as for $\lambda \to 1$ and $1/\lambda \to 0$. A more precise asymptotic statement to this effect follows from (7.7.20), (7.7.21), (7.7.22), where for type I_0^* we have $\alpha = -1/2$ and c a complex number with $\operatorname{Im} c > 0$.

It follows from the description of the tangent lattice in Definition 7.7.5 that the monodromy of the tangent lattice is equal to the monodromy around the singular points as given in Table 6.2.40. Around each of the singular points $\lambda = 0$, $\lambda = 1$, $\lambda = \lambda_0$, and $\lambda = \infty$, this monodromy can easily be read off from Figure 11.2.5, even around $\lambda = 0$, where the density is small. I find it remarkable how the asymptotic

behavior of the set $\mathbb{P}^1_{\tau^k}$ for $k \to \infty$ is already clearly visible in pictures of $\mathbb{P}^1_{\tau^k}$ for relatively small values of the period k.

Figure 11.2.6 In order to see the behavior of the set $\mathbb{P}^1_{\rho^k}$ near $\lambda = \infty$, over which we also have a complex singular fiber of type I_2, we have performed, still for $\lambda_0 = 2$, the substitution of variables $z_0 = -z_0' + z_1'$, $z_1 = 2\,z_0'$, or $\lambda := -z_1/z_0 = 2\,z_0'/(z_0' - z_1') = 2/(1 + \lambda')$ if $\lambda' := -z_1'/z_0'$. Here $\lambda = 0,\ 1,\ 2,\ \infty$ correspond to $\lambda' = \infty,\ 1,\ 0,\ -1$, respectively. This leads to the picture in Figure 11.2.6 of the set $\mathbb{P}^1_{\tau^k}$ in the complex λ'-plane, which has the symmetry $\lambda' \mapsto -\lambda'$, where the other symmetry $\lambda' \mapsto \overline{\lambda'}$ is explained by the real structure. The symmetry $\lambda \mapsto -\lambda'$ can be explained by observing that the biquadratic polynomial on the left-hand side of (2.5.3), with the matrices A^0 and A^1 as in (11.2.11) with $\lambda_0 = 2$, is equal to $4\,z_1'\,(x_1{}^2\,y_1{}^2 + x_0{}^2\,y_0{}^2) + 2\,z_0'\,(x_1\,y_0 - x_0\,y_1)^2$, which is invariant under the substitutions $x_1 \mapsto \mathrm{i}\,x_1$, $y_1 \mapsto \mathrm{i}\,y_1$, $z_0' \mapsto -z_0'$. Because the elliptic surface S and the billiard map ρ^S, hence $\tau^S = (\rho^S)^2$, are defined in terms of (2.5.3), this leads to an automorphism of the rational elliptic surface S that commutes with the billiard map and maps the fiber over λ' to the fiber over $-\lambda'$. See Section 9.2.1 for a discussion of automorphisms of the elliptic surfaces that permute the fibers.

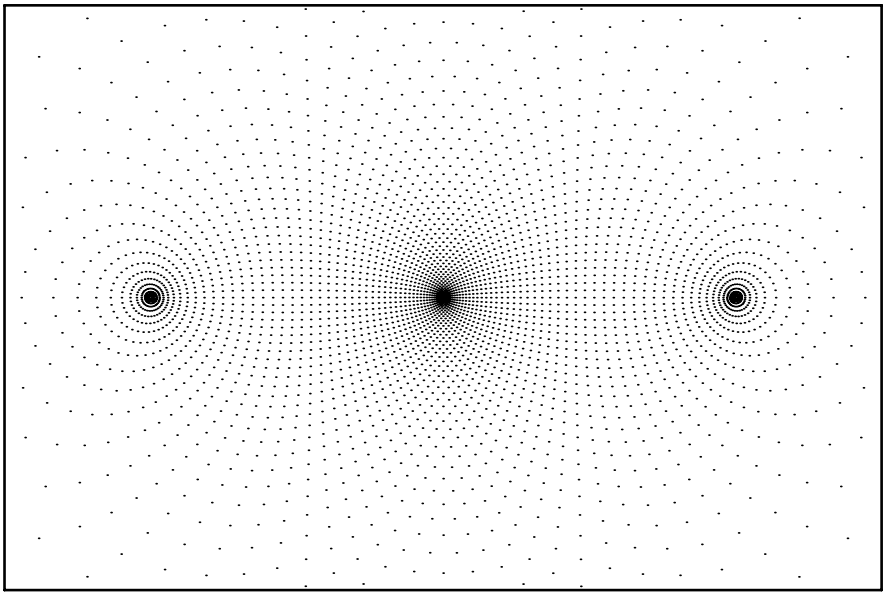


Fig. 11.2.6 Symmetry for $\lambda_0 = 2$.

Figure 11.2.7 is a picture on the complex λ-plane of the k-periodic fibers of the billiard QRT map τ for $k = 67$ and $\lambda_0 = 3$, where we have used the same substitution of variables $z_0 = -z_0' + z_1'$, $z_1 = 2\,z_0'$ as in the case $\lambda_0 = 2$. This time $-z_1/z_0 = \lambda = 0,\ 1,\ \lambda_0,\ \infty$ correspond to $-z_1'/z_0' = \lambda' = \infty,\ 1, -1 + 2/\lambda_0, -1$, and again

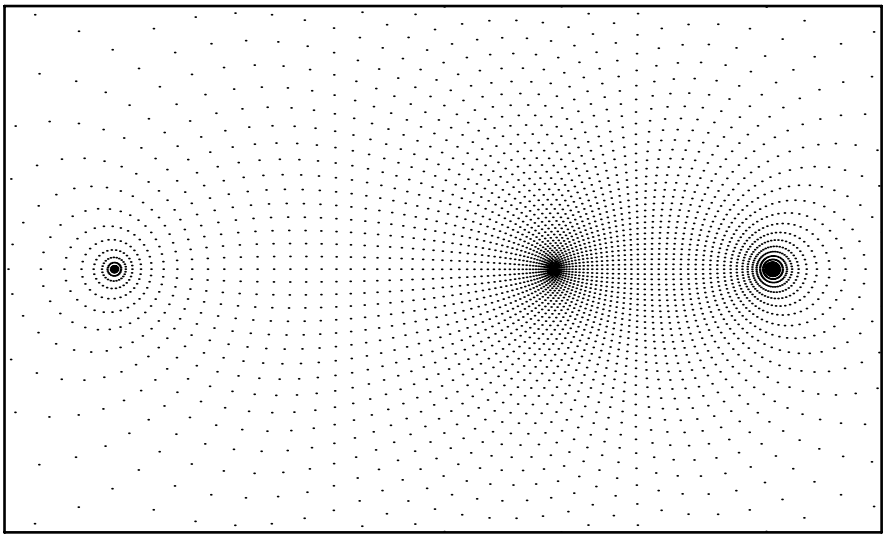


Fig. 11.2.7 Asymmetry for $\lambda_0 = 3$.

both points $\lambda' = 1$ and $\lambda' = -1$ over which we have a hyperbolic singular fiber of Kodaira type I_2 have been made visible. The asymptotic behaviors of the set $\mathbb{P}^1_{\tau^k}$ near $\lambda' = 1$ and $\lambda' = -1$ are quite different, caused by the different values of a in the exponentially small radius $\approx C\, a^{k/k_2}$ of the k_2th approximate circle about each of the points $\lambda' = 1$ and $\lambda' = -1$.

Made curious by the symmetry $\lambda' \mapsto -\lambda'$ when $\lambda_0 = 2$, I realized that the elliptic surface has three complex singular fibers of Kodaira type I_2 over $\lambda = 0, 1, \infty$, and one of type I_0^* over the point λ_0, which is the value of the parameter of the elliptic billiard. The group Σ of all projective linear transformations of the complex projective λ-axis that permute $\lambda = 0, 1, \infty$ is generated by the transformations $\pi_1 : \lambda \mapsto 1 - \lambda$ and $\pi_2 : \lambda \mapsto 1/\lambda$, and this group is isomorphic to the group of all six permutations of $\{0, 1, \infty\}$. The substitution of variables $z_0 = z_0'$, $z_1 = -z_0' - z_1'$, $\lambda_0 = 1 - \lambda_0'$ in the Weierstrass invariants leads to $g_2(z_0, z_1, \lambda_0) = g_2(z_0', z_1', \lambda_0')$, $g_3(z_0, z_1, \lambda_0) = g_3(z_0', z_1', \lambda_0')$, $X_\rho(z_0, z_1, \lambda_0) = X_\rho(z_0', z_1', \lambda_0')$, and $Y_\rho(z_0, z_1, \lambda_0) = Y_\rho(z_0', z_1', \lambda_0')$, and the same statement is true for the substitution of variables $z_0 = (\lambda_0')^2\, z_1'$, $z_1 = (\lambda_0')^2\, z_0'$, $\lambda_0 = 1/\lambda_0'$. It therefore follows from Theorem 6.3.6 that there is, for every $\Psi \in \Sigma$, a complex analytic diffeomorphism Φ from the elliptic surface S for the parameter λ_0 onto the elliptic surface S' for the parameter λ_0' such that $\pi' \circ \Phi = \Psi \circ \pi$ and Φ conjugates the billiard map $\rho : S \to S$ for the parameter λ_0 with the billiard map $\rho' : S' \to S'$ for the parameter λ_0', that is, $\rho' \circ \Phi = \Phi \circ \rho$. Here λ_0' is determined by $[1 : -\lambda_0'] = \Psi([1 : -\lambda_0])$. Furthermore, $\pi : S \to \mathbb{P}^1$ and $\pi' : S' \to \mathbb{P}^1$ are the complex analytic mappings to the complex projective line that define the respective elliptic fibrations. In particular, it follows that Ψ maps, for every period k, the set $\mathbb{P}^1_{\rho^k}$ of k-periodic fibers for ρ bijectively onto the set $\mathbb{P}^1_{(\rho')^k}$ of k-periodic fibers for ρ'.

In the following table we have listed, for each nontrivial element of Σ, in the first column how it acts on $[1 : -\lambda]$, in the second column how it permutes the set $\{0,\ 1,\ \infty\}$, and in the third column its set of fixed points in $\mathbb{P}^1$, where $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ corresponds to $[1 : -\lambda_0] \in \mathbb{P}^1$:

$\lambda \mapsto \lambda/(\lambda - 1)$	$(1 \quad \infty)$	$\lambda_0 = 0,\ 2$
$\lambda \mapsto 1 - \lambda$	$(1 \quad 0)$	$\lambda_0 = 1/2,\ \infty$
$\lambda \mapsto 1/\lambda$	$(0 \quad \infty)$	$\lambda_0 = 1, -1$
$\lambda \mapsto 1/(1 - \lambda)$	$(\infty \quad 0 \quad 1)$	${\lambda_0}^2 - \lambda_0 + 1 = 0$
$\lambda \mapsto (\lambda - 1)/\lambda$	$(1 \quad \infty \quad 0)$	${\lambda_0}^2 - \lambda_0 + 1 = 0$

That is, if λ_0 does not satisfy one of the equations in the third column, then the orbit of λ_0 under the action of Σ consists of six complex numbers λ_0' such that the respective billiard maps ρ and ρ' for the parameters λ_0 and λ_0' are conjugate by an isomorphism over the projective linear transformation $\Psi \in \Sigma$ mentioned in the first column, and Ψ maps $\mathbb{P}^1_{\rho^k}$ bijectively onto $\mathbb{P}^1_{(\rho')^k}$. On the other hand, if $\lambda_0 = 2,\ 1/2$, or -1, then it is fixed under the involution $\Psi \in \Sigma$ mentioned in the first column; we have $\lambda_0' = \lambda_0$, hence $S' = S$, $\rho' = \rho$; and the aforementioned isomorphism $\Phi : S \to S'$ now is an automorphism of the elliptic surface S that commutes with the billiard map ρ, and the set $\mathbb{P}^1_{\rho^k}$ of k-periodic fibers for ρ is invariant under Ψ. Moreover, the set $\{2,\ 1/2, -1\}$ is an orbit for the action of the group Σ, which means that the corresponding billiard maps are conjugate by means of the corresponding complex isomorphisms Φ, and the sets of k-periodic fibers are mapped to each other by the corresponding projective linear transformations Ψ. In other words, the picture for $\lambda_0 = 2$ also serves as a picture for $\lambda_0 = 1/2$ and for $\lambda_0 = -1$.

For $\lambda_0 = 0$, $\lambda_0 = 1$, and $\lambda_0 = \infty$, all the biquadratic curves are singular, and we do not have true elliptic surfaces. For this reason we ignore the orbit $\{0,\ 1,\ \infty\}$ of Σ.

However, if ${\lambda_0}^2 - \lambda_0 + 1 = 0$, that is, $\lambda_0 = \lambda_0^{\pm} := (1 \pm \mathrm{i}\sqrt{3})/2$, then the substitution $z_0 = z_0' - \lambda_0 z_1'$, $z_1 = -\lambda_0 z_0' + z_1'$ maps the three points over which we have a singular fiber of type I_2 to the three roots of unity 1, ω, ω^2, where $\omega = (-1 + \mathrm{i}\sqrt{3})/2$. Because $[z_0' : z_1'] = [1 : 0]$ corresponds to $\lambda = -z_1/z_0 = \lambda_0$, this substitution maps the point λ_0 over which we have the singular fiber of type I_0^* to the origin in the complex plane. The rotation $\Psi : \lambda' \mapsto \omega\,\lambda'$ of order three is covered by a complex automorphism of the elliptic surface that commutes with the billiard map, and therefore it leaves the set of k-periodic fibers invariant.

Figure 11.2.8 shows this threefold symmetry. Moreover, the set $\{\lambda_0^+,\ \lambda_0^-\}$ is an orbit under Σ; hence the sets of k-periodic fibers for $\lambda_0 = \lambda_0^+$ and for $\lambda_0 = \lambda_0^-$ in the corresponding complex projective $[z_0' : z_1']$ lines are the same. Therefore Figure 11.2.8 illustrates both sets.

Remark 11.2.7. The substitution (11.2.10) was chosen in such a way that it is real if $\lambda_0 \in \mathbb{R}_{>1}$. If $0 < \lambda_0 < 1$, when the confocal quadric Q_0^* for $\lambda = \lambda_0$ is a hyperbola, nothing changes in the description when we work over $\mathbb{C}$. However, if we are interested in the behavior in the real domain, then the substitution (11.2.10) may be replaced by the substitution of variables

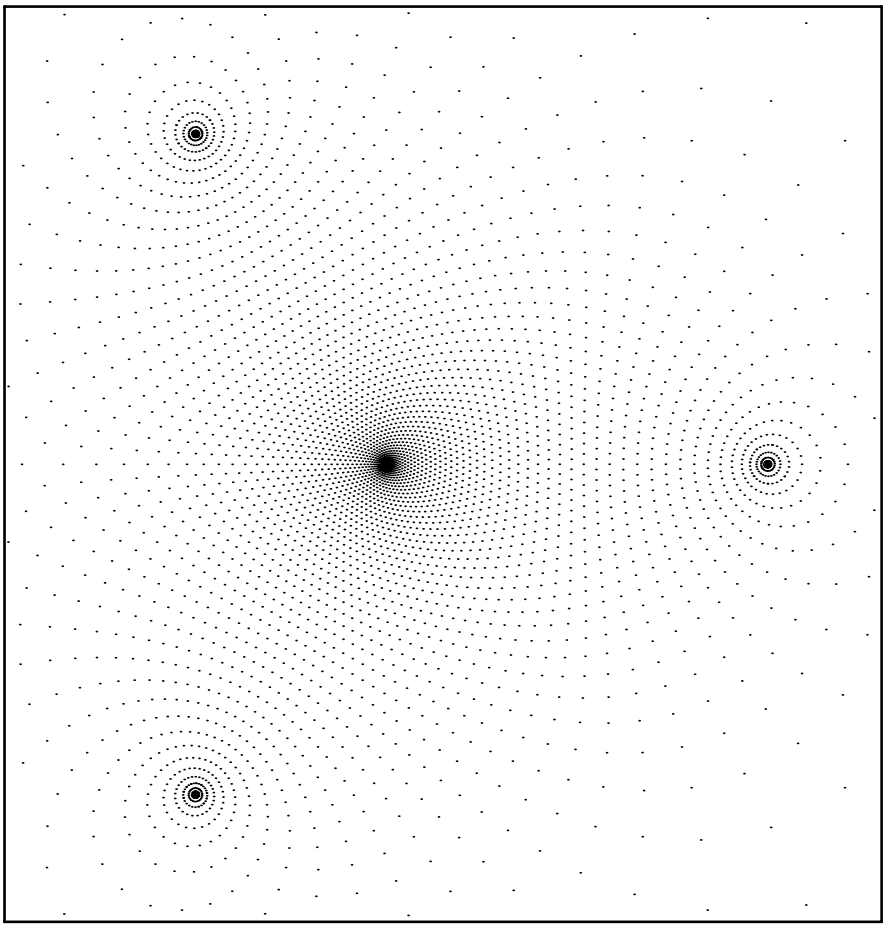


Fig. 11.2.8 Threefold symmetry for $\lambda_0 = (1 \pm \mathrm{i}\sqrt{3})/2$.

$$\xi_0 = u_2 - u_0,\ \xi_1 = {\lambda_0}^{-1/2}(u_2 + u_0),\ \xi_2 = (1 - \lambda_0)^{1/2} u_1,$$

which is a real linear substitution of variables that turns Q_0 into the quadric P in (10.2.5). This leads to conclusions for the *hyperbolic billiard* that are quite analogous to those for the elliptic billiard. ⊘

11.2.4 Historical Remarks

The statements in Corollary 11.2.2, at least for a family of confocal quadrics in the real affine plane have very old historical roots. I found the statement that through each point in the plane, not equal to one of the foci, there passes one confocal ellipse and

one confocal hyperbola, which moreover are orthogonal to each other, in Jacobi [96, 27. Vorlesung, pp. 207, 208].

Let x be a point outside an ellipse E with foci $f_\pm$, and let $l_\pm$ denote the lines through x that are tangent to E, where $l_\pm$ intersects the long axis A of E at the side of $f_\pm$ if $x \notin A$. If E_x is the confocal ellipse through x, then the lines $x\, f_\mp$ make equal angles with the tangent line at x to E_x. It follows that the statement that l_+ and l_- make equal angles with the tangent line at x to E_x, which follows from Corollary 11.2.2, is equivalent to the statement that the angle between l_+ and $x\, f_+$ is equal to the angle between l_- and $x\, f_-$. The latter statement, and the analogous statement in which E is replaced by a confocal hyperbola, is formulated only in terms of the quadric E and the point x in the affine plane. When x lies on one of the tangent lines to E at one of the intersection points of E with its long axis A, then the identity $\angle(l_+,\, x\, f_+) = \angle(l_-,\, x\, f_-)$ can be found in Apollonius [2, Book III, Proposition XLVI, p. 265]. For general points x relative to the quadric E the identity $\angle(l_+,\, x\, f_+) = \angle(l_-,\, x\, f_-)$ goes back at least to Poncelet [165, t. 1, p. 268].

Guillemin and Melrose [78, Appendix] referred to Boscovich [21, Section 184], but the latter expresses the angle between l_+ and l_- as an average of the angle between $t_+\, f_+$ and $t_-\, f_+$ and the angle between $t_+\, f_-$ and $t_-\, f_-$, where $t_\pm$ are the points of E where the lines $l_\pm$ touch E, and appropriate choices have to be made of the numerical values of the angles between straight lines. I could not find the identity $\angle(l_+,\, x\, f_+) = \angle(l_-,\, x\, f_-)$ in [21], but this could be due to my lack of knowledge of Latin.

Birkhoff [17, Chapter VI, Sections 6–9] introduced the billiard system as the motion of a free particle inside a closed convex curve C, reflecting at the boundary with equal angles. In [17, Chapter VI, Section 7], the bouncing states are parametrized by $(\theta,\, \varphi)$, where $0 < \theta < \pi$ is the angle between the outgoing ray with the oriented tangent line to C at the point p, and $\varphi \in \mathbb{R}/\mathbb{Z}\, l$ is the arc length along C from an initial point p_0 to p, and l is the length of C. In [17, Chapter VI, Section 8] it is proved that the billiard map preserves the area form $\sin\theta\ \mathrm{d}\theta \wedge \mathrm{d}\varphi$. Consequences of the existence of an invariant area form for the existence of periodic points are then discussed in [17, Chapter VI, Section 8]. In [17, Chapter 6, Section 6], Birkhoff had observed that "If C happens to be an ellipse an integrable system results, namely as a limiting case of the geodesics on an ellipsoid treated by Jacobi." In [17, Chapter VIII, Section 12], Birkhoff gave more details on the elliptic billiard, which are related to our description of the action of the billiard map on the real parts of the elliptic curves. The picture on [17, p. 249] can be viewed as a nonnumerical version of our Figure 11.2.2. The analyticity of the rotation function, when avoiding the singular invariant curves in the state space, is discussed in [17, Chapter VIII, Section 12], but there are no statements in [17, Chapter VIII, Section 12] concerning, for instance, its monotonicity or its limit values at the singular curves.

Some more recent papers on the elliptic billiard, including further references to the literature, are Chang and Friedberg [34] and Levi and Tabachnikov [119]. The quantum elliptic billiard is discussed in Waalkens, Wiersig, and Dullin [202]. Guillemin and Melrose [78] treated the same partial differential equation in the

interior of the ellipse with the Neumann boundary condition, where in [202] also the Dirichlet boundary condition has been given a separate discussion.

11.3 The Planar Four-Bar Link

The configurations of the *planar four-bar link* are cycles of four points p_i, $1 \le i \le 4$, in the Euclidean plane for which the distances $a := p_1\,p_2$, $b := p_2\,p_3$, $c := p_3\,p_4$, and $d := p_4\,p_1$ are fixed positive real numbers.

By means of a rigid motion we can bring p_1 and p_2 into fixed positions, at distance a. Let $C(p, r)$ denote the circle with center at p and radius equal to r. Then p_3 and p_4 have to lie on the circles $C(p_2, b)$ and $C(p_1, d)$, respectively, and the pair $(p_3, p_4) \in C(p_2, b) \times C(p_1, d)$ is subject to the distance equation $p_3\,p_4 = c$. The set of solutions of this equation is a curve $\Gamma(\mathbb{R})$ that can be empty or have singularities. If it is nonempty and smooth, then it has one or two connected components.

In his colloquium talk in Utrecht on 9 February 2006, Giorgi Khimshiasvili discussed the self map $\tau = \iota_2 \circ \iota_1$ of the curve $\Gamma(\mathbb{R})$ where ι_1 interchanges the two possible positions of p_3 given the position of p_4, and ι_2 interchanges the two possible positions of p_4 given the position of p_3. Also, at the end of the talk, Tonnie Springer conjectured that $\Gamma(\mathbb{R})$ is the real part of an elliptic curve Γ. The planar four-bar link has an extensive literature, see for instance the article of Gibson and Newstead [66], which Khimshiashvili brought to my attention. This article in turn contains a reference to Darboux [44], where it is shown that the complex curve Γ is isomorphic to a cubic curve in the complex projective plane, hence elliptic. The involutions ι_1 and ι_2 of the curve appear in Darboux [44, pp. 120, 121], together with the Poncelet porism for $\tau = \iota_2 \circ \iota_1$, the statement that if one point of the curve is k-periodic for τ, then every point of the curve is k-periodic for τ. The proof, in terms of elliptic functions, is equivalent to showing that ι_1 and ι_2 are inversions, and hence τ is a translation on the elliptic curve. Because of this reference I propose to call τ the *Darboux transformation* of the planar four-bar link.

If we write $p_1 = (0, 0)$, $p_2 = (a, 0)$, $p_3 = (a + b\cos\phi,\, b\sin\phi)$, $p_4 = (d\cos\psi,\, d\sin\psi)$, then the distance equation for (p_3, p_4) is equivalent to

$$\begin{aligned} c^2 &= (a + b\cos\phi - d\cos\psi)^2 + (b\sin\phi - d\sin\psi)^2 \\ &= a^2 + b^2 + d^2 + 2a\,(b\cos\phi - d\cos\psi) - 2bd\,(\cos\phi\cos\psi + \sin\phi\sin\psi). \end{aligned}$$

With the rational parametrizations

$$\begin{aligned} \cos\phi &= (u^2 - 1)/(u^2 + 1) \quad \sin\phi = 2u/(u^2 + 1), \\ \cos\psi &= -(v^2 - 1)/(v^2 + 1) \quad \sin\psi = -2v/(v^2 + 1), \end{aligned}$$

of $p_3 \in C(p_2, b)$ and $p_4 \in C(p_1, d)$, the distance equation is equivalent to

$$0 = ((a+b+d)^2 - c^2)\,u^2\,v^2 + ((a+b-d)^2 - c^2)\,u^2 + ((a-b+d)^2 - c^2)\,v^2 + 8\,b\,d\,u\,v + (-a+b+d)^2 - c^2. \quad (11.3.1)$$

The curve (11.3.1) in the $(u,\ v)$-plane is biquadratic, and therefore, if it is smooth as a curve in the Cartesian product of the complex projective line with itself, it is an elliptic curve. Moreover, the Darboux transformation corresponds to the horizontal switch $(u,\ v) \mapsto (u',\ v)$ followed by the vertical switch $(u',\ v) \mapsto (u',\ v')$ in the $(u,\ v)$-plane. Therefore the Darboux transformation corresponds to the QRT transformation on the biquadratic curve. In particular, if the curve (11.3.1) in $\mathbb{P}^1 \times \mathbb{P}^1$ is smooth, then it is an elliptic curve and the Darboux transformation acts on it as a translation.

Remark 11.3.1. If we fix the point $p_4 \in C(p_1,\ d)$, then the point p_3 is determined by the condition that it lie in the intersection of the circles $C(p_2,\ b)$ and $C(p_4,\ c)$. As quadrics in the projective plane, two circles have four intersection points, whereas in the $(u,\ v)$-plane we find only two points $(u,\ v)$, $(u',\ v)$ on the biquadratic curve (11.3.1) for any given v. The explanation is that in projective coordinates $[x_0 : x_1 : x_2]$ the circle in the affine plane with center $(z_1 : z_2)$ and radius r is the quadric determined by the homegeneous equation

$$x_1{}^2 + x_2{}^2 - 2\,(x_1\,z_1 + x_2\,z_2)\,x_0 + (z_1{}^2 + z_2{}^2 - r^2)\,x_0{}^2 = 0.$$

All the quadrics intersect the complex projective line $x_0 = 0$, the line at infinity, at the same two points $[0 :\ 1 :\ \mathrm{i}]$ and $[0 :\ 1 :\ -\mathrm{i}]$.

Gibson and Newstead [66, p. 118] defined the complex projective configuration space of the planar four-bar linkage as a complex algebraic curve in a six dimensional complex projective space, called the "linkage curve" C. They showed that C contains two complex projective lines $L, \overline{L}$, and a third component, called the "residual curve" R. According to [66, bottom of p. 119], R is elliptic when smooth. In [66, Section 2] it is also proved that R is isomorphic to the cubic curve R' in $\mathbb{P}^2$ that Darboux [44] introduced as the configuration space for the planar four-bar link. The irreducible components L, $\overline{L}$ of the linkage curve C correspond to the known two points at infinity on each circle, viewed as a quadric in the complex projective plane.

The discriminant of the partial discriminant of the biquadratic polynomial in (11.3.1), as defined in Section 2.3.4, is equal to

$$\Delta = 2^{24} \prod_{j=1}^{4} {r_j}^4 \prod_{\epsilon:\{1,\,2,\,3\}\to\{-1,\,+1\}} \left(r_4 - \sum_{k=1}^{3} \epsilon(k)\,r_k \right),$$

where $r_1 = a$, $r_2 = b$, $r_3 = c$, $r_4 = d$. It follows that D is smooth unless one of the complex numbers a, b, c, d is equal to zero, or one has a *Grashof identity*

$$r_4 = \sum_{k=1}^{3} \epsilon(k)\,r_k,$$

where for each $1 \le k \le 3$, we have chosen an $\epsilon(k) = \pm 1$. See Gibson and Newstead [66, (5)].

The modulus of the biquadratic curve (11.3.1) is equal to ${g_2}^3/\Delta$, where $g_2 =$ the Eisenstein invariant D is equal to $64/3$ times

$$g_2 = \sum_{j=1}^{4} {r_j}^8 - 4 \sum_{j \neq k} {r_j}^6 {r_k}^2 + 6 \sum_{j<k} {r_j}^4 {r_k}^4 + 4 \sum_{j=1}^{4} \sum_{k \neq j,\, l \neq j,\, k<l} {r_j}^4 {r_k}^2 {r_l}^2 - 24 \prod_{j=1}^{4} {r_j}^2.$$

Note that g_2 and Δ are homogeneous polynomials of degree 4 and 12, respectively, in the variables ${r_j}^2$, $1 \le j \le 4$.

Because the right-hand side of (11.3.1) depends linearly on c^2, we obtain, if we write $c^2 = -z_1/z_0$, a pencil of biquadratic curves as in (2.5.3), with

$$A^0 = \begin{pmatrix} (a+b+d)^2 & 0 & (a+b-d)^2 \\ 0 & 8\,b\,d & 0 \\ (a-b+d)^2 & 0 & (-a+b+d)^2 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Although it is debatable whether it is natural to vary only c, while keeping a, b, and d constant, I could not resist giving a short discussion of the elliptic surface that is defined by this pencil.

With this substitution, the discriminant Δ is equal to $2^{24}\, a^4\, b^4\, d^4$ times

$${z_0}^6\, {z_1}^2\, ((a+b+d)^2\, z_0+z_1)\, ((a+b-d)^2\, z_0+z_1)\, ((a-b+d)^2\, z_0+z_1)\, ((-a+b+d)^2\, z_0+z_1). \tag{11.3.2}$$

In the sequel, we will assume that $a \neq 0$, $b \neq 0$, and $d \neq 0$, which is equivalent to the assumption that the pencil has at least one smooth member. For $[z_0 : z_1] = [0 : 1]$, the member of the pencil is given by the equation

$$({u_1}^2 + {u_0}^2)({v_1}^2 + {v_0}^2) = 0,$$

which is the union of the two horizontal axes $[v_0 : v_1] = [1 : \pm \mathrm{i}]$ and the two vertical axes $[u_0 : u_1] = [1 : \pm \mathrm{i}]$. This resembles the situation for the sine–Gordon map in Section 11.7. For $[z_0 : z_1] = [1 : 0]$, the member is equal to the union of the two curves

$$(a+b+d)\, u_1\, v_1 + \mathrm{i}\,(a+b-d)\, u_1\, v_0 + \mathrm{i}\,(a-b+d)\, u_0\, v_1 + (-a+b+d)\, u_0\, v_0 = 0$$

and

$$(a+b+d)\, u_1\, v_1 - \mathrm{i}\,(a+b-d)\, u_1\, v_0 - \mathrm{i}\,(a-b+d)\, u_0\, v_1 + (-a+b+d)\, u_0\, v_0 = 0,$$

each of which is bihomogeneous of bidegree $(1,\ 1)$.

In projective coordinates $([u_0 : u_1],\ [v_0 : v_1])$, where $u = u_1/u_0$ and $v = v_1/v_0$, the pencil has the base points

$$\begin{gathered}([1 : \mathrm{i}],\ [a+d : -(a-d)\,\mathrm{i}]),\quad ([1 : -\mathrm{i}],\ [a+d : (a-d)\,\mathrm{i}]),\\ ([a+b : -(a-b)\,\mathrm{i}],\ [1 : \mathrm{i}]),\quad ([a+b : (a-b)\,\mathrm{i}],\ [1 : -\mathrm{i}]),\\ ([1 : \mathrm{i}],\ [1 : \mathrm{i}]),\quad ([1 : -\mathrm{i}],\ [1 : -\mathrm{i}]),\end{gathered}$$

where each of the first four base points has multiplicity one and each of the last two base points has multiplicity two. If $\kappa : S \to \mathbb{P}^1$ is the rational elliptic surface that is obtained by successively blowing up the base points of the anticanonical pencils, then the singular fiber corresponding to $z_0 = 0$ is of type I_6, where over each of the base points of multiplicity two an irreducible component has been inserted in the singular fiber, equal to the proper transform of the -1 curve that appeared at the first blowing up over that base point.

The lift ι_1^S to S of the horizontal switch ι_1 maps the proper transform of the vertical axis $[u_0 : u_1] = [1 : -\mathrm{i}]$ to the vertical axis $[u_0 : u_1] = [1 : \mathrm{i}]$, after which the lift ι_2^S to S of the vertical switch ι_2 maps the proper transform of the vertical axis $[u_0 : u_1] = [1 : \mathrm{i}]$ to the irreducible component that appeared over $[\mathrm{i} : \mathrm{i}]$. It follows that the lift τ^S to S of the QRT map shifts the cycle of the irreducible components of the singular fiber of type I_6 over two units, and the contribution to τ^S of the singular fiber of type I_6 is equal to $2 \times 4/6 = 4/3$. The singular fiber corresponding to $z_1 = 0$ is of type I_2, its irreduducible components are switched by both ι_1^S and ι_2^S, and therefore the contribution to τ^S of this singular fiber is equal to zero.

If $a+b+d \neq 0$, $a+b-d \neq 0$, $a-b+d \neq 0$, $-a+b+d \neq 0$, $a+b \neq 0$, $a+d \neq 0$, $b+d \neq 0$, $b \neq d$, $a \neq d$, and $a \neq b$, then (11.3.2) has four simples zeros, corresponding to four singular fibers of type I_1. It follows that in this case the configuration of the singular fibers is $\mathrm{I}_6\ \mathrm{I}_2\ 4\,\mathrm{I}_1$. According to Lemma 9.2.6, the Mordell–Weil group $\simeq \mathrm{Aut}(S)_\kappa^+$ is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/2\,\mathbb{Z})$ or to $\mathbb{Z}^2$.

However, with the planar four-bar link we are in luck. The orthogonal reflection about the straight line through the fixed points p_1 and p_2 is a symmetry of configuration space. This symmetry corresponds to the mapping $(u,\ v) \mapsto (-u,\ -v)$ or, in projective coordinates,

$$\eta : ([u_0 : u_1],\ [v_0 : v_1]) \mapsto ([u_0 : -u_1],\ [v_0 : -v_1]).$$

This automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ leaves each member (11.3.1) of our pencil of biquadratic curves invariant. Because the fixed points $([1 : 0],\ [1 : 0])$, $([1 : 0],\ [0 : 1])$, $([0 : 1],\ [1 : 0])$, $([0 : 1],\ [0 : 1])$ do not lie on the smooth members of the pencil, η acts as a translation on each smooth member. Therefore η induces an automorphism η^S of the elliptic surface $\kappa : S \to \mathbb{P}^1$ that acts as a translation on each smooth fiber of κ, and $\eta^S \circ \eta^S = (\eta \circ \eta)^S = 1^S$ is equal to the identity. In other words, η^S is an element of the Mordell–Weil group $\sim \mathrm{Aut}(S)_\kappa^+$ of order two. This implies that the Mordell–Weil group is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/2\,\mathbb{Z})$, corresponding to No. 28 in the list of Oguisa and Shioda [155, p. 85].

It follows from (4.3.2) that the number of k-periodic fibers of the QRT automorphism, counted with multiplicities, is equal to

$$\nu((\tau^S)^k) = \begin{cases} 3n^2 - 1 & \text{when} \quad k = 3n, \\ 3n^2 + 2n & \text{when } k = 3n+1, \\ 3n^2 + 4n + 1 & \text{when } k = 3n+2, \end{cases}$$

for every integer n.

When $b = d$, the biquadratic curves are symmetric with respect to the symmetry switch $\sigma : (u,\ v) \mapsto (v,\ u)$, which corresponds to the orthogonal reflection of the planar four-bar link about the axis through the midpoint between p_1 and p_2 and orthogonal to the straight line through p_1 and p_2. In this case the QRT map is equal to $\rho \circ \rho$, where $\rho = \sigma \circ \iota_1 = \iota_2 \circ \sigma$ is the QRT root. In this case (11.3.2) becomes

$$z_0{}^6\, z_1{}^2\, (a^2\, z_0 + z_1)^2\, ((a+2b)^2\, z_0 + z_1)\, ((-a+2b)^2\, z_0 + z_1).$$

The double zero at $a^2\, z_0 + z_1$ corresponds to putting $b = d$ and $c = a$ in (11.3.1), when the planar four-bar link is a parallelogram. The corresponding biquadratic curve is the union of the $(1,\ 1)$-curves $u_1\, v_1 + u_0\, v_0 = 0$ and $(a+b)\, u_1\, v_1 + (-a+b)\, u_0\, v_0 = 0$, each of which is invariant under the symmetry switch σ.

Therefore, if $b = d$, $a + 2b \neq 0$, $-a + 2b \neq 0$, $a \neq b$, and $a \neq -b$, then the configuration of the singular fibers of $\kappa : S \to \mathbb{P}^1$ is $\mathrm{I}_6\ \ 2\,\mathrm{I}_2\ \ 2\,\mathrm{I}_1$. According to Lemma 9.2.6, the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$, where the element of order two is the previously discussed η^S. It follows that the QRT automorphism τ^S has exactly two square roots in the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$, namely ρ^S and $\eta^S \circ \rho^S = (\eta \circ \sigma \circ \iota_1)^S$; see Remark 10.1.3.

The QRT root ρ^S acts on the cycle of the irreducible components of the singular fiber of type I_6 by means of a shift over one unit. Furthermore, it switches the irreducible components of each of the fibers of type I_2. It follows from (4.3.2) that the number of k-periodic fibers of the QRT root, counted with multiplicities, is equal to

$$\nu((\rho^S)^k) = \begin{cases} 3n^2 - 1 & \text{when} \quad k = 6n, \\ 3n^2 + n & \text{when } k = 6n+1, \\ 3n^2 + 2n & \text{when } k = 6n+2, \\ 3n^2 + 3n + 1 & \text{when } k = 6n+3, \\ 3n^2 + 4n + 1 & \text{when } k = 6n+4, \\ 3n^2 + 5n + 1 & \text{when } k = 6n+5, \end{cases}$$

for every integer n.

The other square root $\alpha = \eta^S \circ \rho^S$ of the QRT automorphism τ^S acts on the cycle of the irreducible components of the singular fiber of type I_6 by means of a shift over $1 + 3 = 4$ units. It leaves each of the irreducible components of the fiber for $[z_0 : z_1] = [1 : 0]$ invariant, and it switches the irreducible components of the other singular fiber of type I_2. It follows that $\nu(\alpha^k) = \nu((\rho^S)^k) - 1$ if $k = 6n+3$, $n \in \mathbb{Z}$, and $\nu(\alpha^k) = \nu((\rho^S)^k)$ in all other cases.

Regarding the real configurations, we recall from Section 2.6 that if an elliptic curve is invariant under a complex conjugation, which in our situation happens when all the parameters a, b, c, and d are real, then the set of real points of the elliptic curve is empty, or isomorphic to a circle, or isomorphic to the union of two disjoint parallel circles in the elliptic curve at a distance of half a period. For the action of a real element of the Mordell–Weil group on the real fibers, see Chapter 8. Note that in, for instance, Lemma 8.1.5, the cases are distinguished at to whether the automorphism leaves each of the two connected components of a real fiber invariant or switches these.

Various cases can occur. Note that the point $p_3 \in C(p_2,\, b)$ is a critical value of the projection from $\Gamma(\mathbb{R})$ to the second component $C(p_2,\, b)$ if and only if $p_1,\, p_3$, and p_4 lie on a straight line. Similarly p_4, is a critical value of the projection from $\Gamma(\mathbb{R})$ to the first component $C(p_1,\, d)$ if and only if $p_2,\, p_3$, and p_4 lie on a straight line. We discuss four examples.

(i) If $0 < c < a - (b + d)$, then the real part

$$\Gamma(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \simeq C(p_2,\, b) \times C(p_1,\, d)$$

of the complex curve Γ is empty.

(ii) If $b + d < a$ and c is slightly larger than $a - (b + d)$, then there are small closed intervals I_1 and I_2 around $(d,\, 0)$ and $(a - b,\, 0)$ in $C(p_1,\, d)$ and $C(p_2,\, b)$, respectively, such that the projection of $\Gamma(\mathbb{R})$ to $C(p_1,\, d)$ and $C(p_2,\, b)$ is a twofold branched covering over I_1 and I_2, where the branches meet over the endpoints of I_1 and I_2, respectively. In this case $\Gamma(\mathbb{R})$ is connected. We could also have used Morse theory in order to prove that $\Gamma(\mathbb{R})$ is diffeomorphic to a small circle.

(iii) If $C(p_1,\, d)$ and $C(p_2, b)$ intersect each other at two points i_+ and i_-, and c is a sufficiently small positive real number, then $\Gamma(\mathbb{R})$ has two connected components $\Gamma(\mathbb{R})_\pm$. There are small closed intervals $I_1^\pm$ and $I_2^\pm$ around $i_\pm$ in $C(p_1,\, d)$ and $C(p_2,\, b)$, where the projection of $\Gamma(\mathbb{R})_\pm$ to $C(p_1,\, d)$ and $C(p_2,\, b)$ is a twofold branched covering over $I_1^\pm$ and $I_2^\pm$, where the branches meet over the endpoints of $I_1^\pm$ and $I_2^\pm$, respectively. Both involutions ι_1 and ι_2 leave each of the connected components $\Gamma(\mathbb{R})_\pm$ of $\Gamma(\mathbb{R})$ invariant. Therefore the Darboux map, equivalently the QRT map, τ leaves each of the two connected components of $\Gamma(\mathbb{R})$ invariant.

(iv) If $0 < a - d < c < a + d$, then $C(p_2,\, c)$ intersects $C(p_1,\, d)$ in two points i_+ and i_-. If b is a sufficiently small positive real number, then $\Gamma(\mathbb{R})$ has two connected components $\Gamma(\mathbb{R})_\pm$. There are small closed intervals $I_\pm$ around $i_\pm$ in $C(p_1,\, d)$ such that the projection of $\Gamma(\mathbb{R})_\pm$ to $C(p_1,\, d)$ is a twofold branched covering over $I^\pm$, where the branches meet over the endpoints of $I^\pm$. The projection to $C(p_2,\, b)$ is a diffeomorphism from $\Gamma(\mathbb{R})_\pm$ onto $C(p_2,\, b)$. While p_3 runs over the circle $C(p_2,\, b)$ with a constant speed, p_4 oscillates in the interval $I^\pm$ back and forth between the endpoints of $I^\pm$. The involution ι_1 switches $\Gamma(\mathbb{R})_+$ and $\Gamma(\mathbb{R})_-$, whereas ι_2 leaves $\Gamma(\mathbb{R})_\pm$ invariant. Therefore the

Darboux map (= the QRT map) τ switches the two connected components of $\Gamma(\mathbb{R})$.

For an analysis of the various cases that is more global in the parameters, it is helpful to observe that the topological situation does not change if the parameters a, b, c, d vary in a connected component of the set of regular values, that is, a, b, c, and d are strictly positive real numbers that do not satisfy any of the Grashof identities.

We finally note that Manin's function $\mu(z)$ of the QRT map, see Sections 2.5.3 and 2.6.3, has a numerator of degree 3 whose coefficients are quite long expressions in a, b, d, and that does not seem to factorize in an obvious way. For this reason we have refrained from analyzing the monotonicity consequences of (2.6.8) for the rotation function $\rho(z)$, for arbitrary values of the parameters a, b, and d of the four-bar link. In the symmetric case $d = b$, we have

$$\begin{aligned}\mu(z) &= (a-b)\,(a+b)\,(a^2-4\,b^2-z)\\ &\qquad /z\,(a^2-z)\,((a-2\,b)^2-z)\,((a+2\,b)^2-z)\,(a^2-4\,b^2+3\,z),\\ 3\,\Gamma(z) &= 2^{22}\,a^4\,b^8\,z\,(a^2-z)\,(a^2-4\,b^2+3\,z),\end{aligned}$$

which looks a lot more manageable. The first four factors of the denominator of $\mu(z)$ are the factors in $\Delta(z)$, whereas the fifth factor is the factor of $\Gamma(z)$ that is not a factor of $\Delta(z)$. Because $\mu(z) \equiv 0$ if and only if $b = \pm a$, it follows from Proposition 2.5.20 that when $d = b$, the QRT map τ is of finite order if and only if $b = \pm a$. In this case the configuration of the singular fibers is $\mathrm{I}_6\ \mathrm{I}_3\ \mathrm{I}_2\ \mathrm{I}_1$, when according to Lemma 9.2.6 the Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$. Because τ shifts the cycle of irreducible components of the I_6 fiber over two units, and $\tau = \rho^2$, it follows that the QRT root ρ has period 6 and τ has period 3 when $d = b = \pm a$. In Section 5.2.5 it is proved that up to isomorphism this ρ is the only order-6 element of the Mordell–Weil group of any rational elliptic surface.

11.4 The Lyness Map

Lyness introduced the second-order recurrence equation

$$u_{n+2}\,u_n = u_{n+1} + a,$$

which, with the substitutions $(u_n,\ u_{n+1}) = (x,\ y)$, $(u_{n+1},\ u_{n+2}) = (x',\ y')$, corresponds to the map

$$\lambda : (x,\ y) \mapsto (x',\ y') := (y,\ (y+a)/x) \tag{11.4.1}$$

from the plane to itself, see Remark 10.1.1. Lyness proved in [126, 1942] that the mapping λ is periodic with period 5 when $a = 1$. The Lyness equation with $a = 1$ occurs in the frieze patterns of Coxeter [42, p. 300]. In [126, 1945] Lyness proved that for any value of the parameter a, the rational function

$$\varphi = (x+1)\,(y+1)\,(x+y+a)/x\,y \tag{11.4.2}$$

is invariant under λ. This is also Theorem 1 in Zeeman [214]. Zeeman [214] observed that in projective coordinates $x = [x_0 : x_1]$, $y = [y_0 : y_1]$, the Lyness map (11.4.1) is equal to the QRT root for the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, parametrized by $[z_0 : z_1] \in \mathbb{P}^1$, which are defined by the equations

$$z_0\,(x_1 + x_0)\,(y_1 + y_0)\,(x_1\,y_0 + x_0\,y_1 + a\,x_0\,y_0) + z_1\,x_0\,x_1\,y_0\,y_1 = 0. \tag{11.4.3}$$

That is, in the notation of (2.5.3), we have the symmetric matrices

$$A^0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & a+2 & a+1 \\ 1 & a+1 & a \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \\ 0\,0\,0 \end{pmatrix}. \tag{11.4.4}$$

For the QRT root defined by any pencil of symmetric biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, see the beginning of Section 10.1.

Figure 1 shows the QRT map on the real part of the Lyness curve (11.4.2) for $a = 0.4$ and $z = 10.58$. Figure 2 shows five iterates of it, whereas Figure 2.4.1 shows the Hamiltonian vector field of the biquadratic polynomial $p = p^0 - z\,p^1$ on the Lyness curve $p = 0$.

The Lyness equation has received considerable attention in the literature. The articles of Zeeman [214] and of Beukers and Cushman [16] have been an important source of inspiration for me. Esch and Rogers [60, Section 4] called the Lyness map "the simplest singular map of the plane," and after [60] the Lyness map is often also called the *screensaver map*. Esch and Rogers give some interesting history in [60, Section 6]. The survey of Bastien and Rogalski [12], although it does not refer to [214] or [16], also contains much interesting information.

11.4.0.1 Reconstruction

Let us pretend that we do not know yet that the Lyness map (11.4.1) is a QRT root, and apply the test of Section 10.1.1. The vector-valued function $f(y)$ is of the form $\phi(y)\,(y+a,\ 0,\ -1)$, where $\phi(y) = \phi_0\,y^3 + \phi_1\,y^2 + \phi_2\,y + \phi_3$. If we compute w in terms of f as in Section 10.1.1, then the equation $w \wedge w = 0$ can be solved as follows. The coefficient of $d_1 \wedge d_2 \wedge d_3 \wedge d_4$ in $w \wedge w$ is equal to $2\,{\phi_0}^2$, and therefore $\phi_0 = 0$. Substituting this, we obtain that the coefficient of $d_2 \wedge d_3 \wedge d_5 \wedge d_6$ in $w \wedge w$ is equal to $-2\,{\phi_3}^2$, and therefore $\phi_3 = 0$. Substituting $\phi_0 = \phi_3 = 0$, we obtain that the coefficient of $d_1 \wedge d_2 \wedge d_5 \wedge d_6$ in $w \wedge w$ is equal to $-(\phi_1 - \phi_2)^2/2$, and therefore $\phi_1 = \phi_2$. If we substitute $\phi_0 = \phi_3 = 0$ and $\phi_1 = \phi_2 = -1$, then $w = (d_2 + d_3 + (a+1)\,d_5 + d_6) \wedge d_4 = (A^0 - (a+2)\,A^1) \wedge A^1 = A^0 \wedge A^1$, and it follows that (11.4.1) is the QRT root defined by the span of (11.4.4).

For the somewhat more general map $(x,\ y) \mapsto (y,\ H(y)/(K(y)\,x))$ where $H(y)$ and $K(y)$ are polynomials of degree ≤ 1, see Remark 11.7.1.

11.4.0.2 The Base Points

Note that as for the pencil of the sine–Gordon map in Section 11.7, the curve defined by the equation $x_0\,x_1\,y_0\,y_1 = 0$ belongs to the Lyness pencil, but otherwise the Lyness map behaves quite differently from the sine–Gordon map. The base points are ([0 : 1], [0 : 1]), ([0 : 1], [1 : −1]), ([1 : 0], [1 : −1]), ([1 : 0], [1 : $-a$]), ([1 : −1], [0 : 1]), ([1 : −1], [1 : 0]), ([1 : $-a$], [1 : 0]), where the first one is double and the remaining six are simple if $a \neq 1$ and $a \neq 0$. Figure 3.1.2 shows the double base point at ([0 : 1], [0 : 1]), for $a = 0.3$.

The rational elliptic surface is obtained by blowing up all the simple base points, and performing two consecutive blowing-up transformations over the double base point at ([0 : 1], [0 : 1]). Because the double base point is a singular point of the curve $x_0\,x_1\,y_0\,y_1 = 0$, this double blowing up turns the square $x_0\,x_1\,y_0\,y_1 = 0$ into a singular fiber of Kodaira type I_5. Because there is at least one real base point, indeed all base points are real, we have that for every real $[z_0 : z_1]$ the real part of the curve $z_0\,p^0 + z_1\,p^1 = 0$ is nonempty.

11.4.0.3 The Singular Curves

The Weierstrass invariants of the pencil of biquadratic polynomials, the QRT map and the QRT root = the Lyness map, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$\begin{aligned}
12\,g_2 &= (a-2)^4\,{z_0}^4 + 4\,(a-2)\,(a^2+2)\,{z_0}^3\,z_1 \\
&\quad + 2\,a\,(3\,a+4)\,{z_0}^2\,{z_1}^2 + 4\,(a+2)\,z_0\,{z_1}^3 + {z_1}^4, \\
-2^3\,3^3\,g_3 &= (a-2)^6\,{z_0}^6 + 6\,(a-2)^3\,(a^2+2)\,{z_0}^5\,z_1 \\
&\quad + 3\,(5\,a^4 - 8\,a^3 + 4\,a^2 + 16\,a + 8)\,{z_0}^4\,{z_1}^2 \\
&\quad + 4\,(a-1)\,(5\,a^2 + 11\,a + 14)\,{z_0}^3\,{z_1}^3 + 3\,(5\,a^2 + 12\,a + 8)\,{z_0}^2\,{z_1}^4 \\
&\quad + 6\,(a+2)\,z_0\,{z_1}^5 + {z_1}^6, \\
\Delta &= -{z_0}^5\,{z_1}^3\,((a-1)\,z_0 + z_1)^2\,((a-2)^3\,{z_0}^2 \\
&\quad + (2\,a^2 + 10\,a - 1)\,z_0\,z_1 + a\,{z_1}^2), \\
12\,X &= (a-2)^2\,{z_0}^2 + 2\,(a+2)\,z_0\,z_1 + {z_1}^2, \\
Y &= {z_0}^2\,z_1, \\
12\,X_\rho &= (a-2)^2\,{z_0}^2 + 2\,(a-4)\,z_0\,z_1 + {z_1}^2, \\
Y_\rho &= -z_0\,z_1\,((a-1)\,z_0 + z_1).
\end{aligned}$$

At a zero $[z_0 : z_1]$ of Δ, when the complex curve is singular, the Kodaira type of the corresponding singular fiber in the rational elliptic surface can be read off from the orders of the zeros of g_2, g_3, and Δ at the point $[z_0 : z_1]$; see Table 6.3.2.

Because every curve has a nonempty real part, it follows from Lemma 2.6.3 that for every real $[z_0 : z_1]$ the real part of the curve $z_0\,p^0 + z_1\,p^1 = 0$ consists of one or two circles, when $\Delta(z_0,\,z_1) < 0$ or $\Delta(z_0,\,z_1) > 0$, respectively.

If $a = 1$, when $\Delta = -{z_0}^5 {z_1}^5 (-{z_0}^2 + 11\, z_0\, z_1 + {z_1}^2)$, the configuration of the singular fibers is $2\ \mathrm{I}_5\ \ 2\ \mathrm{I}_1$. According to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z}/5\,\mathbb{Z}$. This implies the result of Lyness [126, 1942] that λ is periodic with period 5, where we obtain in addition that λ generates the Mordell–Weil group. In Section 5.2.4 it is proved that up to isomorphism this ρ is the only order-5 element of the Mordell–Weil group of any rational elliptic surface.

If $a = 0$, when $\Delta = {z_0}^6 {z_1}^3 (z_1 - z_0)^2 (8\, z_0 + z_1)$, the configuration of the singular fibers is $\mathrm{I}_6\ \ \mathrm{I}_3\ \ \mathrm{I}_2\ \ \mathrm{I}_1$, and according to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$. We recover the result of Zeeman [214, Theorem 5] that λ is periodic with period 6, but obtain in addition that λ generates the Mordell–Weil group. In Section 5.2.5 it is proved that up to isomorphism this ρ is the only order-6 element of the Mordell–Weil group of any rational elliptic surface.

If $a = 2$, when $\Delta = -{z_0}^5 {z_1}^4 (z_0 + z_1)^2 (27\, z_0 + 2\, z_1)$, the configuration of the singular fibers is $\mathrm{I}_5\ \ \mathrm{IV}\ \ \mathrm{I}_2\ \ \mathrm{I}_1$. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}$.

If $a = 3/4$, when $\Delta = -{z_0}^5 {z_1}^3 ((-1/4)\, z_0 + z_1)^3 ((125/16)\, z_0 + (3/4)\, z_1)$, the configuration of the singular fibers is $\mathrm{I}_5\ \ \mathrm{I}_3\ \ \mathrm{III}\ \ \mathrm{I}_1$. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}$.

If $a = -1/4$, when $64\,\Delta = {z_0}^5 {z_1}^3 ((-5/4)\, z_0 + z_1)^2 (27\, z_0 + 4\, z_1)^2$, the configuration of the singular fibers is $\mathrm{I}_5\ \ \mathrm{I}_3\ \ \mathrm{I}_2\ \ \mathrm{II}$. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}$.

In all other cases we have the "generic" Lyness map, for which the configuration of the singular fibers is $\mathrm{I}_5\ \ \mathrm{I}_3\ \ \mathrm{I}_2\ \ 2\ \mathrm{I}_1$. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}$. In all cases the degree of the modulus function J is equal to $j = 12$, except if $a = 2$, $a = 3/4$, or $a = -1/4$, when $j = 8$, $j = 9$, or $j = 10$, respectively.

Remark 11.4.1. Zeeman [214, Theorem 5] also contains the result of Lyness [126, 1942] that for $a = 1$ the Lyness map has order 5, and that for $a = \infty$ the Lyness map has order 4. Here $a = \infty$ corresponds to the pencil in which the biquadratic polynomial $p^0 = z_0\,(x_1 + x_0)\,(y_1 + y_0)\,(x_1\, y_0 + x_0\, y_1 + a\, x_0\, y_0)$ has been replaced by the coefficient $z_0\,(x_1 + x_0)\,(y_1 + y_0)\,x_0\, y_0$ in p^0 of a. Because

$$z_0\,(x_1{+}x_0)\,(y_1{+}y_0)\,x_0\,y_0{+}z_1\,x_0\,x_1\,y_0\,y_1 = (z_0\,(x_1{+}x_0)\,(y_1{+}y_0){+}z_1\,x_1\ y_1)\,x_0\,y_0,$$

all the members of this pencil contain the axes $x_0 = 0$ and $y_0 = 0$ at infinity, and therefore are reducible, equal to the union of three complex projective lines. The two blowing-up transformations over the base point of multiplicity two at ([0 : 1], [0 : 1]) turns it into a cycle of four projective lines, which are shifted over one unit by the Lyness map. Since we do not admit pencils without elliptic curves, we will not include the case $a = \infty$ in our further considerations. ⊘

11.4.0.4 The Number of k-Periodic Complex Fibers

Assume that $a \neq 1$ and $a \neq 0$, thus avoiding the cases that the Lyness map is periodic with period 5 and 6, respectively. That is, the Mordell–Weil group is isomorphic to $\mathbb{Z}$,

with the Lyness map as a generator. A quite remarkable feature of the Lyness map in this case is that its permutation of the set of irreducible components of the reducible fibers has order equal to 30, because for each of the fibers of type I_5, I_3, and I_2, the permutation of the irreducible components is a shift in the cycle over one unit, and the smallest common multiple of 5, 3, and 2 is equal to 30. Furthermore, in view of (4.3.2) and Lemma 7.5.3, this action of ρ^S on the set of the irreducible components of the reducible fibers implies that for every $k \in \mathbb{Z}$, the number of k-periodic fibers in S^{reg}, counted with multiplicities, of the QRT root is given by

$$\begin{aligned} \nu((\rho^S)^k) = \frac{1}{60}\,k^2 + \frac{5}{2}\,\left\{\frac{k}{5}\right\}\left(1-\left\{\frac{k}{5}\right\}\right) + \frac{3}{2}\,\left\{\frac{k}{3}\right\}\left(1-\left\{\frac{k}{3}\right\}\right) \\ + \left\{\frac{k}{2}\right\}\left(1-\left\{\frac{k}{2}\right\}\right) - 1, \end{aligned} \tag{11.4.5}$$

where $\{x\}$ denotes the fractional part of x. Furthermore, we have used that $\nu(\rho^S) = 0$, see Section 10.1, and $\chi(S, \mathcal{O}) = 1$, see Lemma 9.1.2(iii).

Corollary 11.4.2 *The Lyness map ρ^S generates the Mordell–Weil group* $\mathrm{Aut}(S)^+_\kappa$.

Proof. Because $\mathrm{Aut}(S)^+_\kappa \simeq \mathbb{Z}$, there are a generator α of $\mathrm{Aut}(S)^+_\kappa$ and an integer m such that $\rho^S = \alpha^m$. Because ρ^S has no fixed points in S^{reg}, α has no fixed points in S^{reg}, that is, $\nu(\alpha) = 0$. It therefore follows from (7.5.1) with $\chi(S, \mathcal{O}) = 1$ that the sum of the contributions of the reducible singular fibers to α is ≤ 2. According to Lemma 7.5.3, the contribution of the I_5 fiber is equal to 0, 4/5, or 6/5, of the I_3 fibers equal to 0 or 2/3, and that of the I_2 fiber is equal to 0 or 1/2. Therefore the sum s of the contributions, being ≤ 2, $\leq 59/30$. It follows from (4.3.2) that the limit for $l \to \infty$ of $\nu(\alpha^l)/l^2$ is equal to $1 - (s/2) \geq 1/60$. Because $(\rho^S)^k = (\alpha^m)^k = \alpha^{m\,k}$, we obtain from (11.4.5) that $(1/60)/m^2 \geq 1/60$, hence $m^2 \leq 1$. Because ρ^S is not equal to the identity mapping, we conclude that $m = \pm 1$. That is, $\rho^S = \alpha$ or $\rho^S = \alpha^{-1}$, and ρ^S is a generator of $\mathrm{Aut}(S)^+_\kappa$. □

It follows from (11.4.5) that for the Lyness map there are no periodic curves of period k when $1 \leq k \leq 6$, there is exactly one periodic curve of period k when $7 \leq k \leq 10$, there are two of them for $k = 11, 12$, three for $k = 13, 14, 15$, four for $k = 16$, five for $k = 17, 18$, six for $k = 19, 20$, and ≥ 7 for $k \geq 21$. In view of Theorem 4.3.2 we conclude that for $1 \leq k \leq 6$ the kth power of the Lyness map and its inverse are Manin QRT automorphisms, and we have 12 Manin QRT automorphisms in the Mordell–Weil group. Since the degree of the polynomial equation for $[z_0 : z_1]$ over which we have a k-periodic curve is equal to $\nu((\rho^S)^k$, the equations for $[z_0 : z_1]$ are surprisingly simple for these relatively small periods, where for the generic QRT map we have the degree $k^2 - 1$ for the k-periodic curves.

Using (7.8.2), (7.8.3), we can successively compute the coordinates $x = X_k$, $y = Y_k$ of the image point $[1 : x : y]$ of the point $[0 : 0 : 1]$ at infinity under the kth power of the Lyness map in the Weierstrass model. After canceling common factors in the numerators and the denominators of X_k and Y_k, we obtain polynomial expressions for X_k and Y_k when $1 \le k \le 6$, corresponding to the fact that for $1 \le k \le 6$ the kth power of the Lynesss map is a Manin QRT automorphism. See Proposition 7.8.1, where the condition that the sections X and Y have no poles corresponds to the condition that the map has no fixed points in S^{reg}. We have $X_1 = X_\rho$, $Y_1 = Y_\rho$, and $X_2 = X$, $Y_2 = Y$ since $\rho^2 = \tau$ is the QRT map, whereas the X_k, Y_k for $3 \le k \le 6$ are given by the following formulas:

$$
\begin{aligned}
12\,X_3 &= (a^2 + 8\,a - 8)\,{z_0}^2 + 2\,(a+2)\,z_0\,z_1 + {z_1}^2,\\
Y_3 &= -a\,{z_0}^2\,((a-1)\,z_0 + z_1),\\
12\,X_4 &= (a-2)^2\,{z_0}^2 - 2\,(5\,a-2)\,z_0\,z_1 + {z_1}^2,\\
Y_4 &= -z_0\,z_1\,((a-1)^2\,z_0 - a\,z_1),\\
12\,(a-1)^2\,X_5 &= (a-1)^2\,(a-2)^2\,{z_0}^2\\
&\quad + 2\,(a-1)\,(a^2 + a + 4)\,z_0\,z_1 + (a^2 + 10\,a + 1)\,{z_1}^2,\\
(a-1)^3\,Y_5 &= z_1\,((a-1)\,z_0 + z_1)\,((a-1)\,z_0 + a\,(a+1)\,z_1),\\
12\,a^2\,X_6 &= (a^4 - 4\,a^3 + 16\,a^2 - 24\,a + 12)\,{z_0}^2 + 2\,a\,(a^2 + 2\,a - 6)\,z_0\,z_1,\\
a^3\,Y_6 &= -z_0\,((a-1)^2\,(a^2 - 2\,a + 2)\,{z_0}^2\\
&\quad - (a-1)\,(a-3)\,z_0\,z_1 + a^2\,{z_1}^2).
\end{aligned}
$$

Because $X_{-k} = X_k$ and $Y_{-k} = -Y_k$, this leads to explicit formulas for all twelve Manin QRT automorphisms in the Weierstrass model.

For $7 \le k \le 12$ we equate in the next table the denominators of X_k and Y_k to zero, which is the condition that the Lyness map is k-periodic on the curve over $[z_0 : z_1]$.

Period k	Equation for $[z_0 : z_1]$
7	$(a-1)\,z_0 + a\,z_1 = 0,$
8	$(a-1)^2\,z_0 - a\,z_1 = 0,$
9	$(a-1)\,(a^2 - a + 1)\,z_0 + a\,z_1 = 0,$
10	$(a-1)\,z_0 + a\,(a+1)\,z_1 = 0,$
11	$(a-1)^4\,{z_0}^2 - a\,(a-1)\,(2\,a-1)\,z_0\,z_1 - a^3\,{z_1}^2 = 0,$
12	$(a-1)^2\,(a^2 - 2\,a + 2)\,{z_0}^2 - a\,(a-1)\,(a-3)\,z_0\,z_1 + a^2\,{z_1}^2 = 0.$

These are all the periods for which the equation is of degree one or two. For higher periods the formulas become gradually more unwieldy, but they keep confirming that the degree of the equation is equal to the number $\nu((\rho^S)^k)$ in (11.4.5). The equations for $k = 9$ and $k = 11$ are Theorems 10 and 11 of Zeeman [214], respectively, where his proof is more geometric. Zeeman's results prompted me to the above discussion of the kth iterates of the Lyness map for $1 \le k \le 12$.

11.4.0.5 The Real Curves

Assume that a is real. We use the affine coordinate z such that $z_0 = 1$, $z_1 = -z$, that is, z is the value of the rational function φ that is invariant under the Lyness map. Then we have singular fibers of type I_5, I_3, and I_2 for z equal to ∞, 0, and $a-1$, respectively. Because the discriminant of the last, quadratic, factor

$$Q(z) = Q(z,\, a) := a\, z^2 - (2\, a^2 + 10\, a - 1)\, z + (a-2)^3 \tag{11.4.6}$$

in Δ is equal to $(4\, a + 1)^3$, we have two further real singular fibers if $a > -1/4$, which occur for the values

$$z_\pm = z_\pm(a) := ((2\, a^2 + 10\, a - 1) \pm (4\, a + 1)^{3/2})/2\, a \tag{11.4.7}$$

of z. Note that with this notation, $z_- < z_+$ if $a > 0$ and $z_+ < z_-$ if $-1/4 < a < 0$. Furthermore, the function $a \mapsto z_+(a)$ has an analytic continuation to a neighborhood of $a = 0$, whereas $z_(a) \downarrow -\infty$ and $z_(a) \uparrow \infty$ when $a \downarrow 0$ and $a \uparrow 0$, respectively.

There are no other real singular fibers if $a < -1/4$. When $a = -1/4$, then there is one additional real singular fiber of type II, which occurs for $z = 27/4$, and which has an ordinary cusp at the point $x = y = 1/2$.

In the following table we list, for the various a-intervals in which the configuration of the singular fibers does not change, the singular values z of φ in increasing order. In between these singular values we have written (1) or (2) when the number of connected components of the real fiber over the regular value z is equal to one or two, which in view of Lemma 2.6.3 corresponds to $\Delta(z) < 0$ or $\Delta(z) > 0$, respectively. Note that all real fibers are nonempty, because there are real base points. The asterisk in (2)* means that the Lyness map switches the two connected components of the real fiber.

The superscripts $^{\mathrm{ell}}$ and $^{\mathrm{hyp}}$ indicate whether the fiber over z is elliptic or hyperbolic, respectively, where we note that the fiber over $z = -\infty$ is equal to the fiber over $z = \infty$. The proof of all the statements in Table 11.4.3 will be given below.

Table 11.4.3

$a > 2$	$-\infty^{\mathrm{hyp}}$	(1)	0^{hyp}	(2)	$z_-{}^{\mathrm{ell}}$	(1)	$(a-1)^{\mathrm{hyp}}$	(1)	$z_+{}^{\mathrm{ell}}$	(2)	∞^{hyp}	
$1 < a < 2$	$-\infty^{\mathrm{hyp}}$	(1)	$z_-{}^{\mathrm{ell}}$	(2)	0^{hyp}	(1)	$(a-1)^{\mathrm{hyp}}$	(1)	$z_+{}^{\mathrm{ell}}$	(2)	∞^{hyp}	
$3/4 < a < 1$	$-\infty^{\mathrm{hyp}}$	(1)	$z_-{}^{\mathrm{ell}}$	(2)	$(a-1)^{\mathrm{hyp}}$	(2)*	0^{hyp}	(1)	$z_+{}^{\mathrm{ell}}$	(2)	∞^{hyp}	
$0 < a < 3/4$	$-\infty^{\mathrm{hyp}}$	(1)	$z_-{}^{\mathrm{hyp}}$	(2)*	$(a-1)^{\mathrm{ell}}$	(2)*	0^{hyp}	(1)	$z_+{}^{\mathrm{ell}}$	(2)	∞^{hyp}	
$-1/4 < a < 0$	$-\infty^{\mathrm{hyp}}$	(2)*	$(a-1)^{\mathrm{ell}}$	(2)*	0^{hyp}	(1)	$z_+{}^{\mathrm{ell}}$	(2)	$z_-{}^{\mathrm{hyp}}$	(1)	∞^{hyp}	
$a < -1/4$	$-\infty^{\mathrm{hyp}}$	(2)*	$(a-1)^{\mathrm{ell}}$	(2)*	0^{hyp}	(1)	∞^{hyp}					

It is somewhat surprising that the singular values $z = a - 1$ and z_- do not cross at $a = 3/4$, since $z_- < a - 1$ when $0 < a < 3/4$ and when $3/4 < a < 1$, whereas $z_- = a - 1$ when $a = 3/4$. When a passes through 0 from positive to negative, then z_- passes from large negative to large positive. At the same time, z_+ for $a > 0$ continues as an analytic strictly positive function of a when a passes through 0 to $-1/4 < a < 0$.

For $z_0 = 0$, $z_1 = 1$, that is, $z = \pm\infty$, we have $2^3\,3^3\,g_3 = -1$, which in view of Remark 8.3.1 implies that the fiber over $[0 : 1] \in \mathbb{P}^1$ is hyperbolic for any value of a. Recall that it is of type I_5 or I_6 when $a \neq 0$ or $a = 0$, respectively.

For $z = 0$, we have $g_3 = -(a-2)^6$, and the fiber over $z = 0$ is of type I_3 and hyperbolic if $a \neq 2$; see Remark 8.3.1. When $a = 2$, it is of type IV.

For $z = a - 1$, we have $216\,g_3 = -(4\,a-3)^3$. Therefore the fiber over $z = a - 1$ is of type I_2 if $a \neq 3/4$, and of type III if $a = 3/4$. If $a > 3/4$ and $a < 3/4$, then Remark 8.3.1 implies that the fiber of type I_2 over $z = a - 1$ is hyperbolic and elliptic, respectively. The singular points are located at the points $([1 : x], [1 : x])$ such that $x^2 + x + 1 - a = 0$. Since the discriminant of this equation is equal to $1 - 4\,(1 - a) = 4\,a - 3$, we have that the singular points are real if $a > 3/4$, but not real if $a < 3/4$. Therefore, if $a < 3/4$, then the complex fiber over $z = a - 1$ is singular, of Kodaira type I_2, whereas the real part of the fiber is smooth, and actually is isomorphic to the union of two circles. In other words, we are in the case as described in the last paragraph of Lemma 8.1.4.

Because $g_3 < 0$ when $z_0 = 0$ and $z_1 \neq 0$, that is, $z = \infty$, the z_- that came from ∞ if a passes through zero from $a > 0$ to $a < 0$ corresponds to a hyperbolic singular fiber of type I_1 when $-1/4 < a < 0$, see Remark 8.3.1. When $a \downarrow -1/4$, the fibers of type I_1 over z_+ and z_- merge to a singular fiber of type II at $z = 27/4$. At these values, $g_3 = 0$ and the partial derivative of g_3 with respect to z is equal to $-27/2 < 0$, and it follows that for a slightly greater that $-1/4$ the fibers over z_+ and z_- are elliptic and hyperbolic, respectively. Because there are no other bifurcations with the singular fibers of type I_1 when $-1/4 < a < 0$, we conclude that for all a such that $-1/4 < a < 0$ the fibers over z_+ and z_- are elliptic and hyperbolic, respectively.

When a passes through 0 from $a < 0$ to $a < 0$, then z_+ passes from large positive to large negative, and becomes z_-, where the fiber over it remains hyperbolic because as we have seen before, $g_3 < 0$ when $z_0 = 0$, $z_1 \neq 0$. The first bifurcation in which this fiber is involved is when a passes through $3/4$ from $a < 3/4$ to $a > 3/4$, when z_- bumps from the negative side into $a - 1$ and returns to the negative side of $a - 1$. Recall that the fiber of type I_2 over $a - 1$ changes from elliptic to hyperbolic when a passes through $3/4$ from $a < 3/4$ to $a > 3/4$. We have

$$z_- = a - 1 - 2\,(a - 3/4)^2 + 3\,(a - 3/4)^3 + \mathrm{O}((a - 3/4)^4)$$

for $a \to 3/4$, and substituting this in the formula for g_3 we obtain that $g_3 = (a - 3/4)^3/27 + \mathrm{O}((a - 3/4)^4)$ as $a \to 3/4$. It therefore follows from Remark 8.3.1 that the fiber over z_- changes from hyperbolic to elliptic when a passes through $3/4$ from $a < 3/4$ to $a > 3/4$. The next bifurcation in which this fiber is involved for $a > 0$ is when a passes through 2 from $a < 2$ to $a > 2$, when z_- passes through 0 from $z_- < 0$ to $z_- > 0$, where for $a = 2$ the fiber over $z = 0$ is of type IV. We have $z_- = (a - 2)^3/27 + \mathrm{O}((a - 2)^4)$ for $a \to 2$, and substituting this into the formula for g_3 we obtain that $g_3 = (a - 2)^6/5832 + \mathrm{O}((a - 2)^7)$. It follows that the fiber over z_- remains elliptic when a passes through 2 from $a < 2$ to $a > 2$. Because there are no other bifurcations, the conclusion is that the fiber over z_- is hyperbolic

when $0 < a < 3/4$, elliptic when $3/4 < a < 2$, and elliptic when $a > 2$. On the other hand, the z_- for $-1/4 < a < 0$ becomes z_+ when $a > 0$, and the fiber over it remains elliptic, because it is not involved in a bifurcation. The conclusion is that for every $a > 0$ the fiber over z_+ is elliptic. This completes the determination of the elliptic or hyperbolic nature of the singular fibers in all the cases.

The real fiber has two connected components if and only if $\Delta > 0$, which implies that $g_2 > 0$, and the Lyness map switches the two connected components if and only if $X_\rho < (g_2/12)^{1/2}$, that is, $X_\rho < 0$ or $X_\rho^2 < g_2/12$. Because the condition of switching does not change when z varies in an open interval between singular values, it is sufficient to determine the signs of $12\,X_\rho = z^2 - 2\,(a-4)\,z + (a-2)^2$ and $6\,({X_\rho}^2 - g_2/12) = z\,(z-(a-2))\,(z-(a-1))$ near one of the endpoints of the intervals over which the real fibers have two connected components. This leads to the determination of the asterisks in Table 11.4.3. For instance, if $z = a-1$ then $12\,X_\rho = 4\,a - 3$, and it follows that we have asterisks over the intervals adjacent to $a-1$ when $a < 3/4$.

11.4.0.6 The Rotation Function

In Chapter 8, the asymptotics for $k \to \infty$ of the real k-periodic fibers for any element of the Mordell–Weil group are related to the properties of the rotation function $\rho(z)$. In this subsection we collect a number of qualitative properties of the rotation function of the QRT map equal to the square of the Lyness map. Note that $Y = 0$ if and only if $[z_0 : z_1] = [0 : 1]$ or $[z_0 : z_1] = [1 : 0]$, which in view of Corollary 2.5.9 implies that $\rho(z) \neq 1/2$ for every $z \in \mathbb{R}$ over which the real fiber is smooth.

There are quite a few bifurcation values of a for which the qualitative behavior of the rotation function changes.

Figures 11.4.1–11.4.13 are numerical plots of the rotation function for values of a that represent all cases. In each plot, the coordinate on the horizontal axis is $(2/\pi)\,\arctan(z)$, where $z = -z_1/z_0$, and therefore the endpoints correspond to $z = \pm\infty$, that is, $[z_0 : z_1] = [0 : 1]$. Over this point we have the complex singular fiber of Kodaira type I_5, with the exception that it is of type I_6 when $a = 0$. Because the orientation of the Hamiltonian vector field on the biquadratic curve gets reversed if z goes from $z \gg 0$ to $z \ll 0$, the natural continuation of the rotation function ρ is such that $\rho(1+\epsilon) = 1 - \rho(-1+\epsilon)$ for $0 < \epsilon \ll 1$. This is related to the fact that the real Lie algebra bundle $\mathfrak{f}(\mathbb{R})$ over $\mathbb{P}^1(\mathbb{R})$ is a Möbius strip; see Remark 8.1.3. The remaining subsections contain proofs of all the qualitative properties of the rotation function that are suggested by the numerical plots.

Assume that $a \in \mathbb{R}$, not equal to one of the exceptional values $-1/4, 0, 3/4, 1, 2$. At values of z where the real fiber is smooth but has two connected components, the Lyness map λ might interchange these, and then the rotation number is not defined. This happens for z in the intervals indicated by $(2)^*$ in Table 11.4.3. However, the QRT map $\tau = \lambda^2$ always preserves the connected components of the real fibers, and according to Lemma 8.1.5 its rotation function $\rho(z)$ is a real analytic function on

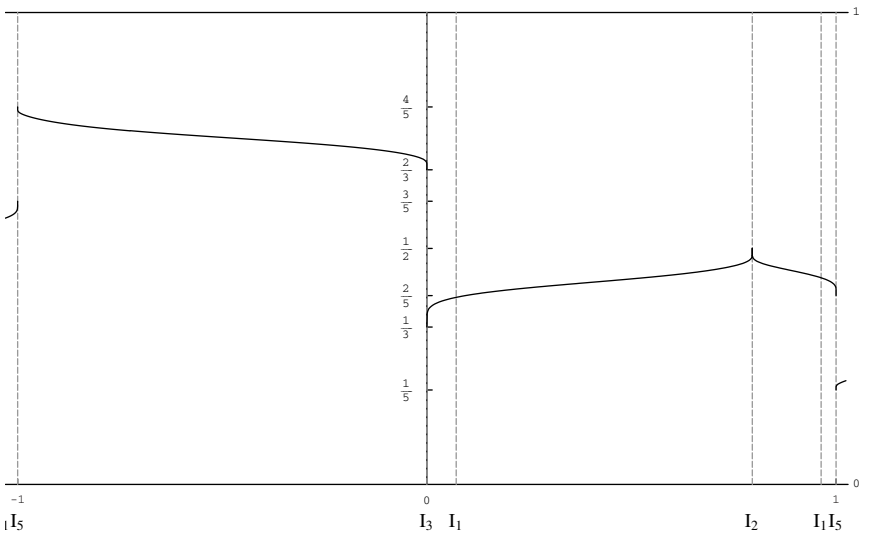


Fig. 11.4.1 The rotation function for $a = 4$.

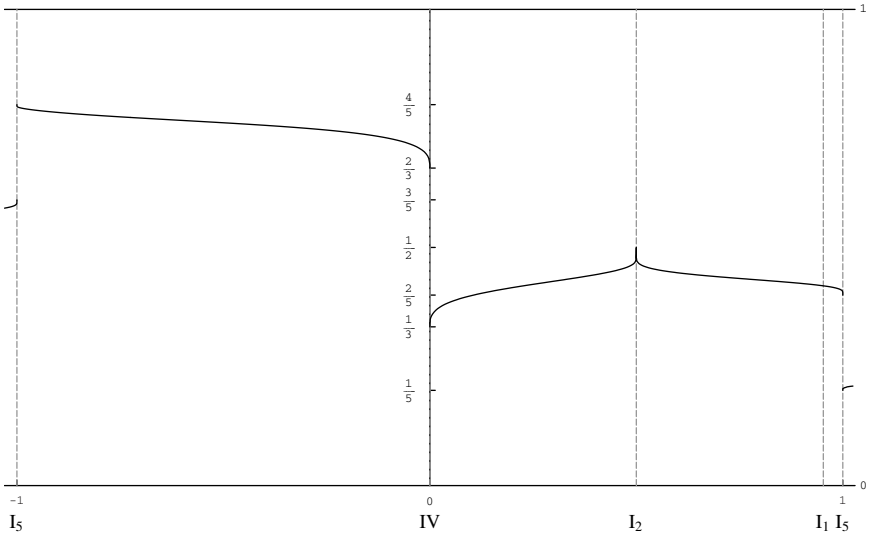


Fig. 11.4.2 The rotation function for $a = 2$.

every open interval between values of z over which the real fiber is singular. It is for this reason that in the sequel we restrict the discussion to the rotation function of the QRT map instead of the Lyness map. Because $\rho_\tau = 2\,\rho_\lambda$ modulo 1, it is easy to reconstruct the rotation function of the Lyness map, where defined, from the rotation function of the QRT map.

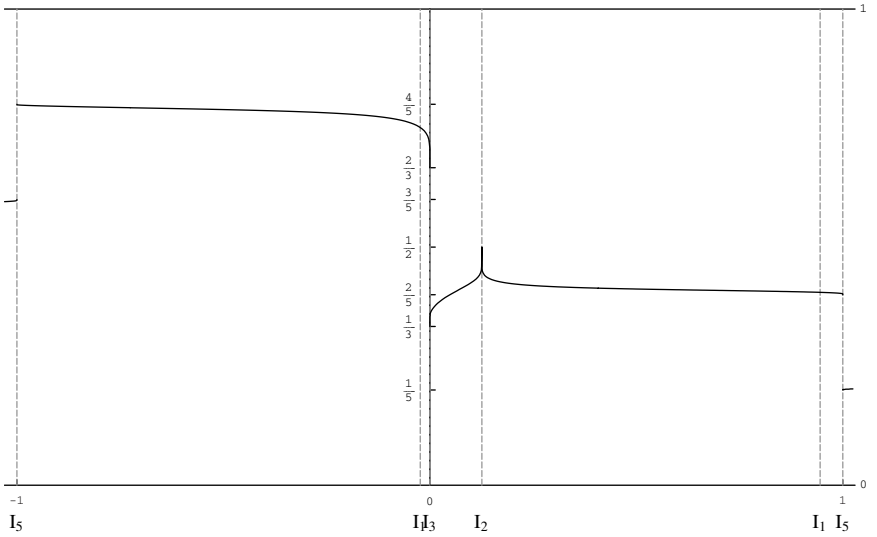


Fig. 11.4.3 The rotation function for $a = 1.2$.

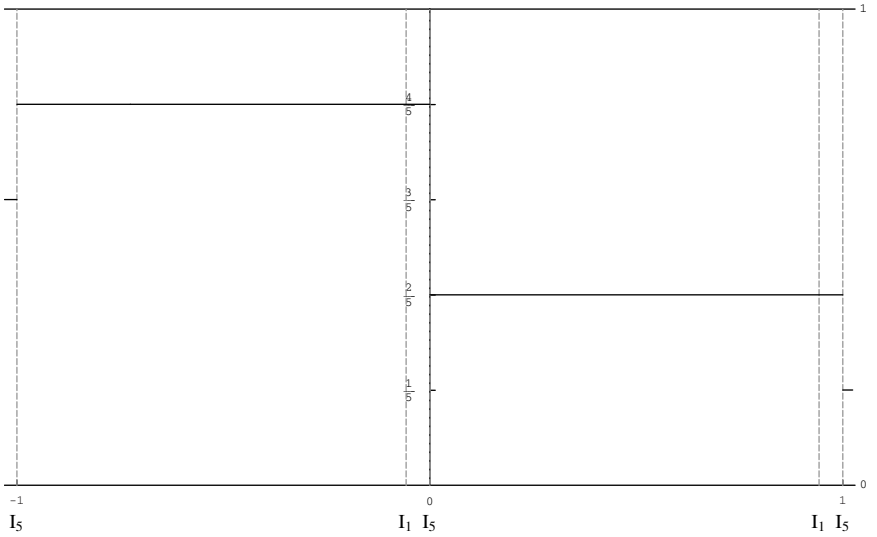


Fig. 11.4.4 The rotation function for $a = 1$.

It follows from Lemma 8.4.1 that the rotation function extends to an analytic function on a neighborhood of any z over which the real fiber is singular, but elliptic with respect to the real structure. That is, $\rho(z)$ extends to an analytic function on the open intervals between the points $z \in \mathbb{R}$ over which the real fiber is hyperbolic of type I_b, $b > 0$. Table 11.4.3 implies that these intervals are the intervals on the real axis separated by the subsequent points 0 and $a-1$ when $a > 1$ and $a \neq 2$; $a-1$ and

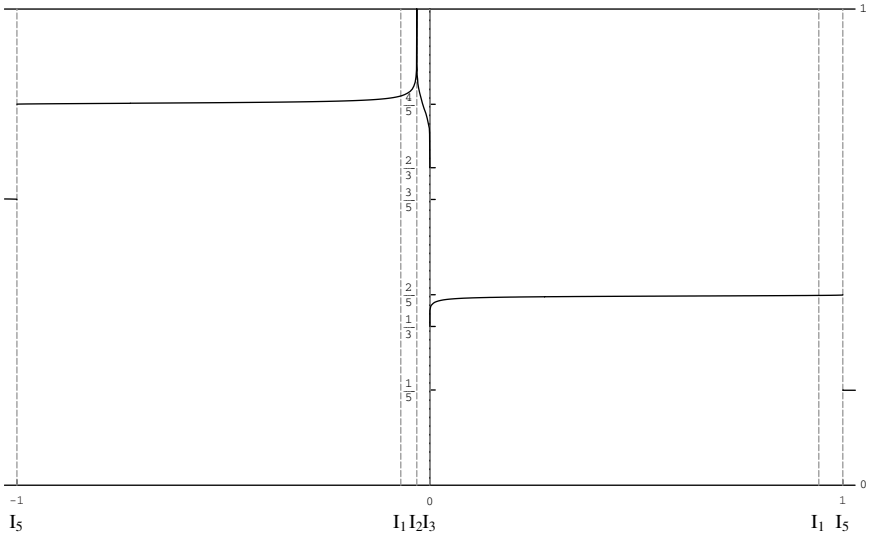


Fig. 11.4.5 The rotation function for $a = 0.95$.

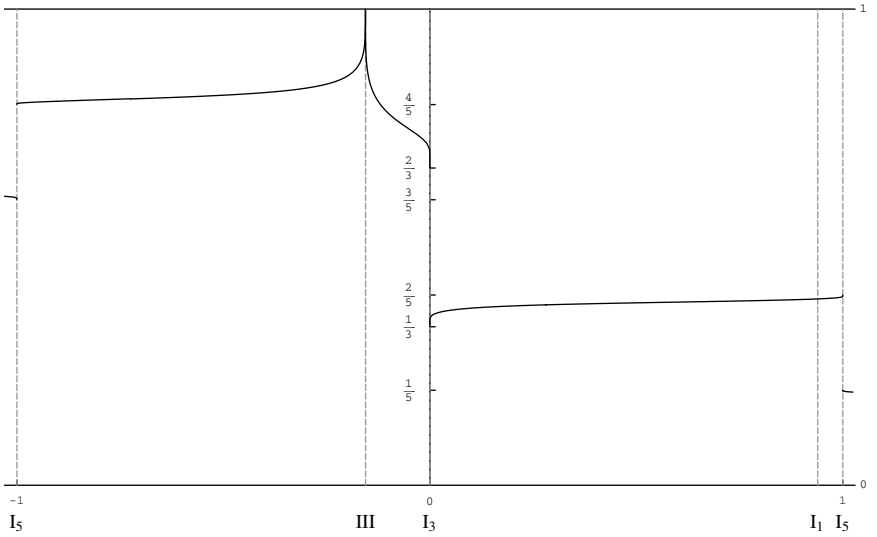


Fig. 11.4.6 The rotation function for $a = 3/4$.

0 when $3/4 < a < 1$; z_- and 0 when $0 < a < 3/4$; 0 and z_- when $-1/4 < a < 0$; and finally, only 0 when $a < -1/4$.

Theorem 8 of Zeeman [214] states that if $a > 0$, then the rotation function $\rho_\lambda(z)$ of the Lyness map is asymptotically equal to $\log z/(5 \log z - \log a)$ as $z \to \infty$, when the fiber over z approaches the singular fiber of hyperbolic type I_5. In the Table 11.4.4 we collect the limit values of the rotation function $\rho_\tau(z) = 2\,\rho_\lambda(z)$

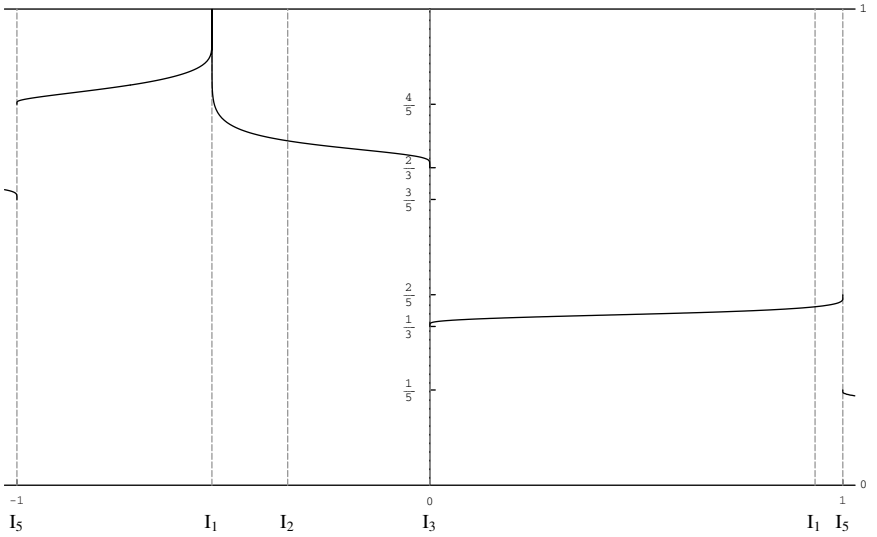


Fig. 11.4.7 The rotation function for $a = 0.4$.

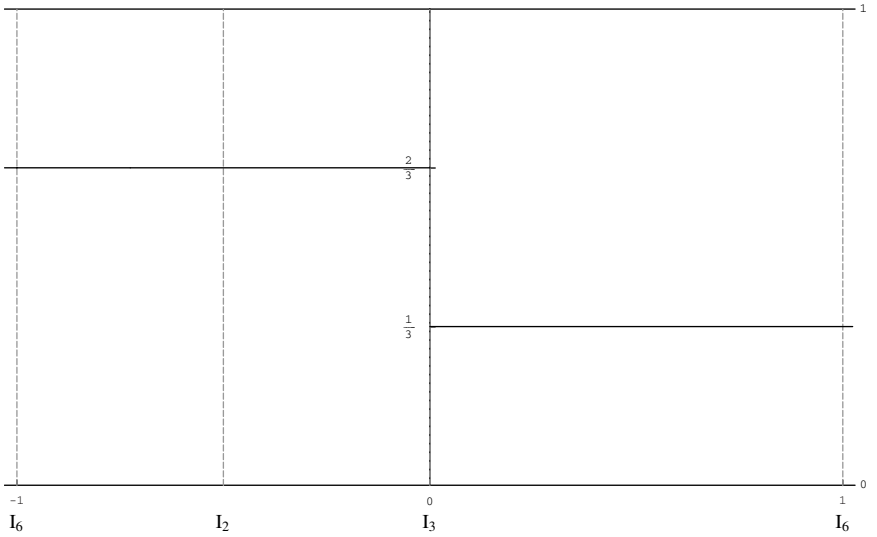


Fig. 11.4.8 The rotation function for $a = 0$.

of the QRT map when it approaches any hyperbolic singular curve, and determine whether these are approached from above or from below. The value of z over which we have a hyperbolic singular fiber is indicated by $\langle z \rangle$, and the limit value of the rotation function when it approaches z from either side is written at the corresponding side of $\langle z \rangle$. The arrows indicate whether the limit value is appraoched from above or below. In almost all cases the approach is infinitely steep, as described in Remark 8.4.2,

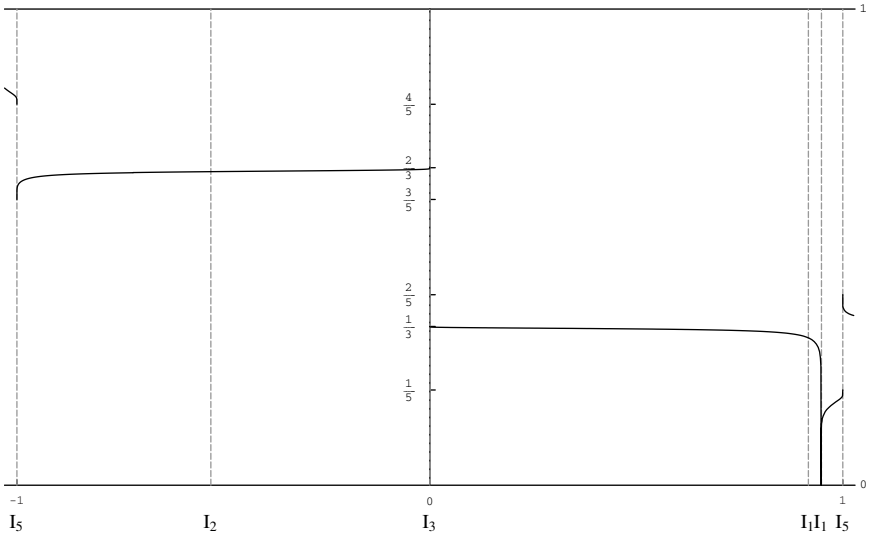


Fig. 11.4.9 The rotation function for $a = -0.1$.

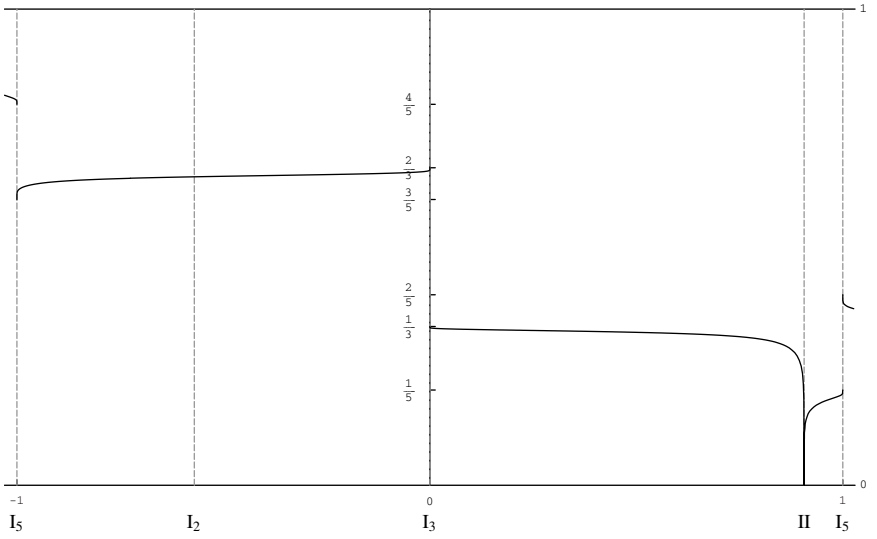


Fig. 11.4.10 The rotation function for $a = -1/4$.

where for $z \to \pm\infty$ we consider the rotation function as a function of the affine coordinate $\zeta = 1/z \to \pm 0$. The only exception occurs for $a = -1$, when the approach of $\rho(1/\zeta)$ to its limit values for $\zeta \to \pm 0$ is horizontal.

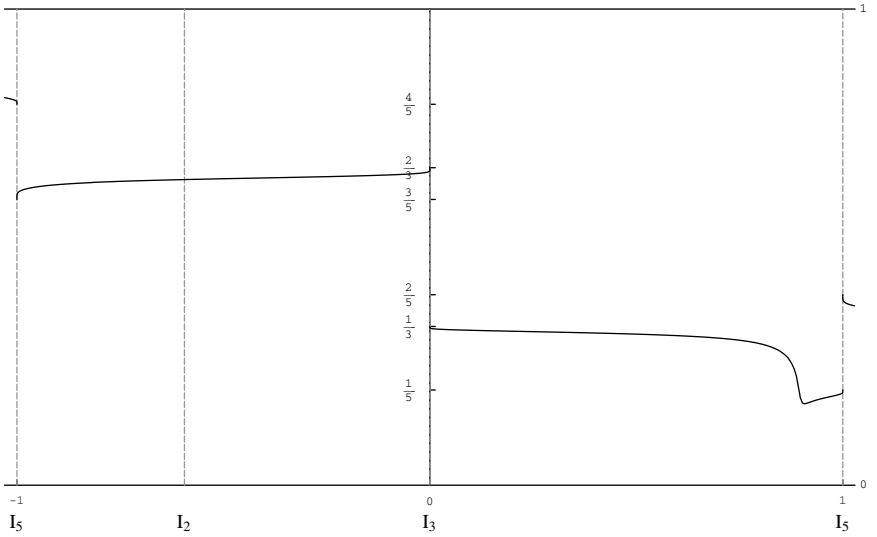


Fig. 11.4.11 The rotation function for $a = -0.35$.

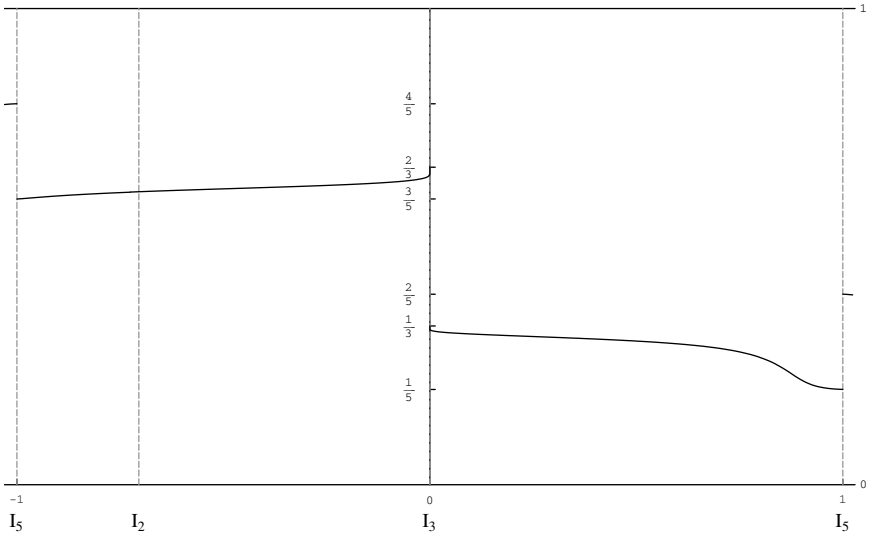


Fig. 11.4.12 The rotation function for $a = -1$.

Table 11.4.4

$a > 1$ and $a \neq 2$	$-\infty\rangle\ 4/5\ \nwarrow \ldots \searrow\ 2/3\ \langle 0\rangle\ 1/3\ \swarrow \ldots \nearrow\ 1/2\ \langle a-1\rangle\ 1/2\ \nwarrow \ldots \searrow\ 2/5\ \langle\infty$
$3/4 < a < 1$	$-\infty\rangle\ 4/5\ \swarrow \ldots \nearrow\ 1\ \langle a-1\rangle\ 1\ \nwarrow \ldots \searrow\ 2/3\ \langle 0\rangle\ 1/3\ \swarrow \ldots \nearrow\ 2/5\ \langle\infty$
$0 < a < 3/4$	$-\infty\rangle\ 4/5\ \swarrow \ldots \nearrow\ 1\ \langle z_-\rangle\ 1\ \nwarrow \ldots \searrow\ 2/3\ \langle 0\rangle\ 1/3\ \swarrow \ldots \nearrow\ 2/5\ \langle\infty$
$-1/4 < a < 0$	$-\infty\rangle\ 3/5\ \swarrow \ldots \nearrow\ 2/3\ \langle 0\rangle\ 1/3\ \nwarrow \ldots \searrow\ 0\ \langle z_-\rangle\ 0\ \swarrow \ldots \nearrow\ 1/5\ \langle\infty$
$-1 < a < -1/4$	$-\infty\rangle\ 3/5\ \swarrow \ldots \nearrow\ 2/3\ \langle 0\rangle\ 1/3\ \nwarrow \ldots \nearrow\ 1/5\ \langle\infty$

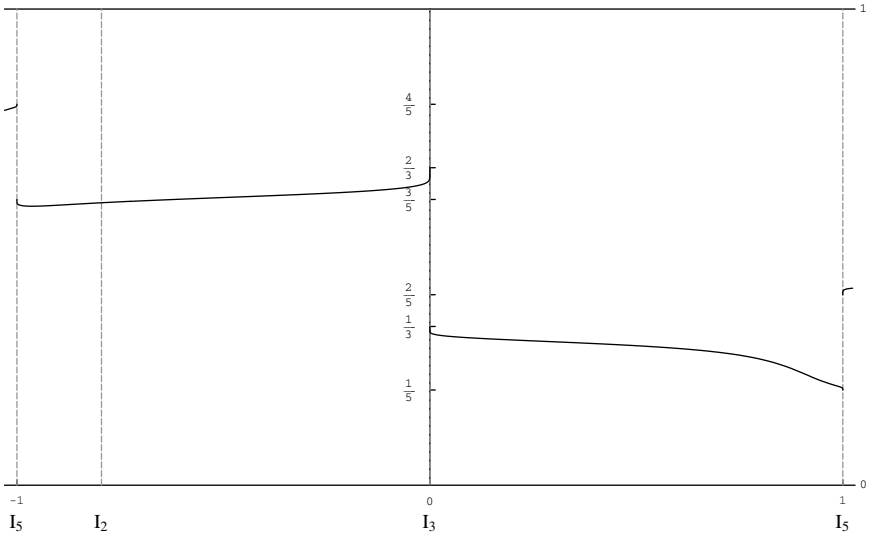


Fig. 11.4.13 The rotation function for $a = -2$.

$a = -1$	$-\infty\rangle\ 3/5\ \swarrow \ldots \nearrow\ 2/3\ \langle 0\rangle\ 1/3\ \nwarrow \ldots \searrow\ 1/5\ \langle\infty$
$a < -1$	$-\infty\rangle\ 3/5\ \nwarrow \ldots \nearrow\ 2/3\ \langle 0\rangle\ 1/3\ \nwarrow \ldots \searrow\ 1/5\ \langle\infty$

Proof. We apply Remark 8.4.3, in combination with Table 11.4.3.

The fiber over $z = \pm\infty$ is of Kodaira type I_5, and the QRT map shifts the cycle of its irreducible components over ± 2 units, where the sign depends on the orientation of the cycle. For $|z| \gg 0$ the biquadratic function is dominated by $-z\,x\,y$ in the affine $(x,\ y)$-plane, with the Hamiltonian system $\mathrm{d}x/\,\mathrm{d}t = -z\,x$, $\mathrm{d}y/\,\mathrm{d}t = z\,y$. Let the five singular points s_1, s_2, s_3, s_4, and s_5 in the cycle be projected by the blowing up map to ([1 : 0], [1 : 0]), ([1 : 0], [0 : 1]), ([0 : 1], [0 : 1]), ([0 : 1], [0 : 1]), and ([0 : 1], [1 : 0]), respectively, where we recall that ([0 : 1], [0 : 1]) is a base point of multiplicity two, over which we have one of the irreducible components of the cycle, and therefore two singular points, s_3 and s_4. Then the orientation coming from the Hamiltonian vector field for $z \gg 0$ or $z \ll 0$ is such that the QRT maps shift the cycle over 2 or -2 units, respectively, if the ordering in the cycle is that of the s_i for increasing i. It therefore follows from Remark 8.4.3 that $\rho(+\infty) = 2/5$ and $\rho(-\infty) = 1 - 1/5 = 4/5$ when $a > 0$, where we note that in Remark 8.4.3 the limit values were determined only modulo $\mathbb{Z}$. On the other hand, if $a < 0$, then $\rho(+\infty) = 1/5$ and $\rho(-\infty) = 1 - 2/5 = 3/5$.

In order to determine from which side $\rho(z)$ approaches its limit value for $z \to \pm\infty$, we observe that the x-coordinate of $\lambda^5(x,\ y)$ is equal to

$$x\,(y + a + a\,x + a\,x\,y + a\,y\,(y + a))/(y + a)\,(y + a + a\,x),$$

and it follows that the fifth power of the Lyness map acts on the axis $y = 0$ by sending x to x/a. Therefore the QRT map τ^5 acts on the axis $y = 0$ by sending x

to x/a^2, and $x = 0$ is an attracting or repelling fixed point if $a^2 < 1$ or $a^2 > 1$, respectively. Because $x = 0$ is an attractor or repeller for the Hamiltonian system if $z \gg 0$ or $z \ll 0$, respectively, it follows that the time needed for the solution of the Hamiltonian system to go from $(x, 0)$ to $\tau^5(x, 0)$ is positive if $z \gg 0$ and $a^2 < 1$ and if $z \ll 0$ and $a^2 > 1$, whereas it is positive if $z \gg 0$ and $a^2 < 1$ and if $z \ll 0$ and $a^2 < 1$. The side from which $\rho(z)$ approaches its limit value for $z \to \pm\infty$ has the same sign as this time.

The fiber over $z = 0$ is of Kodaira type I_3, also hyperbolic with respect to the real structure, and the QRT map shifts the cyle over ± 2 units, where the sign depends on the ordering of the cycle. In affine coordinates the Hamiltonian system of the function $(x + 1)(y + 1)(x + y + a)$ on the axis $y + 1 = 0$ is equal to $\mathrm{d}x/\mathrm{d}t = (x+1)(x-1+a)$. The derivative at $x = -1$ with respect to x of the right-hand side is equal to $a - 2$. Therefore, if $a < 2$, then the ordering induced by the Hamiltonian vector field is from the axis $y + 1 = 0$ to the axis $x + 1 = 0$ to the $(1, 1)$-curve $x + y + a = 0$ and then back to the axis $y + 1 = 0$. The QRT map shifts over 2 units in the same direction, and it follows in view of Remark 8.4.3 and Table 11.4.3 that $\rho(-0) = 2/3$ and $\rho(+0) = 1/3$. When $a > 2$, we have $\rho(+0) = 1 - 2/3 = 1/3$ and $\rho(-0) = 1 - 1/3 = 2/3$, which happen to be the same limit values as for $a < 2$.

In order to determine from which side $\rho(z)$ approaches its limit value for $z \to \pm 0$, we observe that the x-coordinate of $\lambda^3(x, y)$ is equal to $(y+a+a\,x)/x\,y$. Therefore on the component $y + 1 = 0$ of the curve $(x + 1)(y + 1)(x + y + a) = 0$, +a+a*x) parametrized by x, λ^3 acts by sending x to $-(-1 + a + a\,x)/x$, whose derivative with respect to x at the fixed point $x = -1$ is equal to $a - 1$. It follows that the derivative at $x = -1$ of the action of $\tau^3 = (\lambda^2)^3 = \lambda^3)^2$ on the axis $y + 1 = 0$ is equal to $(a - 1)^2$; hence τ^3 on this axis contracts to or expands from this point according to whether $0 < a < 2$ or $a \notin [0, 2]$, respectively. Because the derivative of the Hamiltonian vector field on the axis $y + 1 = 0$ with respect to x at $x = -1$ is equal to $a - 2$, it follows that the time needed for the solution of the Hamiltonian system to go from $(x, -1)$ to $\tau^3(x, -1)$ is positive if $a > 2$ or $0 < a < 2$, and negative if $a < 0$. The side from which $\rho(z)$ approaches its limit value for $z \to \pm 0$ has the same sign as this time.

The fiber over $z = a-1$, corresponding to the curve $(x+y+1)(x\,y+x+y+a) = 0$, is of Kodaira type I_2, but hyperbolic with respect to the real structure only when $a > 3/4$; see Table 11.4.3. The Lyness map interchanges its two irreducible components, and therefore the QRT map τ leaves these invariant. On the component $x+y+1 = 0$, parametrized by x, $\tau : (x, y) \mapsto ((y+a)/x, (y+a+a\,x)/x\,y)$ acts by sending x to $(-1-x+a)/x$. The two singular points of the curve, equal to the fixed points of τ, correspond to the x such that $x^2+x+1-a = 0$, and at these points the derivative of $x \mapsto (-1-x+a)/x$ is equal to $(1-a)/x^2$. It follows, in the notation of Remark 8.4.3, that $\tau(A_i(\mathbb{R})^+) = A_i(\mathbb{R})^+$ if $3/4 < a < 1$, and $\tau(A_i(\mathbb{R})^+) = A_i(\mathbb{R})^-$ when $a > 1$. At both sides of $z = a - 1$ the real fibers have two connected components if $3/4 < a < 1$, and are connected when $a > 1$; see Table 11.4.3. Therefore Remark 8.4.3 implies, if $3/4 < a < 1$, that $\rho(a - 1 - 0) = \rho(a - 1 + 0) = 0$ or 1, where we conclude that $\rho(a - 1 - 0) = \rho(a - 1 + 0) = 1$, because $\rho(-0) = 2/3$

and $\rho(z) \neq 1/2$ for all regular z. On the other hand, if $a > 1$, then Remark 8.4.3 implies that $\rho(a-1-0) = \rho(a-1+0) = 1/2$.

We now determine from which side $\rho(z)$ approaches its limit value for $z \to a-1\pm 0$. On the component $x+y+1=0$, parametrized by x, the derivative of τ^2 with respect to x is equal to $((1-a)/x^2)^2$, which is < 1 or > 1 when $x^4-(1-a)^2 > 0$ or $x^4-(1-a)^2 < 0$, respectively. The Hamiltonian system on the same component is given by $\mathrm{d}x/\mathrm{d}t = -x^2 - x - 1 + a$, where the derivative of the right-hand side with respect to x is equal to $-2x-1$. The polynomial remainder of $(x^4-(1-a)^2)(-2x-1)$ after division by $x^2+x+1-a$ is equal to $(4a-3)(-x-1+a)$, which is equal to $(4a-3)x^2$ modulo $x^2+x+1-a=0$. Therefore, recalling that $a > 3/4$, we have at both fixed points, determined by the equation $x^2+x+1-a=0$, that the sign of $x^4-(1-a)^2$ is equal to the sign of $-2x-1$, and therefore τ^2 is contracting or expanding at the fixed points if and only if the flow of the Hamiltonian system for positive time is repelling or contracting, respectively. It follows that the time needed for the solution of the Hamiltonian system to go from a point on $x+y+1=0$ to its image point under τ^2 is negative, and therefore $\rho(z)$ approaches its limit value from below when $z \to a-1\pm 0$, whenever $a > 3/4$ and $a \neq 1$.

For a singular fiber of Kodaira type I_1, over $z=z_\pm$, when hyperbolic, we observe that the Lyness map λ might interchange $A_i(\mathbb{R})^+$ with $A_i(\mathbb{R})^-$, but then the QRT map $\tau = \lambda^2$ leaves $A_i(\mathbb{R})^+$ invariant. Therefore Remark 8.4.3 implies that $\rho(z_\pm - 0) = \rho(z_\pm + 0) = 0$ or 1. It follows from Table 11.4.3 that we only have a hyperbolic I_1 at $z=z_-$ when $-1/4 < a < 0$ or at $z=z_-$ when $0 < a < 3/4$. In the first case $z=z_-$ is adjacent to $z=+\infty$, and we conclude from $\rho(+\infty)=1/5$ and the fact that ρ cannot cross $1/2$ that $\rho(z_- - 0) = \rho(z_- + 0) = 0$. In the second case, $z=z_-$ is adjacent to $z=-\infty$, and we conclude from $\rho(-\infty)=3/5$ and the fact that ρ cannot cross $1/2$ that $\rho(z_- - 0) = \rho(z_- + 0) = 1$. Because τ has no fixed points on the regular part of the curve, the approach of $\rho(z)$ to its limit value is infinitely steep. □

11.4.0.7 Monotonicity of the Rotation Function

As a function of z, the Manin function $\mu = \mathcal{L}T$ of the QRT map, see Proposition 2.5.20, is equal to

$$\mu(z) = 2a(a-1)((2a+1)z - (a-2)(2a-1))/E(z), \quad \text{where}$$
$$E(z) = -z(z-(a-1))Q(z)(5az + 3(a-1)(a-2)),$$

and $Q(z)$ is equal to second-order factor in $\Delta(z)$ given by (11.4.6), whose two zeros $z_\pm$ given by (11.4.7) correspond to the singular fibers of type I_1. Our $\mu(z)$ is exactly 2 times the function $v(h)$, where $h=z$, found by Beukers and Cushman [16, (17)], where their Δ is equal to our E. The explanation of the factor 2 is that the transformation in [16] is the QRT root. It follows from Proposition 2.5.20 that the Lyness map λ has finite order if and only if $\mu \equiv 0$, which happens if and only if $a=0$ or $a=1$. We have seen before that λ has order 6 and 5 when $a=0$ and $a=1$, respectively.

Because

$$\Gamma = (2/3)\, z^2\, (z - (a-1))\, (5\, a\, z + 3\, (a-1)\, (a-2)) \quad \text{and}$$
$$\Delta = z^3\, (z - (a-1))^2\, Q(z),$$

we recognize the last factor in the denominator of μ as the only factor in Γ that is not a factor in Δ. We also note that

$$P = 3\, \Gamma/2\, \Delta = (5\, a\, z + 3\, (a-1)\, (a-2))/z\, (z - (a-1))\, Q(z) \qquad (11.4.8)$$

has simple poles at $z = 0$, $z = a - 1$, and the two zeros of Δ that correspond to the singular fibers of type I_3, I_2, and I_1, respectively. The singular fiber of type I_5 lies over $z = \infty$, and because the degree of the numerator of μ is equal to the degree of the denominator minus 4, we have $m = 1$ in Proposition 2.5.20. P has a simple zero at

$$z = z_P := -3\, (a-1)\, (a-2)/5\, a,$$

where $J' = 0$, and the numerator of μ has a simple zero at

$$z = z_\mu := (a-2)\, (2\, a - 1)/(2\, a + 1).$$

In order to apply (2.6.8), we determine the location of z_P and z_μ relative to the zeros of Δ, the $z \in \mathbb{R}$ over which the fibers are singular; see Table 11.4.3. The results are listed in Table 11.4.5.

Table 11.4.5

$a > 2$	$-\infty < z_P < 0 < z_- < z_\mu < a - 1 < z_+ < \infty$
$1 < a < 2$	$-\infty < z_\mu < z_- < 0 < z_P < a - 1 < z_+ < \infty$
$3/4 < a < 1$	$-\infty < z_\mu < z_- < a - 1 < z_P < 0 < z_+ < \infty$
$1/2 < a < 3/4$	$-\infty < z_P < z_- < a - 1 < z_\mu < 0 < z_+ < \infty$
$0 < a < 1/2$	$-\infty < z_P < z_- < a - 1 < 0 < z_\mu < z_+ < \infty$
$-1/4 < a < 0$	$-\infty < a - 1 < 0 < z_\mu < z_+ < z_- < z_P < \infty$
$-1/2 < a < -1/4$	$-\infty < a - 1 < 0 < z_P < z_\mu < \infty$
$a < -1/2$	$-\infty < z_\mu < a - 1 < 0 < z_P < \infty$

In the determination of the position of z_P and z_μ relative to the zeros $z_\pm$ of Q, we have used that $Q(z_P) = (a-2)\, (4\, a - 3)\, (4\, a + 1)^2/25\, a$ and $Q(z_\mu) = -(a - 2)\, (4\, a - 3)\, (4\, a + 1)^2/(2\, a + 1)^2$. Because the coefficient of z^2 in $Q(z)$ is equal to a, it follows that z lies between z_- and z_+ if and only if $a\, Q(z) < 0$.

Figures 11.4.14 and 11.4.15 In Figure 11.4.14 we have plotted the horizontal coordinate $\zeta = (2/\pi)\, \arctan(z)$ as a function of the vertical coordinate a, where $z = 0$, $z = a - 1$, $z = z_-$, $z = z_+$, $z = z_P$, and $z = z_\mu$. Note that $\zeta = \pm 1$ corresponds to $z = \pm\infty$, over which we have the singular fiber of type I_5, whereas over $z = 0$, $z = a_1$, $z = z_-$, and $z = z_+$ we have the singular fiber of type I_3, I_2, I_1 and I_1, respectively. This plot is a graphic representation of Table 11.4.5. If follows from (11.4.7) that the two zeros z_+ and z_- of Q come together in an ordinary cusp at $a = -1/4$, $z = 27/4$. Because in Figure 11.4.14 the lower leg of the cusp is dif-

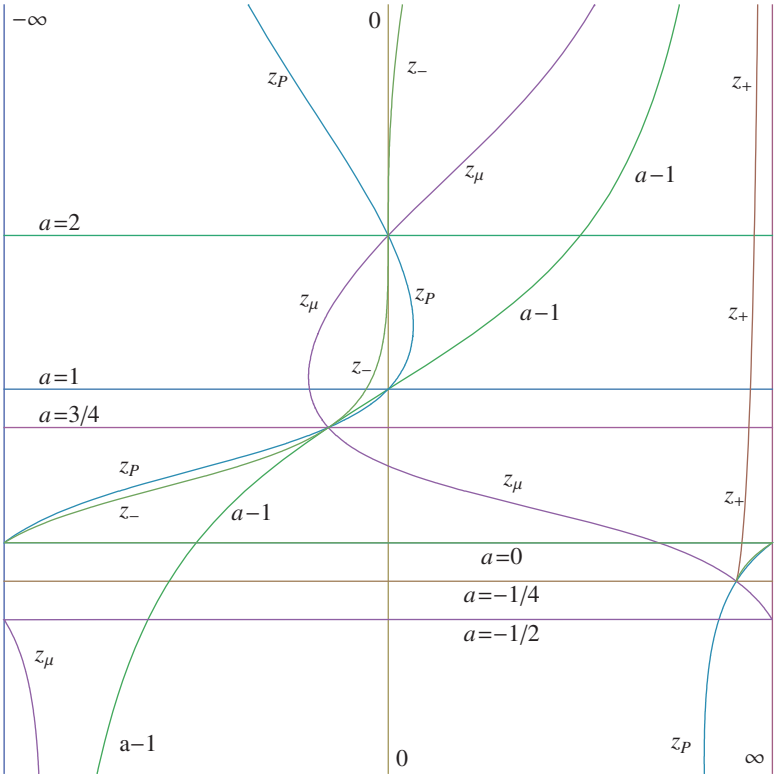


Fig. 11.4.14 Bifurcation diagram

ficult to distinguish from $z = z_P$, we have added an enlargement of the cusp in Figure 11.4.15.

Let $a > 0$. Then it follows from Table 11.4.3 that the singular curve for $z = z_+$ is elliptic with respect to the real structure. On the other hand, Table 11.4.5 shows that there are no zeros of Δ, P, or μ in $]\,z_+,\ \infty[$. Furthermore,

$$\mu(z) \sim 2\,a\,(a-1)\,(2\,a+1)\,z/(-5\,a^2\,z^5), \quad z \to \infty,$$

shows that for $z > z_+$, we have $\mu(z) < 0$ when $a > 1$ and $\mu(z) > 0$ when $0 < a < 1$. Therefore the Beukers–Cushman monotonicity criterion of Section 2.6.3 implies that $\rho'(z) < 0$ for all $z > z_+$ when $a > 1$, and $\rho'(z) > 0$ for all $z > z_+$ when $0 < a < 1$. This is the main result of Beukers and Cushman [16], where no knowledge has been used about the limit behavior of $\rho(z)$ for $z \downarrow z_+$ or $z \uparrow \infty$. The statement that ρ is strictly decreasing and increasing on $]z_+^{\mathrm{ell}},\ \infty[$ when $a > 1$ and $0 < a < 1$, respectively, was Conjecture 1 of Zeeman [214].

However, using the information of Table 11.4.4, the sign behavior of the derivative of the rotation function of the QRT map can be determined on all the intervals on which the rotation function is analytic, and for all values of the parameter a.

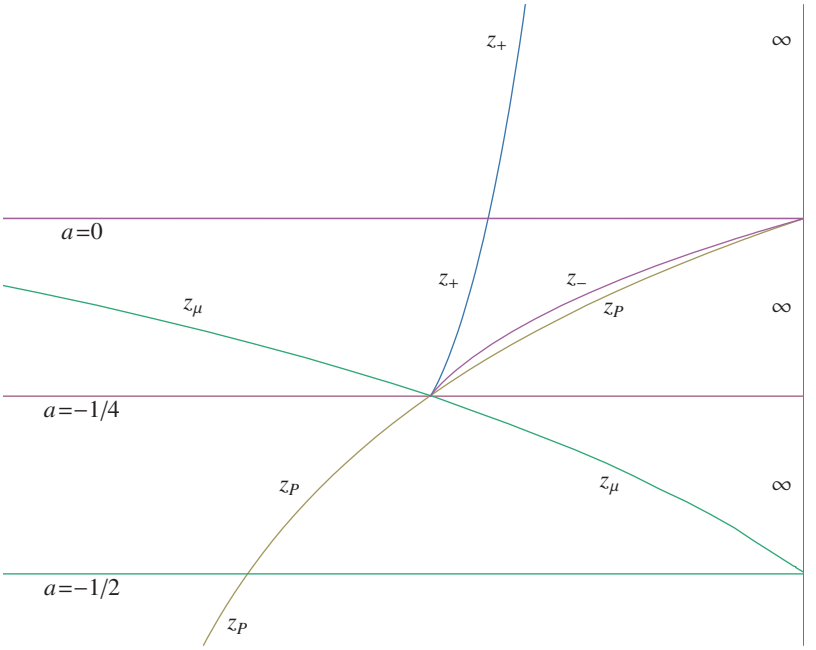


Fig. 11.4.15 Enlargement near $a = -1/4$, $z = 27/4$

Proposition 11.4.6 *If $a = 1$, then the rotation function of the QRT map is constant, equal to $4/5$ and $2/5$ on $]-\infty, 0[$ and $]0, \infty[$, respectively. If $a = 0$, then the rotation function of the QRT map is constant, equal to $2/3$ and $1/3$ on $]-\infty, 0[$ and $]0, \infty[$, respectively.*

If $-1 < a < -1/4$, then the derivative of the rotation function has a simple zero in $]0, \infty[$. If $a < -1$, then the derivative of the rotation function has a simple zero in $]-\infty, 0[$.

For every value of $a \in \mathbb{R}$, there are no other zeros of the derivative of the rotation function in any of the maximal open intervals on which the rotation function is real analytic.

Proof. It has already been observed that the Lyness map has order 5 and 6 when $a = 1$ and $a = 0$, respectively. Therefore the QRT map has order 5 and 3 in these cases, which implies that the rotation function is constant on each maximal open interval on which it is real analytic. The values on these intervals follow from the proof of Table 11.4.4.

Assume that a is not equal to one of the bifurcation values $-1/2$, $-1/4$, 0, $1/2$, $3/4$, 1, and 2, when the zeros of Δ, z_P, and z_μ are all different from each other, and the zeros of Δ are simple. The zeros of Δ that correspond to the hyperbolic singular fibers are the boundary points of the maximal open intervals I on which the rotation function is real analytic. Let $Z = \Delta^{-1}(\{0\}) \cup \{z_P\}$.

We combine the equation (2.6.8), which takes the form

$$\mathrm{d}(\sigma(z)/P(z))/\,\mathrm{d}z = -2\,p(z)\,(a-1)\,(2\,a+1)\,(z-z_\mu)/25\,a\,(z-z_P)^2, \quad (11.4.9)$$

with the information that can be read off from Table 11.4.4 about the sign of the derivative ρ' of the rotation function near the boundary points of the intervals I. Because the period function $p(z)$ is a strictly positive real analytic function on I, the right-hand side of (11.4.9) has a simple zero and a double pole at z_μ and z_P, respectively, if these points lie in I, and no other zeros of poles.

It follows from (11.4.8) and $\sigma(z) = \rho'(z)\,p(z)^2$, see (2.6.7), that

$$\sigma(z)/P(z) = \rho'(z)\,p(z)^2\,z\,(z-(a-1))\,Q(z)/5\,a\,(z-z_P). \quad (11.4.10)$$

In combination with (11.4.9) this implies that the function σ/P has simple zeros at the zeros of Δ, and a simple pole at z_P, if these points lie in I. This in turn implies that $\rho'(z) \neq 0$ when $z \in I \cap Z$. Furthermore, if $z \in I \setminus Z$ and $z \neq z_\mu$, then σ/P, hence ρ', has at most a simple zero at z. If $z_\mu \in I$, then $z_\mu \notin Z$, and either $\rho'(z_\mu) \neq 0$ or $\rho'(z)$ has a double zero at $z = z_\mu$.

It follows from (11.4.10) that the sign ± 1 of $\sigma(z)/P(z)$ is, modulo $2\,\mathbb{Z}$, equal to the sign of $\rho'(z)$ plus the number of elements of Z to the right of z, where we note that the leading coefficient of $Q(z)/a$ is equal to 1. It follows from (11.4.9) that the sign change of σ/P at the pole z_P is equal to the sign of $(a-1)\,a\,(2\,a+1)\,(z_P - z_\mu)$, whereas the sign change of σ/P at a zero $z_0 \in I$ of Δ is equal to the sign of $-(a-1)\,a\,(2\,a+1)\,(z_0 - z_\mu)$. The sign of $(\sigma/P)''(z_\mu)$ is equal to the sign of $-(a-1)\,a\,(2\,a+2)$. This leads to the following information about the sign behavior of the function σ/P. The plus or minus sign refers to the sign of σ/P near the point of Z adjacent to it, whereas $\nearrow z_\mu \nwarrow$ and $\searrow z_\mu \swarrow$ mean that $(\sigma/P)''(z_\mu) < 0$ and $(\sigma/P)''(z_\mu) > 0$, respectively.

Table 11.4.7

$a > 2$	$-\infty^{\mathrm{hyp}} + + z_P - - 0^{\mathrm{hyp}} - - z_-{}^{\mathrm{ell}} + \nearrow z_\mu \nwarrow + (a-1)^{\mathrm{hyp}} + + z_+{}^{\mathrm{ell}} - - \infty^{\mathrm{hyp}}$
$a = 2$	$-\infty^{\mathrm{hyp}} + + 0^{\mathrm{IV}} + + (a-1)^{\mathrm{hyp}} + + z_+{}^{\mathrm{ell}} - - \infty^{\mathrm{hyp}}$
$1 < a < 2$	$-\infty^{\mathrm{hyp}} + \nearrow z_\mu \nwarrow + z_-{}^{\mathrm{ell}} - - 0^{\mathrm{hyp}} - - z_P + + (a-1)^{\mathrm{hyp}} + + z_+{}^{\mathrm{ell}} - - \infty^{\mathrm{hyp}}$
$3/4 < a < 1$	$-\infty^{\mathrm{hyp}} - \searrow z_\mu \swarrow - z_-{}^{\mathrm{ell}} + + (a-1)^{\mathrm{hyp}} + + z_P - - 0^{\mathrm{hyp}} - - z_+{}^{\mathrm{ell}} + + \infty^{\mathrm{hyp}}$
$a = 3/4$	$-\infty^{\mathrm{hyp}} - - (a-1)^{\mathrm{III}} - - 0^{\mathrm{hyp}} - - z_+{}^{\mathrm{ell}} + + \infty^{\mathrm{hyp}}$
$1/2 < a < 3/4$	$-\infty^{\mathrm{hyp}} - - z_P + + z_-{}^{\mathrm{hyp}} + + (a-1)^{\mathrm{ell}} - \searrow z_\mu \swarrow - 0^{\mathrm{hyp}} - - z_+{}^{\mathrm{ell}} + + \infty^{\mathrm{hyp}}$
$a = 1/2$	$-\infty^{\mathrm{hyp}} - - z_P + + z_-{}^{\mathrm{hyp}} + + (a-1)^{\mathrm{ell}} - - 0^{\mathrm{hyp}} - - z_+{}^{\mathrm{ell}} + + \infty^{\mathrm{hyp}}$
$0 < a < 1/2$	$-\infty^{\mathrm{hyp}} - - z_P + + z_-{}^{\mathrm{hyp}} + + (a-1)^{\mathrm{ell}} - - 0^{\mathrm{hyp}} - \searrow z_\mu \swarrow - z_+{}^{\mathrm{ell}} + + \infty^{\mathrm{hyp}}$
$-1/4 < a < 0$	$-\infty^{\mathrm{hyp}} - - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{hyp}} + \nearrow z_\mu \nwarrow + z_+{}^{\mathrm{ell}} - - z_-{}^{\mathrm{hyp}} - - z_P + + \infty^{\mathrm{hyp}}$
$-1/2 < a < -1/4$	$-\infty - - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{ell}} + + z_P - \nearrow z_\mu \nwarrow + \infty$
$a = -1/2$	$-\infty^{\mathrm{hyp}} - - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{hyp}} + + z_P - + \infty^{\mathrm{hyp}}$
$-1 < a < -1/2$	$-\infty^{\mathrm{hyp}} - \searrow z_\mu \swarrow - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{hyp}} + + z_P - + \infty^{\mathrm{hyp}}$
$a = -1$	$-\infty^{\mathrm{hyp}} - \searrow z_\mu \swarrow - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{hyp}} + + z_P - - \infty^{\mathrm{hyp}}$
$a < -1$	$-\infty^{\mathrm{hyp}} + \searrow z_\mu \swarrow - (a-1)^{\mathrm{ell}} + + 0^{\mathrm{hyp}} + + z_P - - \infty^{\mathrm{hyp}}$

For $a = 2$ we have $z_P = 0 = z_- = z_\mu$, over which the fiber is singular of type IV, where $\rho(-0) = 2/3$, $\rho(+0) = 1/3$. The information near this point can be obtained by means of continuity arguments from the information for $a > 2$ and for $1 < a < 2$. For $a = 1$ we have the bifurcation $z_- = 0 = z_P = a - 1$, over which the fiber is singular of type I_5. Because the derivative of the rotation function is identically zero, we have not inserted this case in the table.

For $a = 3/4$ we have $z_\mu = z_- = a - 1 = z_P$, over which the fiber is singular of type III, where $\rho(-0) = \rho(+0) = 1$. The information near this point can be obtained by means of continuity arguments from the information for $3/4 < a < 1$ and for $1/2 < a < 3/4$.

For $a = 1/2$ we have the rather innocuous bifurcation that z_μ passes through 0. For $a = 0$ one of the zeros of Q has run to ∞, as well as z_P, and we have a singular fiber over ∞ of type I_6. Because the derivative of the rotation function is identically zero, we have not inserted this case in the table.

For $a = -1/4$ we have $z_\mu = z_- = z_+ = z_P$, over which the fiber is singular of type II, and the rotation function goes down to 0, as follows from a continuity argument for $a \downarrow -1/4$ and $a \uparrow -1/4$.

For $a = -1/2$ we have the rather innocuous bifurcation that z_μ passes for decreasing a from very large positive to very large negative. When a passes downward through -1, the direction from which the rotation function approaches its limit value at $\pm\infty$ is reversed, from $\rho'(z) > 0$ for large $|z|$ to $\rho'(z) < 0$ for large $|z|$. This forces the point $z = z_0(a)$ where $\rho'(z) =$ to move from very large positive to very large negative values.

Note that the derivative of σ/P is nonzero, and therefore σ/P is strictly monotonous, on every interval between the points indicated in Table 11.4.7. It follows that the only zeros $z = z_0(a)$ of ρ' occur for $-1 < a < -1/4$ and for $a < -1$, when $z_0(a) \in]0,\ \infty[$ and $z_0(a) \in]-\infty,\ 0[$, respectively. More precisely, $z_0(a) \in]z_P,\ z_\mu[$, $z_0(a) \in]z_P,\ \infty[$, and $z_0(a) \in]-\infty,\ z_\mu[$ when $-1/2 < a < -1/4, -1 < a \le -1/2$, and $a < -1$, respectively. □

The combination of Tables 11.4.3, 11.4.4, 11.4.5 with Proposition 11.4.6 leads to a very detailed qualitative description of the rotation function of the Lyness map. The following sequence of plots of the rotation function represents all the cases. If $a = 1$, when the Lyness map and the QRT map have period 5, the rotation function $\rho(z)$ of the QRT map is constant, equal to $2/5$ and $4/5$ for $z > 0$ and $z < 0$, respectively. If $a = 0$, when the Lyness map has order 6 and the QRT map has order 3, the rotation function $\rho(z)$ is constant, equal to $1/3$ and $2/3$ for $z > 0$ and $z < 0$, respectively.

Remark 11.4.8. Table 11.4.4 and Proposition 11.4.6 lead to the following conclusions about the total variation $V(\rho)$ of the rotation function, see (8.4.2).

If $a > 1$, then $V(\rho) = 2/5$.
If $0 < a < 1$, then $V(\rho) = 3/5$.
If $-1/4 \le a < 0$, then $V(\rho) = 3/5$.
If $-1 < a < -1/4$, then $1/5 < V(\rho) < 3/5$.
If $a = -1$, then $V(\rho) = 1/5$.
If $a < -1$, then $1/5 < V(\rho) < 2/5$.

Here I have used a continuity argument for $a = 2$. Let $a < -1/4$, $a \neq -1$, and let $z_0(a)$ be the unique z such that $\rho'(z) = 0$. Then $V(\rho) = 3/5 - 2\,\rho(z_0(a))$ if $-1 < a < -1/4$, and $V(\rho) = 7/5 - 2\,\rho(z_0(a))$ if $a < -1$. If $-1 < a < -1/4$, then $0 < \rho(z_0(a)) < 1/5$; hence $1/5 < V(\rho) < 3/5$. Because always $\rho(z) \neq 1/2$, we have $1/2 < \rho(z_0(a)) < 3/5$, hence $1/5 < V(\rho) < 2/5$ if $a < -1$. I conjecture

that not only $z_0(a) \to -\infty$, but also $\rho(z_0(a)) \to 3/5$, hence $V(\rho) \to 1/5$, as $a \to -\infty$. I have no explict formula for the critical value $\rho(z_0(a))$ of the rotation function. ⊘

Remark 11.4.9. When $a \downarrow -1/4$, the two zeros z_- and z_+ of Q merge to a double zero of Δ at $z = 27/4$, over which we have a singular fiber of Kodaira type II, that is, with a cusp singularity. Note that over z_+ and z_- we had a singular fiber of type I_1 that respectively is elliptic and hyperbolic with respect to the real structure; see Table 11.4.3. It follows from (8.2.4) that if $a = -1/4$, the real period has a $-1/6$ power law, with coefficients in front of the power law that differ by a factor $\sqrt{3}$ or $1/\sqrt{3}$ when approaching $z = 27/4$ from either side. Because for $a > -1/4$ the rotation function goes down to zero in an infinitely steep way when approaching z_-, whereas it is real analytic over $z = z_+$, it is natural to conjecture that at $a = -1/4$ the rotation function has a sixth root behavior when $z \to 27/4$, where the coefficient when approaching from the left is $\sqrt{3}$ times the coefficient when approaching from the right. The computer plot in Figure 11.4.10 appears to confirm this conjecture.

After a has passed the value $-1/4$ in the negative direction, the two zeros of Q have disappeared from the real z-axis. Because the rotation function has a minimum value 0 at $z = 27/4$ when $a = -1/4$, it follows by continuity that for $a < -1/4$ and a close to $-1/4$, the rotation function has a minimum at a point z_0 near $z = 27/4$, where $\rho(z_0)$ is close to 0. Note that at $z = z_0$ we have $\rho'(z) = 0$. The bifurcation argument was the first proof that I had of the existence of a point where the derivative of the rotation function of a QRT map is equal to zero, which in turn implies that the set D introduced in Lemma 7.7.3 is not empty; see Remark 7.7.14. My attention had been drawn to this bifurcation by a computer plot of the rotation map for $a = -0.35$, made by Jaap Eldering.

This existence proof for a zero of ρ' works only for $a < -1/4$ that are close to $-1/4$, whereas Proposition 11.4.6 gave a unique zero $z = z_0(a)$ of ρ' for every $a < -1/4$ with $a \neq -1$. The bifurcation argument shows that $z_0(a) \to 27/4$ and $\rho(z_0(a)) \to 0$ as $a \uparrow -1/4$. Actually, Table 11.4.7 yields that $z_P(a) < z_0(a) < z_\mu(a)$ when $-1/2 < a < -1/4$, which also implies that $z_0(a) \to 27/4$ as $a \uparrow -1/4$, because $z_P(a)$ and $z_\mu(a)$ both converge to $27/4$ when $a \uparrow -1/4$.

This leads to the following description of the zero of ρ' as a function of a. When a passes in the negative direction through $a = -1/4$, $z_0(a)$ bifurcates from the point $z = 27/4$ where $z_-(a)$ and $z_+(a)$ merged when $a \downarrow -1/4$, and disappeared when $a < -1/4$. When a passes in the negative direction through $a = -1$, $z_0(a)$ runs to ∞ and reappears on the negative z-axis with $z_0(a) \to -\infty$ as $a \uparrow -1$. Recall that over $z = \pm\infty$ we have a hyperbolic singular fiber of type I_5. Because it follows from Table 11.4.7 that $z_0(a) < z_\mu(a)$ when $a < -1$, and $z_\mu(a) \to -\infty$ as $a \to -\infty$, we have that $z_0(a) \to -\infty$ as $a \to -\infty$. We have $z_\mu(a) < -12$, hence $z_0(a) < -12$ for all $a < -1$. ⊘

Figures 11.4.16, 11.4.17, and 11.4.18 show the points in $\mathbb{P}^1(\mathbb{C})$ over which the Lyness map has period 200, for $a = -1/5$, $a = -1/4$, and $a = -1/3$, respectively. The points are shown in a neighborhood, close to the point $[z_0 : z_1] = [0 : 1]$ at infinity, of the point over which two singular fibers of type I_1 merge to a singular

fiber of type II over $z := -z_1/z_0 = 27/4$ when $a \to -1/4$. For the asymptotic behavior of the set of points for large periods, see Remark 7.7.11. Each picture shows the approximate circles around the singular point at infinity over which we have a hyperbolic singular fiber of Kodaira type I_5. The smallest approximate circle around it that is still visible contains 10 points. Since each larger circle contains five more points, there actually is one more, much smaller, approximate circle around $z = 0$ that contains five points. When $a = -1.5$, we have two singular points on the real z-axis near $z = 27/4$, the right one of which is hyperbolic of type I_1, and the one to the left is elliptic of type I_1. Around the right one we have small approximate circles, on which the numbers of points differ by 1, whereas around the left one the density of points tends to zero, which makes this elliptic singular point difficult to identify. Around both singular points the monodromy matrix is conjugate to the one in Table 6.2.40 for type I_b with $b = 1$. In Figure 11.4.17 the two singular points have merged to a single one at $z = 27/4$, over which we have a singular fiber of type II, with monodromy matrix conjugate to the one in Table 6.2.40 for type II, which has order 6. In Figure 11.4.18 the double zero of Δ has split into two nearby complex conjugate zeros of Δ, over each of which we have a singular fiber of Kodaira type I_1. These two complex conjugate singular points are just visible in Figure 11.4.18. Nearby on the real axis, there is a zero $z = \zeta$ of the derivative of the rotation function. According to the last statement in Lemma 8.1.5, ζ is a point of the set D mentioned in Lemma 7.7.7, where Figure 7.7.1 shows the prototypical low-density asymptotic behavior for large periods around points of D. It appears that the period in Figure 11.4.18 is not large enough to show the behavior of Figure 7.7.1.

11.4.0.8 The Rotation Number on the Elliptic Singular Fibers

It follows from Table 11.4.3 that for $a > 3/4$ we have two points $z = z_\pm$ over which the fiber is singular of elliptic type I_1. When $0 < a < 3/4$, the I_1 fiber over z_- becomes hyperbolic, whereas the I_2 fiber over $a - 1$ has become elliptic, and stays that way for all $a < 3/4$. The I_1 fiber over z_+ stays elliptic for all $a > -1/4$ even after a has passed downward through 0, when the hyperbolic I_1 fiber over $z = z_-$ has moved from z_- large negative to z_- large positive. For $a \le -1/4$ the only elliptic fiber is the I_2 fiber over $a - 1$.

In each case, (8.4.1) leads to an explicit formula for the limit value of the rotation number $\rho(z)$ when $z \to z^{\mathrm{ell}}$ and the fiber over z^{ell} is elliptic. Let $a > -1/4$, $w^2 = 4\,a + 1$, be such that

$$z^{\mathrm{ell}} = z_+^{\mathrm{ell}} = (a^2 + 10\,a - 1 + w^3)/2\,a = (w + 3)^3/4\,(w + 1)$$

is the zero of Q in (11.4.7) with the plus sign, over which we have an elliptic singular fiber of type I_1. Substituting $z_0 = 1$ and $z_1 = -z^{\mathrm{ell}}$ in the formulas for g_2, X, and Y, we obtain $g_2 = w^2\,(w + 2)^2\,(w + 3)^4/12\,(w + 1)^4$, $X = (w + 3)^2\,(2\,w^2 + 4\,w + 3)/12\,(w + 1)^2$, and $Y = -(w + 3)^3/4\,(w + 1)$. Substituting this into (8.4.1) leads to the formula

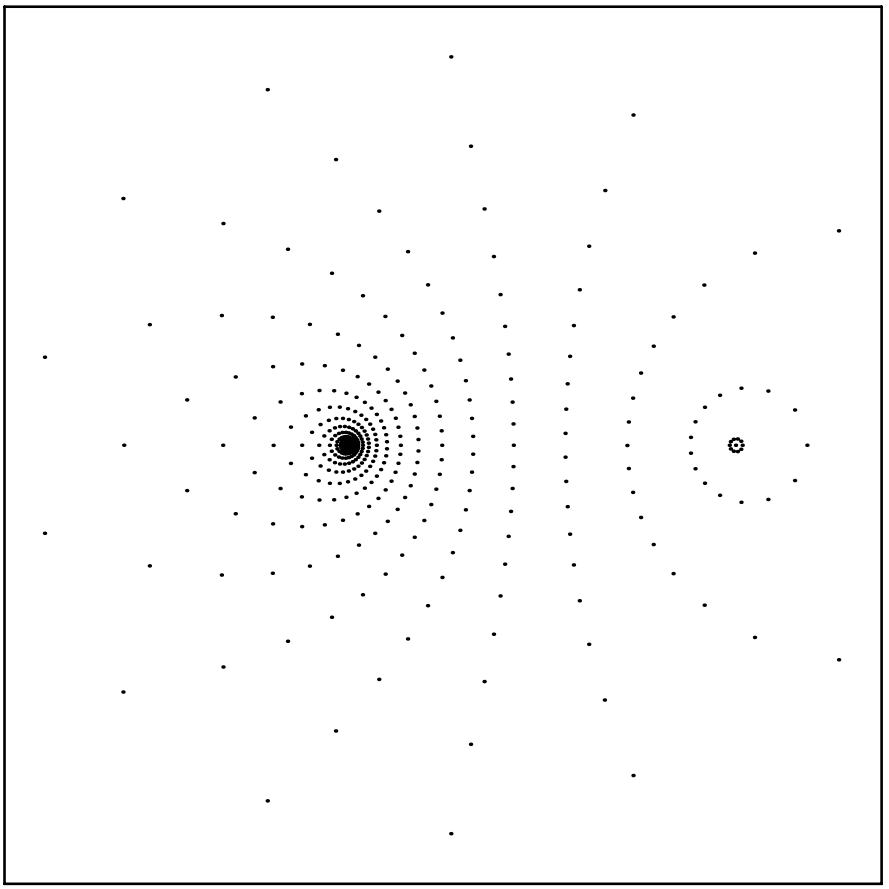


Fig. 11.4.16 Points in $\mathbb{P}^1(\mathbb{C})$ over which the Lyness map with $a = -1/5$ has period 200.

$$\rho(z_+^{\mathrm{ell}}) = \frac{1}{\pi}\,\arccos\left(\frac{1}{w+1}\right), \quad w = \sqrt{4\,a+1}, \tag{11.4.11}$$

for the rotation number of the QRT map on the elliptic fiber over z_+^{ell} when $a > -1/4$. If $z \downarrow -1/4$, when the elliptic I_1 fiber merges with the hyperbolic I_1 fiber over z_-, the rotation number goes down to zero, which is the limit value of the rotation number at the hyperbolic I_1 fiber. Because the QRT map is the square of the Lyness map, the formula (11.4.11) implies the formula $(1/2\pi)\,\arccos(1/(w+1))$ for the limit for $z \downarrow z_+^{\mathrm{ell}}$ of the rotation number of the Lyness map, which for $a > 0$ is Theorem 7 of Zeeman [214]. It is always very satisfactory to observe an agreement between formulas that have been obtained in completely different manners.

On the other hand, if $w = -\sqrt{4\,a+1}$, then we have

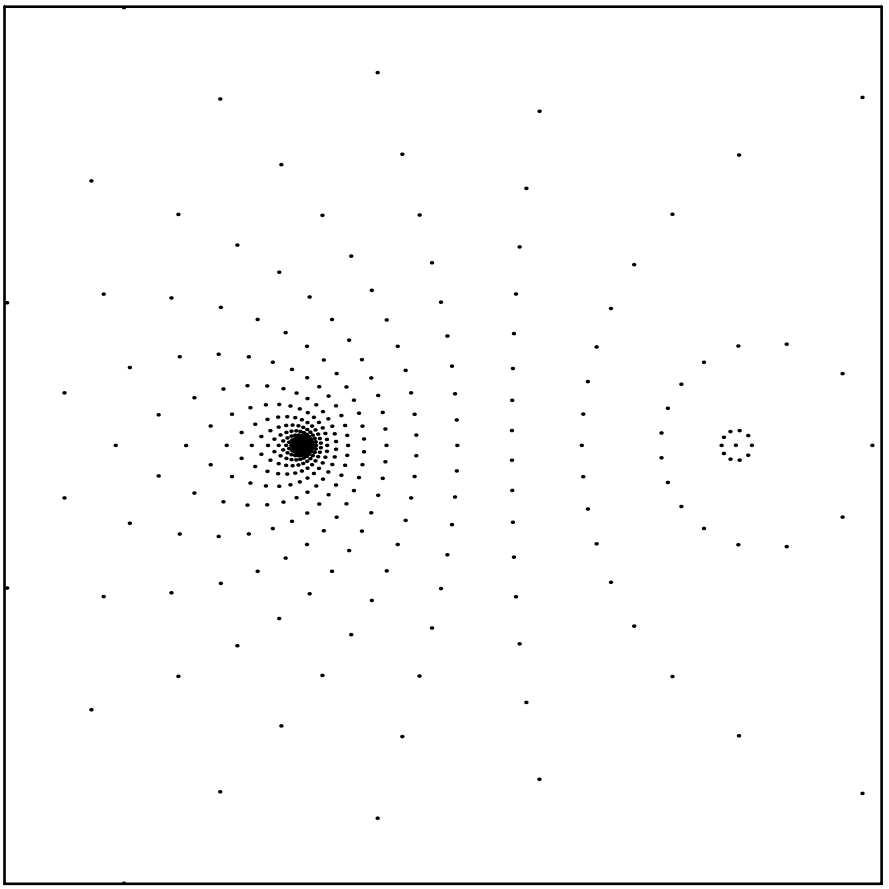


Fig. 11.4.17 Points in $\mathbb{P}^1(\mathbb{C})$ over which the Lyness map with $a = -1/4$ has period 200.

$$\rho(z_-^{\text{ell}}) = \begin{cases} \frac{1}{\pi} \arccos \frac{-1}{w+1} \text{ if } & a > 2, \\ \\ \frac{1}{\pi} \arccos \frac{1}{w+1} \text{ if } & 3/4 < a < 2, \end{cases} \tag{11.4.12}$$

where we note that the fiber over z_- is hyperbolic if $-1/4 < a < 3/4$.

It follows from Table 11.4.3 that we also have an elliptic fiber of Kodaira type I_2 over z if $a < 3/4$. Because $g_2 = (4\,a - 3)^2/12$, $X = (9 - 8\,a)/12$, and $Y = 1 - a$ when $z = a - 1$, the formula (8.4.1) implies that

$$\rho((a-1)^{\text{ell}}) = \frac{1}{\pi} \arccos \frac{-1}{2\,(1-a)^{1/2}}, \quad a < 3/4. \tag{11.4.13}$$

Figure 11.4.19 is a combined plot of (11.4.11), (11.4.12), and (11.4.13), the rotation numbers at the elliptic fibers, viewed as functions of the parameter a. In the picture

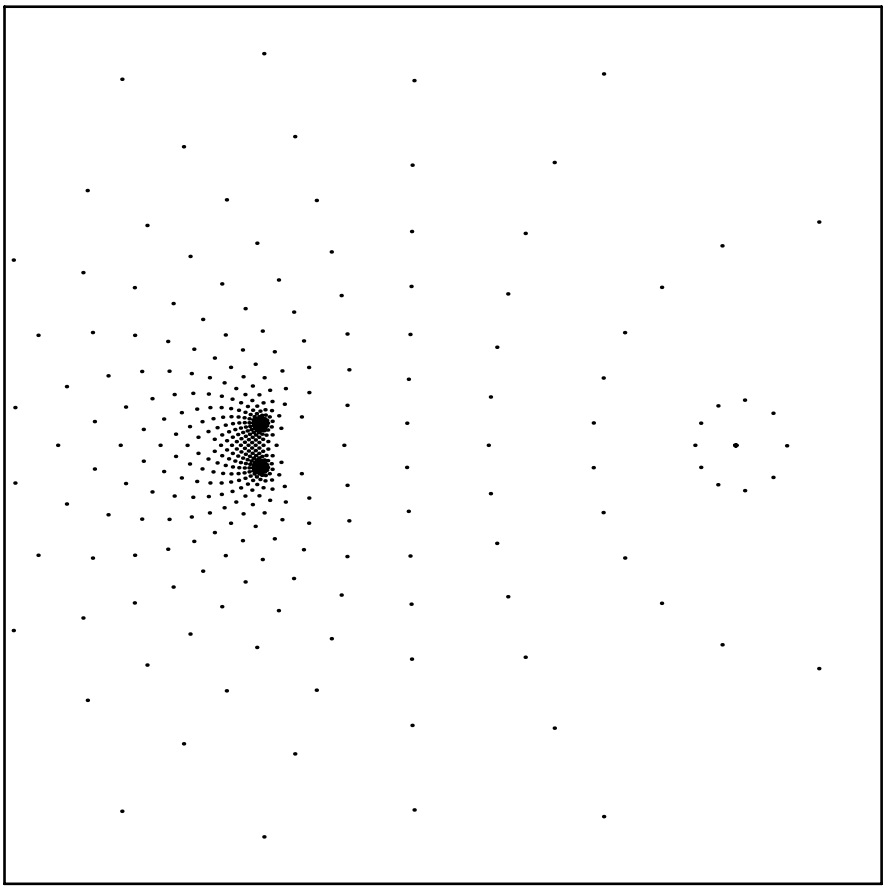


Fig. 11.4.18 Points in $\mathbb{P}^1(\mathbb{C})$ over which the Lyness map with $a = -1/3$ has period 200.

we have marked the bifurcation values $a = -1/4$, $a = 0$, $a = 3/4$, $a = 1$, and $a = 2$ as vertical lines, and also, as horizontal lines, the limit values 0, 1/3, 1/2, 3/5, 2/3, 4/5, and 1 of the rotation numbers.

When a goes downward through 2, the point z_-^{ell} passes 0 from positive to negative, and the values $\rho(z_-^{\mathrm{ell}}) \to 1/3$ for $a \downarrow 2$ and $\rho(z_-^{\mathrm{ell}}) \to 2/3$ for $a \uparrow 2$ agree with $\rho(+0) = 1/3$ and $\rho(-0) = 2/3$; see Table 11.4.4. These values correspond to $\cos(\pi/3) = 1/2$ and $\cos(2\pi/3) = -1/2$, where $w = -3$ in (11.4.12) when $a = 2$.

When $a = 1$, the values $\rho(z_+^{\mathrm{ell}}) = 2/5$ and $\rho(z_-^{\mathrm{ell}}) = 4/5$ follow from the fact that $\rho(z)$ is constant, equal to 2/5 and 4/5 for $z > 0$ and $z < 0$, respectively. These values correspond to $\cos(2\pi/5) = 1/(\sqrt{5}+1) = (\sqrt{5}-1)/4$ and $\cos(4\pi/5) = 1/(-\sqrt{5}+1) = -(\sqrt{5}+1)/4$, where $w = \pm\sqrt{5}$ in (11.4.11) and (11.4.12) when $a = 1$.

When $a = 0$, the values $\rho(z_+^{\mathrm{ell}}) = 1/3$ and $\rho((a-1)^{\mathrm{ell}}) = 2/3$ follow from the fact that $\rho(z)$ is constant, equal to 1/3 and 2/3 for $z > 0$ and $z < 0$, respectively.

These values again correspond to $\cos(\pi/3) = 1/2$ and $\cos(2\pi/3) = -1/2$, where w and $1 - a$ both are equal 1 when $a = 0$.

When $a \to \infty$, both $\rho(z_+^{\mathrm{ell}})$ and $\rho(z_-^{\mathrm{ell}})$ increase to $1/2$, whereas $\rho((a-1)^{\mathrm{ell}})$ decreases to $1/2$ as $a \to -\infty$. In order to show these limits, Figure 11.4.19 should be extended to a much larger a-interval, but then the behavior for $-1/4 \le a \le 2$ would become less clearly visible.

For the location of the points z^{ell} over which we have an elliptic fiber, relative to the other points over which we have singular fibers, and for the limit values of the rotation function at these other singular points, see Tables 11.4.3 and 11.4.4, respectively.

11.5 The McMillan Map

McMillan's map [133] is the QRT root for a pencil of symmetric biquadratic curves where in affine coordinates $([1 : x],\ [1 : y])$ one of the biquadratic polynomials, say p^1, is equal to 1, when the invariant rational function p^0/p^1 a polynomial function of the affine coordinates $(x,\ y)$. In projective coordinates we have $p^1 = {x_0}^2\, {y_0}^2$, whose zero-set is equal to the union of the horizontal axis $\mathbb{P}^1 \times \{[0 : 1]\}$ and the vertical axis $\{[0 : 1]\} \times \mathbb{P}^1$ at infinity, each counted twice. We will also use the name McMillan map for any QRT map defined by a pencil of biquadratic curves with $p^1 = {x_0}^2\, {y_0}^2$, when the other biquadratic polynomial need not be symmetric.

Note that we are in case (7d) in Section 12.1, and therefore the fiber of the elliptic surface S which corresponds to the singular curve $p^1 = 0$ is a singular fiber of Kodaira type I_1^*, I_2^*, I_3^*, IV*, or III*, when the base points are as in the cases (7d1), (7d2), (7d3), (7d4), and (7d5), respectively. For symmetric McMillan maps, we have only the cases (7d1), (7d3), (7d4).

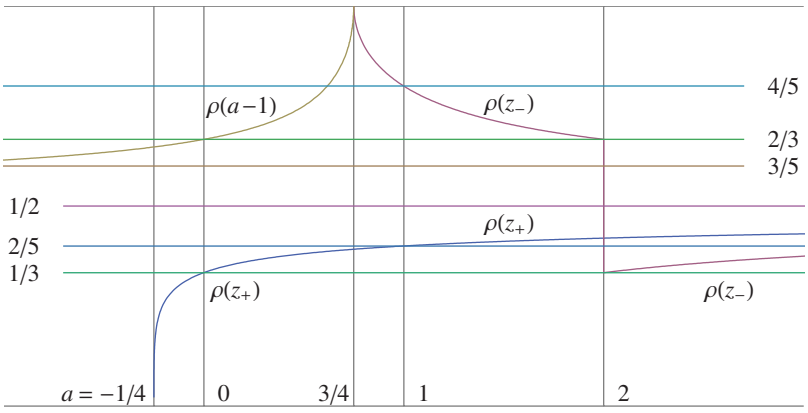


Fig. 11.4.19 The rotation number at the elliptic singular fibers, as a function of a.

In terms of the coefficient matrix in (2.4.1) the equation $p^1 = {x_0}^2\, {y_0}^2$ means that $A^1_{22} = 1$, whereas all other matrix coefficients of A^1 are equal to zero. Then (1.1.3), (1.1.4), (1.1.5) yield

$$\xi(x,\ y) = -x - (A^0_{10}\, y^2 + A^0_{11}\, y + A^0_{12})/(A^0_{00}\, y^2 + A^0_{01}\, y + A^0_{02}) \quad \text{and}$$
$$\eta(x,\ y) = -y - (A^0_{01}\, x^2 + A^0_{11}\, x + A^0_{21})/(A^0_{00}\, x^2 + A^0_{10}\, x + A^0_{20}).$$

In the symmetric case the QRT root is $(x,\ y) \mapsto (y,\ \xi(x,\ y))$, where

$$\xi(x,\ y) = -x - (h_0\, y^2 + h_1\, y + h_2)/(k_0\, y^2 + k_1\, y + k_2)$$

and the coefficients h_i, k_j satisfy $h_0 = k_1$. In the nonsymmetric case the QRT map is $\tau = \iota_2 \circ \iota_1$, where $\iota_1(x,\ y) = (\xi(x,\ y),\ y)$, $\iota_2(x,\ y) = (x,\ \eta(x,\ y))$,

$$\xi(x,\ y) = -x - (h_0\, y^2 + h_1\, y + h_2)/(k_0\, y^2 + k_1\, y + k_2),$$
$$\eta(x,\ y) = -y - (h'_0\, x^2 + h'_1\, x + h'_2)/(k'_0\, x^2 + k'_1\, x + k'_2),$$

where $h_0 = k'_1$, $h_1 = h'_1$, $k_0 = k'_0$, and $k_1 = h'_0$.

Because $\xi(x,\ y)$, hence ι_1, can easily be read off from τ, and $\eta(x,\ y)$ can easily be read off from $\iota_2 = \tau \circ \iota_1$, the test whether a map is a McMillan map and, if so, the computation of the pencil of biquadratic polynomials, are straightforward.

Remark 11.5.1. In a recent paper, Bastien and Rogalski [13] studied real McMillan maps. They also refer to Hone [89], who arrived at the subject coming from the theory of elliptic divisibility sequences. ⊘

11.5.1 The KdV Mapping

The DΔKdV mapping in Quispel, Roberts, and Thompson [169, p. 185], obtained by means of a reduction from a soliton equation for the Korteweg–de Vries equation, is

$$(x,\ y) \mapsto (y,\ -x - (\omega - y)/(2\, p^2\, y^2)),$$

where ω and p are constants and $p \neq 0$. The DΔKdV mapping is the QRT root for the pencil (2.5.3) of symmetric biquadratic curves with the coefficient matrices

$$A^0 = \begin{pmatrix} a & 0 & 0 \\ 0 & -1 & \omega \\ 0 & \omega & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = 2\, p^2$.

The Weierstrass invariants of the pencil of biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$
\begin{aligned}
12\,g_2 &= {z_0}^2\,((1-24\,a\,\omega^2)\,{z_0}^2 - 8\,a\,z_0\,z_1 + 16\,a^2\,{z_1}^2),\\
216\,g_3 &= {z_0}^3\,((-1+36\,a\,\omega^2 - 216\,a^2\,\omega^4)\,{z_0}^3\\
&\qquad + 12\,a\,(1-12\,a\,\omega^2)\,{z_0}^2\,z_1 - 48\,a^2\,z_0\,{z_1}^2 + 64\,a^3\,{z_1}^3),\\
\Delta &= a^3\,\omega^4\,{z_0}^9\,((\omega^2 - 27\,a\,\omega^4)\,{z_0}^3\\
&\qquad + (1-36\,a\,\omega^2)\,{z_0}^2\,z_1 - 8\,a\,z_0\,{z_1}^2 + 16\,a^2\,{z_1}^3),\\
12\,X &= z_0\,(z_0 + 8\,a\,z_1),\\
Y &= a\,{z_0}^2\,(\omega^2\,z_0 + z_1),\\
12\,X_\rho &= z_0\,(z_0 - 4\,a\,z_1),\\
Y_\rho &= -a\,\omega^2\,{z_0}^3.
\end{aligned}
$$

The pencil has at least one smooth member if and only if $p \neq 0$ and $\omega \neq 0$, which we assume in the sequel. Table 6.3.2 implies that there is a singular fiber of Kodaira type I_3^* over the point $[z_0 : z_1] = [0 : 1]$ at infinity. The discriminant of the last factor in Δ is equal to $256\,a^3\,\omega^2(27\,a\,\omega^2 - 2)^3$, and it follows that the configuration of the singular fibers is $\mathrm{I}_3^*\ 3\ \mathrm{I}_1$ if and only if $p \neq 0$, $\omega \neq 0$, and $27\,p^2\,\omega^2 \neq 1$. It follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}$ in this case. The modulus function J has degree $j = 6$, see (6.2.48).

If $27\,p^2\,\omega^2 = 1$, then two of the singular fibers of type I_1 have merged to a singular fiber of type II over $[z_0 : z_1] = [1 : -9\,\omega^2/8]$, with a remaining singular fiber of type I_1 over $[z_0 : z_1] = [1 : 9\,\omega^2]$. Therefore the configuration of singular fibers is $\mathrm{I}_3^*\ \mathrm{II}\ \mathrm{I}_1$, and according to Lemma 9.2.6, the Mordell–Weil group is again isomorphic to $\mathbb{Z}$. However, the modulus function J has degree $j = 4$ in this case; see (6.2.48).

The base points are ([0 : 1], [1 : 0]) and ([1 : 0], [0 : 1]), each of mulitplicity 4. Therefore the fiber in the elliptic surface S that is projected to ${x_0}^2\,{y_0}^2 = 0$ belongs to case (7d3) in Section 12.1, and we recover the fiber of Kodaira type I_3^*. The QRT root ρ interchanges at least two of the irreducible components of multiplicity two, and it follows from the description at the end of Section 6.3.6 that the action of ρ on the set of irreducible components is a generator of the group $\mathfrak{F}_c/{\mathfrak{F}_c}^\circ \simeq \mathbb{Z}/4\,\mathbb{Z}$. The other singular fibers are irreducible, and according to Lemma 7.5.1 the contribution of ρ^k is equal to $0, 7/4, 1$, and $7/4$ if k is equal to $0, 1, 2$, and 3 modulo 4, respectively. It therefore follows from (4.3.2) that the number of k-periodic fibers, counted with multiplicities, is equal to $-1 + k^2/8$, $-1/8 + k^2/8$, $-1/2 + k^2/8$, and $-1/8 + k^2/8$ if k is equal to 0, 1, 2, and 3 modulo 4, respectively.

Let α be a generator of the Mordell–Weil group $\mathrm{Aut}(S)_\kappa^+$. Then $\rho = \alpha^m$ for some $m \in \mathbb{Z}$, and it follows from (4.3.2) that $0 = \nu(\rho) = \nu(\alpha^m) = -1 + 7/8 + [1 + \nu(\alpha) - \mathrm{contr}(\alpha)/2]\,m^2$, and therefore $1 = [8 + 8\,\nu(\alpha) - 4\,\mathrm{contr}(\alpha)]\,m^2$. Because $4\,\mathrm{contr}(\alpha) \in \mathbb{Z}$, it follows that $m = \pm 1$, and we conclude that the DΔKdV map generates the Mordell–Weil group of the elliptic surface S that is obtained by blowing up the base points.

With the substitution $z_0 = 1$, $z_1 = -z$, the rational function $P = 3\,\Gamma/2\,\Delta$ and Manin's function $\mu(z)$ for the QRT map, see Sections 2.5.3 and 2.6.3, are given by $P(z) = (1 - 12\,a\,z)/\Delta_0(z)$ and $\mu(z) = 2\,a\,(7 - 108\,a\,\omega^2 + 12\,a\,z)/((1 - 12\,a\,z)\,\Delta_0(z))$, where

$$\Delta_0(z) = -\omega^2 + 27\,a\,\omega^4 + (1 - 36\,a\,\omega^2)\,z + 8\,a\,z^2 + 16\,a^2\,z^3$$

is the third degree factor in $\Delta(z)$ over the zeros whose singular fibers are not of type I_3^*. Note that $p \neq 0$ implies that $a = 2\,p^2 \neq 0$, hence $\mu(z)$ is not identically zero, and the last statement in Proposition 7.8.8 confirms that the DΔKdV map has infinite order, for any values of the parameters p and μ. The simple structure of the functions P and μ is an invitation to use (2.6.8) in order to analyze the monotonicity properties of the rotation function, when $a = 2\,p^2 \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$.

11.5.2 The Modified KdV Mapping

The DΔMKdV mapping in Quispel, Roberts, and Thompson [169, p. 185], obtained by means of a reduction from a soliton equation for a modified Korteweg-de Vries equation, is

$$(x,\ y) \mapsto (y,\ -x - (y^2 + 1/2 - \omega)/(y - y^2)),$$

where ω is a constant. The DΔMKdV mapping is the QRT root for the pencil (2.5.3) of symmetric biquadratic curves with the coefficient matrices

$$A^0 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & a \\ 0 & a & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = 1/2 - \omega$.

The Weierstrass invariants of the pencil of biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$\begin{aligned}
3\,g_2 &= 4\,{z_0}^2\,(a\,z_0 + z_1)^2, \\
27\,g_3 &= -{z_0}^3\,((8\,a^3 + 27\,a^4)\,{z_0}^3 - 30\,a^2\,{z_0}^2\,z_1 + (27 + 24\,a)\,z_0\,{z_1}^2 + 8\,{z_1}^3), \\
\Delta &= -{z_0}^7\,(a^2\,z_0 - z_1)^2\,((16\,a^3 + 27\,a^4)\,{z_0}^3 - 6\,a^2\,{z_0}^2\,z_1 \\
&\quad + (27 + 48\,a)\,z_0\,{z_1}^2 + 8\,{z_1}^3), \\
3\,X &= -2\,z_0\,(a\,z_0 + z_1), \\
Y &= -{z_0}^2\,(a^2\,z_0 - z_1), \\
3\,X_\rho &= z_0\,(a\,z_0 + z_1), \\
Y_\rho &= {z_0}^2\,(a^2\,z_0 - z_1).
\end{aligned}$$

The pencil always has at least one smooth member. Table 6.3.2 implies that there is a singular fiber of type I_1^* over the point $[z_0 : z_1] = [0 : 1]$ at infinity. The discriminant of the last factor F in Δ is equal to $-2^6\,3^9\,a^3\,(1 + a)^3\,(1 + 2\,a)^2$, and substitution of $z_0 = 1, z_1 = a^2$ in F and g_2 yields $16\,a^3\,(1+a)^3$ and $(4/3)\,a^2\,(1+a)^2$, respectively. Therefore the configuration of the singular fibers is $\mathrm{I}_1^*\ \mathrm{I}_2\ 3\ \mathrm{I}_1$ if and only if $a \neq 0$, $a \neq -1$, and $a \neq -1/2$. One can show that $\mathrm{I}_1^*\ \mathrm{I}_2\ 3\ \mathrm{I}_1$ is the generic configuration of singular fibers for a symmetric McMillan map. In view of

Lemma 9.2.6 the Mordell–Weil group is isomorphic to $\mathbb{Z}^2$. The modulus function J has degree $j = 6$; see (6.2.48).

If $a = -1/2$, then two of the three singular fibers of type I_1 have merged to a singular fiber of type I_2 over $[z_0 : z_1] = [1 : -1/4]$, and we have the configuration of singular fibers $\mathrm{I}_1^*\ 2\ \mathrm{I}_2\ \mathrm{I}_1$, with Mordell–Weil group isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$, see Lemma 9.2.6, and degree of the modulus function equal to $j = 6$; see (6.2.48).

If $a = 0$ or $a = -1$, then we have the configuration of singular fibers $\mathrm{I}_1^*\ \mathrm{IV}\ \mathrm{I}_1$, Mordell–Weil group isomorphic to $\mathbb{Z}$, and degree of the modulus function equal to $j = 2$. As for the DΔKdV map, it can be proved that when $a = 0$ or $a = -1$, the DΔMKdV map generates the Mordell–Weil group of the elliptic surface S that is obtained by blowing up the base points.

We leave it as an exercise to compute the number of k-periodic fibers, counted with multiplicities, in the various cases. With the substitution $z_0 = 1$, $z_1 = -z$, the rational function $P = 3\,\Gamma/2\,\Delta$ and Manin's function $\mu(z)$ for the QRT map, see Sections 2.5.3 and 2.6.3, are given by $P(z) = 4\,(a - z)\,(2\,a + 3\,a^3 + z)/\Delta_0(z)$ and $\mu(z) = -2\,(2\,a + 3\,a^2 + z)^2/((a - z)\,\Delta_0(z))$, where

$$\Delta_0(z) = (a^2 + z)\,(16\,a^3 + 27\,a^4 + 6\,a^2\,z + (27 + 48\,a)\,z^2 - 16\,z^3)$$

is the primary factor of $\Delta(z)$. Note that $\mu(z)$ is not identically zero, and the last statement in Proposition 7.8.8 confirms that the DΔMKdV map has infinite order, for any value of the parameters ω. The simple structure of the functions P and μ is an invitation to use (2.6.8) in order to analyze the monotonicity properties of the rotation function when $\omega \in \mathbb{R}$.

11.5.3 The Nonlinear Schrödinger Mapping

The DΔNLS mapping in Quispel, Roberts, and Thompson [169, p. 185], obtained by means of a reduction from a soliton equation for a nonlinear Schrödinger equation, is

$$(x,\ y) \mapsto (y,\ -x + (\omega + 2)\,y/(1 + y^2/2)),$$

where ω is a constant. The DΔNLS mapping is the QRT root for the pencil (2.5.3) of symmetric biquadratic curves with the coefficient matrices

$$A^0 = \begin{pmatrix} 1/2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = -\omega - 2$.

The Weierstrass invariants of the pencil of biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$12\,g_2 = {z_0}^2\,((4 - a^2)^2\,{z_0}^2 + 4\,(28 - a^2)\,z_0\,z_1 + 4\,{z_1}^2),$$

$$
\begin{aligned}
216\, g_3 &= {z_0}^3\,((4-a^2)\,z_0+2\,z_1)\,((4-a^2)^2\,{z_0}^2-4\,(68+a^2)\,z_0\,z_1+4\,{z_1}^2),\\
2\,\Delta &= {z_0}^7\,((2-a)^2\,z_0-2\,z_1)^2\,((2+a)^2\,z_0-2\,z_1)^2\,z_1,\\
12\,X &= z_0\,((8+a^2)\,z_0+4\,z_1),\\
2\,Y &= a\,{z_0}^2\,(2\,z_0-z_1),\\
12\,X_\rho &= z_0\,((10-a)\,(2-a)\,z_0-2\,z_1),\\
Y_\rho &= -{z_0}^2\,((2-a)^2\,z_0-2\,z_1).
\end{aligned}
$$

The pencil always has at least one smooth member. Table 6.3.2 implies that there is a singular fiber of type I_1^* over the point $[z_0 : z_1] = [0 : 1]$ at infinity.

If $a \neq 0$ and $a \neq \pm 2$, then the configuration of the singular fibers is $\mathrm{I}_1^*\ 2\ \mathrm{I}_2\ \mathrm{I}_1$, As for the D$\Delta$MKdV mapping with $a = -1/2$, the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$, and the modulus function J has degree $j = 6$.

If $a = \pm 2$, then the configuration of the singular fibers is $\mathrm{I}_1^*\ \mathrm{III}\ \mathrm{I}_2$. According to Lemma 9.2.6, the Mordell–Weil group is still isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$, whereas (6.2.48) implies that the modulus function this time has degree $j = 3$.

If $a = 0$, then the configuration of the singular fibers is $\mathrm{I}_1^*\ \mathrm{I}_4\ \mathrm{I}_1$. According to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$, and because for $\omega + 2 = 0$ the DΔNLS mapping is equal to the simple transformation $(x,\ y) \mapsto (y,\ -x)$ of order four, it follows that it generates the Mordell–Weil group.

We leave it as an exercise to compute the number of k-periodic fibers, counted with multiplicities, in the various cases. With the substitution $z_0 = 1$, $z_1 = -z$, the rational function $P = 3\,\Gamma/2\,\Delta$ and Manin's function $\mu(z)$ for the QRT map, see Sections 2.5.3 and 2.6.3, are given by $P(z) = (-4 + a^2 - 2\,z)/\Delta_0(z)$ and $\mu(z) = a\,(-12 + 3\,a^2 + 2\,z)/((-4 + a^2 - 2\,z)\,\Delta_0(z))$, where

$$\Delta_0(z) = ((2-a)^2 + 2\,z)\,((2+a)^2 + 2\,z)\,z$$

is the primary factor of $\Delta(z)$. Note that if $a \neq 0$, then $\mu(z)$ is not identically zero, and the last statement in Proposition 7.8.8 implies that the DΔMKdV map has infinite order. The simple structure of the functions P and μ is an invitation to use (2.6.8) in order to analyze the monotonicity properties of the rotation function when $\omega \in \mathbb{R}$.

11.5.4 A Nonsymmetric McMillan QRT Transformation

The mapping $(x,\ y) \mapsto (\xi(x,\ y),\ \eta(\xi(x,\ y),\ y))$, where

$$\xi(x,\ y) = -x - (2\,y + 2\,K_2)/(1 + y^2), \quad \eta(\xi,\ y) = -y - (2\,\xi + 2\,K_1)/(1 + \xi^2),$$

has been presented by Quispel, Roberts, and Thompson [168, (7)] as describing the stationary solutions of a discrete modified Korteweg–de Vries equation, where K_1 and K_2 are arbitrary integration constants. We recognize this mapping as the QRT map for the pencil (2.5.3) of biquadratic curves with the coefficient matrices

$$A^0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & a \\ 1 & b & 0 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = 2\,K_2$ and $b = 2\,K_1$. If $K_1 \neq K_2$, then this is a nonsymmetric McMillan QRT map.

The Weierstrass invariants of the pencil of biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$\begin{aligned}
3\,g_2 &= 4\,{z_0}^2\,(3\,(-a^2 + a\,b - b^2)\,{z_0}^2 + 12\,z_0\,z_1 + {z_1}^2),\\
27\,g_3 &= {z_0}^3\,(27\,(4\,a^2 - 8\,a\,b + 4\,b^2 - a^2\,b^2)\,{z_0}^3 + 36\,(a^2 + a\,b + b^2)\,{z_0}^2\,z_1\\
&\quad - 288\,z_0\,{z_1}^2 + 8\,{z_1}^3),\\
\Delta &= {z_0}^7\,(d_0\,{z_0}^5 + d_1\,{z_0}^4\,z_1 + d_2\,{z_0}^3\,{z_1}^2 + d_3\,{z_0}^2\,{z_1}^3 + d_4\,z_0\,{z_1}^4 + d_5\,{z_1}^5),\\
3\,X &= z_0\,(3\,z_0 + 2\,z_1),\\
Y &= {z_0}^2\,((2 + a\,b)\,z_0 - 2\,z_1).
\end{aligned}$$

Here the coefficients d_j in Δ are given by

$$\begin{aligned}
d_0 &= -432\,(a^4 + b^4) + 1728\,a\,b\,(a^2 + b^2) - 2592\,a^2\,b^2 - 64\,(a^6 + b^6)\\
&\quad + 192\,a\,b\,(a^4 + b^4), -168\,a^2\,b^2\,(a^2 + b^2) + 16\,a^3\,b^3 - 27\,a^4\,b^4,\\
d_1 &= 24\,(8\,(a^4 + b^4) - 28\,a\,b\,(a^2 + b^2) + 72\,a^2\,b^2 + 6\,a^2\,b^2\,(a^2 + b^2) + 3\,a^3\,b^3),\\
d_2 &= -16\,(48\,(a + b)^2 + 8\,(a^4 + b^4) + 20\,a\,b\,(a^2 + b^2) + 51\,a^2\,b^2),\\
d_3 &= 16\,(256 + 60\,(a^2 + b^2) + 88\,a\,b + a^2\,b^2),\\
d_4 &= -36\,(32 + a^2 + b^2),\\
d_5 &= 256.
\end{aligned}$$

The pencil always has at least one smooth member. Table 6.3.2 implies that there is a singular fiber of type I_1^* over the point $[z_0 : z_1] = [0 : 1]$ at infinity. The discriminant of $\Delta/{z_0}^7$ is equal to $2^{28}\,((4 + a^2)\,(a - b)\,(a + b)\,(4 + b^2))^2\,\delta^3$, where

$$\begin{aligned}
\delta &= 2^{13}\,3^3\,(a - b)^2 + 2^4\,3^3\,31\,(a^4 + b^4) + 2^8\,3^4\,a\,b\,(a^2 + b^2) - 2^5\,3^3\,79\,a^2\,b^2\\
&\quad + 2^{10}\,(a^6 + b^6) - 2^8\,3\,a\,b\,(a^4 + b^4) - 2^3\,3\,17\,a^2\,b^2\,(a^2 + b^2) + 2^4\,11\,a^3\,b^3\\
&\quad - 3^3\,a^4\,b^4.
\end{aligned}$$

It follows that the configuration of singular fibers is equal to $\mathrm{I}_1^*\ 5\ \mathrm{I}_1$ if and only if $a - b \neq 0, a + b \neq 0, a^2 + 4 \neq 0, b^2 + 4 \neq 0$, and $\delta \neq 0$. According to Lemma 9.2.6, the Mordell–Weil group in this case is isomorphic to $\mathbb{Z}^3$, whereas (6.2.48) implies that the modulus function this time has degree $j = 6$.

This is the generic configuration of singular fibers for any pencil of biquadratic curves with a member equal to the union of a horizontal and a vertical axis, both of multiplicity two. We note that it follows from the proof of Proposition 10.1.2 that a rational elliptic surface with the configuration of singular fibers $\mathrm{I}_1^*\ 5\ \mathrm{I}_1$ does not admit any abstract QRT root.

11.6 Heisenberg Spin Chain Maps

11.6.1 The Isotropic Case

The mapping DΔIHSC in Quispel, Roberts, and Thompson [169, p. 186] is the transformation $(x,\ y) \mapsto (y,\ (f_0(y) - f_1(y)\,x)/(f_1(y) - f_2(y)\,x))$, where

$$\begin{aligned} f_0(y) &= -2\,y^3 - \omega\,y^2 - 2\,y + \omega, \\ f_1(y) &= y^4 + \omega\,y^3 - \omega\,y - 1, \\ f_2(y) &= -\omega\,y^4 + 2\,y^3 + \omega\,y^2 + 2\,y, \end{aligned}$$

and ω is a constant.

11.6.1.1 Reconstruction

Application of Section 10.1.1 leads to

$$\begin{aligned} w &= -\omega\,d_1 \wedge d_2 - d_1 \wedge d_3 + 2\,d_1 \wedge d_4 - \omega\,d_1 \wedge d_5 + 2\,\omega\,d_2 \wedge d_4 + \omega\,d_2 \wedge d_6 \\ &\quad +2\,d_3 \wedge d_4 + d_3 \wedge d_6 - 2\,\omega\,d_4 \wedge d_5 - 2\,d_4 \wedge d_6 + \omega\,d_5 \wedge d_6 \\ &= (d_1 + \omega\,d_2 + d_3 + \omega\,d_5 + d_6) \wedge (d_1 + 2\,d_4 + d_6). \end{aligned}$$

It follows that the transformation is equal to the QRT root defined by the symmetric matrices

$$A^0 = \begin{pmatrix} 1 & \omega & 1 \\ \omega & 0 & \omega \\ 1 & \omega & 1 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The pencil of biquadratic curves (2.5.3) has two base points, $([1 : \pm\mathrm{i}],\ [1 : \pm\mathrm{i}])$, each of multiplicity 4. The proper image of the curve $(x_0\,y_0 + x_1\,y_1)^2 = 0$ in the elliptic surface that is obtained after blowing up the base points is a singular fiber of type I_2^*; see case 3(c3) in Section 12.1.

11.6.1.2 The Weierstrass Invariants

The Weierstrass invariants, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$\begin{aligned} 3\,g_2 &= 4\,{z_0}^2\,((16 - 16\,\omega^2 + \omega^4)\,{z_0}^2 + (32 - 4\,\omega^2)\,z_0\,z_1 + 16\,{z_1}^2), \\ 27\,g_3 &= 8\,{z_0}^3\,((2 - \omega^2)\,z_0 + 2\,z_1)\,((-32 + 32\,\omega^2 + \omega^4)\,{z_0}^2 - (64 + 4\,\omega^2)\,z_0\,z_1 \\ &\quad - 32\,{z_1}^2), \\ \Delta &= 256\,\omega^4\,{z_0}^8\,(\omega^2\,z_0 - 4\,z_1)^2\,((1 - \omega)\,z_0 + z_1)\,((1 + \omega)\,z_0 + z_1), \end{aligned}$$

$$
\begin{aligned}
3\,X &= (4-2\,\omega^2)\,{z_0}^2 + 4\,z_0\,z_1 + 3\,{z_1}^2,\\
Y &= 2\,z_1\,(\omega^2\,{z_0}^2 - 2\,z_0\,z_1 - {z_1}^2),\\
3\,X_\rho &= z_0\,((4+\omega^2)\,z_0 - 8\,z_1),\\
Y_\rho &= -4\,{z_0}^2\,(\omega^2\,z_0 - 4\,z_1).
\end{aligned}
$$

It follows from the formula for the discriminant Δ that all the members of the pencil of biquadratic curves are singular if and only if $\omega = 0$. Therefore we assume in the sequel that $\omega \neq 0$.

The horizontal and the vertical switches both have degree 4, and it follows from Corollary 5.1.9 that the number of k-periodic fibers, counted with multiplicities, is equal to $k^2 - 1$.

11.6.1.3 Configuration of the Singular Fibers

It follows from Table 6.3.2 that over $[z_0 : z_1] = [0 : 1]$ we have singular fiber of type I_2^*.

If $\omega \neq \pm 2$, then we have over $[z_0 : z_1] = [1 : \omega^2/4]$ a singular fiber of type I_2 and over each of the points $[z_0 : z_1] = [1 : \pm\omega - 1]$ a singular fiber of type I_1. Therefore the configuration of singular fibers is $\mathrm{I}_2^*\ \mathrm{I}_2\ 2\ \mathrm{I}_1$, when Lemma 9.2.6 implies that the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$. Then the modulus function $J = {g_2}^3/\Delta$ has degree $j = 6$; see (6.2.48).

If $\omega = \pm 2$, then the singular fiber of type I_2 has merged with one of the singular fibers of type I_1 to a singular fiber of type III, and we obtain the confiuration of singular fibers $\mathrm{I}_2^*\ \mathrm{III}\ \mathrm{I}_1$. It follows from Lemma 9.2.6 that again the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$. On the other hand, the modulus function $J = {g_2}^3/\Delta$ has degree $j = 3$; see (6.2.48).

11.6.1.4 Picard–Fuchs Equations

In the affine coordinate $z_0 = 1$, $z_1 = -z$, the discriminant takes the form

$$\Delta(z) = 256\,\omega^4\,(z-1-\omega)\,(z-1+\omega)\,(4\,z+\omega^2)^2.$$

The polynomial $\Gamma(z)$ in (2.5.9) is given by $3\,\Gamma(z) = 256\,\omega^4\,(z+1)\,(4\,z+\omega^2)$, and therefore

$$P(z) := \frac{3\,\Gamma(z)}{2\,\Delta(z)} = \frac{z+1}{2\,(z-1-\omega)\,(z-1+\omega)\,(4\,z+\omega^2)};$$

see (2.5.8). Furthermore, Manin's function $\mu(z) = (\mathcal{L}T)(z)$ in (2.5.16) is equal to

$$\mu = \frac{-z+3-\omega^2}{(z-1-\omega)\,(z-1+\omega)\,(4\,z+\omega^2)\,(z+1)}.$$

Because $\mu(z)$ is never identically zero, it follows from the last statement in Proposition 7.8.8 that the DΔISHC map is not of finite order, for any value of the parameter ω. The simple form of the expressions for P and μ is an invitation to use equation (2.6.8) in the investigation of the monotonicity of the period function.

11.6.2 The Anisotropic Case

The mapping DΔAHSC in Quispel, Roberts, and Thompson [169, pp. 186, 187] is the transformation $(x,\ y) \mapsto (y,\ (f_0(y) - f_1(y)\,x)/(f_1(y) - f_2(y)\,x))$, where $f_0(y) = 2\,\lambda\,y$, $f_1(y) = 1 - \lambda^2\,y^2$, $f_2(y) = -2\,\lambda\,y$, and λ is a constant.

11.6.2.1 Reconstruction

The vector $(1,\ 0,\ 1)$ is orthogonal to $f(y)$, and $(a,\ b,\ c) \times (1,\ 0,\ 1) = f(y)$ if and only if $b = 2\,\lambda\,y$ and $c - a = 1 - \lambda^2\,y^2$, which are satisfied if $a = \lambda^2\,y^2$, $b = 2\,\lambda\,y$, and $c = 1$. The transformation is equal to the QRT root defined by the symmetric matrices

$$A^0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

guessed from the above considerations.

If $\lambda \neq \pm 1$, then the pencil of biquadratic curves (2.5.3) has four base points, $([\pm\mathrm{i} :\ 1],\ [\pm\lambda\ \mathrm{i} :\ 1])$ and $([\pm\lambda\ \mathrm{i} :\ 1],\ [\pm\mathrm{i} :\ 1])$, each of multiplicity 2. If in addition $\lambda \neq 0$, then the proper image of the curve $(x_0\,y_0 + \lambda\,x_1\,y_1)^2 = 0$ in the elliptic surface that is obtained after blowing up the base points is a singular fiber of type I_0^*; see case 3(c1) in Section 12.1.

11.6.2.2 The Weierstrass Invariants

The Weierstrass invariants, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$\begin{aligned}
3\,g_2 &= 4\,{z_0}^2\,(16\,{z_0}^2 + 16\,(1+\lambda^2)\,z_0\,z_1 + (1 + 14\,\lambda^2 + \lambda^4)\,{z_1}^2),\\
27\,g_3 &= -8\,{z_0}^3\,(2\,z_0 + (1+\lambda^2)\,z_1)\,(32\,{z_0}^2 + 32\,(1+\lambda^2)\,z_0\,z_1 - (1 - 34\,\lambda^2 + \lambda^4)\,{z_1}^2),\\
\Delta &= 256\,(1-\lambda^2)^4\,{z_0}^6\,{z_1}^4\,(z_0 + z_1)\,(z_0 + \lambda^2\,z_1),\\
3\,X &= 4\,{z_0}^2 + 2\,(1+\lambda^2)\,z_0\,z_1 + 3\,\lambda^2\,{z_1}^2,\\
Y &= -2\,\lambda\,{z_1}^2\,((1+\lambda^2)\,z_0 + \lambda^2\,z_1),\\
3\,X_\rho &= z_0\,(4\,z_0 - (1 + 6\,\lambda + \lambda^2)\,z_1),\\
Y_\rho &= 4\,(1+\lambda)^2\,{z_0}^2\,z_1.
\end{aligned}$$

It follows from the formula for the discriminant Δ that all the members of the pencil of biquadratic curves are singular if and only if $\lambda = \pm 1$. Therefore we assume in the sequel that $\lambda \neq \pm 1$.

11.6.2.3 Configuration of the Singular Fibers

It follows from Table 6.3.2 that over $[z_0 : z_1] = [1 : 0]$ and $[z_0 : z_1] = [-1 : 1]$ we always have a singular fiber of type I_4 and I_1, respectively.

If $\lambda \neq 0$, then we have a singular fiber of type I_0^* and I_1 over $[z_0 : z_1] = [0 : 1]$ and $[z_0 : z_1] = [-\lambda^2 : 1]$, respectively. The configuration of singular fibers is $\mathrm{I}_0^*\ \mathrm{I}_4\ 2\,\mathrm{I}_1$, when Lemma 9.2.6 implies that the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$. The modulus function $J = {g_2}^3/\Delta$ has degree $j = 6$, see (6.2.48).

If $\lambda = 0$, then the singular fibers of type I_0^* and I_1 over $[z_0 : z_1] = [0 : 1]$ and $[z_0 : z_1] = [-\lambda^2 : 1]$ have merged to a singular fiber of type I_1^*. In this case the configuration of singular fibers is $\mathrm{I}_1^*\ \mathrm{I}_4\ \mathrm{I}_1$, the D$\Delta$AHSC transformation is the very simple transformation $(x,\ y) \mapsto (y,\ -x)$ of order four, and we are in the same situation as for the DΔNLS mapping in Section 11.5.3 for $a = -\omega - 2 = 0$.

11.6.2.4 Picard–Fuchs Equations

In the affine coordinate $z_0 = 1$, $z_1 = -z$, the discriminant takes the form

$$\Delta(z) = 256\,(1 - \lambda^2)^4\, z^4\, (z - 1)\, (\lambda^2\, z - 1).$$

The polynomial $\Gamma(z)$ in (2.5.9) is given by $3\,\Gamma(z) = 128\,(1 - \lambda^2)^4\, z^3$, and therefore

$$P(z) := \frac{3\,\Gamma(z)}{2\,\Delta(z)} = \frac{1}{4\,z\,(z-1)\,(\lambda^2\,z-1)z^4\,(z-1)\,(\lambda^2\,z-1)};$$

see (2.5.8). Furthermore, Manin's function $\mu(z) = (\mathcal{L}T)(z)$ in (2.5.16) is equal to

$$\mu = \frac{\lambda}{4\,z\,(z-1)\,(\lambda^2\,z-1)}.$$

If $\lambda \neq 0$, then $\mu(z)$ is not identically zero, and the last statement in Proposition 7.8.8 implies that the DΔASHC map is not of finite order. If $\lambda = 0$, then it is equal to the mapping $(x,\ y) \mapsto (y,\ -x)$ of order 4. The relatively simple form of the expressions for P and μ is an invitation to use equation (2.6.8) in the investigation of the monotonicity of the period function.

11.7 The Sine–Gordon map

The transformation

$$(x,\ y) \mapsto \left(y,\ \frac{\theta_3 - \theta_1\, y^2}{(\theta_2\, y^2 - \theta_1)\, x}\right) \tag{11.7.1}$$

of the plane, where θ_1, θ_2, θ_3 are nonzero constants, is a reduction of a discrete sine–Gordon equation; see Tuwankotta and Quispel [197], or Quispel, Capel, Papageorgiou, and Nijhoff [167]. This is the transformation that was shown to me by Theo Tuwankotta, and which was the point of departure of this book.

11.7.0.5 Reconstruction

We apply the method described in the last paragraph of Section 10.1.1 to the somewhat more general transformation $\rho : (x,\ y) \mapsto (y,\ \xi(x,\ y))$ of the plane, where $\xi(x,\ y) = h(y)/(k(y)\, x)$, and $h(y), k(y)$ are polynomials without a common zero. In homogeneous coordinates, $\xi([x_0 : x_1],\ [y_0 : y_1]) = [x_1\, k(y_0,\ y_1) : x_0\, h(x_0,\ x_1)]$, where $k(y_0,\ y_1)$ and $h(y_0,\ y_1)$ are homogeneous polynomials of the same degree $d \in \mathbb{Z}_{\geq 0}$ without a common factor of strictly positive degree. We also assume that h and k both have at least two distinct zeros $[a_0^{\pm} : a_1^{\pm}]$ and $[b_0^{\pm} : b_1^{\pm}]$, respectively, where $[a_0^{\pm} : a_1^{\pm}] \neq [0 : 1]$ or $[b_0^{\pm} : b_1^{\pm}] \neq [1 : 0]$.

The rational function $\xi(x,\ y)$ is undetermined at the points $([x_0 : x_1],\ [y_0 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ where $x_1\, k(y) = x_0\, h(y) = 0$, that is, $x_1 = h(y) = 0$ or $x_0 = k(y) = 0$. The vanishing of the symmetric biquadratic polynomial $p((x_0,\ x_1),\ (y_0,\ y_1))$ at the four points of indeterminacy $([1 : 0],\ [a_0^{\pm} : a_1 \pm],\ ([0 : 1],\ [b_0^{\pm} : b_1^{\pm}]$ of ξ leads to four linearly independent linear equations for the symmetric 3×3 matrix $(A_{ij})_{i,\ j=0}^{2}$ in (2.4.1), whose solutions are the linear combinations of

$$A^0 = \begin{pmatrix} k_0 & k_1 & k_2 \\ k_1 & 0 & h_1 \\ h_0 & h_1 & h_2 \end{pmatrix} \quad \text{with} \quad h_0 = k_2, \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.7.2}$$

Therefore, if ρ is a QRT root, then (1.1.3) and (1.1.4) imply that $d = 2$ and

$$\xi(x,\ y) = (h_0\, y^2 + h_1\, y + h_2)/((k_0\, y^2 + k_1\, y + k_2)\, x) \quad \text{with} \quad h_0 = k_2, \tag{11.7.3}$$

where the conditions on the zeros of $h(y)$ and $k(y)$ correspond to the inequalities $h_0 = k_2 \neq 0$, ${h_1}^2 - 4\, h_0\, h_2 \neq 0$, and ${k_1}^2 - 4\, k_0\, k_2 \neq 0$. The mapping (11.7.1) is of this form with $h_0 = -\theta_1 = k_2$, $h_1 = k_1 = 0$, $h_2 = \theta_3$, and $k_0 = \theta_2$. Conversely every map $(x,\ y) \mapsto (y,\ \xi(x,\ y))$ with $\xi(x,\ y)$ as in (11.7.3) is the QRT root of the pencil of symmetric biquadratic curves defined by (11.7.2).

Remark 11.7.1. If we substitute $h_0 = H_0\, K_1$, $h_1 = H_1\, K_1 + {H_0}^2$, $h_2 = H_0\, H_1$, $k_0 = K_0\, K_1$, $k_1 = H_0\, K_0 + {K_1}^2$, $k_2 = H_0\, K_1$, that is,

$$A^0 = \begin{pmatrix} K_0\, K_1 & H_0\, K_0 + {K_1}^2 & H_0\, K_1 \\ H_0\, K_0 + {K_1}^2 & 0 & H_1\, K_1 + {H_0}^2 \\ H_0\, K_1 & H_1\, K_1 + {H_0}^2 & H_0\, H_1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{11.7.4}$$

and assume that not both H_0 and K_1 are equal to zero, then we obtain that the transformation

$$(x,\ y) \mapsto (y,\ (H_0\, y + H_1)/((K_0\, y + K_1)\, x)) \tag{11.7.5}$$

is equal to the QRT root defined by (11.7.5). In the exceptional case that $H_0 = K_1 = 0$ and $H_1 \neq 0$, $K_0 \neq 0$, (11.7.5) is the transformation $(x,\ y) \mapsto (y,\ c/(y\, x))$ for a nonzero constant $c = H_1/K_0$. This transformation has order three and it leaves the biquadratic curves

$$z_0\,({x_1}^2\, {y_1}^2 + c\,(x_0\, x_1\, {y_0}^2 + {x_0}^2\, y_0\, y_1)) + z_1\,({x_1}^2\, y_0\, y_1 + x_0\, x_1\, {y_1}^2 + c\, {x_0}^2\, {y_0}^2) \\ + z_2\, x_0\, x_1\, y_0\, y_1 = 0,$$

for arbitrary $(z_0,\ z_1,\ z_2) \neq (0,\ 0,\ 0)$, invariant. These biquadratic curves form a net instead of a pencil, and therefore $(x,\ y) \mapsto (y,\ c/(y\, x))$ is a QRT root for a whole $\mathbb{P}^2$ of pencils of symmetric biquadratic curves, one pencil for each two-dimensional linear subspace of the $(z_0,\ z_1,\ z_2)$-space. The mappings (11.7.5) are the transformations as above with $d = 1$ when $H_0\, K_1 - H_1\, K_0 \neq 0$ and $d = 0$ when $H_0\, K_1 - H_1\, K_0 = 0$. We have the Lyness map (11.4.1) if $H_0 = 1$, $H_1 = a$, $K_0 = 0$, and $K_1 = 1$, when the span of (11.7.4) is equal to the span of (11.4.4). ⊘

11.7.0.6 Scaling

The scaling $x = c\,\xi$, $y = c\,\eta$ conjugates the mapping (11.7.1) with the same mapping with $x,\ y,\ \theta_1,\ \theta_2,\ \theta_3$ replaced by $\xi,\ \eta,\ c\,\theta_1,\ c^3\,\theta_2,\ \theta_3/c$, respectively. Since multiplication of the θ_j by a common nonzero factor d does not change (11.7.1), we can replace θ_1, θ_2, and θ_3 by $c\,d\,\theta_1$, $c^3\,d\,\theta_2$, and $(d/c)\,\theta_3$, respectively. If we work over $\mathbb{R}$, then the signs of θ_1/θ_2, θ_2/θ_3, and θ_3/θ_1 cannot be changed by means of these scalings, but we can arrive at $\theta_3 = 1$, $\theta_1 = -\vartheta$, and $\theta_2 = s = \pm 1 = \mathrm{sgn}(\theta_2/\theta_3)$. If we work over $\mathbb{C}$ then we can arrange in addition that $s = 1$.

After these scalings, which we will use throughout this section, the transformation (11.7.1) is equal to $(x,\ y) \mapsto (y,\ (1 + \vartheta\, y^2)/((s\, y^2 + 1)x))$, and has the invariant curves (2.5.3) with

$$p = z_0\,(\vartheta\,({x_1}^2\, {y_0}^2 + {x_0}^2\, {y_1}^2) + {x_1}^2\, {y_1}^2 + s\, {x_0}^2\, {y_0}^2) + z_1\, x_0\, x_1\, y_0\, y_1, \tag{11.7.6}$$

where $s = \pm 1$. That is, in the notation of (2.5.3) we have the symmetric matrices

$$A^0 = \begin{pmatrix} 1 & 0 & \vartheta \\ 0 & 0 & 0 \\ \vartheta & 0 & s \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The pencil of biquadratic curves has a smooth member if and only if $\vartheta \neq 0$. Note that (11.7.6) has the symmetries $((z_0,\ z_1),\ ((x_0,\ x_1),\ (y_0,\ y_1))) \mapsto ((z_0,\ -z_1),\ ((x_0,$

$-x_1)$, $(y_0, y_1)))$ and $((z_0, z_1), ((x_0, x_1), (y_0, y_1))) \mapsto ((z_0, -z_1), ((x_0, x_1), (y_0, -y_1)))$.

11.7.0.7 The Base Points

The base points of the pencil of biquadratic curves $p = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are $b_1 = ([1 : (-s\,\vartheta)^{-1/2}], [1 : 0])$, $b_2 = ([1 : -(-s\,\vartheta)^{-1/2}], [1 : 0])$, $b_3 = ([0 : 1], [1 : (-\vartheta)^{1/2}])$, $b_4 = ([0 : 1], [1 : -(-\vartheta)^{1/2}])$, $b_5 = ([1 : (-\vartheta)^{1/2}], [0 : 1])$, $b_6 = ([1 : -(-\vartheta)^{1/2}], [0 : 1])$, $b_7 = ([1 : 0], [1 : (-s\,\vartheta)^{-1/2}])$, and $b_8 = ([1 : 0], [1 : -(-s\,\vartheta)^{-1/2}])$. If $\vartheta \neq 0$, which we assume in the sequel, then these are eight distinct points in the complex surface $\mathbb{P}^1 \times \mathbb{P}^1$, and it follows that all base points are simple, the surface $S = S^p$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined in Section 3.1 is smooth, see Corollary 3.1.6, and $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is equal to the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ at these base points, see Corollary 3.3.1.

If ϑ is real, then we have eight real base points if $s\,\vartheta < 0$ and $\vartheta < 0$. See Figure 3.1.1, where $s = 1$, $\vartheta = -3$, and the affine coordinates have been chosen in such a way that all the real base points are visible. We have four real base points if $s\,\vartheta < 0$ and $\vartheta > 0$, or $s\,\vartheta > 0$ and $\vartheta < 0$. Finally, we have no real base points at all if $s\,\vartheta > 0$ and $\vartheta > 0$. In particular, if $s = 1$, then we have eight or no real base points, whereas for $s = -1$ we have four real base points. If there are real base points, then for every real $[z_0 : z_1]$ the biquadratic curve $p = 0$ has a nonempty real part. On the other hand, if $s = 1$ and $\vartheta > 0$, when there are no real base points, then the biquadratic curve $p = 0$ has no real points if and only if ${z_1}^2 < 4\,(\vartheta + 1)^2\,{z_0}^2$.

Because $A^0\,Y = (y^2 + \vartheta,\ 0,\ \vartheta\,y^2 + s)$ and $A^1\,Y = (0,\ y,\ 0)$, the functions $f_i = g_i$ defined in (1.1.3) are equal to

$$f_0(y) = -(\vartheta\,y^2 + s)\,y, \quad f_1(y) = 0, \quad f_2(y) = (y^2 + \vartheta)\,y,$$

in affine coordinates $y_0 = 1$, $y_1 = y$. Therefore the QRT map $\tau : (x,\ y) \mapsto (x',\ y')$, defined in (1.1.4), (1.1.5), (1.1.6), takes the form

$$x' = \frac{\vartheta\,y^2 + s}{(y^2 + \vartheta)\,x}, \quad y' = \frac{\vartheta\,x'^2 + s}{(x'^2 + \vartheta)\,y}, \tag{11.7.7}$$

where we have divided out the common factors $y = y_0\,y_1$ and $x' = x'_0\,x'_1$, respectively. In particular, if $\vartheta^2 \neq s$, then the degrees of ι_1 and ι_2 are equal to 2. That is, $d_1 = d_2 = 2$ in the notation of Definition 5.1.3. If on the other hand $\vartheta^2 = s$, then $d_1 = d_2 = 0$, $x' = \vartheta/x$, $y' = \vartheta/y$, and the QRT map is periodic of period 2.

11.7.0.8 The Singular Curves

The Weierstrass invariants of the pencil of symmetric biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$
\begin{aligned}
12\,g_2 &= 16\,(\vartheta^4 + 14\,s\,\vartheta^2 + 1)\,{z_0}^4 - 8\,(\vartheta^2 + s)\,{z_0}^2\,{z_1}^2 + {z_1}^4,\\
216\,g_3 &= (4\,(\vartheta^2 + s)\,{z_0}^2 - {z_1}^2)\,(16\,(\vartheta^4 - 34\,s\,\vartheta^2 + 1)\,{z_0}^4\\
&\quad - 8\,(\vartheta^2 + s)\,{z_0}^2\,{z_1}^2 + {z_1}^4),\\
\Delta &= s\,\vartheta^2\,{z_0}^4\,((z_1 + 2\,\vartheta\,z_0)^2 - 4\,s\,{z_0}^2)^2\,((z_1 - 2\,\vartheta\,z_0)^2 - 4\,s\,{z_0}^2)^2,\\
12\,X &= 8\,(\vartheta^2 + s)\,{z_0}^2 + {z_1}^2,\\
Y &= (\vartheta^2 - s)\,{z_0}^2\,z_1,\\
12\,X_\rho &= 4\,(5\,\vartheta^2 - s)\,{z_0}^2 - 12\,\vartheta\,z_0\,z_1 + {z_1}^2,\\
Y_\rho &= -\vartheta\,z_0\,(4\,(\vartheta^2 - s)\,{z_0}^2 - 4\,\vartheta\,z_0\,z_1 + {z_1}^2).
\end{aligned}
$$

We have at least one nonsingular fiber if and only if $\vartheta \neq 0$, which we assume from now on. We recall that the modulus of the elliptic curves is equal to $J = {g_2}^3/\Delta$; see (2.3.10). As a mapping from the complex projective $[z_0 : z]$-line $\mathbb{P}^1$ to $\mathbb{P}^1$, it has degree 12 for every nonzero value of ϑ.

We have a singular fiber of type I_4 if $z_0 = 0$. We have a singular fiber of type I_2 when $z_1 = 2\,(\vartheta + \sqrt{s})\,z_0$, $z_1 = 2\,(\vartheta - \sqrt{s})\,z_0$, $z_1 = -2\,(\vartheta + \sqrt{s})\,z_0$, and $z_1 = -2\,(\vartheta - \sqrt{s})\,z_0$. That is, we have the configuration of singular fibers $\mathrm{I}_4\ 4\,\mathrm{I}_2$ if and only if $\vartheta \neq 0$ and ${\vartheta}^2 \neq s$. This also follows from Table 6.3.2. According to Lemma 9.2.6, the Mordell–Weil group $\simeq \mathrm{Aut}(S)^+_\kappa$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. The degree of the modulus function J is equal to the maximal $j = 12$; see (6.2.48).

If ${\vartheta}^2 = s$, then

$$\Delta = s\,{z_0}^4\,{z_1}^4\,(z_1 - 4\,\vartheta\,z_0)^2\,(z_1 + 4\,\vartheta\,z_0)^2,$$

and the configuration of the singular fibers is $2\ \mathrm{I}_4\ 2\,\mathrm{I}_2$. Lemma 9.2.6 yields that the Mordell–Weil group is isomorphic to $(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$. Again the degree of J is equal to 12.

Because the QRT automorphism τ^S, which is the square of the QRT root ρ^S, is nontrivial, it corresponds to the element $(2+4\,\mathbb{Z},\ 0+2\,\mathbb{Z})$ of $(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$. The QRT root corresponds to an element of $(\mathbb{Z}/4\,\mathbb{Z}) \times (\mathbb{Z}/2\,\mathbb{Z})$ whose first component is equal to $1 + 4\,\mathbb{Z} \in \mathbb{Z}/4\,\mathbb{Z}$.

Assume that $\vartheta \neq 0$ and ${\vartheta}^2 \neq s$. Then the QRT root ρ^S, see Section 10.1, permutes the irreducible components of the singular fiber of type I_4 by a cyclic shift over one unit, which is a permutation of order 4. For the zeros $\neq [0 : 1]$ of Δ, the biquadratic curve is the union of two $(1,\ 1)$-curves, and its proper transform, which is the corresponding fiber of $\kappa : S \to P$, has Kodaira type I_2. When $z_1 = 2\,(\vartheta + \sqrt{s})\,z_0$ or $z_1 = 2\,(\vartheta - \sqrt{s})\,z_0$, the two $(1,\ 1)$ curves are

$$x\,y + \sqrt{s} + \sqrt{-\vartheta}\,(x + y) = 0, \quad x\,y + \sqrt{s} - \sqrt{-\vartheta}\,(x + y) = 0,$$

or the same with the other sign of $\sqrt{s}$. These curves are invariant under the symmetry switch $(x,\ y) \mapsto (y,\ x)$, and interchanged by the horizontal switch, and therefore the $(1,\ 1)$ curves are interchanged by the QRT root. When $z_1 = -2\,(\vartheta + \sqrt{s})\,z_0$ or $z_1 = -2\,(\vartheta - \sqrt{s})\,z_0$, the two $(1,\ 1)$ curves are

$$x\,y+\sqrt{s}+\sqrt{-\vartheta}\,(x-y)=0,\quad x\,y+\sqrt{s}-\sqrt{-\vartheta}\,(x-y)=0,$$

or the same with the other sign of $\sqrt{s}$. These curves are interchanged by both the symmetry switch and the horizontal switch, and therefore are invariant under the QRT root.

11.7.0.9 The Number of k-Periodic Fibers

In view of the way the QRT root permutes the irreducible components of the singular fibers, we obtain from (4.3.2) that for every $k\in\mathbb{Z}$ the number of k-periodic fibers in S^{reg}, counted with multiplicities, of the QRT root is given by

$$\begin{aligned}\nu((\rho^S)^k)&=\frac{1}{8}k^2+2\left\{\frac{k}{4}\right\}\left(1-\left\{\frac{k}{4}\right\}\right)+2\left\{\frac{k}{2}\right\}\left(1-\left\{\frac{k}{2}\right\}\right)-1\\&=\begin{cases}2n^2-1, & k=4n,\\ 2n^2+n, & k=4n+1,\\ 2n^2+2n, & k=4n+2,\\ 2n^2+3n+1, & k=4n+3,\end{cases}\end{aligned}\tag{11.7.8}$$

where $\{x\}$ denotes the fractional part of x. We have used that $\nu(\rho^S)=0$, see Section 10.1, and $\chi(S,\mathcal{O})=1$, see Lemma 9.1.2(iii). Furthermore, the contribution to $(\rho^S)^k$ of the fiber of type I_4 is equal to 0 if k is equal to 0 modulo 4, equal to 3/4 if k is equal to 1 or 3 modulo 4, and equal to 1 if k is equal to 2 modulo 4. On the other hand, the contribution to $(\rho^S)^k$ of each of the two fibers of type I_2 is equal to zero unless k is odd, when contribution is equal to $1/2$, because the irreducible components of the biquadratic curve are switched by the QRT root.

A bit surprisingly, the number of k-periodic fibers, counted with multiplicities, is the same as for the generic McMillan map; see Section 11.5. It may also be noted that $\nu((\tau^S)^2)=\nu((\rho^S)^4)=1$, and because the 2-periodic fibers for the QRT map correspond to $Y=0$, see Corollary 2.5.9, the fiber over $z_1=0$ is the only 2-periodic one.

If we write $v=(\rho^S)^4(e_1)-e_1$, then $v\cdot v=-4$, which shows that $(\rho^S)^4$ is a primitive element of the narrow Mordell Weil lattice, in the sense that if $(\rho^S)^4=\beta^k$ for any other element of the narrow Mordell–Weil lattice, then $k=\pm1$. Because the Mordell–Weil lattice modulo the narrow Mordell–Weil lattice is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, we conclude that *the QRT root is a generator of the Mordell–Weil lattice*, and actually its unique generator up to sign, because the rank of the Mordell–Weil lattice is equal to 1.

11.7.0.10 The Real Curves

Let $\vartheta\in\mathbb{R}$, $\vartheta\neq0$, and $\vartheta^2\neq s$. The singular fiber of type I_4, which occurs for $z_0=0$, is always hyperbolic with respect to the real structure, because its real part

is a square. We also could have argued that $216\, g_3 = -{z_1}^6 < 0$ when $z_0 = 0$; see Remark 8.3.1. In the sequel we take $z_0 = 1$ and $z := -z_1 \in \mathbb{R}$.

Let $s = 1$ and $\vartheta > 0$. Then all the zeros of Δ are real, but none of the four singular fibers of I_2 have real nonsingular points. When $-2\,(\vartheta + 1) < z < 2\,(\vartheta + 1)$, the real fiber is empty. This implies that there are no real base points, as we already have established above. When $z < -2\,(\vartheta + 1)$ or $z > 2\,(\vartheta + 1)$, the real fiber is nonempty, and because $\Delta(z) > 0$ it follows from Lemma 2.6.3 that it has two connected components, each diffeomorphic to a circle. When $z \uparrow -2\,(\vartheta + 1)$ or $z \downarrow 2\,(\vartheta + 1)$, these circles shrink to the singular points $([1 : 1], [1 : -1])$, $([1 : -1], [1 : 1])$ or $([1 : 1], [1 : 1])$, $([1 : -1], [1 : -1])$ of the fiber of type I_2, respectively, each of which is elliptic with respect to the real structure. The ellipticity also follows from $g_3 = 512\,\vartheta^3/27 > 0$ at the limit value for z, see Remark 8.3.1.

Let $s = 1$ and $\vartheta < 0$. Then we have eight real base points, and therefore all real fibers are nonempty, and because $\Delta(z) > 0$ it follows from Lemma 2.6.3 that each nonsingular real fiber has two connected components. We have a singular fiber of type I_2 when z is equal to $2\,(\vartheta - 1)$, $2\,(\vartheta + 1)$, $-2\,(\vartheta + 1)$, or $-2\,(\vartheta - 1)$, when $27\, g_3/512$ is equal to $-\vartheta^3$, ϑ^3, ϑ^3, or $-\vartheta^3$, respectively. It therefore follows from Remark 8.3.1 that the middle two fibers of Kodaira type I_2 are hyperbolic with respect to the real structure, and the first and last ones are elliptic. The singular points are $([1 : \mathrm{i}], [1 : -\mathrm{i}])$ and $([1 : -\mathrm{i}], [1 : \mathrm{i}])$ for $z = 2\,(\vartheta - 1)$; $([1 : 1], [1 : -1])$ and $([1 : -1], [1 : 1])$ for $z = 2\,(\vartheta + 1)$, $([1 : 1], [1 : 1])$ and $([1 : -1], [1 : -1])$ for $z = -2\,(\vartheta + 1)$; and finally, $([1 : \mathrm{i}], [1 : \mathrm{i}])$ and $([1 : -\mathrm{i}], [1 : -\mathrm{i}])$ for $z = -2\,(\vartheta - 1)$. It follows that of the two hyperbolic singular fibers of type I_2 the singular points are real, as is always the case for hyperbolic singular fibers. On the other hand, the singular points of the two elliptic singular fibers of type I_2 are not real. That is, the two elliptic singular fibers are as in the last paragraph of Lemma 8.1.4.

Let $s = -1$. Then we have four real base points; hence all real fibers are nonempty. For every $z \in \mathbb{R}$ we have $\Delta(z) < 0$; hence the complex fiber over z is nonsingular, whereas Lemma 2.6.3 implies that the real fiber over z is connected, diffeomorphic to a circle. Note that in the case $s = -1$ the QRT map has order two if and only if $\vartheta^2 = -1$, which does not occur for any real value of ϑ.

Figures 3.1.1, 11.7.1, and 11.7.2 Figure 3.1.1 shows the real pencil for $s = 1$ and $\vartheta = -3$, where the real affine coordintes have be chosen in such a way that all the eight real base points are visible. In Figure 11.7.1 we have $s = 1$, $\vartheta = 0.5$ in the left picture, and $s = 1$, $\vartheta = 2$ in the right picture. Since there are no real base points when $s > 0$ and $\vartheta > 0$, these pictures look like level curves of a function. In Figure 11.7.2 we have $s = -1$, $\vartheta = -0.5$, in the left picture, with the origin at $([0 : 1], [0 : 1])$, showing the four real base points on the axes at infinity. In the right picture, where $s = -1$, $\vartheta = 2$, the four real base points lie on the finite coordinate axes. The picture for $s = -1$, $\vartheta = 0.5$ looks the same as the picture on the left for $s = -1$, $\vartheta = -0.5$, but in inverted affine coordinates. This might be related to the symmetry $\vartheta \mapsto -\vartheta$ in the Weierstrass invariants, listed above under the heading "The singular curves."

For the choice of the members of the real pencils that appear in the pictures, see the text under the heading "Figure 2.3.2," starting with "In all pictures in this book of real pencils …."

11.7.0.11 The Rotation Function

In Chapter 8, the asymptotics for $k \to \infty$ of the real k-periodic fibers for any element of the Mordell–Weil group are related to the properties of the rotation function. In this subsection we collect a number of qualitative properties of the rotation function of the QRT map of the discrete sine–Gordon equation in each of the various cases for the sign of s and the value of ϑ. We will summarize the results under the headings "When … and …" at the end of this subsection.

Figures 11.7.3, 11.7.4, 11.7.5, 11.7.6, and 11.7.7 are numerical plots of the rotation function for representative choices of s and ϑ. In each plot, the coordinate on the horizontal axis is $(2/\pi)\arctan(z)$, where $z = -z_1/z_0$, and therefore the endpoints ± 1 correspond to $z = \pm\infty$, that is, $[z_0 : z_1] = [0 : 1]$, over which we have the singular fiber of Kodaira type I_4. Because the orientation of the Hamiltonian vector field on the biquadratic curve gets reversed if z goes from $z \gg 0$ to $z \ll 0$, the natural continuation of the rotation function ρ is such that $\rho(1+\epsilon) = 1 - \rho(-1+\epsilon)$ for $0 < \epsilon \ll 1$. This is related to the fact that the real Lie algebra bundle $\mathfrak{f}(\mathbb{R})$ over $\mathbb{P}^1(\mathbb{R})$ is a Möbius strip; see Remark 8.1.3.

Assume that $\vartheta \in \mathbb{R}$, $\vartheta \neq 0$, and $\vartheta^2 \neq 1$. At real values of z where the real fiber is smooth but has two connected components, the QRT root might interchange these, and then the rotation number is not defined. However, the QRT map τ always preserves the connected components of the real fibers, and according to Lemma 8.1.5 its rotation function $\rho(z)$ is a real analytic function on every open interval I between values of z over which the real fiber is singular, and such that over I the real fibers are

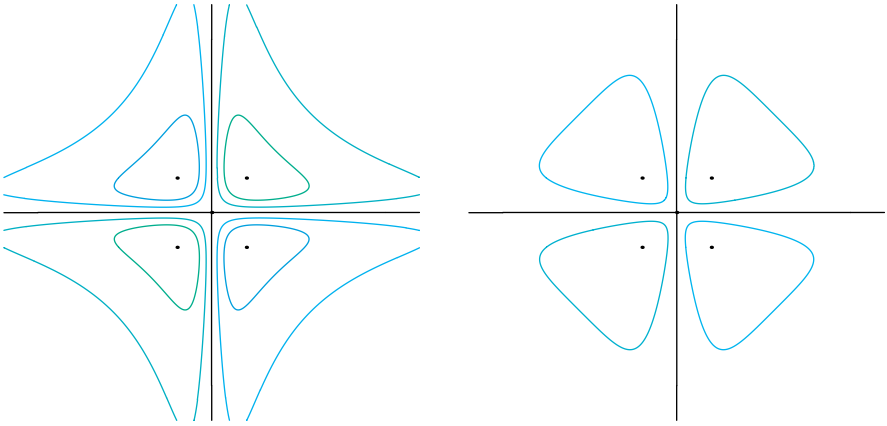


Fig. 11.7.1 The real pencil for $s = 1$, $\vartheta = 0.5$ (left), and $s = 1$, $\vartheta = 2$ (right).

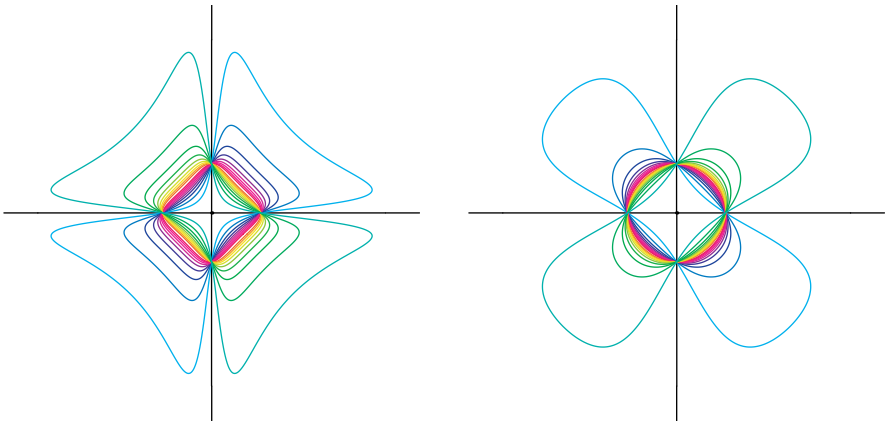


Fig. 11.7.2 The real pencil for $s = -1$, $\vartheta = -0.5$ (left), and $s = -1$, $\vartheta = 2$ (right).

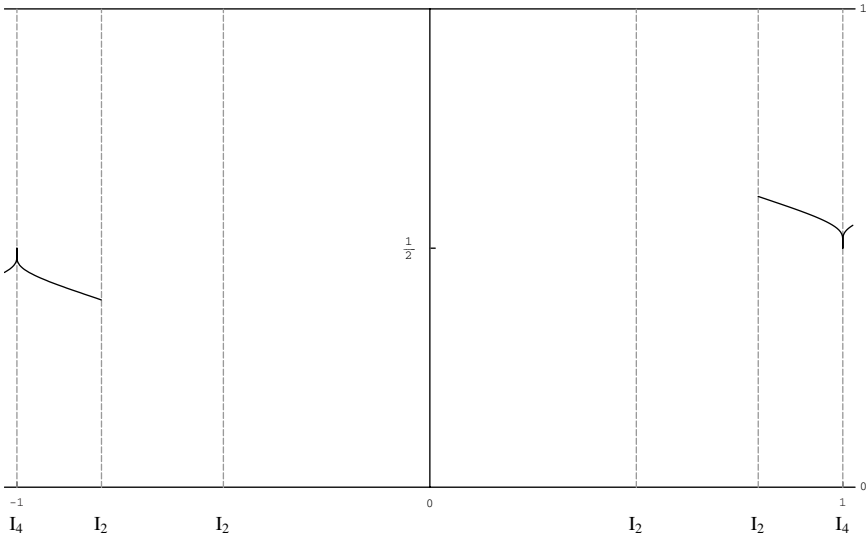


Fig. 11.7.3 Rotation function for $s = 1$ and $\vartheta = 0.5$.

nonempty. Note that if $s = 1$ and $\vartheta > 0$, then $\rho(z)$ is not defined for $-2(\vartheta + 1) < z < 2(\vartheta + 1)$, because the real fibers over these z are empty.

The symmetry mentioned at the beginning of Section 11.7 inverts the orientation on the biquadratic curves, and therefore the rotation function $\rho(z)$ has the symmetry $\rho(-z) = 1 - \rho(z)$. In particular, $\rho(0) = 1/2$. Because the fiber over $z = 0$ is the only 2-periodic one for the QRT map, we have that $\rho(z) \neq 1/2$ for every $z \in \mathbb{R} \setminus \{0\}$.

The Weierstrass invariants g_2, g_3, X, Y, listed above under the heading "The singular curves," are invariant under the sign change $\vartheta \mapsto -\vartheta$ in the parameter ϑ, when keeping s, z_0, and z_1 fixed. This implies that on the regular $[z_0 : z_1]$-intervals

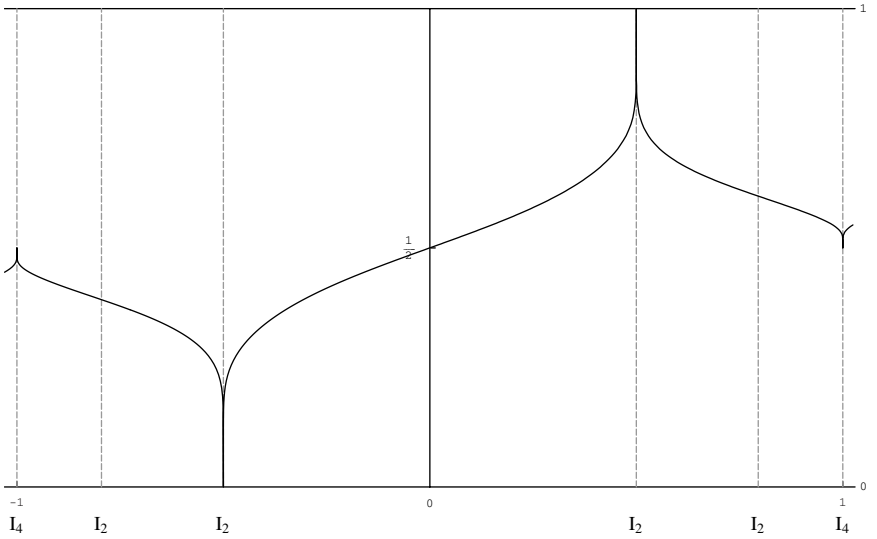


Fig. 11.7.4 Rotation function for $s = 1$ and $\vartheta = -0.5$.

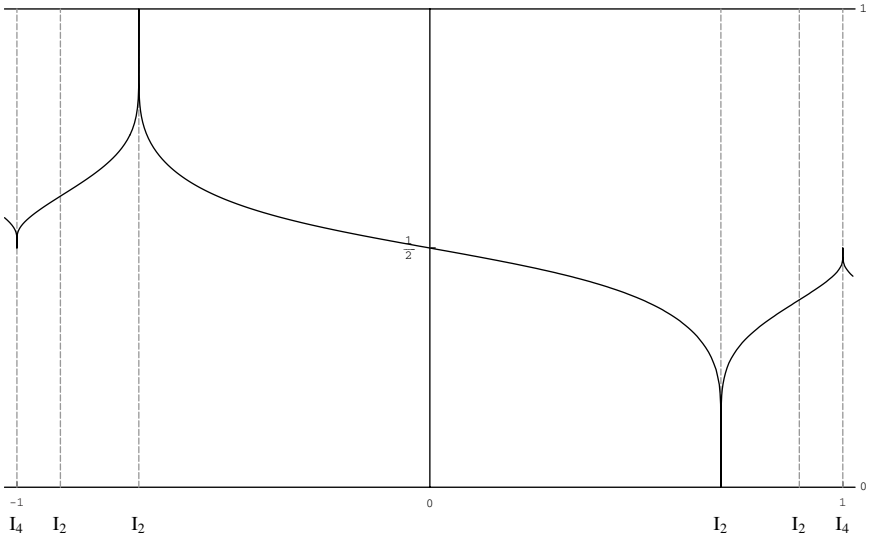


Fig. 11.7.5 Rotation function for $s = 1$ and $\vartheta = -2$.

over which the real fibers are nonempty, the rotation functions remain the same if we replace ϑ by $-\vartheta$. As we have seen in the previous subsection, the real fiber is empty if and only if $s = 1$, $\vartheta > 0$, and $-2(\vartheta + 1) < z < 2(\vartheta + 1)$, where $z = -z_1/z_0$. This corresponds to the empty part in the plot of the rotation function in Figure 11.7.3. For the opposite value of ϑ, see Figure 11.7.4, the plot of the rotation function on the complement of the interval $-2(\vartheta + 1) < z < 2(\vartheta + 1)$ is the same.

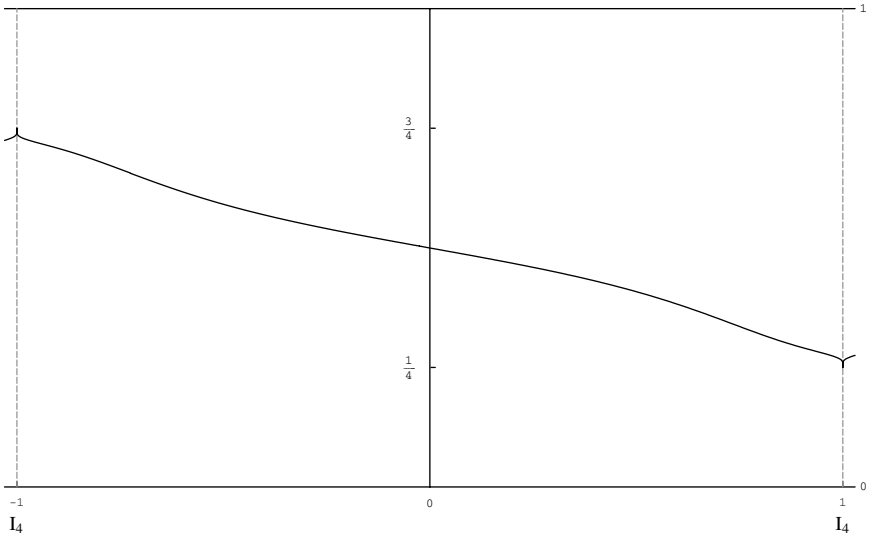


Fig. 11.7.6 Rotation function for $s = -1$ and $\vartheta = 0.5$.

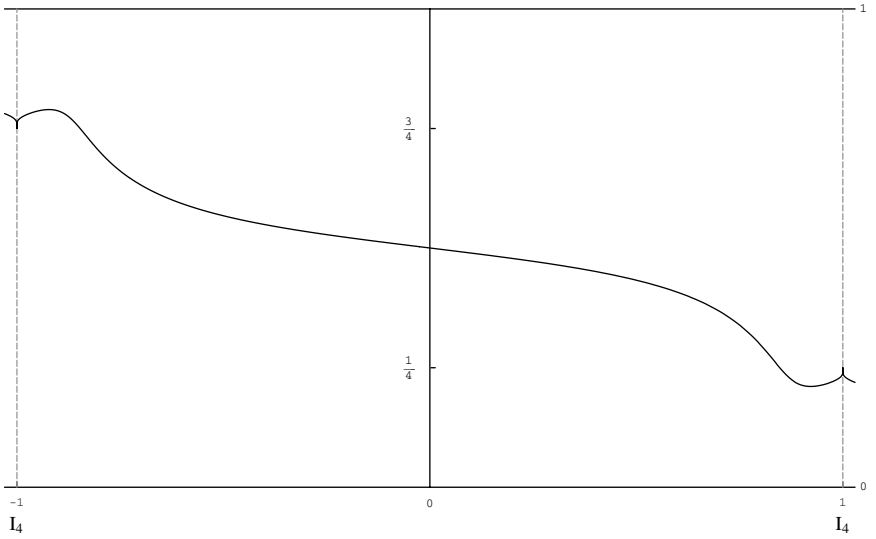


Fig. 11.7.7 Rotation function for $s = -1$ and $\vartheta = 2$.

For $s = 1$ and $\vartheta < 0$ the rotation function extends to the whole z-interval, where we have elliptic real singular fibers of Kodaira type I_2 over $z = \pm 2\,(\vartheta + 1)$, and hyperbolic real singular fibers of Kodaira type I_2 over $z = \pm 1/2$. If $s = -1$, then all the real fibers are nonempty, and we have presented the plots of the rotation function in Figures 11.7.6 and 11.7.7 only for $\vartheta > 0$, since the plots for the opposite sign of ϑ are the same.

We first determine the asymptotic behavior of $\rho(z)$ for $z \to \infty$ when we approach the singular fiber of type I_4 from one side, where we note that the symmetry $\rho(-z) = 1-\rho(z)$ then yields the asymptotic behavior of $\rho(z)$ for $z \to -\infty$ when we approach the singular fiber from the other side. The QRT map (= the square of the QRT root) maps $([1 : x], [1 : 0])$ to $([1 : s/\vartheta\, x], [0 : 1])$, shifting the cycle of four lines of the I_4 fiber over two units. The square of the QRT map (11.7.7) maps $([1 : x], [1 : 0])$ to $([1 : s\,\vartheta^2\, x], [1 : 0])$. On the horizontal axis of all $([1 : x], [1 : 0])$, $x \in \mathbb{R}$, the limit for $z \to \infty$ of $1/z$ times the Hamiltonian vector field of the function p corresponds to $\mathrm{d}x/\,\mathrm{d}t = -x$, which means that its solutions move toward $x = 0$.

Let $s = 1$. Then the nearby real fibers have two connected components, and Remark 8.4.3 implies that $\rho(z)$ converges to $1/2$ when $z \to \infty$, and we have the same limit $1 - 1/2 = 1/2$ when $z \to -\infty$. Because on the $y_1 = 0$ axis the square of the QRT map sends x to $\vartheta^2\, x$, whereas the Hamiltonian flow contracts to $x = 0$, the square of the QRT map is equal to the Hamiltonian flow after a negative or positive time when $\vartheta^2 > 1$ or $\vartheta^2 < 1$, respectively. Therefore, if $|\vartheta| > 1$, then $\rho(1/\zeta)$ increases in an infinitely steep way toward $1/2$ as $\zeta \downarrow 0$; see Remark 8.4.2. The symmetry yields that $\rho(1/\zeta)$ decreases in an infinitely steep way toward $1/2$ as $\zeta \uparrow 0$. On the other hand, if $|\vartheta| < 1$, then $\rho(1/\zeta)$ decreases in an infinitely steep way toward $1/2$ as $\zeta \downarrow 0$, and $\rho(1/\zeta)$ increases in an infinitely steep way toward $1/2$ as $\zeta \uparrow 0$. Note that if $|\theta| = 1$, then the QRT map has order two, and $\rho(z) \equiv 1/2$.

Let $s = -1$. Then the nearby real fibers are connected, and Remark 8.4.3 implies that $\rho(z)$ converges to $3/4$ or to $1 - 3/4 = 1/4$ when $z \to \infty$. Note that in Remark 8.4.3 the rotation number is determined only up to sign and modulo integers. The solution curve of the Hamiltonian vector field on the curve $p = 0$ for $z \gg 1$ crosses the real part of the square $x_0\, x_1\, y_0\, y_1 = 0$ at the real base points, which are $([1 : \vartheta^{1/2}], [1 : 0])$, $([1 : -\vartheta^{-1/2}], [1 : 0])$, $([1 : 0], [1 : \vartheta^{-1/2}])$, and $([1 : 0], [1 : -\vartheta^{-1/2}])$ if $\vartheta > 0$, and $([0 : 1], [1 : (-\vartheta)^{1/2}])$, $([0 : 1], [1 : -(-\vartheta)^{1/2}])$, $([1 : (-\vartheta)^{1/2}], [0 : 1])$, $([1 : -(-\vartheta)^{1/2}], [0 : 1])$ when $\vartheta < 0$. Because the QRT map sends $([1 : x], [1 : 0])$ to $([1 : -1/\vartheta\, x], [0 : 1])$, we conclude, in combination with the orientation of the limit Hamiltonian vector field, that the QRT map is equal to the flow after a positive time, which for both sign choices of ϑ is asymptotically equal to $1/4$ times the period. Because the fourth power of the QRT map sends $([1 : x], [1 : 0])$ to $([1 : \vartheta^4\, x], [1 : 0])$, it follows that $\rho(1/\zeta)$ increases or decreases in an infinitely steep way toward $1/4$ as $\zeta \downarrow 0$, if $\vartheta > 1$ or $0 < \vartheta < 1$, respectively. The symmetry yields that $\rho(1/\zeta)$ decreases or increases in an infinitely steep way toward $3/4$ as $\zeta \uparrow 0$, if $|\vartheta| > 1$ or $0 < |\vartheta| < 1$, respectively. Because there are no singular fibers for $z \in \mathbb{R}$, the rotation function $\rho(z)$ is a real analytic function on $\mathbb{R}$, where the limit $3/4$ for $z \to -\infty$ is different from the limit $1/4$ for $z \to \infty$, implying that for no nonzero real value of ϑ is the rotation function constant.

We only have other real singular fibers of type I_2 only if $s = 1$. If $\vartheta > 0$, then the only ones are the elliptic fibers that occur at the boundary points $z = -2\,(\vartheta + 1)$ and $z = 2\,(\vartheta + 1)$ of the interval over which the real fibers are empty. At these boundary points the rotation function has an extension to a real analytic function on a neighborhood in $\mathbb{R}$ of these boundary points, see Lemma 8.4.1. If $s = 1$ and $\vartheta < 0$,

then we have elliptic singular fibers over $z = 2\,(\vartheta - 1)$ and over $z = 2\,(-\vartheta + 1)$, where again the rotation function has a real analytic extension to a neighborhood in $\mathbb{R}$ of these points, and hyperbolic singular fibers over $z = 2\,(\vartheta + 1)$ and over $z = -2\,(\vartheta + 1)$. Because the QRT map preserves each of the two irreducible components of each singular fiber of type I_2, Remark 8.4.3 implies that the limits of $\rho(z)$, as z approaches the singular value from below or above, are the same for both sides, and both limits are equal to 0 or equal to 1. An analysis of the QRT map and the Hamiltonian vector field on the hyperbolic fibers, as done for the hyperbolic singular fiber of type I_4, leads to the following conclusions. If $s = 1$ and $\vartheta < -1$, when $2\,(\vartheta + 1) < 0$ and $-2\,(\vartheta + 1) > 0$, then $\rho(z)$ increases in an infinitely steep way to 1 if $z \to 2\,(\vartheta + 1)$, from both sides, whereas $\rho(z)$ decreases in an infinitely steep way to 0 if $z \to -2\,(\vartheta + 1)$, again from both sides. If $s = 1$ and $-1 < \vartheta < 0$, when $-2\,(\vartheta + 1) < 0$ and $2\,(\vartheta + 1) > 0$, then $\rho(z)$ decreases in an infinitely steep way to 0 if $z \to -2\,(\vartheta + 1)$, from both sides, whereas $\rho(z)$ increases in an infinitely steep way to 1 if $z \to 2\,(\vartheta + 1)$, again from both sides.

In order to obtain more information about the rotation function in the intervals between the singular values, we apply the basic formula (2.6.8) of the Beukers–Cushman monotonicity criterion. The rational function $\mu = \mathcal{L}\,T$ of the QRT map and the rational function $P = 3\,\Gamma/2\,\Delta$ are given by

$$\mu(z) = 8\,(\vartheta^2 - s)/z\,((z - 2\,\vartheta)^2 - 4\,s)\,((z + 2\,\vartheta)^2 - 4\,s),$$
$$P(z) = -4\,z/((z - 2\,\vartheta)^2 - 4\,s)\,((z + 2\,\vartheta)^2 - 4\,s).$$

We have $\mu(z) \equiv 0$ if and only if $\theta^2 = s$, which according to Proposition 2.5.20 is equivalent to the condition that the QRT map is periodic. This agrees with our previous observation that the QRT map is periodic of period 2 when $\theta^2 = s$.

Let $\vartheta \neq 0$ and $\vartheta^2 \neq s$. Then $\mu(z)$ has simple poles at $z = 0$ and at the finite values of z for which the complex fiber is singular of type I_2, whereas $\mu(z)$ has no zeros. On the other hand, $1/P(z)$ has a simple pole at $z = 0$ and simple zeros at the finite values of z for which the fiber is singular of type I_2, where $\mu(z)/P(z) = -2\,(\vartheta^2 - s)/z^2$. Note that $z = 0$, which is a singular point for (2.6.8), is a regular point for $p(z)$ and $T(z)$ where $0 < T(z) < p(z)$, and the same is true for the points z over which we have an elliptic singular fiber. A straightforward computation with (2.6.8), where $\sigma(z) = \rho'(z)\,p(z)^2$ has the same sign as $\rho'(z)$, shows that at each of these points z we have $\rho'(z) \neq 0$, where for instance the sign of $\rho'(0)$ is equal to minus the sign of $\vartheta^2 - s$. This leads to the following conclusions.

When $s = 1$ and $\vartheta > 0$

The rotation function ρ on the open interval $]2\,(\vartheta + 1),\ \infty[$ has a real analytic extension to a neighborhood in $\mathbb{R}$ of $[2\,(\vartheta + 1),\ \infty[$, denoted by the same letter.

If $\vartheta > 1$, then $0 < \rho(z) < 1/2$ and $\rho'(z) > 0$ for all $z \geq 2\,(\vartheta + 1)$, and $\rho(1/\zeta)$ increases in an infinitely steep way toward $1/2$ when $\zeta \downarrow 0$. If $0 < \vartheta < 1$, then $1/2 < \rho(z) < 1$ and $\rho'(z) < 0$ for all $z \geq 2\,(\vartheta + 1)$, and $\rho(1/\zeta)$ decreases in an infinitely steep way toward $1/2$ when $\zeta \downarrow 0$. If $\vartheta = 1$, then $\rho(z) \equiv 1/2$ on the indicated intervals.

The symmetry $\rho(-z) = 1-\rho(z)$ yields corresponding conclusions for the rotation function on $]-\infty,\ -2\,(\vartheta+1)[$, whereas on the interval $]-2\,(\vartheta+1),\ 2\,(\vartheta+1)[$, over which the real fibers are empty, we have no rotation function. The total variation $V(\rho)$ of the rotation function, see (8.4.2), satisfies $0 \le V(\rho) < 1$, where $V(\rho) = 0$ if and only if $\vartheta = 1$. See Figure 11.7.3.

When $s = 1$ and $\vartheta < 0$

If $\vartheta < -1$, then $\rho'(z) > 0$ for $z < 2\,(\vartheta+1)$, $\rho(1/\zeta)$ decreases in an infinitely steep way to $1/2$ as $\zeta \uparrow 0$ and $\rho(z)$ increases in an infinitely steep way to 1 as $z \uparrow 2\,(\vartheta+1)$. Furthermore, $\rho'(z) < 0$ for $2\,(\vartheta+1) < z < -2\,(\vartheta+1)$, and $\rho(z)$ increases or decreases in an infinitely steep way to 1 or 0 as $z \downarrow 2\,(\vartheta+1)$ or $z \uparrow -2\,(\vartheta+1)$, respectively. The behavior on $]-2\,(\vartheta+1),\ \infty[$ is obtained from the behavior on $]-\infty,\ 2\,(\vartheta+1)[$ by means of the symmetry $\rho(z) = 1-\rho(-z)$. See Figure 11.7.5.

If $-1 < \vartheta < 0$, then $\rho'(z) < 0$ for $z < -2\,(\vartheta+1)$, $\rho(1/\zeta)$ increases in an infinitely steep way to $1/2$ as $\zeta \uparrow 0$ and $\rho(z)$ decreases in an infinitely steep way to 0 as $z \uparrow -2\,(\vartheta+1)$. Furthermore, $\rho'(z) > 0$ for $-2\,(\vartheta+1) < z < 2\,(\vartheta+1)$, and $\rho(z)$ decreases or increases in an infinitely steep way to 0 or 1 as $z \downarrow -2\,(\vartheta+1)$ or $z \uparrow 2\,(\vartheta+1)$, respectively. The behavior on $]2\,(\vartheta+1),\ \infty[$ is obtained from the behavior on $]-\infty,\ -2\,(\vartheta+1)[$ by means of the symmetry $\rho(z) = 1-\rho(-z)$. See Figure 11.7.4.

The total variation of the rotation function, see (8.4.2), is equal to $V(\rho) = 2$ when $s = 1$, $\vartheta < 0$, and $\vartheta \neq -1$. If $\vartheta = -1$, then $\rho(z) = 1/2$ for every $z \neq 0$, where the fiber over $z = 0$ is a hyperbolic singular fiber of type I_4 in the same way as the fiber over $z = \infty$.

When $s = -1$ and $\vartheta \neq 0$

The rotation function $\rho(z)$ is a real analytic function on $\mathbb{R}$, which converges to $3/4$ as $z \to -\infty$ and to $1/4$ as $z \to \infty$.

If $0 < |\vartheta| \le 1$, then $\rho'(z) < 0$ for every $z \in \mathbb{R}$. If $0 < |\vartheta| < 1$, then $\rho(1/\zeta)$ increases or decreases in an infinitely steep way to $3/4$ or $1/4$ as $\zeta \uparrow 0$ or $\zeta \downarrow 0$, respectively. The total variation of the rotation function, see (8.4.2), is equal to $V(\rho) = 1/2$. See Figure 11.7.6.

If $|\vartheta| > 1$, then there is a unique $z_+ > 0$ such that $\rho'(z) > 0$ for $z < -z_+$, $\rho'(-z_+) = 0$ and $\rho''(-z_+) < 0$, $\rho'(z) < 0$ for $-z_+ < z < z_+$, $\rho'(z_+) = 0$ and $\rho''(z_+) > 0$, and finally $\rho'(z) > 0$ for $z > z_+$. Moreover, $\rho(1/\zeta)$ decreases or increases in an infinitely steep way toward $3/4$ or $1/4$ as $\zeta \uparrow 0$ or $\zeta \downarrow 0$, respectively. The total variation $V(\rho)$ of the rotation function satisfies $1/2 < V(\rho) < 3/2$. See Figure 11.7.7.

The Rotation Number on the Elliptic Singular Fibers

Let $s = 1$ and $\vartheta > 0$. Then we have elliptic singular fibers of type I_2 over the points $z^{\mathrm{ell}} := \pm 2\,(\vartheta+1)$, where $g_2 = 64\,\vartheta^2/3$, $X = (3\,\vartheta^2 + 2\,\vartheta + 3)/3$, and $Y = \mp 2\,(\vartheta+1)^2\,(\vartheta-1)$. It therefore follows from (8.4.1) that

$$\rho(z^{\mathrm{ell}}) = \frac{1}{\pi} \arccos\left(\pm\frac{\vartheta - 1}{\vartheta + 1}\right). \tag{11.7.9}$$

If $s = 1$ and $\vartheta < 0$, then we have elliptic singular fibers of type I_2 over the points $z^{\mathrm{ell}} := \pm 2\,(\vartheta - 1)$, and a similar computation yields that

$$\rho(z^{\mathrm{ell}}) = \frac{1}{\pi} \arccos\left(\pm\frac{1 + \vartheta}{1 - \vartheta}\right). \tag{11.7.10}$$

Remark 11.7.2. The sine–Gordon map is a one-parameter subfamily of the family of pencils of symmetric biquadratic curves that have the square $x_0\,x_1\,y_0\,y_1 = 0$ as a member. The identity component of the group of all linear transformations $(x,\ y) \mapsto (L\,x,\ L\,y)$ that leave the square invariant is equal to the set of all such transformations for which L is a diagonal matrix, and it follows that there are $6-2 = 4$ essential parameters in the family of pencils of symmetric biquadratic curves that have the square as a member.

If the pencil contains another square $(a\,x_0 - x_1)\,(b\,x_0 - x_1)\,(a\,y_0 - y_1)\,(b\,y_0 - y_1)$, which for the sine–Gordon map happens if $\vartheta = s$, then there is one modulus for the configuration of the four points $[1 : 0]$, $[0 : 1]$, $[1 : a]$, $[1 : b]$ on the complex projective line. Therefore the family of pencils of symmetric biquadratic curves that have two squares as members has one essential parameter. ⊘

Remark 11.7.3. The one-parameter family of elliptic curves introduced by Edwards [55] is the pencil of biquadratic curves generated by ${x_1}^2\,{y_0}^2 + {x_0}^2\,{y_1}^2 = 0$ and ${x_0}^2\,{y_0}^2 + {x_1}^2\,{y_1}^2 = 0$. This pencil is isomorphic to the pencil for the sine–Gordon map with $s = 1$ and $\vartheta^2 = 1$. In [55] the object of study is the pencil of elliptic curves, not the QRT map or its root, which in the case $s = 1$ and $\vartheta^2 = 1$ is not particularly interesting. In this way [55] is quite complementary to our discussion of the sine–Gordon map. ⊘

11.7.1 The Hard Hexagon Model

Quispel, Roberts, and Thompson [169, Section 4.2] mentioned that the mapping $(x,\ y) \mapsto (y,\ \xi(x,\ y))$, where $\xi(x,\ y) = (\delta^2\,y - \delta)/(y^2\,x)$, where δ is a constant, occurs in the solution of the Yang–Baxter equations for the hard hexagon model in statistical mechanics, and recognized this mapping as a QRT root. Indeed, we have (11.7.3) with $h_0 = 0$, $h_1 = \delta^2$, $h_2 = -\delta$, $k_0 = 1$, $k_1 = 0$, and $k_2 = 0$. Therefore the mapping is the QRT root defined by

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \delta^2 \\ 0 & \delta^2 & -\delta \end{pmatrix} \quad \text{and} \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Weierstrass invariants of the pencil of biquadratic polynomials, see Corollary 2.5.10, Corollary 2.5.13, and Proposition 10.1.6, are computed as

$$
\begin{aligned}
12\,g_2 &= 16\,\delta^2\,{z_0}^4 + 24\,\delta^4\,{z_0}^3\,z_1 + 8\,\delta\,{z_0}^2\,{z_1}^2 + {z_1}^4,\\
-216\,g_3 &= 8\,\delta^3\,(8 + 27\,\delta^5)\,{z_0}^6 + 144\,\delta^5\,{z_0}^5\,z_1 + 48\,\delta^2\,{z_0}^4\,{z_1}^2 + 36\,\delta^4\,{z_0}^3\,{z_1}^3\\
&\quad + 12\,\delta\,{z_0}^2\,{z_1}^4 + {z_1}^6,\\
\Delta &= -\,\delta^9\,{z_0}^8\,(\delta^2\,(16 + 27\,\delta^5)\,{z_0}^4 + 36\,\delta^4\,{z_0}^3\,z_1 + 8\,\delta\,{z_0}^2\,{z_1}^2 + \delta^3\,z_0\,{z_1}^3\\
&\quad + {z_1}^4),\\
12\,X &= -\,8\,\delta\,{z_0}^2 + {z_1}^2,\\
Y &= \delta\,{z_0}^2\,(\delta^3\,z_0 + z_1),\\
12\,X_\rho &= 4\,\delta\,{z_0}^2 + {z_1}^2,\\
Y_\rho &= -\,\delta^4\,{z_0}^3.
\end{aligned}
$$

We have $\delta = 0$ if and only if $\Delta \equiv 0$, when every biquadratic curve is singular. For this reason we assume that $\delta \neq 0$ in the sequel. It follows from Table 6.3.2 that the factor ${z_0}^8 = 0$ in $\Delta = 0$ corresponds to a singular fiber of type I_8. The discriminant of the remaining factor of degree 4 in Δ is equal to $-8\,\delta^3\,(8 + 27\,\delta^5)$. It follows that the configuration of the singular fibers is $\mathrm{I}_8\ 4\ \mathrm{I}_1$ if and only if $\delta \neq 0$ and $\delta^5 \neq -8/27$. Lemma 9.2.6 yields that the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$ or to $\mathbb{Z}$. The degree of the modulus function J is equal to the maximal one $j = 12$; see (6.2.48).

If we use the affine coordinate $z_0 = 1$, $z_1 = -z$, then the rational functions $P = 3\,\Gamma/2\,\Delta$ and the Manin function $\mu = \mathcal{L}\,T$ of the QRT map, as defined in the Sections 2.5.3 and 2.6.3, are given by $P(z) = -(8\,z - 9\,\delta^3)/\Delta_0(z)(z)$ and $\mu(z) = -2\,\delta\,(3\,\delta^2\,z - 32)(8\,z - 9\,\delta^3)/\Delta_0(z)$, where

$$\Delta_0(z) = z^4 - \delta^3\,z^3 + 8\,\delta\,z^2 - 36\,\delta^4\,z + 27\,\delta^7 + 16\,\delta^2$$

is the factor of $\Delta(z)$ whose zeros correspond to the four singular fibers of type I_1. The simple form of these functions is an invitation to apply the formula (2.6.8) to study the monotonicity properties of the rotation function when $\delta \in \mathbb{R}$.

In order to understand how the singular fiber of type I_8 comes about, we observe that, as in the sine–Gordon map, the curve $q^1 := x_0\,x_1\,y_0\,y_1 = 0$ is a square of two horizontal and two vertical axes. The curve

$$q^0 := {x_1}^2\,{y_1}^2 + x_0\,y_0\,(\delta^2\,(x_1\,y_0 + x_0\,y_1) - \delta\,x_0\,y_0)$$

passes through the corner points $b_1 = ([1:0],\,[0:1])$ and $b_2 = ([1:0],\,[0:1])$, and therefore b_1 and b_2 are base points of the pencil $z_0\,p^0 + z_1\,p^1 = 0$ of biquadratic curves. The other base points are $b_3 = ([0:1],\,[1:1/\delta])$ and $b_4 = ([1:1/\delta],\,[0:1])$. At b_1 and b_2, the curve $p^0 = 0$ has a contact of order two with the horizontal and the vertical axes, respectively. Therefore b_1 and b_2 are base points of order 3, and the -1 curves that appear after the first two blowing-up transformations over these base points give rise to $2 \times 2 = 4$ additional complex projective lines in the cycle of complex projective lines that is the proper image of the curve $p^1 = 0$ in the elliptic surface. In other words, the singular fiber of type I_8 comes from the biquadratic curve

$p^1 = 0$ as in case 7(a8) in Section 12.1. If $0 \le k < b$, then the k-power of the QRT root ρ^S shifts the complex projective lines in the cycle over k units, and therefore the contribution of the singular fiber I_8, see Lemma 7.5.3, is equal to $k\,(8-k)/8$. All the other singular fibers are of type I_1, hence irreducible, and therefore do not contribute to the powers of the QRT root. It therefore follows from (4.3.2) that for any $k \in \mathbb{Z}$, the number of k-periodic fibers in S^{reg}, counted with multiplicities, of the QRT root is given by

$$\begin{aligned} \nu((\rho^S)^k) &= \frac{9}{16}\,k^2 + 4\,\left\{\frac{k}{8}\right\}\left(1 - \left\{\frac{k}{8}\right\}\right) - 1 \qquad (11.7.11)\\ &= 36\,n^2 + 9\,r\,n + \frac{1}{2}\,r\,(r+1) - 1 \quad \text{if} \quad k = 8\,n + r,\ 0 \le r < 8, \end{aligned}$$

where $\{x\}$ denotes the fractional part of x. We also have used that $\nu(\rho^S) = 0$, see Section 10.1, and $\chi(S, \mathcal{O}) = 1$; see Lemma 9.1.2(iii).

11.8 Jogia's Example

In Jogia [98, Example 3.11] the pencil is defined by the biquadratic polynomials

$$\begin{aligned} p^0 &= {x_1}^2\,{y_1}^2 + x_0\,x_1\,{y_0}^2 - 3\,{x_0}^2\,y_0\,y_1,\\ p^1 &= {x_1}^2\,{y_1}^2 + {x_1}^2\,{y_0}^2 + {x_0}^2\,{y_1}^2 + 3\,x_0\,x_1\,y_0\,y_1, \end{aligned} \qquad (11.8.1)$$

where p^0 is not symmetric. Let τ denote the QRT transformation defined by (11.8.1).

The point $o = ([1 : 0], [1 : 0])$ is a base point of multiplicity two, where the common tangent line of the curves is not horizontal. Because the horizontal switch maps o to $([z_1 : -z_0], [1 : 0])$, there are no other base points on the horizontal axis through o. It follows that the proper image H in the elliptic surfaces S of the horizontal axis through o is a section, which moreover is disjoint from the section E equal to the -1 curve that appears at the second blowing up over o. According to Lemma 7.1.1, there is a unique $\tau_H \in \mathrm{Aut}(S)^+_\kappa$ that maps E to H. According to Theorem 4.3.2, τ' can be obtained as the QRT map of some other pencil of biquadratic curves, which we will not pursue here. One of the goals in Jogia [98, Example 3.11] is to compare the rotation function of the QRT map τ with the rotation function of the QRT-like map τ_H that accompanies τ.

The Weierstrass invariants of the pencil of symmetric biquadratic polynomials, see Corollaries 2.5.10 and 2.5.13, are computed as

$$\begin{aligned} 12\,g_2 &= z_1\,(-696\,{z_0}^3 - 696\,{z_0}^2\,z_1 + 25\,{z_1}^3),\\ 216\,g_3 &= -1944\,{z_0}^6 - 3888\,{z_0}^5\,z_1 - 1944\,{z_0}^4\,{z_1}^2 + 21636\,{z_0}^3\,{z_1}^3\\ &\quad + 21636\,{z_0}^2\,{z_1}^4 - 125\,{z_1}^6,\\ \Delta &= {z_0}^2\,(z_0 + z_1)\,(-2187\,{z_0}^9 - 6561\,{z_0}^8 - 6561\,{z_0}^7\,{z_1}^2 - 148618\,{z_0}^6\,{z_1}^3 \end{aligned}$$

$$
\begin{aligned}
&\quad - 292862\, z_0{}^5\, z_1{}^4 - 146431\, z_0{}^4\, z_1{}^5 - 250157\, z_0{}^3\, z_1{}^6 - 250157\, z_0{}^2\, z_1{}^7 \\
&\quad + 2375\, z_1{}^9), \\
X_\tau &= (17/12)\, z_1{}^2, \\
Y_\tau &= 3\,(z_1{}^3 - z_0{}^2\, z_1 - z_0{}^3), \\
X_{\tau_H} &= (233/12)\, z_1{}^2, \\
Y_{\tau_H} &= 3\,(-57\, z_1{}^3 - z_0{}^2\, z_1 - 1),
\end{aligned}
$$

where X_{τ_H} and Y_{τ_H} denote the affine coordinates of the image of the point at infinity under the transformation τ_H acting on the Weierstrass curve. For the latter, we have used Lemma 2.4.13, together with the second-order Taylor expansion at $y = 0$ with respect to y of the solution $x = x(y, z)$ of $p_z(x,\ y) = 0$ such that $x(0, z) = 1/z$, if we use the affine coordinates $x_0 = 1$, $x_1 = x$, $y_0 = 1$, $y_1 = y$, and $z_0 = 1$, $z_1 = -z$. Note that X_{τ_H} and Y_{τ_H} are polynomials in z. Jogia used a differently scaled Weierstrass form, such that our x and y coordinates are equal to -1 and -2 times the respective x and y coordinates of Jogia [98, Example 3.11].

For the determination of the Kodaira types of the singular fibers we use Table 6.3.2, and for the description of the real fibers we use Sections 2.6.2 and 8.3, and in particular Remark 8.3.1. The discriminant Δ has a double zero at $z_0 = 0$, where $g_3 < 0$, and therefore the corresponding singular fiber is of Kodaira type I_2, hyperbolic with respect to the real structure. In terms of the usual affine coordinate $z_0 = 1$, $z_1 = -z$, the other ten zeros of Δ occur at $z = -10.759\ldots$, $-0.303242 \pm 0.918014\ \mathrm{i}\ldots$, $-0.113965 \pm 0.233027\ \mathrm{i}\ldots$, $0.22793\ldots$, $0.606483\ldots$, 1, $1.00972\ldots$, and $9.74925\ldots$. All these zeros are simple and therefore correspond to singular fibers of Kodaira type I_1. It follows that the configuration of the singular fibers is $\mathrm{I}_2\ \ 10\ \mathrm{I}_1$, and according to Lemma 9.2.6, the Mordell–Weil group is isomorphic to $\mathbb{Z}^7$.

Because all singular curves in the pencil of biquadratic curves are irreducible, none of the curves in the pencil contains a horizontal or vertical axis. It follows that the action on $\mathrm{H}_2(S,\ \mathbb{Z})$ of the QRT automorphism τ^S of the rational elliptic surface S, defined by the pencil of biquadratic curves, is as in Corollary 5.1.9. In particular, for each $k \in \mathbb{Z}$ the number of k-periodic fibers, counted with multiplicities, is equal to $k^2 - 1$.

Because the base point $o = ([1 : 0], [1 : 0])$ is real, all the real curves are nonempty, and have one or two components, when $\Delta < 0$ or $\Delta > 0$, respectively. At $z = 0$ we have $\Delta < 0$, and therefore we have $\Delta < 0$ and connected real fibers when $-10.759\ldots < z < 0.22793\ldots$, $0.606483\ldots < z < 1$, and $1.00972\ldots < z < 9.74925\ldots$, whereas $\Delta > 0$ and the real fibers have two connected components when $z < -10.759\ldots$, $0.22793\ldots < z < 0.606483\ldots$, $1 < z < 1.00972\ldots$, and $z > 9.74925\ldots$.

The six zeros of g_3 are at $z = -13.6269\ldots$, $-0.514437\ldots$, $0.257218 \pm 0,329525\ \mathrm{i}\ldots$, $1.00588\ldots$, and $12.6211\ldots$. All these zeros are simple, and because $g_3 < 0$ at $z = 0$, this leads to a complete determination of the sign behavior of g_3 on the real z-axis. In particular, we have $g_3 < 0$ and a hyperbolic real singular

fiber at $z = 0.22793\ldots$, $0.606483\ldots$, and 1, whereas we have $g_3 > 0$ and an elliptic real singular fiber at $z = -10.759\ldots$, $1.00972\ldots$, and $9.74925\ldots$.

In the following discussion of the rotation functions of τ and τ_H, we use Lemma 8.1.5 and Section 8.4. We have four open z-intervals bounded by hyperbolic singular points: $I_1 =]-\infty,\ 0.22793\ldots[$, $I_2 =]0.22793\ldots,\ 0.606483\ldots[$, $I_3 =]0.606483\ldots,\ 1[$, and $I_4 =]1,\ \infty[$. In I_1 we have the elliptic singular point $z = -10.759\ldots$. Over $]-\infty, -10.759\ldots[$ the real fibers have two connected components, whereas over $]-10.759\ldots,\ 0.22793\ldots[$ the real fibers are connected. It follows that both τ and τ_H preserve the two connected components of the real fibers over $]-\infty, -10.759\ldots[$, and both τ and τ_H have well-defined respective rotation functions ρ_τ and ρ_{τ_H}, which extend to real analytic functions on I_1. Because over I_3, which does not contain singular z-values, the real fibers are connected, both τ and τ_H have well-defined rotation functions on I_3, which are real analytic. In I_4 we have the elliptic singular points $z = 1.00972\ldots$ and $z = 9.74925\ldots$, and a similar reasoning as for I_1 leads to the conclusion that both τ and τ_H preserve the two connected components of the real fibers over the z-intervals $]1,\ 1.00972\ldots[$ and $]9.74925\ldots,\ \infty[$. It follows that both τ and τ_H have well-defined rotation functions that extend to real analytic function on I_4.

The interval $I_2 =]0.22793\ldots,\ 0.606483\ldots[$ does not contain elliptic singular points, whereas over it each real fiber has two connected components. For this reason the above arguments that τ and τ_H preserve the connected components of the real fibers do not apply. In order to determine what happens, we observe that if the Weierstrass curve $y^2 - 4\,x^3 + g_2\,x + g_3$ has a hyperbolic singular point, then $\Delta := {g_2}^3 - 27\,{g_3}^2 = 0$ with $g_3 < 0$, hence $g_2 > 0$, and the singular point occurs at $y = 0$, $x = \sqrt{g_2/12}$. For a smooth perturbation of the Weierstrass curve with two connected components, the points with $x > \sqrt{g_2/12}$ and $x < \sqrt{g_2/12}$ lie on the unbounded and bounded connected component, respectively. It follows that the translation that maps the point at infinity to the point $[1 :\ X :\ Y]$, on the smooth perturbation of the Weierstrass curve with two connected components, preserves and interchanges the connected components if and only if $X > \sqrt{g_2/12}$ and $X < \sqrt{g_2/12}$, respectively. At the left endpoint $z = 0.22793\ldots$ we have $X_\tau = 0.0735988\ldots < \sqrt{g_2/12} = 0.922512\ldots < X_{\tau_H} = 1.00874\ldots$. It follows that τ interchanges the two connected components of the real fibers over I_2, and τ has no well-defined rotation function on I_2, whereas τ_H preserves the connected components of the real fibers over I_2, and τ_H has a well-defined real analytic rotation function on I_2. As an additional check, at the other endpoint $z = 0.606483\ldots$ of I_2 we have $X_\tau = 0.521081\ldots < \sqrt{g_2/12} = 1.0849\ldots < X_{\tau_H} = 7.14188\ldots$.

The fact that over I_2 the QRT map interchanges the connected components of the real fibers implies that both for $z \uparrow 0.22793\ldots$ and for $z \downarrow 0.606483\ldots$, the rotation number $\rho_\tau(z)$ of the QRT map has the limit value $1/2$. This is the only example in the literature that I have seen in which a limit value not equal to 0 modulo 1 occurs at a hyperbolic singular fiber of type I_1. See Remark 8.4.3.

Figures 11.8.1 and 11.8.3 are computer plots of the respective rotation functions ρ_τ and ρ_{τ_H} of the QRT map τ and its QRT-like companion τ_H. The horizontal coordinate in the plots is $(2/\pi)\ \arctan z$, and therefore the endpoints ± 1 correspond to $z = \pm\infty$,

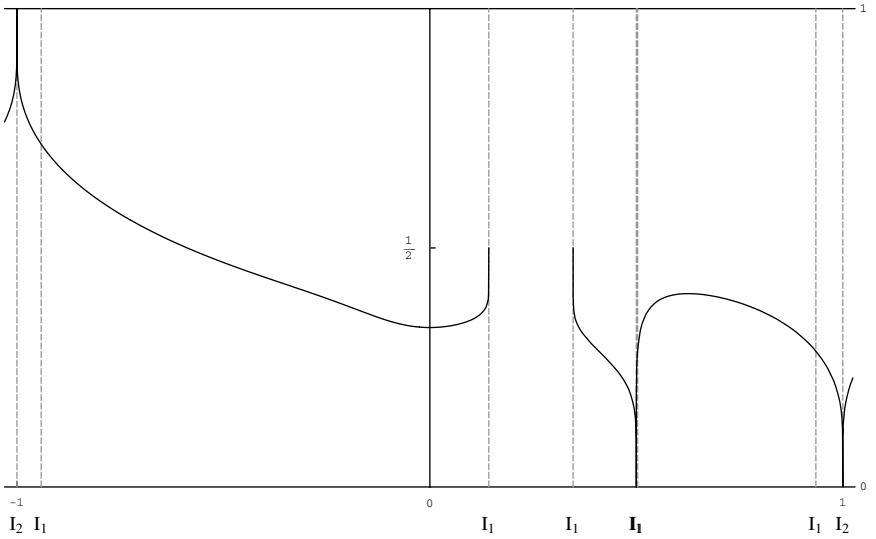


Fig. 11.8.1 Rotation function for the QRT map.

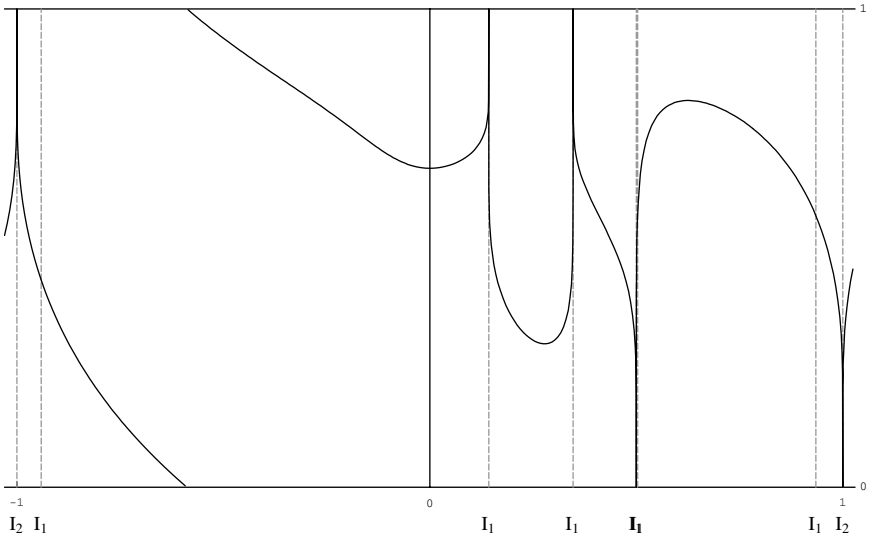


Fig. 11.8.2 Rotation function for the square of the QRT map.

that is, $[z_0 : z_1] = [0 : 1]$, over which we have the complex singular fiber of Kodaira type I_2. Because the orientation of the Hamiltonian vector field on the biquadratic curve gets reversed if z goes from $z \gg 0$ to $z \ll 0$, the natural continuation of the rotation function ρ is such that $\rho(1+\epsilon) = 1 - \rho(-1+\epsilon)$ for $0 < \epsilon \ll 1$. This is related to the fact that the real Lie algebra bundle $\mathfrak{f}(\mathbb{R})$ over $\mathbb{P}^1(\mathbb{R})$ is a Möbius strip; see Remark 8.1.3.

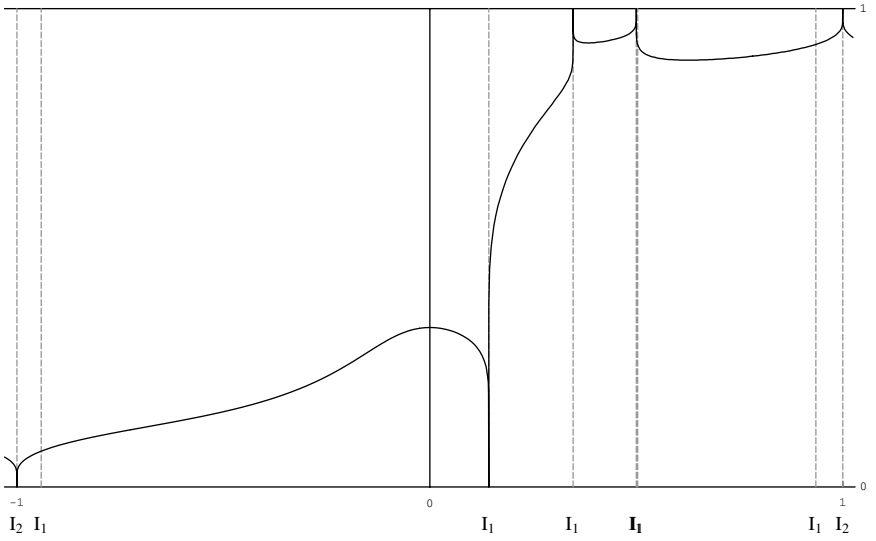


Fig. 11.8.3 Rotation function for τ_H.

Because the singular values $z = 1$ and $z = 1.00972\ldots$ are so close to each other, the captions I_1 indicating the Kodaira type of the singular fibers almost coincide. The derivative $\rho'_\tau(z)$ of the rotation function of the QRT map has two simple zeros. The derivative $\rho'_{\tau_H}(z)$ of the rotation function of the companion τ_H of the QRT map has three simple zeros. This is the largest number of zeros of the derivative of the rotation function of a real QRT transformation that I have seen in the examples in the literature.

In Jogia [98, Example 3.11] the computer plots of ρ_τ and ρ_{τ_H} have been compared on the interval $]9.74925\ldots,\ \infty[$, where $z = 9.74925\ldots$ is the largest finite singular value of z. Over $z = 9.74925\ldots$ we have an elliptic singular fiber of Kodaira type I_1. Recall that over $z = \infty$ we have a hyperbolic singular fiber of type I_2. Over the interval $]9.74925\ldots,\ \infty[$, contained in the much larger interval $I_4 =]1,\ \infty[$ to which the rotation functions of τ and τ_H have a real analytic extension, the real fibers have two connected components, preserved both by τ and by τ_H. One of these connected components shrinks to a point as $z \downarrow 9.74925\ldots$, whereas the other survives as a connected real fiber over the interval $]1.00972\ldots,\ 9.74925\ldots[$. Over the small interval $]1,\ 1.00972\ldots[$ we again have two connected components, each of which is preserved by both τ and τ_H.

Figure 11.8.2 The square τ^2 of the QRT map preserves the connected components of the real fibers over I_2, and therefore its rotation function ρ_{τ^2} is a well-defined real analytic function on I_2. On the other intervals I_1, I_3, I_4, we have $\rho_{\tau^2} = 2\,\rho_\tau$ modulo 1. We have added a computer plot of ρ_{τ^2} in Figure 11.8.2. Its derivative has three simple zeros, since, in comparison to the rotation function of the QRT map, one more zero of the derivative appears in the interval I_2, on which ρ_{τ^2} is defined

and on which ρ_τ is not defined. The situation is similar to that for the "generic QRT map" in Figures 9.2.4 and 9.2.5.

It appears that $\rho_\tau(z)$, for increasing z, increases through the value $1/2$ once in the regular interval I_1. At this value ζ of z the rotation function of τ^2 decreases, for increasing z, to 0, where it jumps to 1, after which it continues to decrease. The real fiber over $z = \zeta$ is a fixed-point fiber for τ^2, that is, a 2-periodic fiber for τ. For regular values of z the 2-periodic fibers are determined by the third-order equation $Y = 3\,(-z^3 + z - 1) = 0$, corresponding to the $2^2 - 1 = 3$ complex 2-periodic fibers for τ. The equation $Y = 0$ has one real solution $z = -1.32472\ldots$, which indeed lies in $I_1 =]-\infty,\ 0.22793\ldots[$. The other two solutions of $Y = 0$ are $z = 0.662359 \pm 0.56228\ \mathrm{i}\ldots$, the two 2-periodic fibers over that have no real points and are complex conjugate to each other.

11.9 A Non-QRT Map with the Weierstrass Data of a QRT Map

Viallet, Grammaticos, and Ramani [200, Section 2] presented the rational transformation

$$A : (x,\ y) \mapsto (y,\ Y(x,\ y)), \tag{11.9.1}$$

where

$$Y(x,\ y) := \left(\frac{(y^2 - b\,y + 1)\,(y^2 - 1)}{(x\,y - 1)\,(q\,y^2 - 1)} + 1 \right) / y. \tag{11.9.2}$$

Because $Y(Y(x,\ y),\ y) \equiv x$, the mapping $\iota_1 : (x,\ y) \mapsto (Y(x,\ y),\ y)$ is an involution, and $A = \sigma \circ \iota_1$ is the composition of the involution ι_1 with the symmetry switch $\sigma : (x,\ y) \mapsto (y,\ x)$, which is another involution.

Actually, instead of b and q, [200] used the parameters $a \neq 0$ and p such that

$$b = a + \frac{1}{a}, \quad q = p^2, \tag{11.9.3}$$

which turn out to be useful in the computations. They stated that the transformation A is of infinite order and leaves the rational function $p^0(x,\ y)/p^1(x,\ y)$ invariant, where

$$p^1(x,\ y) := (x\,y - 1)^2 \quad \text{and} \tag{11.9.4}$$

$$p^0(x,\ y) := \left((x - y)^2 - q\,(x\,y - 1)^2\right)\left((x + y - b)^2 - q\,(x\,y - 1)^2\right). \tag{11.9.5}$$

They remarked that the polynomial $p^0(x,\ y)$ in (11.9.5) is biquartic, and not biquadratic as would be the case for a QRT transformation. However, using the algorithm of van Hoeij [88], they checked that the level curves of p_0/p_1 are of genus one, and for various choices of these, they computed their Weierstrass normal form. They also observed that on these elliptic level curves the map acts as translations, just as it does for QRT transformations. Finally, on [200, p. 189] they wrote:

At this point one may wonder whether all integrable second order mappings with a rational invariant can be brought to a QRT form by a birational change of coordinates of the 2-plane. This question is open at the moment.

The answer to this question is no, because Proposition 11.9.1 and Lemma 11.9.2 imply that their transformation (11.9.1) is not birationally conjugate to a QRT map if $q \neq 0$, $q \neq 1$, and $(b,\ q) \neq (0, -1)$.

In Section 11.9.3 we present the Weierstrass data of the map (11.9.1), that is, a Weierstrass normal form of the curves and of the mapping acting on the Weierstrass curves. At the same time that I obtained the Weierstrass data, Viallet independently determined g_2 and g_3 by means of the algorithm of van Hoeij [88], as a polynomial expression in a, p, and z, and checked that after a suitable rescaling his formulas agree with the formulas for g_2 and g_3 given here.

It turns out that under the sole condition that $q \neq 0$, the Weierstrass model of the transformation (11.9.1) is equal to the Weierstrass model of a QRT transformation. I have no explicit formula for the QRT transformation, which is not unique. Therefore, although (11.9.1) is not birationally conjugate to a QRT transformation, it behaves very much like a QRT transformation.

11.9.1 The Elliptic Fibration

In order to obtain a pencil $\mathfrak{P}$ of biquartic curves $z_0\, p^0 + z_1\, p^1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$, parametrized by $[z_0 :\ z_1] \in \mathbb{P}^1$, where $z := p^0/p^1 = -z_1/z_0$, we use homogeneous coordinates $([x_0 :\ x_1],\ [y_0 :\ y_1])$, where $x = x_1/x_0$ and $y = y_1/y_0$ in (11.9.5) and (11.9.4). With the abbreviation

$$H := x_1\, y_1 - x_0\, y_0, \tag{11.9.6}$$

the bihomogeneous versions of bidegree $(4,\ 4)$ of p^0 and p^1 are

$$p^0 = \Big((x_1\, y_0 - x_0\, y_1)^2 - q\, H^2\Big)\ \Big((x_1\, y_0 + x_0\, y_1 - b\, x_0\, y_0)^2 - q\, H^2\Big) \tag{11.9.7}$$

and

$$p^1 = r^2, \quad r := (x_1\, y_1 - x_0\, y_0)\, x_0\, y_0 = H\, x_0\, y_0. \tag{11.9.8}$$

The base points of our pencil of biquartic curves are the four points

$$b_1 := ([0 :\ 1],\ [p,\ 1]),\ \ b_2 := ([0 :\ 1],\ [-p,\ 1]),$$
$$b_3 := ([p :\ 1],\ [0 :\ 1]),\ b_4 := ([-p :\ 1],\ [0 :\ 1]),$$

"at infinity," and the four "finite" points

$$b_5 := ([1 :\ 1],\ [1,\ 1]), \qquad b_6 := ([1 : -1],\ [1, -1]),$$
$$b_7 := ([1 :\ a],\ [1 :\ 1/a]),\ b_8 := ([1 :\ 1/a],\ [1 :\ a]),$$

on the curve $x\, y - 1 = 0$.

We assume first that the eight base points are distinct, that is, $q \neq 0$ and $b^2-4 \neq 0$. At each of the base points b_j, each of the members of the pencil is singular. That is, in the notation of Lemma 3.2.7 we have $k \geq 2$; hence for each $1 \leq j \leq 8$ the multiplicity m_{b_j} of the base point b_j satisfies $m_{b_j} \geq k^2 \geq 4$. On the other hand, Lemma 2.4.1 implies that the intersection number of two biquartic curves without common irreducible components is equal to $4 \cdot 4 + 4 \cdot 4 = 32$, and therefore the number of base points, when counted with multiplicites, is equal to 32. It follows that $m_{b_j} = 4$ and $k = 2$ for each base point. Therefore, if $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ denotes the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ in all the eight base points, then Lemma 3.2.7 implies that the proper transform $\widetilde{\mathfrak{P}}$ of the pencil $\mathfrak{P}$ is a pencil of curves in S without base points, and the rational function $(p^0/p^1) \circ \pi$ on S extends to a surjective complex analytic map (= algebraic morphism) $\varphi : S \to \mathbb{P}^1$. Note that because QRT surfaces are also obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ eight times, S has the same topological type as a QRT surface.

It is readily verified that the generic member C of $\mathfrak{P}$ is irreducible. On the other hand, if K denotes the canonical line bundle of $\mathbb{P}^1 \times \mathbb{P}^1$, then $-K \cdot C = K^* \cdot C$ is equal to the intersection number of C with the generic biquadratic curve, which according to Lemma 2.4.1 is equal to $2 \cdot 4 + 2 \cdot 4 = 16$. It follows that the virtual genus $\mathrm{vg}(C)$ of C, which is equal to the genus of any smooth biquartic curve, is equal to $1 + (-16 + 32)/2 = 1 + 8 = 9$; see (6.2.8). This implies in view of (6.2.9) that the genus of the smooth proper transform $\widetilde{C}$ of C is equal to $9 - 8 \cdot 1 = 1$, that is, $\varphi : S \to \mathbb{P}^1$ is an elliptic fibration.

Lemma 3.3.3 implies that the space of all biquadratic polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ is canonically isomorphic to the space of all holomorphic sections of $\mathrm{K}^*_{\mathbb{P}^1 \times \mathbb{P}^1}$, and therefore the space of all biquartic polynomials is canonically isomorphic to the space of all holomorphic sections of $(\mathrm{K}^*_{\mathbb{P}^1 \times \mathbb{P}^1})^2$. Because the proper transform of the latter bundle is equal to $(\mathrm{K}^*_S)^2$, see Proposition 3.3.7, it follows that the fibers of $\varphi : S \to \mathbb{P}^1$ are the $\mathrm{Div}(w)$, where the w are nonzero elements in a two-dimensional vector space W of holomorphic sections of $(\mathrm{K}^*_S)^2$.

It follows from Lemma 3.2.7 with $k = 2, m_b = k^2 = 4$, that none of the -1 curves E that appear at the blowing up over any of the base points is contained in a member of the pencil $\widetilde{\mathfrak{P}}$. Therefore, if C' is an irreducible component of a fiber of φ, it is equal to the proper transform $\pi'(C)$ of an irreducible component C of a member of $\mathfrak{P}$. In view of (3.2.8) we have $C' \cdot C' = C \cdot C - \sum_{j=1}^{8} \mathrm{ord}_{b_j}(C)^2$, where $C \cdot C = 2\,d_1\,d_2$ if C is a $(d_1,\ d_2)$-curve; see Lemma 2.4.1. On the other hand, if $D' = \pi'(D)$ is a fiber of φ that is disjoint from C', then another application of (3.2.8) and Lemma 2.4.1 yields $0 = C' \cdot D' = C\,\dot{D} - \sum_{j=1}^{8} \mathrm{ord}_{b_j}(C)\ \mathrm{ord}_{b_j}(D) = 4\,(d_1 + d_2) - 2\ \sum_{j=1}^{8} \mathrm{ord}_{b_j}(C)$. Because $\mathrm{ord}_{b_j}(C)$ is equal to 0, 1, or 2, it follows from the latter equation that the number of j's such that $\mathrm{ord}_{b_j}(C) = 1$ is even, and then the first equation implies that $C' \cdot C'$ is even. This implies in turn that C' is not a -1 curve, that is, the elliptic fibration $\varphi : S \to \mathbb{P}^1$ is relatively minimal.

Using Lemma 3.4.1 we also conclude that the birational transformation $\pi^{-1} \circ S \circ \pi$ on S extends to a complex analytic diffeomorphism (an automorphism α of S), which preserves each fiber of φ. Because the involutions ι_1 and σ have fixed points on each

curve, these are inversions, and it follows that α acts as a translation on each smooth fiber of φ, which is an elliptic curve. In other words, α is an element of the Mordell–Weil group $\mathrm{Aut}(S)^+_\varphi$ of the elliptic fibration $\varphi : S \to \mathbb{P}^1$. Furthermore, $(x,\ y)$ is a fixed point of A if and only if $x = y$ and $q\,x^3 - q\,x - 2\,x + b = 0$, and it follows that no fiber of φ consists entirely of fixed points of α.

If $q \neq 0$ and $b^2 - 4 = 0$, for instance when $a = 1$, then the three base points b_5, b_7, and b_8 merge to a base point $*$ of multiplicity 12, but still $k = 2$ in Lemma 3.2.7. After one blowing up at $*$ the proper transform of the pencil has a unique base point $*'$ over $*$ of multiplicity $12 - 2^2 = 8$, with again $k = 2$. After blowing up $*'$ the proper transform of the pencil once more has a unique base point $*''$, of multiplicity $8 - 2^2 = 4$ and $k = 2$. However, after blowing up $*''$ the proper transform of the pencil no longer has base points over $*$. After also blowing up the remaining five base points b_1, b_2, b_3, b_4, b_6, we arrive at an elliptic fibration with all the properties of $\varphi : S \to \mathbb{P}^1$ as in the case that $q \neq 0$ and $b^2 - 4 \neq 0$.

On the other hand, if $q = 0$, then all the members of $\mathfrak{P}$ are reducible, and as in Section 3.3.1 we arrive after blowing up at a fibration at $\mathbb{P}^1$'s or unions of $\mathbb{P}^1$, on which the map acts as projective linear transformations. It is for this reason that in the sequel we will always assume that $q \neq 0$.

Until now we have collected properties of the map (11.9.1) that it shares with QRT maps. However, the member of $\mathfrak{P}$ defined by the equation $p^1 := r^2 = 0$, see (11.9.8), has multiplicity two, and therefore the corresponding fiber of $\varphi : S \to \mathbb{P}^1$ has multiplicity two as well. More precisely, this is a singular fiber of Kodaira type ${}_2\,\mathrm{I}_3$. In view of Lemma 6.2.13, it follows that the fibration $\varphi : S \to \mathbb{P}^1$ does not have a holomorphic section, and because every rational elliptic surface has such a section, see Theorem 9.1.3, we conclude that $\varphi : S \to \mathbb{P}^1$ *is not isomorphic to a rational elliptic surface*.

This, in turn, implies the following proposition, where it follows from Lemma 11.9.2 that A has infinite order if and only if $q \neq 1$ and $(b,\ q) \neq (0, -1)$.

Proposition 11.9.1 *Assume that $q \neq 0$ and that the transformation A defined by* (11.9.1) *has infinite order. Then A is not birationally conjugate to a QRT transformation.*

Proof. Assume that C is a birational transformation of the plane such that $B := C \circ A \circ C^{-1}$ is a QRT map, which implies that B preserves a nonconstant rational function of the form q^0/q^1, where q^0 and q^1 are biquadratic polynomials. We will first prove that the fact that A has infinite order then implies that C, where defined, maps each irreducible level curve of p^0/p^1 to a level curve of q^0/q^1.

For the proof of this, it is convenient to take the complex projective plane $\mathbb{P}^2$ as the completion of the affine plane. Then all the level curves of p^0/p^1 and $(q^0/q^1) \circ C$ have a common finite degree d and e, respectively. Let $r \in \mathbb{P}^2$ be such that for every $0 \leq j \leq d\,e$, $A^j(r)$ is defined and not equal to one of the finitely many points s where $p^0/p^1, C, (q^0/q^1) \circ C$, and $B \circ C$ are not defined. Furthermore, assume that all the points $A^j(r)$ are distinct. Write $z := (p^0/p^1)(r)$ and $\zeta := (q^0/q^1)(C(r))$, and assume that the z-level curve of p^0/p^1 is irreducible. Then, for every $0 \leq j \leq d\,e$, we have $(p^0/p^1)(A^j(r)) = z$ and $(q^0/q^1)(C(A^j(r)) = (q^0/q^1)(B^j(C(r))) =$

$(q^0/q^1)(C(r)) = \zeta$. It follows that the z-level curve of p^0/p^1 has at least $d\,e + 1$ points in common with the ζ-level curve of $(q^0/q^1) \circ C$, which in view of Bézout's theorem implies that the z-level curve of p^0/p^1 is contained in the ζ-level curve of $(q^0/q^1) \circ C$.

For any $k \in \mathbb{Z}_{>0}$, the set of k-periodic points of A corresponds to an algebraic subset $S_{\mathrm{per},\,k}$ of S, not equal to S because A is not of finite order, and therefore equal to the union of finitely many algebraic curves in S. Furthermore, if S_z is a nonsingular fiber of φ, then the fact that α acts as a translation on the elliptic curve S_z implies the Poncelet porism that $S_{\mathrm{per},\,k} \cap S_z = \emptyset$ or $S_{\mathrm{per},\,k} \cap S_z = \emptyset$. Because φ has only finitely many nonsingular, hence irreducible, fibers, we have that there are only finitely many z such that the A-orbits on the z-level curve of p^0/p^1 have $\leq d\,e$ elements. In view of the previous paragraph it follows that for each z except possibly finitely many, the z-level curve of p^0/p^1 is contained in a level curve of $(q^0/q^1) \circ C$. Using a continuity argument we conclude that C maps each level curve of p^0/p^1 to a level curve of q^0/q^1.

Let $\pi' : S' \to \mathbb{P}^1 \times \mathbb{P}^1$ be the iterated blowup in base points of the pencil of biquadratic curves defined by q^0 and q^1. Then q^0/q^1 defines a rational elliptic surface $\varphi' : S' \to \mathbb{P}^1$, and it follows from the previous paragraph that $(\pi')^{-1} \circ \pi$, where defined, maps φ-fibers in S to φ'-fibers in S'. Therefore Lemma 3.4.1, with α replaced by $(\pi')^{-1} \circ \pi$, implies that $(\pi')^{-1} \circ \pi$ extends to a unique isomorphism from S onto S' that maps φ-fibers to φ'-fibers. That is, the elliptic fibration $\varphi : S \to \mathbb{P}^1$ is isomorphic, as an elliptic fibration, to the rational elliptic surface $\varphi' : S' \to \mathbb{P}^1$, and we have arrived at a contradiction. □

11.9.2 The Elliptic K3 Surface

Kodaira [109, Theorem 6.3] has given a reduction of elliptic fibrations with multiple singular fibers to elliptic fibrations without multiple singular fibers. A local version of this reduction, over a neighborhood of one multiple singular fiber, has been described in more detail in the text preceding Definition 6.1.4. In this subsection we will carry out an explicit version of Kodaira's reduction, and show that it leads to an elliptic K3 surface. The transformation A defined by (11.9.1) is covered by an element of the Mordell–Weil group of the elliptic K3 surface that maps each section to a disjoint one.

Assume that $q \neq 0$. Let $*$ be one of the multiplicity 4 base points of the pencil of quartic curves $z_0\,p^0 + z_1\,p^1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $E = \pi^{-1}(\{*\} \simeq \mathbb{P}^1$ be the -1 curve over $*$ in the blowing up $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ described in Section 11.9.1. If ι_E denotes the identity, viewed as a mapping from E to S, then $\epsilon := \varphi \circ \iota_E$ is a twofold branched covering map from E onto $\mathbb{P}^1$, branching over the point $z = \infty$, corresponding to the multiplicity two fiber, and one other point $z = z_*$ in $\mathbb{P}^1$, where the fiber of φ is tangent to E. We denote the corresponding points in E by e_∞ and e_*, respectively. Consider the projective algebraic surface

$$\epsilon^*(S) := \{(e,\, s) \in E \times S \mid \varphi(s) = \epsilon(e)\}$$

in $E \times S$. The restriction to $\epsilon^*(S)$ of the projection $\pi_1 : (e,\, s) \to e$ is an algebraic morphism from $\epsilon^*(S)$ to E. For each $e \in E$, the restriction to the fiber $\epsilon^*(S)_e =$

$(\pi_1)^{-1}(\{e\}) \cap \epsilon^*(S)$ of the projection $\pi_2 : (e,\ s) \mapsto s$ is an isomorphism from $\epsilon^*(S)_e$ onto the fiber $S_{\epsilon(e)} = \varphi^{-1}(\{\epsilon(e)\})$ of φ over $\epsilon(e)$. In this way the fibration $\pi_1 : \epsilon^*(S) \to E$ over E is the pullback of the fibration $\varphi : S \to \mathbb{P}^1$ over $\mathbb{P}^1$ by means of the mapping $\epsilon : E \to \mathbb{P}^1$. See Section 2.1.4 for the analogous construction in the case of complex line bundles.

The surface $\epsilon^*(S)$ in $E \times S$ is singular at the point $(e,\ s)$ if and only if $\mathrm{d}\varphi(s) = 0$ and $\mathrm{d}\epsilon(e) = 0$. It follows that every point of $\{e_\infty\} \times S_\infty$ is singular, whereas the only other singular points are the points $(e_*,\ s)$ where s is a singular point of S_{z_*}. We will assume in the sequel that *the curve S_{z_*} is smooth*. For instance, if $* = b_7$, then this holds if and only if $b^2 - 4 \neq 0$ and $16\,q^3 - 48\,q^2 + (27\,b^2 - 60)\,q - 16 \neq 0$.

Let $\rho : \widetilde{S} \to \epsilon^*(S)$ be a minimal resolution of singularities of the surface $\epsilon^*(S)$, which is unique up to isomorphism; see Section 6.2.13. We have the mappings $\widetilde{\varphi} := \pi_1 \circ \rho : \widetilde{S} \to E$ and $\gamma := \pi_2 \circ \rho : \widetilde{S} \to S$, where $\varphi \circ \gamma = \varphi \circ \pi_2 \circ \rho = \epsilon \circ \pi_1 \circ \rho = \epsilon \circ \widetilde{\varphi}$. In particular, if $e \in E$, $e \neq e_\infty$, $e \neq e_*$, and $\epsilon(e)$ is a regular value of φ, then $\gamma|_{\widetilde{S}_e}$ is an isomorphism from the fiber $\widetilde{S}_e$ of $\widetilde{\varphi}$ over e onto the fiber $S_{\epsilon(e)}$ of φ over $\epsilon(e)$, which is an elliptic curve. This shows that $\widetilde{\varphi} : \widetilde{S} \to E \simeq \mathbb{P}^1$ is an elliptic fibration. Note that if $z \in \mathbb{P}^1$, $z \neq \infty$, and $z \neq z_*$, then there are distinct $e,\ e' \in E$ such that $\epsilon(e) = z = \epsilon(e')$, hence two disjoint fibers $\widetilde{S}_e$ and $\widetilde{S}_{e'}$ of $\widetilde{\varphi}$, each of which is mapped isomorphically by ρ onto the fiber S_z of φ over z.

As discussed in the text preceding Definition 6.1.4, over a neighborhood of the multiplicity-two fiber S_∞ the surface $\epsilon^*(S)$ is the union of two nonsingular surfaces that intersect each other over S_∞, and the modification $\rho : \widetilde{S} \to \epsilon^*(S)$ consists of *separating the two sheets*. Furthermore, the fiber $\widetilde{S}_{e_\infty} = \gamma^{-1}(S_\infty)$ of $\widetilde{\varphi}$ over e_∞ is a singular fiber of $\widetilde{\varphi}$ of Kodaira type ${}_1\,\mathrm{I}_6 = \mathrm{I}_6$, whose irreducible components are six $\mathbb{P}^1$'s, embedded in $\widetilde{S}$ with self-intersection number equal to -2, see the text preceding Definition 6.1.4 and Section 6.2.6. Because the elliptic fibration $\varphi : S \to \mathbb{P}^1$ is relatively minimal, and the modification $\rho : \widetilde{S} \to S$ does not add any -1 curves to fibers of $\widetilde{\varphi}$, it follows that the elliptic fibration $\widetilde{\varphi} : \widetilde{S} \to E$ is relatively minimal.

Over a neighborhood of E, γ is a twofold branched covering, branching over S_{z_*}. It follows that the restriction to $E' := \gamma^{-1}(E)$ of γ is a twofold branched covering $: E' \to E$, branching only over the point e_*. More precisely, because S_{z_*} is smooth at e_* and has a contact of order two with E, there are local coordinates $(\xi,\ \eta)$ near e_* in S that map e_* to the origin such that E is the η-axis $\xi = 0$ and $\varphi = z_* - \xi + \eta^2$ near e_*. Note that $\epsilon(\zeta) = \varphi(0,\ \zeta) = z_* + \zeta^2$. Then the surface

$$\epsilon^*(S) = \{(\zeta,\ (\xi,\ \eta)) \mid 0 = z_* + \zeta^2 - (z_* - \xi + \eta^2) = \xi - (\eta - \zeta)\,(\eta + \zeta)\}$$

is smooth near $(e_*,\ e_*) \in E \times S$ and can be coordinatized by $(\eta,\ \zeta)$. Because $\epsilon^*(S)$ is smooth near e_*, we can identify it there with $\widetilde{S}$, that is, take ρ equal to the identity. Then $\gamma(\zeta,\ (\xi,\ \eta)) = (\xi,\ \eta) = ((\eta - \zeta)\,(\eta + \zeta),\ \eta)$, and E' is determined by the equation $0 = \xi = (\eta - \zeta)\,(\eta + \zeta)$.

If N denotes the normalization of the curve E', where the two intersecting local components $\zeta = \pm\eta$ have been separated, then γ induces an unbranched twofold covering $: N \to E$. However, $E \simeq \mathbb{P}^1$, the Riemann sphere, is simply connected, and it follows that N is the disjoint union of two $\mathbb{P}^1$'s. In other words, E' has two

irreducible components $C_\pm$, each of which is an embedded complex projective line, where C_+ and C_- intersect each other only at the unique point $e'_* \in \widetilde{S}$ such that $\gamma(e'_*) = e_*$, and the intersection is transversal. Because each fiber of φ intersects E with multiplicity two, each fiber of $\widetilde{\varphi} : \widetilde{S} \to E$ intersects $C_\pm$ exactly once. That is, both C_+ and C_- are global sections of $\widetilde{\varphi}$. In view of Lemma 6.2.13, the existence of at least one section for $\widetilde{\varphi}$ implies that the elliptic fibration $\widetilde{\varphi} : \widetilde{S} \to E$ has no multiple singular fibers.

In the aforementioned local coordinates $(\eta,\ \zeta)$, where the curves $C_\pm$ are determined by the equations $\zeta = \pm\eta$, the mapping $\gamma : (\eta,\ \zeta) \mapsto ((\eta - \zeta)\,(\eta + \zeta),\ \eta)$ maps a curve close and parallel to $C_\pm$ to a curve that intersects E with multiplicity one. Because γ is an isomorphism over the complement of any neighborhood of e'_* in a small open neighborhood of $C_\pm$, it follows that the self-intersction number of E_* is one higher than the self-intersction number of $C_\pm$. Because $E_* \cdot E_* = -1$, we conclude that $C_\pm \cdot C_\pm = -2$. Because the left-hand side is equal to the degree of the Lie bundle ϱ, see Lemma 6.2.34, we conclude from Section 6.3.4 that $\widetilde{\varphi} : \widetilde{S} \to E \simeq \mathbb{P}^1$ *is an elliptic K3 surface*.

There is a unique map $\widetilde{\alpha}$ from the complement $\widetilde{S}'$ in $\widetilde{S}$ of $\widetilde{S}_{e_\infty} \cup \widetilde{S}_{e_*}$ to itself such that $\widetilde{\varphi} \circ \widetilde{\alpha} = \widetilde{\varphi}$ and $\gamma \circ \widetilde{\alpha} = \alpha \circ \gamma$. Indeed, the first equation says that for each $e \in E_* \setminus \{e_\infty,\ e_*\}$ the map $\widetilde{\alpha}$ has to preserve the fiber $\widetilde{S}_e$. Because the restriction γ_e to $\widetilde{S}_e$ of γ is an isomorphism from $\widetilde{S}_e$ onto $S_{\epsilon(e)}$, and $\alpha|_{S_{\epsilon(e)}}$ is an automorphism of $S_{\epsilon(e)}$, the equation

$$\widetilde{\alpha}|_{\widetilde{S}_e} := {\gamma_e}^{-1} \circ (\alpha|_{S_{\epsilon(e)}} \circ \gamma_e$$

defines the unique solution of the second equation.

The map $\widetilde{\alpha}$ is bijective, where the inverse $\widetilde{\alpha}^{-1}$ is defined by the equations $\widetilde{\varphi} \circ \widetilde{\alpha}^{-1} = \widetilde{\varphi}$ and $\gamma \circ \widetilde{\alpha}^{-1} = \alpha^{-1} \widetilde{\circ} \gamma$. Because $\widetilde{\alpha}$ and its inverse are defined by algebraic equations, $\widetilde{\alpha}$ is a birational transformation of $\widetilde{S}$. Because $\widetilde{\alpha}$, where defined, preserves the fibers of $\widetilde{\varphi}$, which have genus 1 when nonsingular, we conclude in view of Lemma 3.4.1 that $\widetilde{\alpha}$ has a unique extension to an automorphism of $\widetilde{S}$, which we denote by the same letter. Because $\widetilde{S}'$ is dense in $\widetilde{S}$, it follows by continuity that the extension also satisfies the algebraic equations $\widetilde{\varphi} \circ \widetilde{\alpha} = \widetilde{\varphi}$ and $\gamma \circ \widetilde{\alpha} = \alpha \circ \gamma$.

If $e \neq e_\infty$, $e \neq e_*$, and $\widetilde{S}_e$ is nonsingular, then the restriction to $\widetilde{S}_e$ of $\widetilde{\alpha}$, which is conjugate by means of γ_e to the translation $\alpha|_{S_{\epsilon(e)}}$ on the elliptic curve $S_{\epsilon(e)}$, is a translation on the elliptic curve $\widetilde{S}_e$. It follows that $\widetilde{\alpha}$ *is an element of the Mordell–Weil group of the elliptic fibration* $\widetilde{\varphi} : \widetilde{S} \to E_*$. Furthermore, because no fiber of φ is left pointwise fixed by α, no fiber of $\widetilde{\varphi}$ is left pointwise fixed by $\widetilde{\alpha}$, or equivalently, $\widetilde{\alpha}$ *maps every global section of* $\widetilde{\varphi}$ *to a disjoint one*.

11.9.3 The Weierstrass Data

For every $(z_0,\ z_1) \in \mathbb{C}^2 \setminus \{(0,\ 0\}$, the rational function $f := (z_0\, p_0 + z_1\, p_1)/r$, see (11.9.8), (11.9.7), is bihomogeneous of bidegree (2, 2). It therefore follows from the proof of Lemma2.4.5, with p replaced by f, that the restriction to the smooth part $Z^{\mathrm{reg}}_{[z_0:\, z_1]}$ of the curve $z_0\, p_0 + z_1\, p_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ of the Hamiltonian vector field $\mathrm{d}x/\,\mathrm{d}t = \partial f(x,\ y)/\partial y$, $\mathrm{d}y/\,\mathrm{d}t = -\partial f(x,\ y)/\partial x$ defined by the function f extends to a holomorphic vector field $v_{(z_0,\, z_1)}$ without any zeros on $Z^{\mathrm{reg}}_{[z_0:\, z_1]}$. Note

that if $[z_0 : z_1] = [0 : 1]$, then $f = r$, and $v_{[0:1]}$ is also nonzero on the nonsingular part of the multiplicity-one curve $r = 0$, despite the fact that the curve $p^1 := r^2 = 0$ has multiplicity two. The Hamiltonian vector field $v_{(z_0, z_1)}$ is equal to zero at the singular points of $Z_{[z_0:z_1]}$, in particular at the base point of the pencil $\mathfrak{P}$, where every member of the pencil is singular. However, if $\widetilde{v}_{(z_0, z_1)}$ denotes the vector field on the fiber $S_{[z_0:z_1]} = S_{-z_1/z_0}$ of $\varphi : S \to \mathbb{P}^1$, which is mapped to $v_{(z_0, z_1)}$ by the tangent map of the blowing up $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$, then $\widetilde{v}_{(z_0, z_1)}$ extends to a holomorphic vector field without zeros on the nonsingular part $S^{\mathrm{reg}}_{[z_0:z_1]}$ of $S_{[z_0:z_1]}$. This reconfirms that the nonsingular fibers of φ are elliptic curves, but we now have equipped these fibers with holomorphic vector fields in a coherent way.

Write $z = -z_1/z_0$ and assume that S_z is smooth. Let $P = P_{(z_0, z_1)}$ denote the period lattice of the vector field $\widetilde{v} = \widetilde{v}_{(z_0, z_1)}$ on S_z, where for each choice of a point $s_z \in S_z$, the mapping $t \mapsto \mathrm{e}^{t\,\widetilde{v}}(c_0)$ defines an isomorphism from $\mathbb{C}/P$ onto S_z; see Section 2.3.1. Under this isomorphism the origin $0 + P$ in $\mathbb{C}/P$ corresponds to the point s_z. On the other hand, the mapping $t \mapsto [1 : \wp(t) : \wp'(t)]$, where $\wp(t)$ is the Weierstrass $\wp$-function, induces an isomorphism from $\mathbb{C}/P$ onto the Weierstrass curve $W_{g_2, g_3} \subset \mathbb{P}^2$ defined by (2.3.6), where $g_2 = g_2(z_0, z_1)$ and $g_3 = g_3(z_0, z_1)$ are given by (2.3.4), and the origin $0+P$ is mapped to the point $[0 : 0 : 1] \in W_{g_2, g_3}$. If $\iota = \iota_{(z_0, z_1)} : S_z \to W_{g_2, g_3}$ denotes the inverse of the isomorphism $\mathbb{C}/P \to S_z$ followed by the isomorphism $\mathbb{C}/P \to W_{g_2, g_3}$, then ι is an isomorphism from S_z onto W_{g_2, g_3} that maps s_z to $[0 : 0 : 1]$.

The functions g_2 and g_3 are holomorphic on the open set of all $(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $-z_1/z_0$ is not one of the finitely many singular values of φ. If $c \in \mathbb{C} \setminus \{0\}$, then $\widetilde{v}_{(c\,z_0, c\,z_1)} = c\,\widetilde{v}_{(z_0, z_1)}$, hence $P_{(c\,z_0, c\,z_1)} = (1/c)\,P_{(z_0, z_1)}$, and therefore (2.3.4) implies that $g_2(c\,z_0, c\,z_1) = c^4\,g_2(z_0, z_1)$ and $g_3(c\,z_0, c\,z_1) = c^6\,g_3(z_0, z_1)$. That is, g_2 and g_3 are homogeneous of degree 4 and 6, respectively. If (z_0, z_1) approaches a point $(z_0', z_1') \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $z' := -z_1'/z_0'$ is a singular value of φ, then it follows from the description of the asymptotic behavior of the periods in Lemma 6.2.38 and Corollary 6.2.47 that g_2 and g_3 remain bounded. It follows that g_2 and g_3 extend to holomorphic functions on $\mathbb{C}^2 \setminus \{(0, 0)\}$, and in view of their homogeneity the conclusion is that $g_2(z_0, z_1)$ *and* $g_3(z_0, z_1)$ *extend to homogeneous polynomials in* (z_0, z_1) *of degree* 4 *and* 6*, respectively*.

The automorphism α of S defined by the transformation (11.9.1) acts as a translation on the elliptic curve S_z, that is, there is a unique element $T+P = T(z_0, z_1)+P \in \mathbb{C}/P$ such that $\alpha(s) = \mathrm{e}^{T\,\widetilde{v}}(s)$ for every $s \in S_z$. That is, the isomorphism from $\mathbb{C}/P$ onto S_z conjugates the translation $t + P \mapsto t + T(z_0, z_1) + P$ on $\mathbb{C}/P$ with $\alpha : S_z \to S_z$. Write $X(z_0, z_1) := \wp(T(z_0, z_1) + P)$ and $Y(z_0, z_1) := \wp'(T(z_0, z_1) + P)$. In other words, $[1 : X(z_0, z_1) : Y(z_0, z_1)]$ is the image point of $[0 : 0 : 1]$ of the translation in W_{g_2, g_3} that is conjugate to $\alpha : S_z \to S_z$ by means of ι. Because $\widetilde{v}_{(c\,z_0, c\,z_1)} = c\,\widetilde{v}_{(z_0, z_1)}$, we have $T(c\,z_0, c\,z_1) = (1/c)\,T(z_0, z_1)$, and it follows from (2.3.3) that the functions X and Y are homogeneous of degree 2 and 3, respectively. Because there are no fixed point fibers for α, we have $[1 : X(z_0, z_1) : Y(z_0 : z_1)] \neq [0 : 0 : 1]$ for every $(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, that is, the functions $X(z_0, z_1)$ and $Y(z_0 : z_1)$ have no poles, and a similar argument as for g_2 and g_3 leads to the conclusion that $X(z_0, z_1)$ *and* $Y(z_0, z_1)$ *extend to homo-*

geneous polynomials in $(z_0,\ z_1)$ *of degrees* 2 *and* 3*, respectively*. The homogeneous polynomials $g_2(z_0,\ z_1)$, $g_3(z_0,\ z_1)$, $X(z_0,\ z_1)$, and $Y(z_0,\ z_1)$, of respective degrees 4, 6, 2, and 3, are the *Weierstrass data of the transformation* (11.9.1) in the title of Section 11.9.

The elliptic fibration $\varphi : S \to \mathbb{P}^1$ has regular values, which are precisely the points where the discriminant $\Delta := {g_2}^3 - 27\,{g_3}^2$ in (2.3.9) is nonzero. Moreover, the asymptotics of the period functions in Lemma 6.2.38 imply that the functions g_2, g_3, and Δ satisfy the conditions on their order of zeros in Table 6.3.2. It therefore follows from Remark 5 with $N = 1$ that the functions $g_2(z_0,\ z_1)$ and $g_3(z_0,\ z_1)$ are the Weierstrass data of a rational elliptic surface $\kappa : R \to \mathbb{P}^1$. Furthermore, Proposition 7.8.1 implies that there is an element $\tau \in \mathrm{Aut}(R)^+_\kappa$ of the Mordell–Weil group of $\kappa : R \to \mathbb{P}^1$ that has Weierstrass data X and Y. Because X and Y have no poles, τ has no fixed-point fibers. In view of (vi) $\Rightarrow$ (i) in Theorem 4.3.2, we conclude that there exists a pencil $\mathfrak{B}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that R is equal to the successive blowing up at base points of anticanoincal pencils starting with $\mathfrak{B}$, and τ is equal to the QRT automorphism $\tau^S_{\mathfrak{B}}$ of S defined by $\mathfrak{B}$. It is in this sense that we have proved: *If* $q \neq 0$*, then the Weierstrass data of the transformation A in (11.9.1) are equal to the Weierstrass data of a QRT map*.

Below follow the explicit formulas for the Weierstrass data g_2, g_3, Δ, X, and Y. The homogeneous forms in $(z_0,\ z_1)$ of the respective degrees 4, 6, 12, 2, and 3 are obtained by the substitution $z = -z_1/z_0$, followed by multiplication by ${z_0}^4, {z_0}^6, {z_0}^{12}$, ${z_0}^2$, and ${z_0}^3$, respectively. We have not displayed the homogeneous forms, because these only take up more space. The strategy for the computation of the Weierstrass data by means of a formula manipulation computer program will be described in Section 11.9.4.

$$
\begin{aligned}
(3/4)\,g_2 &= z^4 + 4\,(b^2\,q+2)\,z^3 + 2\,\Big((3\,b^4 - 8\,b^2 - 8)\,q^2 - 4\,b^2\,q + 8\Big)\,z^2 \\
&\quad +4\,(b^2-4)^2\,(b^2\,q-4)\,q^2\,z + (b^2-4)^4\,q^4,
\end{aligned}
$$

$$
\begin{aligned}
-(27/8)\,g_3 &= z^6 + 6\,(b^2\,q+2)\,z^5 + \Big(3\,(5\,b^4 - 8\,b^2 - 8)\,q^2 + 12\,b^2\,q + 48\Big)\,z^4 \\
&\quad \Big(4\,b^2\,(5\,b^4 - 24\,b^2 + 12)\,q^3 - 12\,(3\,b^4 - 8\,b^2 - 32)\,q^2 - 48\,b^2\,q + 64\Big)\,z^3 \\
&\quad +3\,(b^2-4)^2\,\Big((5\,b^4 - 8\,b^2 - 8)\,q^2 - 20\,b^2\,q + 40\Big)\,q^2\,z^2 \\
&\quad +6\,(b^2-4)^4\,(b^2\,q-4)\,q^4\,z + (b^2-4)^6\,q^6,
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= -2^{12}\,q^2\,z^4\,\Big[z + (b^2-4)\,(q-1)\Big]^2\,\Big[z^3 + \Big(-q^2 + (3\,b^2+8)\,q + 8\Big)\,z^2 \\
&\quad + \Big(-2\,(b^2+4)\,q^3 + (3\,b^4 - 2\,b^2 + 8)\,q^2 - 4\,(5\,b^2-8)\,q + 16\Big)\,z \\
&\quad -(b^2-4)^2\,q^2\,(q^2 - (b^2-2)\,q + 1)\Big],
\end{aligned}
$$

$$
3\,X = z^2 + 2\,\Big((b^2-6)\,q + 2\Big)\,z + (b^2-4)^2\,q^2,
$$

$$
Y = -8\,q\,z\,\Big(z + (b^2-4)\,(q-1)\Big).
$$

The fact that all the Weierstrass data are even in b is explained by the symmetry $(x,\ y,\ b,\ q) \mapsto (-x,\ -y,\ -b,\ q)$ in the transformation (11.9.1) and the functions p^0 and p^1 in (11.9.5) and (11.9.4), respectively.

The formula for Δ shows that Δ is not identically zero as a function of z if and only if $q \neq 0$, which reconfirms that $\varphi : S \to \mathbb{P}^1$ is an elliptic fibration if and only if $q \neq 0$. It can also be checked explicitly from the formulas that if $q \neq 0$, the order of zeros of g_2, g_3, and Δ satisfy Table 6.3.2, and therefore the homogeneous polynomials $g_2(z_0,\ z_1)$ and $g_3(z_0,\ z_1)$ in $(z_0,\ z_1)$ of the respective degrees 4 and 6 are the Weierstrass data of a rational elliptic surface.

The explicit formulas for g_2, g_3, and Δ can, in combination with Table 6.3.2, be used to determine the configuration of the singular fibers of the elliptic fibration $\varphi : S \to \mathbb{P}^1$. The configuration of the singular fibers of the rational elliptic surface with the Weierstrass data g_2 and g_3 is the same as for $\varphi : S \to \mathbb{P}^1$, with the exception of the singular fiber S_∞ of φ of Kodaira type ${}_2\,\mathrm{I}_3$, which is replaced by a singular fiber of Kodaira type I_3. The singular fibers of the elliptic fibration $\widetilde{\varphi} : \widetilde{S} \to E$ of Section 11.9.2 are $\widetilde{S}_{e_\infty}$, the singular fiber of type ${}_1\,\mathrm{I}_6 = \mathrm{I}_6$ that is mapped by γ to the singular fiber S_∞ of φ of type ${}_2\,\mathrm{I}_3$, whereas each other singular fiber of φ is replaced by two singular fibers of $\widetilde{\varphi}$ of the same Kodaira type.

Because the respective coefficients of z and z^3 in the multiplicity-two factor in Δ and the factor of degree three in Δ are equal to one, the ${}_2\,\mathrm{I}_3$ fiber over $z = \infty$ never coalesces with any of the other singular fibers. The I_4 fiber over $z = 0$ coalesces with the I_2 fiber if and only if $(b^2-4)\,(q-1) = 0$, that is, $b^2 = 4$ or $q = 1$. The I_4 fiber over $z = 0$ coalesces with an I_1 fiber if and only if $(b^2-4)^2\,q^2\,(q^2-(b^2-2)\,q+1) = 0$, that is, $b^2 = 4$ or $q = 0$ or $q^2-(b^2-2)\,q+1 = 0$. The resultant of the multiplicity-two factor and the degree-three factor in Δ is equal to $(4-b^2)\,(b^2-8\,q+4\,q^2)^2$, and therefore the I_2 fiber over $z = (4-b^2)\,(q-1)$ coalesces with an I_1 fiber if and only if $b^2-4 = 0$ or $b^2-8\,q+4\,q^2 = 0$. Finally, the discriminant of the degree-three factor in Δ is equal to $b^2\,q\,(4\,q^3+24\,q^2-3\,(9\,b^2-16)\,q+32)^3$, and therefore some of the I_1 fibers coalesce if and only if $b = 0$ or $q = 0$ or $4\,q^3+24\,q^2-3\,(9\,b^2-16)\,q+32 = 0$. Therefore the configuration of the singular fibers is equal to ${}_2\,\mathrm{I}_3,\ \mathrm{I}_4,\ \mathrm{I}_2,\ \mathrm{I}_1,\ \mathrm{I}_1,\ \mathrm{I}_1$, if and only if $q \neq 0$, $b^2-4 \neq 0$, $q \neq 1$, $q^2-(b^2-2)\,q+1 \neq 0$, $b^2-8\,q+4\,q^2 \neq 0$, $b \neq 0$, and $4\,q^3+24\,q^2-3\,(9\,b^2-16)\,q+32 \neq 0$.

The degree of the modulus function $J := {g_2}^3/\Delta : \mathbb{P}^1 \to \mathbb{P}^1$, see (2.3.10), can be computed as the number of points where $J = \infty$, counted with multiplicities. If $J = \infty$, then $\Delta = 0$. Because the coefficient of the highest degree term z^4 in g_2 is equal to $1 \neq 0$, and that of the highest-degree term z^9 in Δ is equal to $-4096\,q^2 \neq 0$, we have $J = \infty$ with multiplicity $3 \cdot 4 - 9 = 3$ at $z = \infty$, corresponding to the singular fiber of type ${}_2\,\mathrm{I}_3$. The resultant of $(3/4)\,g_2$ and z, the linear multiplicity two factor in Δ, and the cubic multiplicity one factor in Δ is equal to $(b^2-4)^4\,q^4$, $(b^2-4)^2\,(b^2-8\,q+4\,q^2)^2$, and $(b^2-4)^4\,q^6\,(b^2-8\,q+4\,q^2)^2\,(4\,q^3+24\,q^2-3\,(9\,b^2-16)\,q+32)^2$, respectively. We conclude that the degree of J is equal to 12 if and only if $g_2 \neq 0$ at all the finite zeros z of Δ if and only if $q \neq 0$, $b^2-4 \neq 0$, $b^2-8\,q+4\,q^2 \neq 0$, and $4\,q^3+24\,q^2-3\,(9\,b^2-16)\,q+32 \neq 0$. If the modulus function J has degree 12 and j is a regular value of J, then there are exactly 12

$z \in \mathbb{P}^1$ such that $J(z) = \mathrm{j}$, where the corresponding 12 fibers S_z are isomorphic elliptic curves, and not isomorphic to any fiber $S_{z'}$ such that $J(z') \neq \mathrm{j}$.

The explicit form of the Weierstrass data g_2, g_3, X, and Y also allows the explicit computation of Manin's function $\mu(z)$ defined by (2.5.16). We have $\mu(z) = ((1 - q)\, f(z))/(2\, z\, g(z)\, h(z)\, k(z))$. Here the factor $f(z)$ in the numerator is the polynomial of degree three in z given by

$$\begin{aligned} f(z) = {} & (q + b^2 + 1)\, z^3 + \left((8 - b^2)\, q^2 + (3\, b^4 - 17\, b^2 + 12)\, q + 4\right) z^2 \\ & -(b^2 - 4)\, (3\, b^2 + 4)\, (q - 1)\, (q^2 - (b^2 - 2)\, q + 1)\, z \\ & -(b^2 - 4)^2 \left(b^2\, q^4 - (b^4 + b^2 - 4)\, q^3 + 3\, (b^4 - 3\, b^2 + 4)\, q^2 \right. \\ & \left. -(7\, b^2 - 12)\, q + 4\right). \end{aligned}$$

Furthermore, z, $g(z)$, and $h(z)$ are the factors that occur in $\Delta(z)$ with multiplicity 3, 2, and 1, respectively, but in the denominator of μ only with multiplicity one. Finally,

$$\begin{aligned} k(z) = -3\, z^2 + \left(4\, q^2 + (b^2 - 12)\, q + 5\, b^2 - 16\right) z \\ - 4\, (b^2 - 4)\, (q - 1)\, \left(q^2 - (b^2 - 2)\, q + 1\right) \end{aligned}$$

is an additional factor, of degree two in z, in the denominator of μ.

A straightforward computation shows that Manin's function $\mu(z)$ is identically equal to zero if and only if $q = 1$ or $(b,\ q) = (0, -1)$. Therefore the last statement in Proposition 7.8.8 implies the following:

Lemma 11.9.2 *The birational transformation A defined by* (11.9.1) *has finite order if and only if $q = 1$ or $(b,\ q) = (0, -1)$.*

Another straightforward computation shows that A has order three and six if $q = 1$ and $(b,\ q) = (0, -1)$, respectively.

11.9.4 The Computation

The idea is to compute, for every $z \in \mathbb{P}^1$, a rational function $P = P_z(x,\ y)$ of two variables $(x,\ y)$ such that the restriction of P to the level curve $p^0/p^1 = z$, via the blowing up $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ in the base points, is equal to a Weierstrass function on S_z defined by the vector field $\widetilde{v}_z$ as in Section 11.9.3. For this we have to choose the point $s_z \in S_z$ at which the Weierstrass function $(P \circ \pi)|_{S_z}$ has its pole. Because the fibration $\varphi : S \to \mathbb{P}^1$ does not have a global section, there is no coherent way of choosing a unique $s_z \in S_z$. Instead, we choose a base point $*$ of the pencil $\mathcal{P}$ of biquartic curves, and parametrize the pole points by the elements $e \in E$, where E is the -1 curve that appears at the blowing up over $*$. Because $\varphi|_E$ is a twofold branched covering $E \to \mathbb{P}^1$, branching over $z = \infty$ and $z = z_*$, this means that for every $z \in \mathbb{P}^1 \setminus \{\infty,\ z_*\}$ we compute two Weierstrass functions on S_z instead of one. In this way the computation mimics the determination of the Weierstrass model of

Theorem 6.3.6 for the elliptic fibration with a global section $\widetilde{\varphi} : \widetilde{S} \to E$, the elliptic K3 surface of Section 11.9.2.

I have chosen the base point $* = b_7$, where $z_* = -p^2\,(1-a^2)/a^2$. Blowing up at $*$, with an affine coordinate λ on E_*, is done explicitly by means of a substitution of variables of the form $x = a + t\,\xi + u\,\lambda\,\xi$, $y = (1/a) + v\,\xi + w\,\lambda\,\xi$, where $t,\ u,\ v,\ w$ are suitable coefficients. We write $(x,\ y) = \Pi(\xi,\ \lambda)$. In order that the restriction ϵ of φ to E take the simple form $\lambda \mapsto z = z_* + \lambda^2$, I took $t = 1 - a^2$, $u = -a^2$, $v = a^2 - 1$, and $w = 1$. The pole point of the Weierstrass function on $S_{\epsilon(e)}$ is the point $e \in E$ with coordinates $\xi = 0$, $\lambda = e$.

In the computation I used the rational formulas in terms of the parameters a and p, rather than b and q, for all the base points. For this reason, in the sequel the word "compute the expression ε" will mean that the value of ε is returned by the formula manipulation computer program as a rational function of a, p, and e. In the end, the Weierstrass data will actually turn out to be polynomials in $b = a + 1/a$, $q = p^2$, and $z = z_* + e^2$.

The biquadratic polynomial r, the one for which $p_1 = r^2$, vanishes at all the base points, and conversely, any biquadratic polynomial that vanishes at all the eight base points is a scalar multiple of r. We note in passing that this means that the points b_j, $1 \le j \le 8$, are not the base points of a pencil of biquadratic curves. Because the space of all biquadratic polynomials is nine-dimensional, it follows that the linear equations for biquadratic polynomials to vanish at b_j are linearly independent. It follows that the space $\mathcal{Q}$ of all biquadratic poynomials that vanish at the base points b_j such that $1 \le j \le 6$ is three-dimensional, and actually $\mathcal{Q}$ is spanned by the biquadratic polynomials

$$\begin{aligned} r &:= (x_1\,y_1 - x_0\,y_0)\,x_0\,y_0, \\ q_1 &:= x_0\,x_1\,{y_0}^2 - {x_0}^2\,y_0\,y_1, \quad \text{and} \\ q_2 &:= -p^2\,{x_1}^2\,{y_1}^2 + {x_1}^2\,{y_0}^2 + {x_0}^2\,{y_1}^2 + (p^2-2)\,{x_0}^2\,{y_0}^2. \end{aligned}$$

Note that because any two elements $n,\ d \in \mathcal{Q}$ are bihomogeneous of the same bidegree $(2,\ 2)$, the quotient n/d defines a rational function on $\mathbb{P}^1 \times \mathbb{P}^1$, undetermined only at the points where both n and d vanish.

Let δ and δ_1 be the coefficients such that the biquadratic polynomial $d := \delta\,r + \delta_1\,q_1 + q_2 \in \mathcal{Q}$ vanishes at b_7, and moreover $(d \circ \Pi)/\xi$ vanishes at $(\xi,\ \lambda) = (0,\ e)$. δ and δ_1 are computed as the solutions of the two inhomogeneous linear equations $d(b_7) = 0$ and $((d \circ \Pi)/\xi)(0,\ e) = 0$, where I instructed the formula manipulation computer program first to substitute Π into d and then to factorize $(d \circ \Pi)/\xi$, before taking its value at $(0,\ e)$. The function $d \circ \Pi$, which has ξ as a factor, has a zero of order two at $(0,\ e)$. Therefore, for a generic $n \in \mathcal{Q}$ the restriction of $(n/d) \circ \pi$ to $S_{\varphi(e)}$ will have a pole of order two at e.

For each $1 \le j \le 6$, the biquadratic curve $d = 0$ passes through b_j, whereas the quartic curve $p^0 - \varphi(e)\,p^1 = 0$ has a double point at b_j, and therefore the intersection number of $d = 0$ and $p^0 - \varphi(e)\,p^1 = 0$ at b_j is ≥ 2. Furthermore, the curve $d = 0$ passes through b^7, and at b^7 it is tangent to the local component of the curve $p^0 - \varphi(e)\,p^1 = 0$, which has $e \in S_{\varphi(e)}$ as its tangent line. Therefore the intersection

number of $d = 0$ and $p^0 - \varphi(e)\, p^1 = 0$ at b_7 is equal to 3. Because the total intersection number of the $(2,\ 2)$-curve $d = 0$ with the $(4,\ 4)$-curve $p^0 - \varphi(e)\, p^1 = 0$, counted with multiplicities, is equal to $2 \cdot 4 + 2 \cdot 4 = 16$, see Lemma 2.4.1, there are at most $16 - 6 \cdot 2 - 3 = 1$ remaining intersection points.

The factorized resultant of the polynomials $p^0(x,\ y) - \varphi(e)\, p^1(x,\ y)$ and $d(x,\ y)$, as polynomials in y, is computed as $(p\,x - 1)^2\,(p\,x + 1)^2\,(x - 1)^2\,(x + 1)^2\,(x - a)^3$ times a linear factor in x, where $x = \pm 1/p$, $x = \pm 1$, and $x = a$ correspond to the base points b_j, $3 \le j \le 7$. Let $x = x_{\text{new}}$ be the computed value of the zero of the linear factor. The same computation with the roles of x and y interchanged leads to the coordinate $y = y_{\text{new}}$, and $(x_{\text{new}},\ y_{\text{new}})$ is the intersection point of the curves $p_0 - \varphi(e)\, p_1 = 0$ and $d = 0$, which is not equal to one of the base points b_j, $1 \le j \le 7$. At $(x_{\text{new}},\ y_{\text{new}})$ the intersection has multiplicity one, that is, both curves are nonsingular and transversal to each other at $(x_{\text{new}},\ y_{\text{new}})$.

Write $n := \nu\, r + \nu_1\, q_1 + q_2 \in \mathcal{Q}$ and compute ν and ν_1 as the solutions of the two inhomogeneous linear equations $n(x_{\text{new}},\ y_{\text{new}}) = 0$ and $n(b_8) = 0$ for ν and ν_1. Because n is not a scalar multiple of r, and vanishes at b_j, $1 \le j \le 6$, and at b_8, it does not vanish at b_7. It follows that the restriction to $S_{\varphi(e)}$ of $(n/d) \circ \pi$ has a pole of order two at e, a pole of order one at $-e$, and no other poles, because the fact that $n = 0$ at $(x_{\text{new}},\ y_{\text{new}})$ leads to the elimination of the pole at the point $s_{\text{new}} \in S_{\varphi(e)}$ with the affine coordinates $(x_{\text{new}},\ y_{\text{new}})$. Because $n = 0$ at b_j, $1 \le j \le 6$, and at b_8, and does not vanish at b_7, the restriction to $S_{\varphi(e)}$ of $(n/r) \circ \pi$ has simple poles at e and $-e$. Therefore there is a unique coefficient γ such that the restriction to $S_{\varphi(e)}$ of $g := ((n/d) - \gamma\,(n/r)) \circ \pi$ has no pole at the point $-e$. Because $(n/d) - \gamma\,(n/r) = n\,(r - \gamma\,d)/(d\,r)$, we compute γ as the solution of the linear equation $r' - \gamma\,d' = 0$, where r' and d' denote the respective derivatives of $r \circ \pi$ and $d \circ \pi$ at $-e$ in the direction of the tangent space of $S_{\varphi(e)}$ at $-e$.

Let v denote the Hamiltonian vector field of the function $f = (p^0 - z\, p^1)/r$, computed in the coordinates $(\xi,\ \eta)$; see Section 11.9.3. Let θ be the computed coefficient of t^2 in the Taylor expansion at $t = 0$ of the function $1/g \circ \Pi$, in which we have substituted $(\xi,\ \eta) = t\, v(0,\ 0)$. Then, if we write $h := \theta\, g$, the restriction H to $S_{\varphi(e)}$ of h has a pole of order two at e, no other poles, and $H = t^{-2} + \mathrm{O}(t^{-1})$ for $t \to 0$, where t denotes the time parameter of the solution curve on $S_{\varphi(e)}$ of the aforementioned Hamiltonian vector field with e as its initial point. Because $\wp(t) = t^{-2} + \mathrm{O}(1)$, see (2.3.3), it follows that $H - \wp$ has at most a simple pole at e and no other poles on $S_{\varphi(e)}$. If it had a pole, which then would be simple, then $H - \wp$ would define a holomorphic mapping $S_{\varphi(e)} \to \mathbb{P}^1$ of degree one, which would be an isomorphism, in contradiction to the fact that $S_{\varphi(e)}$ is an elliptic curve. We conclude that $H - \wp$ has no poles on $S_{\varphi(e)}$, and therefore is equal to a constant c. This implies that $K := v\,H = \wp'$, the derivative with respect to t of the Weierstrass $\wp$-function on $S_{\varphi(e)}$.

For the determination of c, g_2 and g_3, I used that the Weierstrass $\wp$ function satisfies the equation

$$\begin{aligned} 0 &= (\wp')^2 - 4\,\wp^3 + g_2\,\wp + g_3 = K^2 - 4\,(H - c)^3 + g_2\,(H - c) + g_3 \\ &= K^2 - 4\,H^3 + 12\,c\,H^2 + (-12\,c^2 + g_2)\,H + (4\,c^3 - g_2\,c + g_3); \end{aligned}$$

see (2.3.5). Write $L = K^2 - 4\,H^3 + \alpha\,H^2 + \beta\,H + \gamma$, in which α, β, γ are free coefficients. If we have three points ζ_i, $i = 1, 2, 3$, on the curve $p^0 - \varphi(e)\,p^1 = 0$ in sufficiently general position, then we can compute α, β, γ from the three inhomogeneous linear equations $L(\zeta_i) = 0$, $1 \le i \le 3$, for α, β, γ, and c, g_2, and g_3 are computed by successively solving the equations $12\,c = \alpha$, $-12\,c^2 + g_2 = \beta$, and $4\,c^3 - g_2\,c + g_3 = \gamma$. This worked by taking $\zeta_1 = \pi(\alpha(e))$, the constant term in the expansion of $A \circ \Pi(\xi, e)$ as $\xi \to 0$, $\zeta_2 = A(\zeta_1)$, and $\zeta_3 = A(\zeta_2)$. Note that the invariance of p^0/p^1 under the transformation (11.9.1) implies that the points ζ_i lie on the curve $p^0 - \varphi(e)\,p^1 = 0$. The Weierstrass data of the map A are then computed as $X = \wp(\zeta_1) = H(\zeta_1) - c$ and $Y = \wp'(\zeta_1) = K(\zeta_1)$.

All the Weierstrass data turned out to be even polynomial functions of e and of p, and replacing every even power $e^{2\,k}$ and $p^{2\,l}$ by $(z - z_*)^k$ and q^l, respectively, I obtained the Weierstrass data as polynomials in z and q. Furthermore, as a function of a, the Weierstrass data turned out to be linear combinations of $a^{2\,m} + a^{-2\,m}$, for $0 \le m \le 6$. Writing $a^{2\,m} + a^{-2\,m}$ as a polynomial of degree m in $a^2 + a^{-2} = b^2 - 2$, where $b = a + 1/a$, I obtained the Weierstrass data g_2, g_3, X, and Y as the explicit polynomial functions of z, q, and b, presented in Section 11.9.3.

Chapter 12
Appendices

12.1 The QRT Mapping on Singular Fibers

According to Corollary 3.3.10, the QRT surface S is a rational elliptic surface, and it follows from Theorem 9.2.4 that the lattice $\Lambda = \mathrm{H}_2(S, \mathbb{Z})$ is equal to the Néron–Severi group $\mathrm{NS}(S)$ of S. According to Lemma 7.3.2, the action of the QRT map on the lattice Λ is equal to an Eichler–Siegel transformation (7.3.1) if and only if τ^S leaves every irreducible component of every reducible fiber invariant. Let P^{red} denote the set of $r \in P$ such that the fiber S_r of κ over r is reducible. For each $r \in P^{\mathrm{red}}$, let S_r^{irr} denote the set of all irreducible components of S_r. Because P^{red} is a subset of the finite set P^{sing} of singular values of κ, and for each $r \in P^{\mathrm{red}}$ the set S_r^{irr} is finite, a finite iterate of the QRT map acts on L as an Eichler–Siegel transformation.

In this subsection we determine the reducible fibers S_r of $\kappa : S \to P$ that can appear after the eight successive blowing-up transformations at the base points of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, and in each case describe how τ^S permutes the irreducible components of these reducible components. It is quite spectacular to observe that in each case the singular fiber that appears after the blowing-up transformations is one of Kodaira's list in Section 6.2.6. Our classification is analogous to the identification of Wall [206, §3] of the singular fibers that can appear after successively blowing up at the base points, nine times, starting with a pencil of cubic curves in $\mathbb{P}^2$. One of the differences, which makes our list longer, is that we also determine the action of the QRT map on the set of the irreducible components of the reducible fibers.

In the analysis of reducible biquadratic curves, we will use Corollary 2.4.2 and Lemma 3.2.7. Each fiber S_r of $\kappa : S \to P$ is the preimage under $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ of a member C of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$, minus each -1 curve that appears at the last blowup over a given base point of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The structure of the fiber, and the way the QRT map $\tau^S : S \to S$ permutes the irreducible components of the fiber, can be determined from the way S_r arises from C by means of the successive blowing-up transformatons, where in each of these we apply Corollary 3.3.8 = Lemma 3.2.8. We also use that the action of the QRT

J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*, Springer Monographs in Mathematics 304, DOI 10.1007/978-0-387-72923-7_12,
© Springer Science+Business Media, LLC 2010

map $\tau^S \in \mathrm{Aut}(S)^+_\kappa$ on the set S_r^{irr} of irreducible components of S_r is determined by its action on any of the multiplicity 1 components. See Corollary 6.3.31 with $c = r$. If the member C of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ is nonsingular, hence an irreducible elliptic curve, then the fiber of $\kappa : S \to P$ that is mapped to C by π is equal to the proper transform $\pi'(C)$ of C in S, which is isomorphic to C, hence irreducible and nonsingular as well. The QRT automorphism τ^S leaves each fiber of κ invariant.

In each case, we mention at the end the Kodaira type of the singular fiber, its intersection diagram, its contribution to the QRT automorphism according to Lemma 7.5.3, and the order of the permutation of the irreducible components. For instance, in the case (1) we have $\mathrm{I}_{\mathrm{m}_d}$, $\mathrm{A}^{(1)}_{\mathrm{m}_d-1}$, 0, 1. This means that the singular fiber S_r is of Kodaira type $\mathrm{I}_{\mathrm{m}_d}$ with intersection diagram $\mathrm{A}^{(1)}_{\mathrm{m}_d-1}$, and that the QRT automorphism τ^S leaves each irreducible component of S_r invariant.

(1) C has a rational curve as normalization with an ordinary double point d. We get a reducible fiber S_r only if d is a base point, of some order $\mathrm{m}_d \leq 8$. Because the order μ_d of C at d is 2, we have $\mathrm{m}_d \geq 2$. All the other members $\widetilde{C}$ are smooth at d and intersect each other at d with multiplicity $\mathrm{m}_d \geq 2$, and therefore have the same tangent line l. Therefore $\widetilde{C}$ intersects one of the local components of C at d transversally, and the other local component with multiplicity $\mathrm{m}_d - 1$. Blowing up m_d times, where the -1 curve that appears at the last blowing up is a holomorphic section and therefore is not contained in the fiber, we obtain a fiber S_r that is a cycle of m_d rational curves, where $\mathrm{m}_d - 1 \leq 7$. One of the irreducible components of S_r is the proper transform $\pi'(C)$ of C. The QRT automorphism τ^S leaves the multiplicity-one component $\pi'(C)$ invariant, and because it acts as a cyclic permutation on the set S_r^{irr} of irreducible components, it leaves each irreducible component invariant.

$\mathrm{I}_{\mathrm{m}_d}$, $\mathrm{A}^{(1)}_{\mathrm{m}_d-1}$, 0, 1. Here $2 \leq \mathrm{m}_d \leq 8$.

(2) C has a rational curve as normalization and an ordinary *cusp* at the point c, with local equation $x^2 + y^3 = 0$ for C at $c = (0, 0)$. We get a reducible fiber S_r only if c is a base point. All other members $\widetilde{C}$ of the pencil are smooth at c with a common tangent line l.

(2a) In the local coordinates l is not equal to $x = 0$. Then $\widetilde{C}$ has a local equation $y = f(x)$ for an analytic germ f at 0 with $f(0) = 0$. The asymptotic expansion $x^2 + y^3 = x^2 + f(x)^3 = x^2 + \mathrm{O}(x^3)$ shows that $\mathrm{m}_c = 2$. Blowing up once with the substitution $x = \xi\, y$ yields $(x^2 + y^3)/y = (\xi^2\, y^2 + y^3)/y = (\xi + y^2)\, y$. Blowing up the second time, the -1 curve that appears is a holomorphic section, which is not contained in the fiber. One of the irreducible components is $\pi'(C)$, which is invariant under τ^S.

III, $\mathrm{A}^{(1)}_1$, 0, 1.

(2b) In the local coordinates, l is equal to $x = 0$. Then $\widetilde{C}$ has a local equation $x = g(y)$ for an analytic germ g at 0 with $g(0) = 0$ and $g'(0) = 0$. The asymptotic expansion $x^2 + y^3 = g(y)^2 + y^3 = y^3 + \mathrm{O}(y^4)$ shows that $\mathrm{m}_c = 3$. After the first blowing up we obtain two curves with a second order contact of order 2, at a base

point of multiplicity 2, whereas after the second blowing up another -1 curve is added and the three curves intersect each other at one point with the tangent lines in general position. One of the irreducible components is $\pi'(C)$, which is invariant under τ^S.

IV, $A_2^{(1)}$, 0, 1.

This completes the cases in which C is irreducible.

If C is reducible, then its defining polynomial p can be written as a product of polynomials p_i, $1 \le i \le h$, where p_i is irreducible and bihomogeneous of bidegree (d_i^1, d_i^2), $d_i^j \in \mathbb{Z}_{\ge 0}$, $d_i^1 + d_i^2 > 0$ for $1 \le i \le h$, and $\sum_{i=1}^h d_i^j = 2$ for $j = 1, 2$. Fortunately, there are not too many cases. On the other hand, there are several different configurations of the base points possible on a reducible member of the pencil of biquadratic curves, which makes the list of reducible fibers on the surface S quite long.

(3) The only case in which $d_i^1 > 0$ and $d_i^2 > 0$ for all $1 \le i \le h$ is that $p = p_1\, p_2$, where both p_1 and p_2 are irreducible of bidegree $(1, 1)$. Let C_i denote the zero-set of p_i. For any biquadratic curve $\widetilde{C}$ we have $C_i \cdot \widetilde{C} = (l_1 + l_2) \cdot (2l_1 + 2l_2) = 4l_1 \cdot l_2 = 4$, whereas $C_1 \cdot C_2 = (l_1 + l_2) \cdot (l_1 + l_2) = 2l_1 \cdot l_2 = 2$. Also note that the fact that C_i has intersection number 1 with each horizontal axis implies that C_i is smooth and that the projection of C_i to the second component $\mathbb{P}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ is an isomorphism. This shows that C_i is a smooth rational curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

Because C_i is a $(1, 1)$-curve, it intersects each horizontal axis and each vertical axis with multiplicity 1. Therefore the involutions ι_1 and ι_2 both interchange C_1 and C_2, which implies that the QRT mapping τ leaves C_1 and C_2 invariant. It follows that the QRT automorphism τ^S leaves the proper transforms of C_1 and C_2 in the QRT surface S invariant. This implies that τ^S leaves every irreducible component of the singular fiber invariant in the cases that $C_1 \ne C_2$.

(3a) C_1 and C_2 have two distinct points of intersection a and b, where the intersection is transversal. We have $\mu_a = \mu_b = 2$.

(3a1) If a and b are not base points, then S_r is equal to the proper transform of C.

I_2, $A_1^{(1)}$, 0, 1.

(3a2) If a is not a base point and b is a base point of multiplicity m_b, then $\mathrm{m}_b \ge \mu_b = 2$. All other members, which are smooth at b, intersect one of the curves C_1, C_2 with multiplicity 1 at b and the other with multiplicity $\mathrm{m}_b - 1 \le 4$. Blowing up m_b times, where the -1 curve that appears at the last blowing up is a holomorphic section that is not contained in the fiber, a chain of $\mathrm{m}_b - 1$ rational curves is inserted between the proper transforms of C_1 and C_2, which intersect each other only in the point over b. The proper transforms of C_1 and C_2 are adjacent rational curves in the cycle of $\mathrm{m}_b + 1$ rational curves.

I_{m_b+1}, $A_{\mathrm{m}_b}^{(1)}$, 0, 1. Here $2 \le \mathrm{m}_b \le 5$.

(3a3) Both a and b are base points of multiplicity $\mathrm{m}_a \ge 2$ and $\mathrm{m}_b \ge 2$, respectively, and each other member intersects C_1 transversally both at a and at b. Then it intersects C_2 at a and b with multiplicity $\mathrm{m}_a - 1$ and $\mathrm{m}_b - 1$, respectively, where $(\mathrm{m}_a - 1) +$

$(\mathrm{m}_b - 1) \leq 4$. The proper transforms of C_1 and C_2 are separated in the cycle of $\mathrm{m}_a + \mathrm{m}_b$ rational curves by one chain of $\mathrm{m}_a - 1$ rational curves and another chain of $\mathrm{m}_b - 1$ rational curves, where $(\mathrm{m}_a - 1) + (\mathrm{m}_b - 1) \leq 4$, $\mathrm{m}_a - 1 \geq 1$, and $\mathrm{m}_b - 1 \geq 1$.

$\mathrm{I}_{\mathrm{m}_a+\mathrm{m}_b}$, $\mathrm{A}^{(1)}_{\mathrm{m}_a+\mathrm{m}_b-1}$, 0, 1. Here $\mathrm{m}_a \geq 2$, $\mathrm{m}_b \geq 2$, and $\mathrm{m}_a + \mathrm{m}_b \leq 6$.

(3a4) Both a and b are base points of multiplicity $\mathrm{m}_a \geq 2$ and $\mathrm{m}_b \geq 2$, respectively, and each other member intersects C_1 transversally at a and intersects C_2 transversally at b. Then it intersects C_2 at a with multiplicity $\mathrm{m}_a - 1$ and C_1 at b with multiplicity $\mathrm{m}_b - 1$, where $\mathrm{m}_a \leq 4$ and $\mathrm{m}_b \leq 4$. The proper transforms of C_1 and C_2 are separated in the cycle of $\mathrm{m}_a + \mathrm{m}_b$ rational curves by one chain of $\mathrm{m}_a - 1$ rational curves and another chain of $\mathrm{m}_b - 1 \leq 3$ rational curves, where $1 \leq \mathrm{m}_a - 1 \leq 3$ and $1 \leq \mathrm{m}_b - 1 \leq 3$.

$\mathrm{I}_{\mathrm{m}_a+\mathrm{m}_b}$, $\mathrm{A}^{(1)}_{\mathrm{m}_a+\mathrm{m}_b-1}$, 0, 1. Here $2 \leq \mathrm{m}_a \leq 4$, $2 \leq \mathrm{m}_b \leq 4$, and $\mathrm{m}_a + \mathrm{m}_b \leq 8$.

(3b) C_1 and C_2 have one intersection point d, of multiplicity 2.

(3b1) If d is not a base point, then S_r is equal to the proper transform $\pi'(C)$ of C, isomorphic to C.

III, $\mathrm{A}^{(1)}_1$, 0, 1.

(3b2) If d is a base point, then it has multiplicity $\mathrm{m}_d \geq \mu_d = 2$. Each other member of the pencil is smooth at d. Assume that its tangent line l at d is not equal to the common tangent line at d of C_1 and C_2. Then $\mathrm{m}_d = 2$. After the first blowing up we get a -1 curve E, intersected by the proper transforms C_1' and C_2' of C_1 and C_2 at one point p, and with the tangent lines at p in general position. The new base point is not equal to p, and after all the blowing-up transformations the components of the singular fiber are the proper transforms of C_1, C_2, and E, intersecting each other at one point and with tangent lines in general position.

IV, $\mathrm{A}^{(1)}_2$, 0, 1.

(3b3) As in (3b2) but with l equal to the common tangent line at d of C_1 and C_2. Then l has contact of order 2 with one of these, say C_1, and of order $2 \leq \mathrm{m}_d - 2 \leq 4$ with C_2. This time the new base point is equal to p, and the members smooth at p have a contact of order 1 and $\mathrm{m}_d - 2$ with C_1' and C_2', respectively. Also, the singular member vanishes to order 3 at p, which means that at the next blowing up we obtain a -1 curve of multiplicity 2. After all the blowing up transformations we arrive at a chain of the $\mathrm{m}_d - 3$ multiplicity-2 components, where the proper transforms of C_1 and C_2 are multiplicity-one components at different ends of the chain.

$\mathrm{I}^*_{\mathrm{m}_d-4}$, $\mathrm{D}^{(1)}_{\mathrm{m}_d}$, 0, 1. Here $4 \leq \mathrm{m}_d \leq 6$.

(3c) $C_1 = C_2 = C$. Each member intersects C with multiplicity 4, where each intersection point of degree d is a base point of degree $2d$. The QRT map τ acts as the identity on the double curve C, and therefore the QRT automorphism τ^S acts as the identity on the proper transform C' of C in S, which is a multiplicity-two component of the fiber over C.

(3c1) If all four intersection points are simple, then after all the blowing up transformations the fiber over C has a multiplicity-two component equal to the proper transform of C, and four multiplicity one components, equal to the proper transforms

of the -1 curves that appear after the first blowing up at each of the four base points of multiplicity 2. Because the QRT map τ leaves each of the base points fixed, the QRT automorphism τ^S leaves each of the multiplicity-one components invariant.

I_0^*, $\mathrm{D}_4^{(1)}$, 0, 1.

(3c2) One of the intersection points a has multiplicity $d = 2$ and the other two, b_1 and b_2, are simple. Let E_1, E_2, E_3 be the proper transforms of the -1 curves that appear at the first three blowing-up transformations at the base point a of multiplicity 4. Let F_1 and F_2 be the proper transforms of the -1 curves that appear after the first blowing up at b_1 and b_2, respectively. Then the singular fiber has the multiplicity two components C' and E_2, the multiplicity-one components E_1 and E_3 intersecting E_2, and the multiplicity-one components F_1 and F_2 intersecting C'. Because τ leaves b_1 and b_2 fixed, τ^S leaves F_1 and F_2 invariant. It follows from the proof of Lemma 5.1.8 that ι_1^S and ι_2^S both interchange E_1 and E_3, and therefore τ^S leaves E_1 and E_3 invariant.

I_1^*, $\mathrm{D}_5^{(1)}$, 0, 1.

(3c3) Two intersection points a and b, each of multiplicity $d = 2$. let E_1, E_2, E_3 and F_1, F_2, F_3 be the proper transforms of the -1 curves that appear at the first three blowing up transformations over a and b, respectively. Then the fiber over C has the chain E_2, C', F_2 of multiplicity-two components, where the multiplicity-one components E_1 and E_3 intersect E_2 and the multiplicity-one components F_1 and F_3 intersect F_2. It follows from the proof of Lemma 5.1.8 that ι_1^S and ι_2^S both interchange E_1 and E_3, and both interchange F_1 and F_3, and therefore τ^S leaves E_1 and E_3 invariant and leaves F_1 and F_3 invariant.

I_2^*, $\mathrm{D}_6^{(1)}$, 0, 1.

(3c4) One of the intersection points a has multiplicity $d = 3$ and the other b is simple. Let E_j, $1 \le j \le 5$ be the proper transforms of the -1 curves that appear at the first five blowing-up transformations over the base point a of multiplicity 6, and let F be the proper transform of the -1 curve that appears at the first blowing up at b. Then the singular fiber has the multiplicity-three component E_3, intersected by the three multiplicity-two components E_2, E_4, C', each of which is intersected by the multiplicity-one component E_1, E_5, F, respectively. ι_1^S and ι_2^S leave F invariant and switch E_1 and E_5, and therefore τ^S leaves the multiplicity-one components F, E_1, and E_5 invariant.

IV^*, $\mathrm{E}_6^{(1)}$, 0, 1.

(3c5) There is only one intersection point with multiplicity $d = 4$, meaning that there is only one base point a of multiplicity 8. Let E_j, $1 \le j \le 7$, be the proper transforms of the -1 curves that appear at the first seven blowing-up transformations over the base point a. Then the singular fiber has the chain of components E_1, E_2, E_3, E_4, E_5, E_6, E_7 of multiplicities 1, 2, 3, 4, 3, 2, 1, respectively, and the multiplicity-two component C' that intersects E_4. It follows from the proof of Lemma 5.1.8 that ι_1^S and ι_2^S both interchange E_1 and E_7, and therefore τ^S leaves both multiplicity-one components E_1 and E_7 invariant.

III^*, $\mathrm{E}_7^{(1)}$, 0, 1.

This completes the cases in which C does not contain a horizontal axis $\mathbb{P}^1 \times \{y\}$ or a vertical axis $\{x\} \times \mathbb{P}^1$. Note that in all these cases τ^S acts trivially on the set S_r^{irr} of irreducible components of the reducible fiber S_r, and therefore the contribution $\mathrm{contr}_r(\tau)$, see Lemma 7.5.3, is equal to zero in each of these cases. Note that the order of the identity permutation is equal to 1.

Remark 12.1.1. The description of the cases (1)–(3) might also be useful if one meets automorphisms $\alpha \in \mathrm{Aut}(S)^+_\kappa$ other than τ^S, which may act in a nontrivial fashion on some S_r^{irr} in the above cases (1)–(3). This can happen, for instance, when the matrices A^0 and A^1 are symmetric, and $\alpha = \sigma^S \circ \iota_1^S = \iota_2^S \circ \sigma^S$, where σ^S is the automorphism of S induced by the switch involution $\sigma : (x, y) \mapsto (y, x)$. See Section 10.1. Then $\tau^S = \alpha^2$, and therefore α acts as a permutation of order at most 2 on the S_r^{irr} in cases (1)–(3). For instance, in cases (3a) and (3b), the switch involution σ either permutes C_1 and C_2 or leaves C_1 and C_2 invariant. In the second case, α permutes the proper transforms of C_1 and C_2, and therefore acts as a permutation of order 2 on S_r^{irr}. ⊘

For the discussion of the cases in which C contains, for instance, a horizontal axis $A = \mathbb{P}^1 \times \{y\}$, we first determine to which curves ι_1^S sends the proper transform of A and the proper transforms of the -1 curves that appear at the blowing-up transformations over base points on A. Note that every smooth member of the pencil of biquadratic curves has intersection number 2 with A, and therefore A contains either two base points where the intersection is transversal or one base point with a second order contact.

Lemma 12.1.2 *Let $a = (x, y)$ and $a' = (x', y)$ be two distinct base points of multiplicity m and m', respectively, of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Let L_1 be the proper transform in S of the horizontal axis $A = \mathbb{P}^1 \times \{y\}$ through both base points, under the blowing up $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$. Let E_{l+j} and $E_{l'+j'}$, for $1 \le j \le m$ and $1 \le j' \le m'$, denote the proper transforms of the -1 curves that appear at the successive blowing-up transformations over the base points a and a', respectively. We may assume that $m \ge m'$. Then $\iota_1^S(E_{l+j}) = E_{l'+m'-m+j}$ for $m - m' + 1 \le j \le m$, $\iota_1^S(E_{l+m-m'}) = L_1$ if $m > m'$, and $\iota_1^S(L_1) = L_1$ if $m = m'$, and finally $\iota_1^S(E_{l+j}) = E_{l+m-m'-j}$ for $1 \le j \le m - m' - 1$.*

With the notation of Lemma 5.1.2*, it follows that $l + j \notin I_1$ and $\iota_1(e_{l+j}) = e_{l'+m'-m+j}$ for every $m - m' + 1 \le j \le m$, whereas $l + j \in I_1$ and $\iota_1(e_{l+j}) = l_1 - e_{l+m'-m-j+1}$ for every $1 \le j \le m - m'$.*

Proof. We refer to the Sections 3.2.2 and 3.3.3 for the basic facts about blowing up that will be used in the proof. Let C be a smooth member of the pencil of biquadratic curves. Then C intersects A at a and a' with multiplicity 1, and the restriction to C of ι_1 interchanges the base points a and a'. It follows that ι_1^S interchanges the intersection points of $\pi'(C)$ with the sections E_{l+m} and $E_{l'+m'}$ over a and a', respectively. Because the set of these intersection points where C ranges over the smooth members of the pencil is dense in E_{l+m} and $E_{l'+m'}$, it follows that ι_1^S interchanges E_{l+m} and $E_{l'+m'}$.

According to Lemma 5.1.7, the total transform of A, which is equal to the union of L_1, E_{l+j}, $1 \le j \le m$, and $E_{l'+j}$, $1 \le j \le m'$, is invariant under ι_1^S. Their

intersection diagram is a chain, in the order

$$E_{l+k},\ \ldots,\ E_{l+1},\ L_1,\ E_{l'+1},\ \ldots,\ E_{l'+k'}.$$

Because the diffeomorphism ι_1^S interchanges the endpoints of the chain, we obtain that $\iota_1^S(E_{l+j}) = E_{l'+k'-k+j}$ for $k-k'+1 \le j \le k$, $\iota_1^S(E_{l+k-k'}) = L_1$ if $k > k'$ and $\iota_1^S(L_1) = L_1$ if $k = k'$, and finally $\iota_1^S(E_{l+j}) = E_{l+k-k'-j}$ for $1 \le j \le k-k'-1$. Because the homology classes e_{l+j} and $e_{l'+j'}$ of the total transform of E_{l+j} and $E_{l'+j'}$ are equal to the sum of the homology classes of the E_{l+h}, $j \le h \le k$, and the $E_{l'+h'}$, $j' \le h' \le k'$, respectively, and l_1 is equal to $[L_1] + e_{l+1} + e_{l'+1}$, the desired equations for $\iota_1(e_{l+j})$ follow. □

Lemma 12.1.3 *Assume that the horizontal axis $A = \mathbb{P}^1 \times \{y\}$ is contained in a member of the pencil, and that A contains only one base point $a = (x, y)$, which then is of multiplicity $m \ge 2$. Let L_1 and E_{l+j}, $1 \le j \le m$, denote the proper transforms in S of A and the -1 curves that appear at the successive blowing-up transformations at the base points over a. Then $\iota_1^S(E_{l+j}) = E_{l+j}$ for $2 \le j \le m$ and ι_1^S either leaves E_{l+1} and L_1 invariant or interchanges E_{l+1} and L_1, where the first case occurs if $m = 2$. If the vertical axis $B = \{x\} \times \mathbb{P}^1$ through a does not contain any other base point, then the involution ι_2^S interchanges E_{l+1} and E_{l+m-1}.*

With the notation of Lemma 5.1.2*, it follows that $l + j \notin I_1$ and $\iota_1(e_{l+j}) = e_{l+j}$ when $2 \le j \le m$. In the first case $l + 1 \notin I_1$ and $\iota_1(e_{l+1}) = e_{l+1}$, whereas in the second case $l + 1 \in I_1$ and $\iota_1(e_{l+1}) = l_1 - e_{l+1}$.*

Proof. We refer to the Sections 3.2.2 and 3.3.3 for the basic facts about blowing up that will be used in the proof. Let C be a smooth member of the pencil of biquadratic curves. Then a is the only point of intersection of C with A, which is of multiplicity 2, and therefore $m = \mathrm{m}_a \ge \mu_a = 2$. It follows that for $\widetilde{y} \ne y$, $\widetilde{y}$ close to y, C has two points of intersection with $\widetilde{A} = \mathbb{P}^1 \times \{\widetilde{y}\}$, both close to close a, which are interchanged by ι_1. Because these points of intersection converge to a as $\widetilde{y} \to y$, it follows that the unique point c in the proper transform $\pi'(C)$ of C such that $\pi(c) = a$, see Corollary 3.3.8 = Lemma 3.2.8, is a fixed point of ι_1^S. Because the -1 curve E_{l+m} that appears at the last blowing up is a holomorphic section, intersecting $\pi'(C)$ exactly once at the point c, and the set of these points c, as C ranges over the smooth members of the pencil, is dense in E_{l+m}, it follows that ι_1^S leaves E_{l+m} pointwise fixed.

Let E_{l+1}^0 denote the -1 curve that appears after the first blowing up at a. Because C intersects the horizontal axis with multiplicity 2, it follows from Corollary 3.3.8 = Lemma 3.2.8 that the proper transform of A after the first blowing up intersects E_{l+1}^0 in the base point e of the second blowing up, but that at the next blowing up it no longer intersects E_{l+2}^0 in a base point, where E_{l+2}^0 denotes the -1 curve that appears after the blowing up at e. This means that L_1 and E_{l+1} both transversally intersect E_{l+2} at two different points, E_{l+j-1} intersects E_{l+j} with multiplicity 1 for $3 \le j \le m$, and there are no other intersections.

The total transform of A is equal to the union of L_1 and the E_{l+j}, $1 \le j \le k$, which according to Lemma 5.1.7 is invariant under ι_1^S. Because of the intersection

structure, we conclude that $\iota_1^S(E_{l+j}) = E_{l+j}$ for $2 \le j \le k$, and ι_1^S either also leaves E_{l+1} and L_1 invariant, or switches them. If $m = 2$, then a is a smooth point of the member to which A belongs; hence E_{l+1} does not belong to the fiber to which L_1 belongs, and therefore cannot be interchanged with L_1.

The smooth members intersect the vertical axis $B = \{x\} \times \mathbb{P}^1$ transversally at a, and therefore have one other simple intersection point with B. If there is no base point on B other than a, then the involution ι_2 maps the base point $a = (x, y)$ on different smooth members to different points on B. The first blowing up at a leads to disjoint proper transforms of A and B, which intersect the -1 curve that appears over a at different points, where the intersection point of the proper transform of A is the base point of the new anticanonical pencil where we will blow up further. It follows that after the m blowing-up transformations over a, the proper transform L_2 of B is a -1 curve, hence a holomorphic section, which intersects the singular fiber at E_{l+1}, and ι_2^S maps the section E_{l+m} onto L_2. Because L_2 and E_{l+m} intersect the singular fiber at E_{l+1} and E_{l+m-1}, respectively, it follows that $\iota_2^S(E_{l+m-1}) = E_{l+1}$. This proves the first three statements in the lemma.

For the last statement we observe that for $1 \le j \le m$, e_{l+j} is equal to the sum over $j \le k \le m$ of the homology classes of the E_{l+k}. The total transform of A after the first blowing up is equal to $A' + E_{l+1}^0$, where A' denotes the proper transform of A after the first blowing up, whereas the total transform of A' after the second blowing up is equal to $A'' + E_{l+2}^0$, where A'' denotes the proper transform of A' after the second blowing up. Because l_1 is equal to the homology class of the total transform of A, it follows that

$$l_1 = [E_{l+1}] + 2 \sum_{j=2}^{m} [E_{l+j}].$$

Therefore, if ι_1^S switches E_{l+1} and L_1, we have

$$\iota_1(e_{l+1}) = [L_1] + \sum_{j=2}^{m} [E_{l+j}] = l_1 - \sum_{j=2}^{m} [E_{l+j}] = l_1 - e_{l+1}.$$

□

Remark 12.1.4. In the situation of Lemma 12.1.3, let C be the member of the pencil of biquadratic curves that contains A. Then C has irreducible components other than A, and we have $m > 2$ if and only if a is contained in at least one of these other irreducible components of C. Whether ι_1^S switches L_1 and E_{l+1} or not depends on the configuration of the irreducible components passing through a. For instance, in case (4a3) below, ι_1^S switches L_1 and E_{l+1}. In case (4b3), ι_1^S leaves L_1 and E_{l+1} invariant if m_d is even, and switches them if m_d is odd. ⊘

Remark 12.1.5. A similar description as in Lemmas 12.1.2 and 12.1.3 is valid with ι_1 replaced by ι_2. If in Lemma 12.1.3 the other intersection point of the smooth members with B is a base point b, then we are, regarding ι_2^S, in the situation of Lemma 12.1.2 with ι_1 replaced by ι_2. ⊘

We now resume our classification of the action of the QRT automorphism τ^S on the set of irreducible components of a reducible fiber, with the cases that the corresponding member of the pencil of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ contains at least one horizontal or vertical axis.

(4) The irreducible components of C are a horiontal axis $L_1 = \mathbb{P}^1 \times \{y\}$, which is a (0, 1) curve, and an irreducible (2, 1) curve C_2. The horizontal axis L_1 and the curve C_2 intersect each other with multiplicity $l_1 \cdot (l_1 + 2l_2) = 2l_1 \cdot l_2 = 2$. Because the intersection number of C_2 with each vertical axis is equal to $(l_1 + 2l_2) \cdot l_2 = l_1 \cdot l_2 = 1$, C_2 is smooth, equal to the graph of a smooth function $y = f(x)$. This also shows that C_2 is rational, that is, isomorphic to a complex projective line. We have the following cases. Also note that each member of the pencil of biquadratic curves has intersection number with L_1 and C_2 equal to $(2l_1 + 2l_2) \cdot l_1 = 2$ and $(2l_1 + 2l_2) \cdot (l_1 + 2l_2) = 4 + 2 = 6$, respectively. Because ι_1 leaves C_2 invariant and ι_2 maps C_2 to L_1, τ^S maps the proper transform of C_2 to the proper transform of L_1.

(4a) L_1 and C_2 have two distinct intersection points a and b, each simple.

(4a1) If a and b are not base points, then S_r is equal to the proper transform of C, where the QRT automorphism τ^S interchanges the two irreducible components of S_r.

I_2, $\mathrm{A}_1^{(1)}$, 1/2, 2.

(4a2) a is not a base point and b is a base point. The other members are smooth at b and all have the same tangent line at b. Suppose that these intersect both L_1 and C_2 transversally at b. Then $\mathrm{m}_b = \mu_b = 2$ and S_r is a cycle of three rational curves. The QRT automorphism τ^S maps the proper transform of C_2 to the proper transform of L_1, and therefore it acts as a rotation on the cycle of rational curves; see Corollary 6.3.31.

I_3, $\mathrm{A}_2^{(1)}$, 2/3, 3.

(4a3) As in (4a2) but now each other member has contact of order 2 with L_1 at b and hence intersects C_2 transversally at b. In this case $\mathrm{m}_b = 3$, and S_r is a cycle of four rational curves. τ^S maps the proper transform of C_2 to the proper transform of L_1, where these proper transforms are adjacent because the intersection point a of L_1 and D is not a base point.

I_4, $\mathrm{A}_3^{(1)}$, 3/4, 4.

(4a4) As in (4a2) but now every other member has contact at b of order 1 and $\mathrm{m}_b - 1$ with L_1 and C_2, respectively, where $2 \le \mathrm{m}_b - 1 \le 4$. In this case S_r is a cycle of $\mathrm{m}_b + 1$ rational curves. τ^S maps the proper transform of C_2 to the proper transform of L_1, which are adjacent in the cycle of $\mathrm{m}_b + 1$ rational curves.

$\mathrm{I}_{\mathrm{m}_b+1}$, $\mathrm{A}_{\mathrm{m}_b}^{(1)}$, $\mathrm{m}_b/(\mathrm{m}_b + 1)$, $\mathrm{m}_b + 1$. Here $4 \le \mathrm{m}_b + 1 \le 6$.

(4a5) a and b are both base points. Then every other member intersects L_1 transversally both at a and at b, and intersects C_2 at a and b with multiplicity $\mathrm{m}_a - 1 \ge 1$ and $\mathrm{m}_b - 1 \ge 1$, respectively, where $(\mathrm{m}_a - 1) + (\mathrm{m}_b - 1) \le 6$. S_r is a cycle of $\mathrm{m}_a + \mathrm{m}_b$ rational curves, where $\mathrm{m}_a + \mathrm{m}_b - 1 \le 7$. The proper transforms of L_1 and C_2 are separated in the cycle of $\mathrm{m}_a + \mathrm{m}_b$ rational curve by one chain of $\mathrm{m}_a - 1 \ge 1$

rational curves and another chain of $\mathrm{m}_b - 1 \geq 1$ rational curves. τ_S maps the proper transform of C_2 to the proper transform of L_1 and acts as a rotation on the cycle S_r^{irr}; see Corollary 6.3.31 with $c = r$. These properties determine the action of τ^S on S_r^{irr}.

$\mathrm{I}_{\mathrm{m}_a+\mathrm{m}_b}$, $\mathrm{A}^{(1)}_{\mathrm{m}_a+\mathrm{m}_b-1}$, $\mathrm{m}_a\ \mathrm{m}_b/(\mathrm{m}_a+\mathrm{m}_b)$, $(\mathrm{m}_a+\mathrm{m}_b)/\gcd(\mathrm{m}_a,\ \mathrm{m}_a+\mathrm{m}_b)$. Here $\mathrm{m}_a \geq 2$, $\mathrm{m}_b \geq 2$, and $\mathrm{m}_a + \mathrm{m}_b \leq 8$.

(4b) L_1 and C_2 have one point of intersection d, of multiplicity 2. The QRT map τ interchanges L_1 and C, and therefore the QRT automorphism τ^S interchanges the proper transforms L_1' and C' of L_1 and C, respectively.

(4b1) If d is no base point, then S_r consists of L_1' and C_2', which are interchanged by the QRT automorphism τ^S.

III, $\mathrm{A}_1^{(1)}$, 1/2, 2.

(4b2) d is a base point. The other members are smooth at d and have the same tangent line l. Suppose that l is not equal to the common tangent line of L_1 and C_2 at d. Then $\mathrm{m}_d = \mu_d = 2$. After one blowing up we obtain a -1 curve E, and the proper transforms of L_1 and C_2 have a common intersection point with E, where the tangent lines are in general position. The new base point on E is not equal to this common intersection point.

IV, $\mathrm{A}_2^{(1)}$, 2/3, 3.

(4b3) d is a base point such that the other members are tangent at d to L_1 and C_2. Then they have contact of order 2 and $\mathrm{m}_d - 2$ with L_1 and C_2 at d, respectively, where $2 \leq \mathrm{m}_d - 2 \leq 6$. Let E_j, $1 \leq j \leq \mathrm{m}_d - 1$, be the proper transforms of the -1 curves that appear after the first m_d blowing-up transformations over d. Then E_j, $2 \leq j \leq \mathrm{m}_d - 2$, is a chain of multiplicity-two components. The multiplicity-one components L_1' and E_1 intersect E_2, and the multiplicity-one components C' and E_{m_d-1} intersect E_{m_d-2}. The singular fiber is of Kodaira type $\mathrm{I}^*_{\mathrm{m}_d-4}$; see Corollary 6.3.31 for the way the irreducible components can be permuted by an element of $\mathrm{Aut}(S)^+_\kappa$.

$\mathrm{I}^*_{\mathrm{m}_d-4}$, $\mathrm{D}^{(1)}_{\mathrm{m}_d}$, $\mathrm{m}_d/4$, the order is 4 when m_d is odd and 2 when m_d is even. Here $4 \leq \mathrm{m}_d \leq 8$.

(5) The irreducible components of C are a vertical axis $L_2 = \{x\} \times \mathbb{P}^1$, which is a $(1,\ 0)$ curve, and an irreducible $(1,\ 2)$ curve C_1. Interchanging the role of horizontal and vertical, which corresponds to replacing the QRT map by its inverse, we can obtain the classification of case (5) from the classification of (4).

(6) The irreducible components of C are a horizontal axis $L_1 = \mathbb{P}^1 \times \{y\}$, which is a $(0,\ 1)$ curve, a vertical axis $L_2 = \{x\} \times \mathbb{P}^1$, which is a $(1,\ 0)$ curve, and a $(1,\ 1)$ curve D. The curve D is smooth and both the projection onto the horizontal axis and the projection onto the vertical axis are isomorphisms from D onto $\mathbb{P}^1$. The axes L_1 and L_2 intersect each other at $a = (x,\ y)$, whereas D intersects L_1 and L_2 in a unique point $a_1 = (x_1,\ y)$ and $a_2 = (x,\ y_2)$, respectively. All these intersections are transversal. The intersection number of each fiber with L_1 and with L_2 is equal to 2, which leaves $(2l_1 + 2l_2) \cdot (l_1 + l_2) = 2 + 2 = 4$ as the intersection number with D. The involution ι_1 maps L_2 to D and ι_2 maps D to L_1; hence τ^S maps the proper

transform of L_2 to L_1. We denote the proper transforms under $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ of L_1, L_2 and D by L_1', L_2', and D', respectively.

(6a) $a \notin D$. Then $a_1 \neq a$, $a_2 \neq a$, and $a_1 \neq a_2$.

(6a1) a is not a base point.

If neither a_1 nor a_2 is a base point, then S_r is of Kodaira type I_3, with intersection diagram $\mathrm{A}_2^{(1)}$.

If a_1 is not a base point and a_2 is a base point, of multiplicity m_2, then the other members are smooth at a_2 and have contact of order k_2 and l_2 at a_2 with D and L_2, respectively, where $1 \leq k_2 \leq 4$, $1 \leq l_2 \leq 2$, $k_2 + l_2 = m_2$, and not $k_2 > 1$ and $l_2 > 1$. The singular fiber S_r is of Kodaira type I_{m_2+2}, with intersection diagram $\mathrm{A}_{m_2+1}^{(1)}$, where L_1' is adjacent to L_2', L_1' is adjacent to D', and D' and L_2' are separated by a chain of $m_2 - 1$ rational curves. Here $2 \leq m_2 \leq 5$.

Similarly, if a_1 is a base point of multiplicity m_1 and a_2 is not a base point, then $2 \leq m_1 \leq 5$, S_r is of Kodaira type I_{m_1+2}, with intersection diagram $\mathrm{A}_{m_1+1}^{(1)}$, where L_1' is adjacent to L_2', L_2' is adjacent to D', and D' and L_1' are separated by a chain of $m_1 - 1$ rational curves.

If a_1 and a_2 are base points of multiplicity m_1 and m_2, respectively, then every other member is smooth at a_i, and has contact of order k_i and l_i at a_i with L_i and D, respectively. Here $1 \leq k_i \leq 2$, $l_i \geq 1$, not $k_i > 1$ and $l_i > 1$, $l_1 + l_2 \leq 4$, and $m_i = k_i + l_i$. The singular fiber S_r is of Kodaira type $\mathrm{I}_{m_1+m_2+1}$, with intersection diagram $\mathrm{A}_{m_1+m_2}^{(1)}$, where L_1' is adjacent to L_2', D' and L_1' are separated by a chain of $m_1 - 1$ rational curves, and D' and L_2' are separated by a chain of $m_2 - 1$ rational curves.

In all cases S_r^{irr} is a cycle of b rational curves, where $3 \leq b \leq 7$. We have L_1', L_2', $D' \in S_r^{\mathrm{irr}}$, where L_1' and L_2' are adjacent in the cycle, and τ^S acts as a rotation on S_r^{irr} sending L_2' to the adjacent rational curve L_1'.

I_b, $\mathrm{A}_{b-1}^{(1)}$, $(b-1)/b$, b. Here $3 \leq b \leq 7$.

(6a2) a is a base point of multiplicity 2, when the other members, which are smooth and have the same tangent line at a, are transversal to both L_1 and L_2 at a.

S_r is of Kodaira type I_b, with intersection diagram $\mathrm{A}_{b-1}^{(1)}$, with one rational curve inserted between L_1' and L_2' in the cycle of b rational curves. D' and L_1' are adjacent in the cycle if a_1 is not a base point. If a_1 is a base point of multiplicity m_1, then at a_1 the other members are transversal to L_1 and have a contact of order $m_1 - 1$ with D, and a chain of $m_1 - 1$ rational curves is inserted between D' and L_1'. We have $1 \leq m_1 - 1 \leq 4$. Similarly D' and L_2' are adjacent in the cycle if a_2 is not a base point. If a_2 is a base point of multiplicity m_2, then at a_2 the other members are transversal to L_2 and have a contact of order $m_2 - 1$ with D, and a chain of $m_2 - 1$ rational curves is inserted between D' and L_2'. We have $1 \leq m_2 - 1 \leq 4$, and $(m_1 - 1) + (m_2 - 1) \leq 4$ if also a_1 is a base point. Therefore at most $1 + 4 = 5$ rational curves have been inserted, and therefore

$$4 = 3 + 1 \le b = 3 + 1 + (m_1 - 1) + (m_2 - 1) \le 3 + 5 = 8.$$

τ^S acts as a rotation on S_r^{irr} sending L_2' to the one but adjacent rational curve L_1'.

I_b, $\mathrm{A}_{b-1}^{(1)}$, $2\,(b-2)/b$, and the order is $b/2$ and b when b is even and odd, respectively. Here $4 \le b \le 8$.

(6a3) a is a base point of multiplicity 3.

The singular fiber is of Kodaira type I_b, with intersection diagram $\mathrm{A}_{b-1}^{(1)}$ with a chain of two rational curves inserted between L_1' and L_2' in the cycle of b rational curves. If the other members intersect L_2 at a with multiplicity 2, then a_2 is not a base point and D' and L_2' are adjacent, and if also a_1 is not a base point, then also D' and L_1' are adjacent, when $b = 5$. The other members intersect L_1 transversally at a. Therefore, if a_1 is a base point of multiplicity $m_1 \ge 2$, then the other members intersect L_1 transversally and have contact of order $m_1 - 1 \le 4$ with D at a_1. This means that D' and L_1' are separated by a chain of $m_1 - 1$ rational curves, and $5 \le b = 3 + 2 + m_1 - 1 \le 9$. The QRT automorphism τ^S acts as a rotation on S_r^{irr} sending L_2' to the two but adjacent rational curves L_1'.

A similar description, with the roles of L_1 and L_2 interchanged, holds if the other members intersect L_1 at a with multiplicity 2.

I_b, $\mathrm{A}_{b-1}^{(1)}$, $3(b-3)/b$, order $b/3$ and b when $b \in 3\,\mathbb{Z}$ and $b \notin 3\,\mathbb{Z}$, respectively. Here $5 \le b \le 9$.

(6b) $a \in D$. Then $a_1 = a_2 = a$.

(6b1) a is not a base point. The singular fiber S_r has the irreducible components L_1', L_2', and D', which form a cycle. τ^S maps L_2' to L_1' to D' to L_2'.

IV, $\mathrm{A}_2^{(1)}$, 2/3, 3.

(6b2) a is a base point, of multiplicity $\mathrm{m}_a \ge \mu_a = 3$. At a, the other members, which are smooth and have a contact of order $\mathrm{m}_a \ge 3$ with each other, are transversal to both L_1 and L_2, and therefore have a contact of order $\mathrm{m}_a - 2$ with D, where $1 \le \mathrm{m}_a - 2 \le 4$. The irreducible components L_1' and L_2' are attached to one end of the chain of $\mathrm{m}_a - 2$ multiplicity-two components, whereas D' is attached to the other end. Because τ^S maps L_2' to L_1', its action on S_r^{irr} is of order 2; see the last part of Section 6.3.6.

$\mathrm{I}_{\mathrm{m}_a-3}^*$, $\mathrm{D}_{\mathrm{m}_a+1}^{(1)}$, 1, 2. Here $4 \le \mathrm{m}_a + 1 \le 7$.

(6b3) a is a base point at which the other members have a contact of order 2 with L_1 or L_2, which implies that they are transversal to D and to L_2 or L_1, respectively. It follows that $\mathrm{m}_a = 4$. The irreducible components L_1' and L_2' are attached to different ends of the chain of two multiplicity two components. The QRT automorphism

τ^S maps L'_2 to L'_1. See the last part of Section 6.3.6 for the permutations of the components of the singular fiber.

I_1^*, $\mathrm{D}_5^{(1)}$, 5/4, 4.

(7) The irreducible components of C are two horizontal axes $L_{1,\pm} = \mathbb{P}^1 \times \{y_\pm\}$, and two vertical axes $L_{2,\pm} = \{x_\pm\} \times \mathbb{P}^1$, where it can happen that $x_+ = x_-$ and/or $y_+ = y_-$. Write $L'_{1,\pm}$, $L'_{2,\pm}$, for the proper transform under $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ of $L_{1,\pm}$, $L_{2,\pm}$, respectively. The intersection number of any smooth biquadratic curve with any of the axes is equal to 2, which implies that each axis contains one or two base points, where the multiplicities depend on the way the other members intersect the axes. ι_1 interchanges $L_{2,+}$ and $L_{2,-}$, keeping the second coordinate fixed, whereas ι_2 interchanges $L_{1,+}$ and $L_{1,-}$, keeping the first coordinate fixed. It follows that ι_1^S interchanges the irreducible components $L'_{2,+}$ and $L'_{2,-}$ of S_r, and ι_2^S interchanges the irreducible components $L'_{1,+}$ and $L'_{1,-}$ of S_r. For the action of ι_1^S on $L'_{1,\pm}$ and of ι_2^S on $L'_{2,\pm}$ a closer examination of the cases is needed.

(7a) $L_{1,+} \neq L_{1,-}$ and $L_{2,+} \neq L_{2,-}$. Let $a_{\sigma\tau}$ be the intersection point of $L_{1,\sigma}$ and $L_{2,\tau}$, where $\sigma \in \{+, -\}$ and $\tau \in \{+, -\}$. In other words, the $a_{\sigma\tau}$ are the four corners of the quadrangle consisting of the two horizontal and the two vertical axes.

(7a1) None of the corner points is a base point. If $L_{1,\pm}$ contains two base points, then these are simple, whereas the multiplicity of the base point is 2 if $L_{1,\pm}$ contains only one base point. It follows from Lemmas 12.1.2 and 12.1.3 that in either case ι_1^S leaves $L'_{1,\pm}$ invariant, and similarly we have that ι_2^S leaves $L'_{2,\pm}$ invariant. The QRT automorphism τ^S interchanges $L_{1,+}$ with $L_{1,-}$ and interchanges $L_{2,+}$ with $L_{2,+}$.

I_4, $\mathrm{A}_3^{(1)}$, 1, 2.

(7a2) One of the corner points, say $c = a_{++}$, is a base point, where the smooth members intersect $L_{1,+}$ and $L_{2,+}$ transversally. $L_{1,+}$ and $L_{2,+}$ contain one other base point, which is simple and not a corner point, whereas $L_{1,-}$ and $L_{2,-}$ contain two base points, which may coincide but are not corner points. Blowing up c leads to a -1 curve E on which the canonical pencil has a simple base point c'. After blowing up c' the proper transform E' of E is an irreducible component in the singular fiber. The singular fiber is a cycle $L'_{1,+}$, E', $L'_{2,+}$, $L'_{1,-}$, $L'_{2,-}$ and back to $L'_{1,+}$. ι_1^S switches $L_{2,+}$ and $L_{2,-}$, and therefore also $L_{1,+}$ and E', leaving $L_{1,-}$ invariant. The involution ι_2^S switches $L_{1,+}$ and $L_{1,-}$, and therefore also $L_{2,+}$ and E', leaving $L_{2,-}$ invariant. It follows that $\tau^S = \iota_2^S \circ \iota_1^S \in \mathrm{Aut}(S)^+_\kappa$ maps $L_{1,+}$ to $L_{2,+}$ to $L_{2,-}$ to E' to $L_{1,-}$ and back to $L_{1,+}$.

I_5, $\mathrm{A}_4^{(1)}$, 6/5, 5.

(7a3) One of the corner points a_{++} is a base point, where the smooth members are tangent to $L_{1,+}$, and hence have second-order contact with $L_{1,+}$. The corner point a_{++} is a base point of order three, we have no other base point on $L_{1,+}$, one other on $L_{2,+}$, and two others on $L1$, $-$ and on $L_{2,-}$, where all these other base points are smooth points of the rectangle. The singular fiber is a cycle $L_{1,+}$, $L_{2,-}$, $L_{1,-}$, $L_{2,+}$, E_1, E_2, and back to $L_{1,+}$. The horizontal involution ι_1^S switches $L_{2,+}$ and

$L_{2,-}$, the vertical involution ι_2^S switches $L_{1,+}$ and $L_{1,-}$, and therefore $\tau^S = \iota_2^S \circ \iota_1^S$ maps $L_{2,+}$ to $L_{2,-}$ to E_2 to $L_{2,+}$ and $L_{1,+}$ to E_1 to $L_{1,-}$ to $L_{1,+}$.

If the smooth members are tangent at a_{++} to $L_{2,+}$, then we have the same situation with E_1 and E_2 interchanged.

I_6, $\mathrm{A}_5^{(1)}$, 4/3, 3.

(7a4) Two of the corner points are base points, not on the same horizontal or vertical axis, say a_{++} and a_{--}, and the smooth members are transversal to the horizontal and vertical axes at each of these corner points. The QRT automorphism τ^S maps $L_{2,+}$ to E_2 to $L_{1,+}$ to $L_{2,+}$ and $L_{1,-}$ to $L_{2,-}$ to E_1 to $L_{1,-}$.

I_6, $\mathrm{A}_5^{(1)}$, 4/3, 3.

(7a5) Two of the corner points are base points, lying on the same horizontal axis, say a_{++} and a_{+-}, and the smooth members are transversal to the horizontal and vertical axes at each of these corner points. The singular fiber is a cycle $L_{2,+}$, E_1, $L_{1,+}$, E_2, $L_{2,-1}$, $L_{1,-}$, and back to $L_{2,+}$. The QRT automorphism τ^S switches $L_{2,+}$ with E_2, E_1 with $L_{2,-}$, and $L_{1,+}$ with $L_{1,-}$.

Similar conclusions hold if instead the corner base points lie on the same vertical axis.

I_6, $\mathrm{A}_5^{(1)}$, 3/2, 2.

(7a6) Two of the corner points are base points, not lying on the same horizontal or vertical axis. At one of the corner points the smooth members are transversal to both axes, whereas at the other corner point the smooth members are tangent to one of the axes.

I_7, $\mathrm{A}_6^{(1)}$, 10/7, 7.

(7a7) Two of the corner points are base points, lying on the same horizontal or vertical axis. At one of the corner points the smooth members are transversal to both axes, whereas at the other corner point the smooth members are tangent to one of the axes.

I_7, $\mathrm{A}_6^{(1)}$, 12/7, 7.

(7a8) Two of the corner points are base points, not lying on the same horizontal or vertical axis. At both corner points the smooth members are tangent to one of the axes.

I_8, $\mathrm{A}_7^{(1)}$, 3/2, 4.

(7a9) Two of the corner points are base points, lying on the same horizontal or vertical axis. At both corner points the smooth members are tangent to the vertical or horizontal axis, respectively.

I_8, $\mathrm{A}_7^{(1)}$, 2, 2.

(7a10) Three of the corner points are base points. At each corner point the smooth members are transversal to the vertical and to the horizontal axis.

I_7, $\mathrm{A}_6^{(1)}$, 12/7, 7.

(7a11) Three of the corner points are base points. At one of these corner points the smooth members are tangent to the vertical or horizontal axis, whereas at the other

two corner base points, which necessarily lie on the other vertical or horizontal axis, the smooth members are transversal to the vertical and horizontal axes.

I_8, $\mathrm{A}_7^{(1)}$, 15/8, 8.

(7a12) Three of the corner points are base points. At a_{++} the smooth members are tangent to the horizontal axis, at a_{--} the smooth members are tangent to the vertical axis, and at a_{-+} the smooth members are transversal to both axes.

I_9, $\mathrm{A}_8^{(1)}$, 2, 3.

(7a13) All four corner points are base points, at each of which the smooth members are transversal to both axes.

I_8, $\mathrm{A}_7^{(1)}$, 2, 2.

(7b) $L_{1,+} \neq L_{1,-}$ and $L_{2,+} = L_{2,-}$. We write $L_2 = L_{2,+} = L_{2,-}$, and $a_\pm$ is the intersection point of $L_{1,\pm}$ with L_2. Note that all points of L_2, which is a double line, are singular points of our member, whereas the points on $L_{1,\pm} \backslash a_\pm$ are smooth points. The QRT map τ interchanges $L_{1,+}$ and $L_{1,-}$ and therefore the QRT automorphism interchanges their proper transforms $L'_{1,+}$ and $L'_{1,-1}$, respectively.

(7b1) a_+ and a_- are not base points and there are two distinct base points on L_2, each of which has multiplicity 2. The singular fiber consists of the proper transform of L_2, which has multiplicity two, the proper transforms of $L_{1,+}$ and $L_{1,-1}$, and the proper transforms E_1 and E_2 of the -1 curves that appeared at the first blowing up at the two base points of multiplicity 2 on L_2. The latter four components have multiplicity one and intersect the multiplicity two component. The QRT automorphism τ^S switches the proper transforms of $L_{1,+}$ $L_{1,-}$ and, according to Lemmas 12.1.2 and 12.1.3, also switches E_1 and E_2.

I_0^*, $\mathrm{D}_4^{(1)}$, 1, 2.

(7b2) a_+ and a_- are not base points and there is only one base point b on L_2, which is a base point of multiplicity 4. Let L'_2 and $L'_{1,\pm}$ denote the proper transforms in S of L_2 and $L_{1,\pm}$, respectively, and let E_i, $1 \leq i \leq 3$, denote the proper transforms of the -1 curves that appear at the ith blowing up over b. These are the irreducible components of the singular fibers, where L'_2 and E_2 have multiplicity 2, $L'_{1,\pm}$ have multplicity 1 and intersect L'_2, and E_1 and E_3 have multiplicity 1 and intersect E_2. The QRT automorphism τ^S switches $L'_{1,+}$ and $L'_{1,-}$ and, according to Lemma 12.1.3, also switches E_1 and E_3.

I_1^*, $\mathrm{D}_5^{(1)}$, 1, 2.

(7b3) One of the points $a_\pm$, say a_+, is a base point, where the smooth members are transversal at a_+ to both L_2 and $L_{1,+}$. The base point a_+ has multiplicity 3, whereas the other base point on L_2 has multiplicity 2. Let E_1 and E_2 be the proper transforms of the -1 curves that appear at the first and the second blowups over a_+, and E_3 the proper transform of the -1 curve at the first blowing up over the other base point on L_2. If L'_2 and $L'_{1,\pm}$ denote the proper transforms of L_2 and $L_{1,\pm}$, then the singular fiber has the irreducible components L'_2 and E_1 of multiplicity 2, the multiplicity-one components $L'_{1,+}$ and E_2 that intersect E_1, and the multiplicity one

components $L'_{1,-}$ and E_3 that intersect L'_2. According to Lemma 12.1.2, ι_1^S switches $L'_{1,+}$ and E_2, whereas ι_2^S switches $L'_{1,+}$ and $L'_{1,-}$.

I_1^*, $\mathrm{D}_5^{(1)}$, 5/4, 4.

(7b4) One of the points $a_\pm$, say a_+, is a base point, where the smooth members are tangent at a_+ to $L_{1,+}$. The base point a_+ has multiplicity 4, whereas the other base point on L_2 has multiplicity 2. Let E_1, E_2, and E_3 be the proper transforms of the -1 curves that appear at the first three blowing up transformations over a_+, and E_4 the proper transform of the -1 curve at the first blowing-up over the other base point on L_2. If L'_2 and $L'_{1,\pm}$ denote the proper transforms of L_2 and $L_{1,\pm}$, then the singular fiber has the irreducible components L'_2, E_1, and E_2 of multiplicity 2, the multiplicity one components $L'_{1,+}$ and E_3 that intersect E_2, and the multiplicity-one components $L'_{1,-}$ and E_4 that intersect L'_2. The horizontal involution ι_1^S leaves $L'_{1,-1}$ invariant, whereas ι_2^S switches $L'_{1,+}$ and $L'_{1,-}$.

I_2^*, $\mathrm{D}_6^{(1)}$, 3/2, 2.

(7b5) One of the points $a_\pm$, say a_+, is a base point, where the smooth members are tangent at a_+ to L_2. The base point a_+ has multiplicity 5, whereas the other base point on L_2 has multiplicity 2. Let E_1, E_2, E_3, and E_4 be the proper transforms of the -1 curves that appear at the first four blowing-up transformations over a_+, and E_5 the proper transform of the -1 curve at the first blowing up over the other base point on L_2. If L'_2 and $L'_{1,\pm}$ denote the proper transforms of L_2 and $L_{1,\pm}$, then the singular fiber has the irreducible component E_2 of multiplicity 3, the irreducible components L'_2, E_1, and E_3 of multiplicity 2, each intersecting E_2, and the multiplicity-one components E_4, $L'_{1,+}$, and $L'_{1,-}$, which intersect E_3, E_1, and L_2, respectively. The horizontal involution ι_1^S leaves $L'_{1,-1}$ invariant, whereas ι_2^S switches $L'_{1,+}$ and $L'_{1,-}$.

IV^*, $\mathrm{E}_6^{(1)}$, 4/3, 3.

(7c) $L_{1,+} = L_{1,-}$ and $L_{2,+} \neq L_{2,-}$. We write $L_1 = L_{1,+} = L_{1,-}$, and $a_\pm$ is the intersection point of L_1 with $L_{2,\pm}$. This is (7b) with the roles of the horizontal and vertical axes interchanged, and therefore we have the same conclusions as in (7b) with τ^S replaced by its inverse $(\tau^S)^{-1}$.

(7d) $L_{1,+} = L_{1,-}$ and $L_{2,+} = L_{2,-}$. We write $L_1 = L_{1,+} = L_{1,-}$ and $L_2 = L_{2,+} = L_{2,-}$, and a is the intersection point of L_1 with L_2. For the determination of the way the QRT automorphism permutes the irreducible components of the singular fiber over $L_1 \cup L_2$, we use Lemmas 12.1.2 and 12.1.3.

(7d1) a is not a base point; we have two distinct base points of multiplicity 2 on L_1 and also on L_2.

I_1^*, $\mathrm{D}_5^{(1)}$, 1, 2.

(7d2) a is not a base point, on one of the axes L_1 and L_2 we have one base point, of multiplicity 4, and on the other we have two base points, each of multiplicity 2.

I_2^*, $\mathrm{D}_6^{(1)}$, 1, 2.

(7d3) a is not a base point, and on each of the axes L_1 and L_2 we have one base point, of multiplicity 4.

I_3^*, $D_7^{(1)}$, 1, 2.

(7d4) a is a base point. The smooth members are transversal at a to both L_1 and L_2, which implies that the base point a has multiplicity 4. On each of the axes L_1 and L_2 there is one base point other than a, which has multiplicity 2.

IV^*, $E_6^{(1)}$, 4/3, 3.

(7d5) a is a base point and the smooth members are tangent at a to one of the axes. The base point a has multiplicity 6, and there is one other base point, lying on the other axis, of multiplicity 2.

III^*, $E_7^{(1)}$, 3/2, 2.

Because a polynomial of bidegree $(0,\ 2)$ and $(2,\ 0)$ is the product of two polynomials of bidegree $(0,\ 1)$ and $(1,\ 0)$, respectively, case (7) completes the classification.

For none of the cases, I see an obvious reason why it could not be realized by a pencil of biquadratic curves with a smooth member, but I have not tried to prove the existence of such a pencil in each case. One could also ask for a normal form of such a pencil in each case.

The singular fibers that occur in our list are of Kodaira type I_b with $1 \le b \le 9$, II, III, IV, I_b^* with $0 \le b \le 4$, III^*, and IV^*. The intersection diagrams of reducible fibers that occur in our list are $A_k^{(1)}$ with $1 \le k \le 8$, $D_k^{(1)}$ with $4 \le k \le 8$, and $E_k^{(1)}$ with $k = 6,\ 7$. These are exactly the types that occur in Persson's list [156, pp. 7–14] of configurations of singular fibers of rational elliptic surfaces, with the exception that in Persson's list also the Kodaira type II^* occurs, with intersection diagram $E_8^{(1)}$. However, for the two configurations of singular fibers in which II^* occurs, the Mordell–Weil group is trivial, and therefore this agrees with Corollary 4.5.6, which implies that these are precisely the rational elliptic surfaces that are not QRT surfaces.

12.1.1 Example: $\mathbf{I_9}$

The only case in our list of Kodaira type I_9 is (6a3) with $b = 9$. We may use projective linear transformations in the two components of $\mathbb{P}^1 \times \mathbb{P}^1$ in order to arrange that $a_1 = ([1 : 0], [1 : 0])$, $a = ([0 : 1], [1 : 0])$, $a_2 = ([0 : 1], [0 : 1])$, and D is given by an equation of the form $x_1\, y_0 - x_0\, y_1 = 0$, or $y = x$ in affine coordinates. A straightforward computation shows that the smooth biquadratic curves that pass through a_1 and a, and have contact at a_1 with D of order 4 and contact with L_2 at a of order 2 are the curves

$$a\, {x_1}^2\, {y_1}^2 + (x_1\, y_0 - x_0\, y_1)\, (b\, x_0\, y_1 + c\, x_0\, y_0) = 0,$$

with $a \neq 0$ and $c \neq 0$. Because C is the curve $x_0\, y_1\, (x_1\, y_0 - x_0\, y_1) = 0$, we may assume that $b = 0$. By rescaling x_1/x_0 and y_1/y_0 we can finally arrange that $a = c = 1$. In other words, we have a singular fiber of type I_9 if the pencil of biquadratic curves in (2.5.3) is defined by the matrices

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{12.1.1}$$

The invariants D, E, and F, see (2.3.20), (2.3.21), of the discriminant of p with respect to $[y_0 : y_1]$ are computed as

$$\begin{aligned} 12D &= 24\, {z_0}^3\, z_1 + {z_1}^4, \\ 216E &= 216\, {z_0}^6 + 36\, {z_0}^3\, {z_1}^3 + {z_1}^6, \\ F &= -{z_0}^9\, (27\, {z_0}^3 + {z_1}^3). \end{aligned}$$

Because the singular fibers correspond to $F = 0$, we recognize the zero $z_0 = 0$ of F of order 9 as corresponding to the fiber of type I_9. The fact that F has three other zeros that are simple shows that the configuration of the singular fibers is $\mathrm{I}_9\ 3\,\mathrm{I}_1$. We have met this configuration before in Lemma 4.5.3.

This is the only configuration in the list of Persson [156, pp. 7–14] in which I_9 occurs, and then it follows from Lemma 9.2.6 that the Mordell–Weil group is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$. Because it follows from Corollary 4.5.6 that a rational elliptic surface is a QRT surface if its Mordell–Weil group is nontrivial, and because we are in the aforementioned case (6a3) with $b = 9$ for a pencil of biquadratic curves leading to a singular fiber of type I_9, we obtain the following conclusion.

Proposition 12.1.6 *Let $\kappa : S \to P$ be a rational elliptic surface. Then the following conditions are equaivalent:*

(a) *S contains a singular fiber of κ of Kodaira type* I_9.
(b) *S is isomorphic to the successive blowing up, nine times, at base points of the anticanonical pencils starting with a pencil of cubic curves in $\mathbb{P}^2$ as in Lemma* 4.5.3.
(c) *S is isomorphic to the successive blowing up, eight times, at base points of the anticanonical pencils starting with the pencil of biquadratic curves in $\mathbb{P}^2$ defined by the matrices* (12.1.1).

Remark 12.1.7. It follows from Lemma 9.2.6 that if a rational elliptic surface $\kappa : S \to P$ has a singular fiber of Kodaira type I_9, then the Mordell–Weil group $\mathrm{Aut}(S)^+_\kappa$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Because (6a3) shows that the QRT map τ^S acts on S^{irr}_r as a cyclic permutation of order 3, it follows that τ^S generates $\mathrm{Aut}(S)^+_\kappa$, and is itself of order 3. ⊘

12.2 Configurations of Singular Fibers in this Book

In this section we list the configurations of singular fibers in rational elliptic surfaces that we have met in this book. These cover about one-seventh of Persson's list [156, pp. 1–14] of the 289 configurations of singular fibers for a rational elliptic surface, together with the structures of their Mordell–Weil groups.

1. $\mathrm{II}^*\ 2\,\mathrm{I}_1$ in Lemma 4.5.1. The Mordell–Weil group is trivial, and therefore there cannot be any Manin QRT automorphism of this rational elliptic surface.
2. $\mathrm{II}^*\ \mathrm{II}$ in Lemma 4.5.1, equal to (9.2.3) in Proposition 9.2.17. The Mordell–Weil group is trivial, and therefore there cannot be any Manin QRT automorphism of this rational elliptic surface. The automorphism group of the rational elliptic surface is complex one-dimensional.
3. $\mathrm{I}_9\ 3\,\mathrm{I}_1$ in Lemma 4.5.3. This rational elliptic surface can be obtained from a pencil of cubic curves in $\mathbb{P}^2$ with only one base point, but the Mordell–Weil group is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$, and therefore the surface can also be obtained from a pencil of cubic curves in $\mathbb{P}^2$ with three base points.
4. $4\,\mathrm{I}_2\ 4\,\mathrm{I}_1$ with Mordell–Weil group isomorphic to $\mathbb{Z}^4 \times (\mathbb{Z}/2\,\mathbb{Z})$. This is the generic QRT map of order two in Section 5.2.1.
5. $3\ \mathrm{I}_3\ 3\ \mathrm{I}_1$ in Section 5.2.2. This is the generic QRT map of order three, where the Mordell–Weil group is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/3\,\mathbb{Z})$.
6. $2\ \mathrm{I}_4\ \mathrm{I}_2\ 2\ \mathrm{I}_1$ in Section 5.2.3. This is the generic QRT automorphism of order four, where the Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/4\,\mathbb{Z})$.
7. $2\ \mathrm{I}_4\ 2\ \mathrm{I}_2$ in Section 5.2.3. A nongeneric QRT automorphism of order four, where the Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/4\,\mathbb{Z})$. It also occurs as the rational elliptic surface for the sine–Gordon map in the degenerate case; see Section 11.7.
8. $\mathrm{I}_8\ \mathrm{I}_2\ 2\,\mathrm{I}_1$ in Section 5.2.3. A nongeneric QRT automorphism of order four, where in Remark 5.2.5 it is argued that the Mordell–Weil group is isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$.
9. $\mathrm{I}_1^*\ \mathrm{I}_4\ \mathrm{I}_1$ in Section 5.2.3. A nongeneric QRT automorphism of order four with Mordell–Weil group isomorphic to $\mathbb{Z}/4\,\mathbb{Z}$. It also occurs in Section 11.5.3 for $a = -\omega - 2 = 0$ and in Section 11.6.2 for $\lambda = 0$. In both cases the map is $(x,\ y) \mapsto (y,\ -x)$, which generates the Mordell–Weil group $\simeq \mathbb{Z}/4\,\mathbb{Z}$.
10. $2\ \mathrm{I}_5\ 2\ \mathrm{I}_1$ in Section 5.2.4, with Mordell–Weil group $\simeq \mathbb{Z}/5\,\mathbb{Z}$. Up to isomorphism, there is only one rational elliptic surface whose Mordell–Weil group contains elements of order five. The Lyness map for $a = 1$ is an example of order 5 that occurs in the literature; see Section 11.4.
11. $\mathrm{I}_6\ \mathrm{I}_3\ \mathrm{I}_2\ \mathrm{I}_1$ in Section 5.2.5, with Mordell–Weil group $\simeq (\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$. Up to isomorphisms, there is only one rational elliptic surface whose Mordell–Weil group contains elements of order six. The Lyness map for $a = 0$ is an example of order 6 that occurs in the literature, see Section 11.4. Another example is the QRT root for the planar four-bar link with $a = b = d$; see the end of Section 11.3.
12. $12\ \mathrm{I}_1$, $\mathrm{II}\ \ 10\ \mathrm{I}_1$, $2\ \mathrm{II}\ \ 8\ \mathrm{I}_1$, $3\ \mathrm{II}\ \ 6\ \mathrm{I}_1$, $4\ \mathrm{II}\ \ 4\ \mathrm{I}_1$, and $6\ \mathrm{II}$, the possible configurations of singular fibers of Manin elliptic surfaces in Section 9.2.4. The Mordell–Weil group is isomorphic to $\mathbb{Z}^8$. There are 8, 7, 6, 5, 4, and 3 moduli, respectively.
13. $12\ \mathrm{I}_1$ in Section 9.2.5, the generic configuration of singular fibers. The Mordell–Weil group is isomorphic to $\mathbb{Z}^8$ and there are eight moduli.

14. III* III = (9.2.4) in Proposition 9.2.17. The Mordell–Weil group is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$, but the automorphism group of the rational elliptic surface is complex one-dimensional.
15. IV* IV = (9.2.5) in Proposition 9.2.17. The Mordell–Weil group is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$, but the automorphism group of the rational elliptic surface is complex one-dimensional.
16. 2 I_0^* = (9.2.6) in Proposition 9.2.17. The Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z})^2$, but the automorphism group of the rational elliptic surface is complex one-dimensional.
17. I_4^* 2 I_1 in Table 9.2.20. The Mordell–Weil group is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$.
18. III* I_2 I_1 in Table 9.2.20. The Mordell–Weil group is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$.
19. IV* I_3 I_1 in Table 9.2.20. The Mordell–Weil group is isomorphic to $\mathbb{Z}/3\,\mathbb{Z}$.
20. I_2^* 2 I_2 in Table 9.2.20. The Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z})^2$.
21. 4 I_3 in Section 11.1. This is the rational elliptic surface defined by the Hesse pencil of cubic curves in $\mathbb{P}^2$. The Mordell–Weil group is isomorphic to $(\mathbb{Z}/3\,\mathbb{Z})^2$, corresponding to the nine common flex points of the members of the pencil.
22. 3 I_2 6 I_1, the generic QRT root in Section 10.1.3. The Mordell–Weil group is isomorphic to $\mathbb{Z}^5$.
23. 2 III I_2 2 II, a nongeneric QRT root in Section 10.1.3 with Mordell–Weil group isomorphic to $\mathbb{Z}^5$.
24. I_0^* 3 I_2 in Section 11.2.2. This is the elliptic billiard. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})^2$.
25. I_5 I_3 I_2 2 I_1 for the Lyness map in Section 11.4 when $a \notin \{-1/4,\ 0,\ 3/4,\ 1,\ 2\}$. The permutation by the Lyness map of the irreducible components of the reducible fibers is of order 30. For any value of the parameter a, the Mordell–Weil group is generated by the Lyness map. The Lyness map has infinite order, when the Mordell–Weil group is isomorphic to $\mathbb{Z}$, unless $a = 0$ or $a = 1$, when it has order 6 or 5, when the Mordell–Weil group is isomorphic to $(\mathbb{Z}/2\,\mathbb{Z}) \times (\mathbb{Z}/3\,\mathbb{Z})$ or $\mathbb{Z}/5\,\mathbb{Z}$, respectively.
26. I_5 I_3 I_2 II for the Lyness map with $a = -1/4$.
27. I_5 I_3 III I_1 for the Lyness map with $a = 3/4$.
28. I_5 IV I_2 I_1 for the Lyness map with $a = 2$.
29. I_6 I_2 4 I_1 for the generic planar four-bar link in Section 11.3. This is another configuration of the singular fibers for which there are two nonisomorphic possibilities for the Mordell–Weil group. For the planar four-bar link the Mordell–Weil group is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/2\,\mathbb{Z})$.
30. I_6 2 I_2 2 I_1 for the symmetric planar four-bar link in Section 11.3. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$.
31. I_1^* I_2 3 I_1 in Section 11.5. This is the rational elliptic surface for the generic symmetric McMillan map. It occurs for the modified KdV map in Section 11.5.2. The Mordell–Weil group is isomorphic to $\mathbb{Z}^2$.
32. I_1^* 5 I_1 in Section 11.5.4. This is the rational elliptic surface for the generic nonsymmetric McMillan map, see Section 11.5.4. The Mordell–Weil group is isomorphic to $\mathbb{Z}^3$.

33. $\mathrm{I}_2^*\ \mathrm{I}_2\ 2\ \mathrm{I}_1$ in Subection 11.6.1. This is the rational elliptic surface for the isotropic Heisenberg spin chain map, in the generic case $\omega \neq 0$, $\omega^2 \neq 4$. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$.
34. $\mathrm{I}_2^*\ \mathrm{III}\ \mathrm{I}_1$ in Section 11.6.1. This is the rational elliptic surface for the isotropic Heisenberg spin chain map, in the degenerate case $\omega^2 = 4$. The Mordell–Weil group is still isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$.
35. $\mathrm{I}_0^*\ \mathrm{I}_4\ 2\ \mathrm{I}_1$ in Subection 11.6.2. This is the rational elliptic surface for the anisotropic Heisenberg spin chain map, in the generic case $\lambda^2 \neq 1$, $\lambda \neq 0$. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$.
36. $\mathrm{I}_4\ 4\ \mathrm{I}_2$ in Section 11.7. This is the rational elliptic surface for the sine–Gordon map, in the generic case. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})^2$.
37. $\mathrm{I}_8\ 4\ \mathrm{I}_1$ in Section 11.7.1. for the generic hard hexagon model. The Mordell–Weil group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\,\mathbb{Z})$ or to $\mathbb{Z}$.
38. $\mathrm{I}_2\ 10\ \mathrm{I}_1$ for Jogia's example in Section 11.8. The Mordell–Weil group is isomorphic to $\mathbb{Z}^7$.
39. ${}_2\mathrm{I}_3\ \ \mathrm{I}_4\ \ \mathrm{I}_2\ \ 3\ \mathrm{I}_1$ is the generic configuration of the singular fibers for the non-QRT example of Viallet, Grammaticos, and Ramani [200, Section 2]. See Section 11.9.3. The QRT surface with the same Weierstrass data has the configuration of singular fibers $\mathrm{I}_3\ \ \mathrm{I}_4\ \ \mathrm{I}_2\ \ 3\ \mathrm{I}_1$, and according to Lemma 9.2.6, its Mordell–Weil group is isomorphic to $\mathbb{Z}^2$.

References

1. V.E. Adler: On a class of third order mappings with two rational invariants. arXiv:nlin/0606056v1 [nlin.SI] 23 Jun 2006
2. Apollonius: Paul Ver Eecke: *Les Coniques d'Apollonius de Perge*. A. Blanchard, Paris, 1963.
3. V.I. Arnol'd: Normal forms of functions with simple critical points, the Weyl groups A_k, D_k, E_k, and Lagrange immersions. *Funct. Anal. and Its Appl.* **6** (1972) 254–272.
4. V.I. Arnol'd: Dynamics of complexity of intersections. *Boll. Soc. Brasil. Mat.* **21** (1990) 1–10.
5. V.I. Arnol'd and A. Avez: *Ergodic Problems of Classical Mechanics*. Translated from the 1967 French edition. W.A. Benjamin, inc., New York, Amsterdam, 1968.
6. S. Aronhold: Zur Theorie der homogenen Functionen dritten Grades von drei Variablen. *J. Reine u. Angew. Math.* **39** (1850) 140–159.
7. M. Artebani and I. Dolgachev: The Hesse pencil of plane cubic curves. arXiv:math.AG/0611590v1 19 Nov 2006.
8. M.F. Atiyah: Complex fibre bundles and ruled surfaces. *Proc. London Math. Soc.* **5** (1955) 407–434.
9. M.F. Atiyah and I.M. Singer: The index of elliptic operators: III. *Annals of Math.* **87** (1968) 546–604.
10. C. Bănică and O. Stănăşilă: *Algebraic Methods in the Global Theory of Complex Spaces*. John Wiley & Sons, New York, 1976.
11. W.P. Barth, K. Hulek, C.A.M. Peters, A. van de Ven: *Compact Complex Surfaces*. Second enlarged edition. Springer, Berlin, etc., 2004.
12. G. Bastien and M. Rogalski: Global behavior of the solutions of Lyness' difference equation $u_{n+2}\,u_n = u_{n+1} + a$. *J. Difference Equ. Appl.* **10** (2004) 977–1003.
13. G. Bastien, M. Rogalski: On the algebraic difference equations $u_{n+2} + u_n = \psi(u_{n+1})$ in $\mathbb{R}$, related to a family of elliptic quartics in the plane. *J. Math. Anal. Appl.* **326** (2007) 822–844.
14. S.R. Bell and R. Narasimhan: Proper holomorphic mappings of complex spaces. pp. 1–38 in *Several Complex Variables VI*. Encyclopaedia Math. Sci. 69, Springer, Berlin, 1990.
15. L. Berzolari: Die singuläre Punkte. Part 4, II in *Encyklopädie der Mathematischen Wissenschaften, Band 3, Teil 2, erster Hälfte*. B.G. Teubner Verlag, Leipzig, 1915.
16. F. Beukers and R. Cushman: Zeeman's monotonicity conjecture. *J. Diff. Eq.* **143** (1998) 191–200.
17. G.D. Birkhoff: *Dynamical Systems*. A.M.S. Colloquium Publications Vol. IX, A.M.S., Providence, 1927.
18. S. Bochner and D. Montgomery: Groups on analytic manifolds. *Annals of Math.* **48** (1947) 659–669.
19. F.A. Bogomolov: Surfaces of class VII_0 and affine geometry. *Izv. Mat. Nauk SSSR* **46**, 4 (1982). translated in *Math. USSR Izvestiya* **21**, 1 (1983) 31–73.
20. H.J.M. Bos, C. Kers, F. Oort, D.W. Raven: Poncelet's closure theorem. *Expositiones Math.* **5** (1987) 289–364.

21. R.J. Boscovich: *Elementorum Universae Mathesos. Vol. III: Sectionum Conicarum Elementa*. Venice, 1757.
22. R. Bott and L.W. Tu: *Differential Forms in Algebraic Topology*. Springer-Verlag, New York, etc., 1982.
23. N. Bourbaki: *Éléments de Mathématique. Groupes et Algèbres de Lie. Chap. 4, 5 et 6*. Masson, Paris, etc., 1981.
24. H. Brandt: Über eine Verallgemeinerung des Gruppenbegriffs. *Mathematische Annalen* **96** (1926) 360–366.
25. E. Brieskorn: Über dei Auflösung gewisser Singularitäten von holomorphen Abbildungen. *Math. Ann.* **166** (1966) 79–102.
26. H. Bruns: Ueber die Perioden der elliptischen Integrale erster und zweiter Gattung. Festschrift, Dorpat, 1875, reprinted in *Math. Ann.* **27** (1886) 234–252.
27. H. Cartan: Quotient d'un espace analytique par un groupe d'automorphismes. pp. 90–102 in: *Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz, April 8–10, 1954*. Princeton University Press, Princeton, New Jersey, 1957.
28. E. Casas-Alvero: *Singularities of Plane Curves*. Cambridge University Press, 2000.
29. G. Castelnuouvo und F. Enriques: Die Theorie der Invarianten. Part 6b, II in *Encyklopädie der Mathematischen Wissenschaften, Band 3, Teil 2, erster Hälfte*. B.G. Teubner Verlag, Leipzig, 1915.
30. A. Cayley: On the theory of linear transformations. *Cambridge Math. J.* **4** (1845) 193–209 = pp. 80–94 in: *Collected Mathematical Papers, vol. I*. Cambridge University Press, 1889.
31. A. Cayley: Note on the porism of the in-and-circumscribed polygon. *Philosophical Magazine* **6** (1853) 99–102, 376, 377 = pp. 87–92 in: *Collected Mathematical Papers, vol. II*. Cambridge University Press, 1889.
32. A. Cayley: On the porism of the in-and-circumscribed polygon and the $(2,\ 2)$ correspondence of points on a conic. *Quarterly Journal of Pure and Applied Mathematics* **11** (1871) 83–91= pp. 14–21 in: *Collected Mathematical Papers, vol. VIII*. Cambridge University Press, 1895.
33. Chin-Li Chai: A note on Manin's theorem of the kernel. *Amer. J. Math.* **113** (1991) 387–389.
34. S.-J Chang and R. Friedberg: Elliptic billiard and Poncelet's theorem. *J. Math. Phys.* **29**, 9 (1988) 1537–1550.
35. C. Chevalley: Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955) 778–782.
36. C. Chevalley: Sur la théorie des variétés algebriques. *Nagoya Math. J.* **8** (1955) 1–43.
37. E.A. Coddington and N. Levinson: *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, 1955.
38. J.H. Conway and N.J.A. Sloane: *Sphere Packings, Lattices and groups*. Springer-Verlag, New York, etc., 1988.
39. D.A. Cox: Mordell–Weil groups over $\mathbb{C}(t)$ with $p_g = 0$ or 1. *Duke Math. J.* **49** (1982) 677–689.
40. D.A. Cox and W.R. Parry: Torsion in elliptic curves over $k(t)$. *Comp. Math.* **41**, 3 (1980) 337–354.
41. D.A. Cox and S. Zucker: Intersection numbers of sections of elliptic surfaces. *Invent. Math.* **53** (1979) 1–44.
42. H.S.M. Coxeter: Frieze patterns. *Acta Arithmetica* **18** (1971) 297–310.
43. R. Cushman and J.J. Duistermaat: The quantum mechanical spherical pendulum. *Bull. Amer. Math. Soc.* **19** (1988) 475–479.
44. G. Darboux: De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan. *Bulletin des Sciences Mathématiques et Astronomiques* **3** (1879) 109–120.
45. M. Demazure: Surfaces de Del Pezzo II–V, especially II - Éclater n points dans $\mathbb{P}^2$. pp. 23–35 in M. Demazure, H. Pinkham et B. Teissier, eds.: *Séminaire sur les Singularités des Surfaces*, Palaiseau, France 1976–77. Lecture Notes in Mathematics 777. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
46. G. Dloussky, K. Oeljeklaus and M. Toma: Class VII_0 surfaces with b_2 curves. *Tôhoku Math. J.* **55** (2003) 283–309.

47. I.V. Dolgachev: Rational surfaces with a pencil of elliptic curves. *Izv. Akad. Nauk SSSR Ser. Mat.* **30** (1966) 1073–1100.
48. J.J. Duistermaat: On global action-angle coordinates. *Comm. Pure Appl. Math.* **33** (1980) 687–706.
49. J.J. Duistermaat and A. Pelayo: Symplectic torus actions with coisotropic principal orbits. *Annales de l'Institut Fourier* **57** (2007) 2239–2327.
50. H.R. Dullin and S. Vũ Ngọc: Vanishing twist near focus-focus points. *Nonlinearity* **17** (2004) 1777–1785.
51. P. Du Val: On isolated singularities of surfaces which do not affect the conditions of adjunction. I, II, III. *Proc. Camb. Phil. Soc.* **30** (1934) 453–459, 460–465, 483–491.
52. P. Du Val: On absolute and non-absolute singularities of algebraic surfaces. *Istanbul Üniv. Fen. Fak. Mecmuasi* **9** Ser. A (1944) 159–215.
53. P. Du Val: *Homographies Quaternions and Rotations*. Oxford University Press, Oxford, 1964.
54. E. Dynkin: Semi-simple subalgebras of semi-simple Lie algebras. *Mat. Sbornik N.S.* **30** (72) (1952) 349–462 = *AMS Transl.* **6** (1957) 111—244.
55. H.M. Edwards: A normal form for elliptic curves. *Bull. Amer. Math. Soc* **44** (2007) 393–422.
56. C. Ehresmann: Sur les espaces fibrés différentiables. *C.R. Acad. Sci. Paris* **224** (1947) 1611–1612 = pp. 326–326 in *Œuvres Complètes et Commentées, I-1 & I-2*, Evrard, Amiens, 1984.
57. M. Eichler: *Quadratische Formen und Orthogonale Gruppen*. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1952.
58. G. Eisenstein: Allgemeine Auflösung der Gleichungen von den ersten vier Graden. *J. Reine u. Angew. Math.* **27** (1844) 81–83.
59. I. Enoki: Surfaces of class VII_0 with curves. *Tôhoku Math. J.* **33** (1981) 453–492.
60. J. Esch and T.D. Rogers: The screensaver map: dynamics on elliptic curves arising from polygonal folding. *Discrete Comput. Geom.* **25** (2001) 477–502.
61. H.M. Farkas and I. Kra: *Riemann Surfaces.* 2nd ed. Springer-Verlag, New York, etc., 1992.
62. L. Flatto: *Poncelet's Theorem*. American Mathematical Society, Providence, 2009.
63. L.R. Ford: *Automorphic Functions*. Chelsea Publishing Company, New York, 1929, 1951.
64. G. Frobenius: Theorie der biquadratischen Formen. *J. Reine u. Angew. Math.* **106** (1890) 125–188.
65. L. Fuchs: *Abelian Groups*. Hungarian Academy of Sciences, Budapest, 1958.
66. C.G. Gibson and P.E. Newstead: On the geometry of the planar 4-bar mechanism. *Acta Applicandae Mathematicae* **7** (1986) 113–135.
67. B. Grammaticos, A. Ramani, and V. Papageorgiou: Do integrable mappings have the Painlevé property? *Phys. Rev. Letters* **67**, 14 (1991) 1825–1828.
68. H. Grauert: On Levi's problem and the embedding of real-analytic manifolds. *Ann. of Math.* **68** (1958) 460–472.
69. H. Grauert: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. *Publ. Math. I.H.E.S.* **5** (1960) 233–292.
70. H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.* **146** (1962) 331–368.
71. H. Grauert und R. Remmert: Komplexe Räume. *Math. Ann* **136** (1958) 245–318.
72. H. Grauert and R. Remmert: *Coherent Analytic Sheaves*. Springer-Verlag, Berlin etc., 1984.
73. P.A. Griffiths: Variations on a theorem of Abel. *Invent. Math.* **35** (1976) 321–390.
74. Ph. Griffiths and J. Harris: *Principles of Algebraic Geometry*. Wiley-Interscience, New York, 1978.
75. Ph. Griffiths and J. Harris: On Cayley's explicit solution to Poncelet's porism. *L'Enseignement Mathématique* **24** (1978) 31–40.
76. A. Grothendieck: Sur la classification des fibres holomorphes sur la sphère de Riemann. Amer. J. Math. **79** (1957) 121–138.
77. A. Grothendieck: Crystals and the de Rham cohomology of schemes. Notes by J. Coates and O. Jussila, pp. 306–358 in: *Dix Exposées sur la Cohomologie des Schémas.* North-Holland, Amsterdam, and Masson, Paris, 1968.
78. V. Guillemin and R. Melrose: An inverse spectral result for elliptical regions in $\mathbb{R}^2$. *Advances in Mathematics* **32** (1979) 128–148.

79. R.C. Gunning: *Lectures on Riemann Surfaces*. Princeton University Press, 1966.
80. R. Hartshorne: *Algebraic Geometry*. Springer-Verlag, New York, Berlin, Heidelberg, 1977.
81. G. Heckman and E. Looijenga: The moduli space of rational elliptic surfaces. pp. 185–248 in *Algebraic Geometry 2000, Azumino (Hotaka)*. Advanced Studies in Pure Mathematics 36, Math. Soc. Japan, Tokyo, 2002.
82. L.O. Hesse: Über die Elimination der Variablen aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln and Über die Wendepuncte der Curven dritter Ordnung. *J. Reine u. Angew. Math.* **28** (1844) 68–96, 97–107.
83. J. Hietarinta and C. Viallet: Singularity confinement and chaos in discrete systems. *Phys. Rev. Letters* **81**, 2 (1998) 325–328.
84. F. Hirzebruch: Über eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten. *Math. Ann.* **124** (1951) 77–86.
85. F. Hirzebruch: Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen. *Math. Ann.* **126** (1953) 1–22.
86. F. Hirzebruch: Übertragung einiger Sätze aus der Theorie der algebraischen Flächen auf komplexe Mannigfaltigkeiten von zwei komplexen Dimensionen. *J. Reine u. Angew. Math.* **191** (1953) 110–124.
87. F. Hirzebruch: *Topological Methods in Algebraic Geometry*. Third enlarged edition. Springer-Verlag, Berlin, Heidelberg, New York, 1966.
88. M. van Hoeij: An algorithm for computing the Weierstrass normal form. pp. 90–95 in: *Proceedings ISSAC 1995*.
89. A.N.W. Hone: Elliptic curves and quadratic recurrence sequences. *Bull. London math. Soc.* **37** (2005) 161–171.
90. H. Hopf: Schlichte Abbildungen und lokale Modifikatione 4-dimensionaler komplexer mannigfaltigkeiten. *Comment. Math. Helv.* **29** (1955) 132–156.
91. L. Hörmander: *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag, Berlin, etc., 1985.
92. M. Inoue: On surfaces of class VII_0. *Invent. Math.* **24** (1974) 269–310.
93. A. Iatriou and J.A.G. Roberts: Integrable mappings of the plane preserving biquadratic invariant curves. *J. Phys. A: Math. Gen.* **34** (2001) 6617–6636. II *Nonlinearity* **15** (2002) 459–489. III *Physica A* bf 326 (2003) 400–411.
94. V.A. Iskovskikh and I.R. Shafarevich: Algebraic Surfaces. pp. 127–262 in I.R. Shafarevich (ed.): *Algebraic Geometry II*. Encyclopaedia of Mathematical Sciences, vol. 35. Springer, Berlin, etc., 1996.
95. C.G.J. Jacobi: De formatione et proprietatibus Determinantium. *J. Reine u. Angew. Math.* **22** (1841) 285–318.
96. C.G.J. Jacobi: *Vorlesungen über Dynamik*. Gehalten an der Universität Königsberg im Wintersemester 1842–43 und nach einem von C.W. Borchardt ausgearbeten Hefte = Gesammelte Werke, Band VIII. Verlag G. Reimer, Berlin 1881. Reprint Chelsea Publ. Co., New York, N.Y., 1969.
97. D. Jogia, J.A.G. Roberts, and F. Vivaldi: An algebraic geometric appoach to integrable maps of the plane. *J. Phys. A: math. gen.* **39** (2006) 1133–1149.
98. D. Jogia: PhD thesis, University of New South Wales, Sydney, 2008.
99. C. Jordan: Mémoire sur les équations différentielles linéaires à intégrale algebrique. *J. Reine u. Angew. Math.* **84** (1877) 89–215.
100. V.G. Kac: Simple graded Lie algebras of finite growth. *Funkt. Analis i ego Prilozh.* **1** (1967) 82, 83. English translation in *Funct. Anal. Appl.* **1** (1967) 328, 329.
101. V.G. Kac: *Infinite dimensional Lie algebras*. 3rd ed., Cambridge University Press, 1990.
102. A. Kas: Weierstrass normal forms and invariants of elliptic surfaces. *Trans. Amer. Math. Soc.* **225** (1977) 259–266.
103. A. Kas: On the deformation types of regular elliptic surfaces = pp. 107–111 in: *Complex Analysis and Algebraic Geometry. A Collection of Papers Dedicated to K. Kodaira*, Iwanami Shoten Publishers, Tokyo, and Cambridge University Press, 1977.
104. A.Ya. Khinchin: *Continued Fractions*. Chicago University Press, 1964.

105. D. Kirby: The structure of an isolated multiple point of a surface I, II, III. *Proc. London Math. Soc.* **6** (1956) 597–609, **7** (1957) 1–28.
106. F. Klein: Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades. *Math. Ann.* **14** (1879) 111–172.
107. F. Klein: *Vorlesungen über das Ikosaeder*. B.G. Teubner Verlag, Leipzig, 1884.
108. K. Kodaira: The theorem of Riemann-Roch on compact analytic surfaces. *Am. J. Math.* **73** (1951) 815–875.
109. K. Kodaira: On compact analytic surfaces, I, II, III. *Ann. of Math.* **71** (1960) 111–152, **77** (1963) 563–626, **78** (1963) 1–40.
110. K. Kodaira: On the structure of compact analytic surfaces, I–IV. *Amer. J. Math.* **86** (1964) 751–798, **88** (1966), 682–721, **90** (1968), 55–83 and 1048–1066.
111. K. Kodaira and D.C. Spencer: On deformations of complex analytic structures, I, II, and III, *Ann. of Math.* **67** (1958) 328–466, and **71** (1960) 43–76.
112. K. Kodaira and J. Morrow: *Complex Manifolds*. Rinehart & Winston, New York, 1971.
113. V.V. Krasnov: Compact complex analytic surfaces without meromorphic function. *Mat. Zametki* **17** (1975) 119–122. English translation in *Math. Notes* **17** (1975) 69–71.
114. L. Kronecker: Ueber die Discriminante algebraischer Functionen einer Variabeln. *J. Reine u. Angew. Math.* **91** (1881) 301–334.
115. D. S. Kubert: Universal bounds on the torsion of elliptic curves. *Proc. London Math. Soc.* **33** (1976) 193–237.
116. H. Künneth: Über die Bettischen Zahlen von Produktmannigfaltigkeiten. *Math. Ann.* **90** (1923) 65–85, and: Über die Torsionszahlen von Produktmannigfaltigkeiten. *Math. Ann.* **91** (1924) 125–134.
117. S. Lefschetz: *L'analysis Situs et la Géometrie Algébrique*. Gauthier-Villars, Paris, 1924. Also contained in [118].
118. S. Lefschetz *Selected Papers*. Chelsea Publ. Co., Bronx, New York, 1971.
119. M. Levi and S. Tabachnikov: The Poncelet porism and billiards in ellipses. *Amer. Math. Monthly* **114** (2007) 895–908.
120. R. Levy: A new proof of the Grauert direct image theorem. *Proc. Amer. Math. Soc.* **99** (1987) 535–542.
121. P. Libermann: Sur les groupoïdes différentiables et la "presque parallelisme." pp. 59–93 in: *Symposia Mathematica, vol. X*, Academic Press, London and New York, 1972.
122. M. Lönne: On the diffeomorphism groups of elliptic surfaces. *Math. Ann.* **310** (1998) 103–117.
123. S. Łojasiewicz: Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa* **18** (1964) 449–474.
124. S. Łojasiewicz: Sur les trajectoires du gradient d'une fonction analytique. pp. 115–117 in *Geometry seminars, 1982–1983 (Bologna, 1982/1983)*. Univ. Stud. Bologna, Bologna, 1984.
125. S. Łojasiewicz: *Introduction to Complex Analytic Geometry*. Birkhäuser Verlag, Basel etc., 1991.
126. R.C. Lyness: Notes 1581, 1847, and 2952, in: *Mathematical Gazette* **26** (1942) p. 62, **29** (1945), p. 231, and **45** (1961), p. 201, respectively.
127. W. Magnus: *Non-Euclidean Tesselations and Their Groups*. Academic Press, New York and London, 1974.
128. Yu.I. Manin: Rational points of algebraic curves over function fields. *Izv. Akad. Nauk SSSR* Ser. Mat. **27** (1963) 1395–1440. English translation in *A.M.S. Translations* Ser. 2 **50** (1966) 189–234.
129. Yu.I. Manin: The Tate height of points on an Abelian variety. *Izv. Akad. Nauk SSSR* Ser. Mat. **28** (1964) 1363–1390. English translation in *A.M.S. Translations* Ser. 2 **59** (1966) 82–110.
130. Yu.I. Manin: Rational surfaces over perfect fields. *Publ. Math. I.H.E.S.* **30** (1966) 55–113. In Russian, with a synopsis in English on pp. 99–113.
131. Yu.I. Manin: *Cubic Forms*, 2nd ed.. North-Holland Math. Library, Elsevier Sc. Publ., 1986.
132. Y. Matsumoto: Diffeomorphism types of elliptic surfaces. *Topology* **25** (1986) 549–563.
133. E.M. McMillan: A problem in the stability of periodic systems. pp. 219–244 in *Topics in Modern Physics. A Tribute to E.U. Condon.* eds. E. Britton and H. Odabasi. Colorado Assoc. Univ. Press, Boulder, 1971.

134. C.T. McMullen: *Complex Dynamics and Renormalization*. Annals of Mathematics Studies, Number 135. Princeton University Press, Princeton, New Jersey, 1994.
135. R. Miranda: On the stability of pencils of cubic curves. *Amer. J. Math.* **102** (1980) 1172–1202.
136. R. Miranda: The moduli of Weierstrass fibrations over $\mathbb{P}^1$. *Math. Ann.* **255** (1981) 379–394.
137. R. Miranda: Persson's list of singular fibers for a rational elliptic surface. *Math. Z.* **205** (1990) 191–211.
138. R. Miranda and U. Persson: On extremal rational elliptic surfaces. *Math. Z.* **193** (1986) 537–558.
139. R.V. Moody: Lie algebras associated with generalized Cartan matrices. *Bull. Amer. Math. Soc.* **73** (1967) 217–221.
140. S. Mori: Threefolds whose canonical bundles are not numerically effective. *Ann. of Math.* **116** (1982) 133–176.
141. D. R. Morrison: On K3 surfaces with large Picard number. *Invent. Math.* **75** (1984) 105–121.
142. I. Morrison and U. Persson: The group of sections on a rational elliptic surface. pp. 321–347 in *Algebraic geometry—open problems (Ravello, 1982)*. Lecture Notes in Math., 997, Springer, Berlin, 1983.
143. I. Morrison and U. Persson: Numerical sections on elliptic surfaces. *Compositio Mathematica* **59** (1986) 323–337.
144. D. Mumford: Topology of normal singularities and a criterion for simplicity. Publ. I.H.E.S. **9** (1961) 5–22.
145. D. Mumford, J. Fogarty, F. Kirwan: *Geometric Invariant Theory*. Third edition. Springer-Verlag, Berlin, 1994.
146. M. Nagata: On rational surfaces I. Irreducible curves of arithmetic genus 0 or 1. *Memoirs of the College of Science, University of Kyoto, Series A* **32** (1960) 351–370.
147. M. Nagata: On rational surfaces, II. *Memoirs of the College of Science, University of Kyoto, Series A* **33** (1960) 271–293.
148. I. Naruki: Cross ratios as moduli of del Pezzo surfaces of degree one. *Proc. Japan Acad. Ser. A* **56** (1980) 328–331.
149. I. Naruki: Configurations related to maximal rational elliptic surfaces. pp. 315–347 in *Complex Analytic Singularities*, Tsukuba Univ. 1984. Advanced Studies in Pure Math. **8**. Kinokuniya Co., Tokyo and North-Holland, Amsterdam etc., 1986.
150. I. Naruki: On confluence of singular fibers in elliptic fibrations. *Publ. RIMS, Kyoto Univ.* **23** (1987) 409–431.
151. I. Newton: *The Mathematical Papers of Isaac Newton*, ed. D.T. Whiteside. Vol. IV. Cambridge University Press, 1971.
152. K-V. Nguyen: Extremal elliptic fibrations and singular $K3$ surfaces. *Tokyo J. Math.* **22** (1999) 415–424.
153. Z. Nguyên Tiên: A note on focus-focus singularities. *Differential Geometry and Applications* **7** (1997) 123–130.
154. M. Noether: Rationale Ausführung der Operationen in der Theorie der algebraischen Functionen. *Math. Ann.* **23** (1884) 311–358.
155. K. Oguiso and T. Shioda: The Mordell–Weil lattice of a rational elliptic surface. *Commentarii Mathematici Universitas Sancti Pauli* **40**, 1 (1991) 83–99.
156. U. Persson: Configurations of Kodaira fibers on rational elliptic surfaces. *Math. Z.* **205** (1990) 1–47.
157. J. Pettigrew: PhD thesis, University of New South Wales, Sydney, 2009.
158. E. Picard et G. Simart: *Théorie des Fonctions Algébriques de Deux Variables Indépendents.* Paris, 1897 – 1906.
159. R. Piene: A proof of Noether's formula for the arithmetic genus of an algebraic surface. *Comp. Math.* bf 38 (1979) 113–119.
160. I.I. Pjateckiĭ-Šapiro and I.R. Šafarevič: A Torelli theorem for algebraic surfaces of type K3. *Mathematics of the USSR – Izvestija*. **5** (1971) 547–588.
161. J. Plücker: Anzeige des "Systems der analytischen Geometrie." *Crelle's J. Reine u. Angew. Math.* **10** (1833) 98–100 = pp. 586–590 in [177].

162. J. Plücker: *System der analytischen Geometrie.* Berlin, 1835.
163. J. Plücker: *System der analytischen Geometrie des Raumes.* Düsseldorf, 1846.
164. H. Poincaré: Sur les propriétés arithmétiques des courbes algébriques. *J. de Liouville* **7** (1901) 161–233.
165. J.-V. Poncelet: *Traité de propriétés projectives des figures*. Gauthiers-Villars, Paris 1822. Second edition in two volumes, Paris, 1865–66.
166. V.A. Puiseux: Recherches sur les fonctions algébriques. *J. Math. Pures Appl.* **15** (1850) 365–480.
167. G.R.W. Quispel, H.W. Capel, V.G. Papageorgiou, F.W. Nijhoff: Integrable mappings derived from soliton equations. *Physica A* **173** (1991) 243–266.
168. G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson: Integrable mappings and soliton equations. *Physics Letters A* **126** (1988) 419–421.
169. G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson: Integrable mappings and soliton equations II. *Physica D* **34** (1989) 183–192.
170. R. Remmert: Holomorphe und meromorphe Abbildungen komplexer Räume. *Math. Ann.* **133** (1957) 328–370.
171. R. Remmert und K. Stein: Über die wesentlichen Singularitäten analytischer Mengen. *Math. Ann.* **126** (1953) 263–306.
172. G. de Rham: *Variétés Differentiables*. Hermann, Paris, 1955. English translation: *Differentiable Manifolds*. Springer-Verlag, Berlin, etc., 1984.
173. B. Riemann: *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. Inauguraldissertation, Göttingen, 1851. pp. 3–48 in *Bernhard Riemann's gesammelte Mathematische Werke*, Dover reprint, New York, 1953.
174. B. Rink: A Cantor set of tori with monodromy near a focus-focus singularity. *Nonlinearity* **17** (2004) 347–356.
175. J.A.G. Roberts and G.R.W. Quispel: Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems. *Physics Reports (Review Section of Physics Letters)* **216** (1992) 63–177.
176. U. Schmickler-Hirzebruch: *Elliptische Flächen über* $\mathbb{P}_1(\mathbb{C})$ *mit drei Ausnahmefasern und die hypergeometrische Differentialgleichung.* Schriftenreihe des Mathematischen Instituts der Universität Münster, No. 33, 1985.
177. A. Schoenflies, ed.: *Gesammelte Mathematische Abhandlungen von Julius Plücker.* B.G. Teubner Verlag, Leipzig, 1895.
178. H. Seifert und W. Threlfall: *Lehrbuch der Topologie*. Teubner Verlag, Leipzig und Berlin, 1934.
179. J.-P. Serre: Un théorème de dualité. *Comm. Math. Helv.* **29** (1955) 9–26.
180. J.-P. Serre: Géometrie algébrique et géométrie analytique. *Ann. Inst. Fourier* **6** (1956) 1–42.
181. J.-P. Serre: *A Course in Arithmetic*. Springer-Verlag, New York, etc., 1973.
182. I.R. Shafarevich: *Basic Algebraic Geometry*. Springer-Verlag, Berlin, etc., 1977.
183. T. Shioda: On elliptic modular surfaces. *J. Math. Soc. Japan* **24** (1972) 20–59.
184. T. Shioda: On the Mordell–Weil lattices. *Commentarii Mathematici Universitas Sancti Pauli* **39**, 2 (1990) 211–240.
185. C.-L. Siegel: Über die analytische Theorie der quadratischen Formen I, II. *Ann. of Math.* **36** (1935) 527–606, **37** (1936) 230–263.
186. J.H. Silverman: *Advanced Topics in the Arithmetic of Elliptic Curves*. Springer-Verlag, New York, etc., 1994.
187. Y.-T. Siu: A simple proof of the surjectivity of the period map. *Manuscripta math.* **35** (1981) 311–321.
188. Y.-T. Siu: Every K3 surface is Kähler. *Invent. math.* **73** (1983) 139–150.
189. P. Slodowy: *Simple Singularities and Simple Algebraic Groups*. Springer Lecture Notes in Math. 815, Springer-Verlag, Berlin, etc., 1980.
190. E.H. Spanier: *Algebraic Topology*. McGraw-Hill, Inc., New York, 1966.
191. J.J. Sylvester: A proof that all the invariants to a cubic ternary form are rational functions of Aronhold's invariants and of a cognate theorem for biquadratic binary forms. *Philosophical Magazine* **5** (1853) 299–303, 367–372 = pp. 599–608 in *Collected Mathematical Papers, Vol. 1*, Cambridge University Press, 1904.

192. T. Takenawa: A geometric approach to singularity confinement and algebraic entropy. *J. Phys. A* **34** (2001) no. 10, L95–L102.
193. T. Takenawa: Discrete dynamical systems associated with root systems of indefinite type. *Commun. Math. Phys.* **224** (2001) 657–681.
194. J. Tate: On the conjecture of Birch and Swinnerton-Dyer and a geometric analogue. Sém. Bourbaki, Exp. 306 (Fév. 1966) 1–26.
195. A.D. Teleman: Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces. *Internat. J. Math.* **5** (1994) 253–264.
196. T. Tsuda: Integrable mappings via rational elliptic surfaces. *J. Phys. A: Math. Gen.* **37** (2004) 2721–2730.
197. J.M. Tuwankotta and G.R.W. Quispel: Dynamics of 2-dimensional maps derived from a discrete Sine-Gordon equation. In preparation, 2005.
198. R. Vakil: Twelve points on the projective line, branched covers, and rational elliptic fibrations. *Math. Ann.* **320** (2001) 33–54.
199. A.P. Veselov: Integrable maps. *Russian Math. Surveys* **46**:5 (1991) 1–51.
200. C.M. Viallet, B. Grammaticos, and A. Ramani: On the integrability of correspondences associated to integral curves. *Phys. Lett. A* **322** (2004) 186–193.
201. S. Vũ Ngọc: On semi-global invariants for focus-focus singularities: *Topology* **42** (2003) 365–380.
202. H. Waalkens, J. Wiersig, and H.R. Dullin: Elliptic quantum billiard. *Annals of Physics* **260** (1997) 50–90.
203. B.L. van der Waerden: Topologische Begründung des Kalküls der abzählende Geometrie. *Math. Ann.* **102** (1930) 337–362.
204. B.L. van der Waerden: *Algebra*, 1er Teil. Springer-Verlag, Berlin, etc., 1966.
205. R.J. Walker: Reduction of singularities of an algebraic surface. *Ann. of Math.* **36** (1935) 336–365.
206. C.T.C. Wall: Pencils of cubic curves and rational elliptic surfaces. pp. 373–405 in D. Mond and J. Montaldi (eds.): *Singularity Theory and Its Applications, Part I*, Warwick 1989. Lecture Notes in Mathematics 1462, Springer-Verlag, Berlin, Heidelberg, 1991.
207. K. Weierstrass: *Vorlesungen über die Theorie der elliptischen Functionen* (1863, 1874–75) = *Mathematische Werke von Karl Weierstrass*, Vol. V, Mayer & Müller, Berlin, 1915.
208. A. Weil: Sur les théorèmes de de Rham. *Comm. Math. Helv.* **26** (1952) 119–145.
209. A. Weil: Fibre-spaces in algebraic geometry. pp. 55–59 in *Algebraic Geometry Conference*. University of Chicago, 1955 = pp. 411–413 in Œuvres Scientifiques Collected Papers, Vol. I. Springer-Verlag, New York, etc., 1979.
210. A. Weil: *Number Theory. An approach through history from Hammurapi to Legendre*. Birkhäuser, Boston–Basel–Stuttgart, 1984.
211. R. Willox, B. Grammaticos and A. Ramani: A study of the antisymmetric QRT mapping. *J. Phys. A: Math. Gen.* **38** (2005) 5227–5236.
212. O. Zariski: *Algebraic Surfaces*. Springer-Verlag, Berlin, 1935.
213. O. Zariski: Reduction of the singularities of algebraic three dimensional varieties. *Ann. of Math.* **45** (1944) 472–542.
214. E.C. Zeeman: Geometric unfolding of a difference equation. Preprint, Hertford College, Oxford, 1996, 42 pp.
http://ww2.warwick.ac.uk/fac/sci/maths/research/preprints/2008
Lectures at UT San Antonio, March 10, 1997 and Trinity University, March 17, 1997, and on March 21, 2000 at the Pacific Institute for the Mathematical Sciences.
http://zakuski.utsa.edu/~gokhman/ecz/gu.html
http://www.pims.maths.ca/activities/
dist_lect/zeeman/zeeman.html

Index

Symbols

$A \cdot B$, 24
$\mathrm{Aut}(S)$, 307
$\mathrm{Aut}(S)_\varphi$, 307
$\mathrm{Aut}(S)_\varphi^1$, 307
$\mathrm{Aut}(S)_\varphi^{+,0}$, 341
$\mathrm{Aut}(S)_\varphi$, 424
C_{α^k}, 355
$\mathbb{C}/P$, 34
$\mathrm{c}(L)$, 16
$\mathrm{contr}_r(\alpha)$, 348
C^{red}, 204
$\mathbb{C}(V)$, 30
$\mathrm{Div}(M)$, 14
$f^\perp$, 336
$\mathcal{F}(J, M)$, 319
Φ^*, 110
Φ_*, 111
$\mathfrak{f}$, 211
$\mathrm{g}(C)$, 19
$\mathcal{G}_{C_0}$, 308
$\mathrm{GL}(V)$, 29
H, 41
$\mathrm{H}^0(M, \mathcal{O}(L))$, 106
$\mathrm{H}^1(M, \mathcal{O}^\times)$, 15.
K_M, 28
K_M^*, 28
$\mathcal{M}^\times$, 13
i_b, 87
$\mathrm{i}_b^{\mathfrak{P}}$, 104
$\mu_b(F)$, 104
μ_Θ, 192
$\mathrm{N}_{\widetilde{A}}^{\sim}(E)$, 27
$\mathrm{N}_S(A)$, 27
$\mathrm{NS}(S)$, 331
$\wp(t)$, 36
$\chi(M)$, 233
$\chi(\mathcal{O}_S)$, 221
$\mathcal{O}^\times$, 13
Ω^p, 27, 220
$\mathrm{p_a}(S)$, 225
$\mathrm{P}(V)$, 28
$\mathrm{p_g}(S)$, 222
$\mathrm{PGL}(V)$, 29
$\mathrm{Pic}(M)$, 15.
$\mathbb{P}^n$, 28
$\mathrm{PSL}(2, \mathbb{Z})$, 42
$Q := f^\perp/\mathbb{Z}\, f$, 336
(q, r)-form, 219
$\mathrm{q}(S)$, 222
$\mathrm{SL}(2, \mathbb{Z})$, 42
S_r^{irr}, 204
Σ_n, 198
Σ_φ, 330
$\Theta \in S_c^{\mathrm{irr}}$, 192
$\mathrm{vg}(C)$, 190
$X\, A\, Y$, 1
$[x_0 : x_1 : \ldots : x_n]$, 28
Z_{p_z}, 64
$\mathrm{Z}_{[w]}$, 107

A

abelian differential, 28
abelian variety, 230
absolute invariant, 49
abstract QRT root, 454
adjunction formula, 187, 189, 110
affine coordinates, 28
affine coordinatization, 29
algebraic variety, 226
algebraic variety, complex projective, 29
ample line bundle, 433
analytic subset, complex, 10
anharmonic elliptic curve, 35

anticanonical bundle, 28
anticanonical fibration, 405
anticanonical system, 405
antisymmetric QRT map, 427
arithmetic genus, 222
Aronhold's invariants, 143
Aronhold morphism, 290
Atyiah–Singer index theorem, 221
automorphism of a surface, 307

B

base locus, 86
base locus of linear system, 107
base point of a loop, 242
base point of a pencil of curves, ix, 86
base line of confocal quadrics, 489
basic member, 319
Bertini's theorem, 105
Betti numbers, 222
Beukers–Cushman criterion, 82
Bézout, theorem of, 25
bifurcation diagram, 445
billiard, elliptic, 487
billiard map, 490
binode, 278
biquadratic curve, 49
biquadratic curves, pencil of, 64,
biquadratic equation, 64
biquadratic polynomial, viii, 49
bimeromorphic transformation, 31
birational transformation, 31
blowup, xi, 97
branched covering, 44
branch point, 44

C

canonical line bundle, 28
Cartan matrix, 205
Castelnuovo–Enriques criterion, 99
categorical quotient, 285
categorical quotient, universal, 285
Cauchy–Riemann operator, 220
Chow's theorem, 30
Chow and Kodaira theorem, 226
Chern class of a line bundle, 16
Chevalley's theorem, 169
circumscribed polygon, 470
closed complex analytic subset, 10
closed projective set, 30
coboundary, 15
cocycle, 15
coherent sheaf, 214
cohomology of sheaves, 15
comatrix, Cramer's, 143
compactification, 29
compact Riemann surface, 18
complete elliptic integral, 57
complex analytic curve, 18
complex analytic manifold, 10
complex analytic mapping, 10
complex analytic space, 262
complex analytic subset, 10
complex analytic surface, 28
complex antilinear, 219
complex area form, 28
complex conjugation, 74
complex Hamiltonian vector field, 217
complex p-form, 27
complex projective algebraic variety, 30
complex projective line, 28
complex projective plane, 28
complex projective space, 28
complex time, flow with, 33
complex torus, 230
complex upper half-plane, 41
complex volume form, 28
complexity of a map, 124
conductor, 189
confocal quadrics, 488
conjugation, complex, 74
conic, 451
conormal bundle, 187
constructible set, 169
contribution of a reducible fiber, 348
Cramer's comatrix, 143
cubic curves, pencil of, 132
cubic form, ternary, 131
cubic planar curve, 131
-1 curve, 99
cusp, ordinary, 592

D

Darboux transformation of 4-bar link, 512
deformation equivalent, 299
degree of a function at a point, 43
degree of a meromorphic one-form, 433
degree of Del Pezzo surface, 433
degree of ι_1, 160
degree of divisor on Riemann surface, 19
degree of line bundle over curve, 19
Del Pezzo surface, 433
density of translational map, 357
derivation, 373
desingularization of plane curve, 190
determinant of a lattice, 351
differential form of type (q, r), 219
dilatation, 97
dimension of linear system, 107

direct image of a sheaf, 214
discrete dynamical system, integrable, 3
discrete Heisenberg spin chain map, 553
discrete KdV map, 547
discrete modified KdV map, 549
discrete nonlinear soliton map, 550
discrete sine–Gordon map, 556
divisor, 14
divisor, effective 13
divisor of meromorphic function, 14
divisor of a section, 16
Dolbeault sequence, 220
dual lattice, 352
dual line bundle, 107
dual projective plane, 467
dual quadric, 468
Du Val singularity, 278
dynamical system, integrable discrete, 3
Dynkin diagram, 205

E

effective divisor, 13
Eichler–Siegel transformation, 339
Eisenstein morphism, 292
Eisenstein invariants, 46
ellipse, gardener's construction, 487
elliptic bifurcation, 80
elliptic billiard, 487
elliptic curve, x, 34
elliptic fibration, 186, 179
elliptic function, 39
elliptic integral, 57
elliptic K3 surface, 296
elliptic singular fiber, 394
elliptic surface, 186
elliptic surface, rational, xii, 408
Enriques surface, 39
Euler number, holomorphic, 221
Euler number of singular fiber, 234
Euler number, topological, 233
Euler–Poincaré characteristic, 233
even lattice, 336
exceptional curve, 265
exceptional curve of the first kind, 99
exceptional fiber at blowup, 97
exceptional unode, 278
extended Dynkin diagram, 207
exterior p-vector field, 28
extremal elliptic surface, 432

F

faithful representation in $\mathfrak{I}$, 426
fiber of elliptic fibration, 179
fiber system of groups, 309
fiber-tangent vector field, 215
fibration, 179
fibration defined by line bundle, 185
finite-order group element, 349
finite-order QRT mapping, 164
fixed part of line bundle, 184
fixed-point fibers, number of, 346
flex point, 134
flow with complex time, 33
focus, perfect, 499
fold, 359
form of type $(q,\ r)$, 219
four-bar link, planar, xviii, 512
full lattice, 34
function field, 30
fundamental group, 242
fundamental weights, 313
fractional linear transformation, 42
fractional part, 492
frequency map, 357
Frobenius invariants, 62
Frobenius invariants, symmetric case, 460
Frobenius morphism, 291

G

G.A.G.A. principle, 33
gardener's construction of ellipse, 487
garland, 392
Gauss–Manin connection, 74
general linear group, 29
generalized Cartan matrix, 205
generic, order two, 167
generic, order three, 170
generic, order four, 172
generic QRT root, 458
generic rational elliptic surface, 441
genus, 19
genus, arithmetic, 225
genus, geometric, 222
genus, topological, 19
geometric genus, 222
geometric invariant theory, 285
geometric singularity confinement, 124
gliding billiard ball, 500
gluing map, 15, 95
Grashof identities, 513
Grassmann manifold, 4
Grauert's exceptional curve criterion, 265
Grauert's ample line bundle criterion, 433
Grauert's direct image theorem, 213
groups, fiber system of, 309

H

half-plane, complex upper, 41

harmonic elliptic curve, 35
Heisenberg spin chain map, discrete, 553
helicoid, 97
Hesse group, 483
Hesse map, 478
Hesse pencil, 482
Hesse surface, 482
Hesse determinant, 144
Hirzebruch index formula, 222
Hirzebruch surface, Σ_n, 198
Hodge decomposition, 224
holomorphic, 10
holomorphic Euler number, 221
holomorphic mapping, 10
holomorphic complex p-form, 27, 220
homological invariant, 24,
horizontal switch, viii, 1
hyperbolic bifurcation, 80
hyperbolic billiard, 510
hyperbolic singular fiber, 392
hyperelliptic curve, 46
hyperelliptic involution, 46
hyperplane section, 225

I

identity component, 34
incidence relation, 94
incomplete elliptic integral, 57
indeterminacy, set of 13, 31
index theorem for surfaces, 224
infinitely steep, 396
infinity, projective space at, 29
inscribed polygon, 470
integrable bi-Hamiltonian system, 218
integrable discrete dynamical system, 3
integrable tangent lattice bundle, 360
integral, 3
integral lattice, 336
intersection diagram, 204
intersection form, 25
intersection matrix, 204
intersection number of cycles, 24
intersection pairing on homology, 24, 24
invariant, absolute 49
invariant, projective 46
invariant area form, 3
invariants, Aronhold's, 143
invariants, Eisenstein's, 46
invariants, Frobenius's, 62, 460
invariants, of biquadratics, 62
invariants, of plane quartics, 46
invariants, of symmetric biquadratics, 460
invariants, of ternary cubics, 143
inversion about a section, 308
inversion on an elliptic curve, 60
involution, viii, 60
involution, reversing, 60
involution, translational, 60
irreducible component of analytic set, 11
irreducible component of analytic space, 262
irreducible component of divisor, 14
irreducible component, local 12
irreducible germ, 12
irregularity of a surface, 222

J

Jogia's example, 572

K

K3 lattice, 301
K3 surface, 295
K3 surface, elliptic, 296
Kähler manifold, 224
KdV map, discrete, 547
KdV map, discrete modified, 549
kernel theorem, Manin's, 375
Klein singularity, 278
Kodaira embedding theorem, 30
Kodaira's numerical invariants, 223
Kodaira–Serre duality theorem, 220
Kodaira–Spencer theorem, 212
Kodaira type of singular fiber, 209
Künneth formula, 50

L

lattice, 336
lattice automorphism, 338
lattice, full, 34
Lefschetz theorem on (1, 1) classes, 225
Levi extension theorem, 14
Lie algebra bundle, 211
lift of monodromy representations, 315
line bundle fibration, 185
linear equivalence of divisors, 332
linear group, general, 29
linear group, projective, 29
linear system, 107
local intersection number, 24
local irreducible component, 12
local trivialization of line bundle, 15
Łojasiewicz triangulation theorem, 23
loop with base point, ix
Lyness equation, 518
Lyness map, 518

M

Manin automorphism, 136
Manin function of a QRT map, 74

Manin homomorphism, 74, 375
Manin QRT automorphism, 140
Manin transformation, 136
Manin surface, 436
Manin's kernel theorem, 375
matrix of lattice, 336
maximal elliptic surface, 432
Mayer–Vietoris sequence, 100, 234
McMillan map, 546
meromorphic function, 13
meromorphic mapping, 31
minimal resolution of singularities, 265
minimal surface, 199
Möbius strip, 97, 380
modification, 87
modification of analytic space, 263
modified KdV map, discrete, 549
modular function, 41
modular group, 43
modular group action, 42
moduli spaces, 285
modulus of elliptic curve, 40
modulus function, xiv, 186
monodromy group, 242,
monodromy matrix, 242, 244
monodromy representation, xiv, 242
monodromy representation of J, 242
monoidal transformation, 97
monotonicity criterion of Beukers–Cushman, 82
Mordell–Weil group, xiii, 307, 330
Mordell–Weil lattice, 347
Mordell–Weil lattice, narrow, 342
Mori, theorem of, 199
morphism, 30
moving part of line bundle, 184
multiple singular fiber, 207,
multiplicity of base point, 87, 104
multiplicity of irreducible component, 14
multivector field, exterior, 28

N

narrow Mordell–Weil lattice, 342
Néron–Severi group, xv, 331
net of linear system, 107
Noether's formula, 222
nonlinear soliton map, discrete, 550
normal bundle, 27, 187
normal space, 264
number of periodic fibers, xiii, 346
numerical adjunction formula, 188, 190
numerically effective, 199
numerical stability criterion, 286, 346

O

Oguiso and Shioda's list, 415
order of a meromorphic function, 14
order of the QRT mapping, 165
order of ramification, 44
order of vanishing, 12
ordinary cusp, 592
oriented n-vector, 28

P

partial discriminant, 52, 52
path, 241,
pencil of biquadratic curves, 64, 86
pencil of biquadratic curves, 64
pencil of cubic curves, 132
pencil of curves, ix
perfect focusing, 499
period of the QRT mapping, 165
period integral, 35
periodic group element, 349
periodic fibers, number of 346
periodic point, 61
periodic QRT mapping, 164
period group, 34
Persson's list, 418
$\wp$-function, Weierstrass, 36
Picard–Fuchs equation, 73
Picard–Fuchs operator, 373
Picard group, 15
planar four-bar link, xviii, 512
Plücker embedding, 5
Poincaré duality, 24
Poincaré residue, 110
Poisson brackets, 218
pole along hypersurface, 14
Poncelet closure theorem, 469
Poncelet curve, 469
Poncelet mapping, xviii, 469
Poncelet polygon, 470
Poncelet porism, 35, 61
principal $\mathbb{C}/P$-bundle, 325
product of line bundles, 16
projective algebraic variety, complex, 30
projective coordinates, 28
projective invariant, 46
projective line, complex, 28
projective linear group, 29
projective plane, complex, 28
projective set, closed, 30
projective space, 28
projective space at infinity, 29
proper action, 179
proper mapping, 93
proper transform, 101

proper transform of a pencil, 103
pullback of line bundle, 17
pullback of volume form, 110
Puiseux theorem, 188
pushforward of n-vector field, 111

Q

quadric, planar, 451
quadric transformation, 97
QRT map, viii, 2
QRT root, 452, 453
QRT root, abstract, 454

R

ramification point, 44
ramification order, 44
ranging together of maps on curves, 30
rank of abelian group, 336
rank of lattice, 336
rational curve, 99
rational double point, 278
rational elliptic surface, xii, 408
rational function, 30
rational map, 30
rational surface, 407
real structure, 378
real automorphism, 378
recurrence equation, second order, 452
regular fiber, 179
regular function, 30
regular lift, 124
regular point of analytic set, 10
regular point of mapping, 179
regular singular point, 68
regular unode, 278
regular value of a mapping, 179
relative quotient, 188
relatively minimal elliptic surface, 186
removable singularity theorem, 18
representation in $\mathfrak{I}$, 426
resolution of singularities, 265
retrivialization of line bundle, 15
reversible automorphism, 331
reversible system, 3
reversing involution, 60
Riemann–Hurwitz formula, 44
Riemann–Roch formula, 21
Riemann–Roch formula for surfaces, 222
Riemann's removable singularity theorem, 18
Riemann surface, 18
root lattice, 313
roots of unity, 154
rotation function, 82, 384
rotation map, xvi, 356
rotation number, 81, 384
ruled surface, 186
ruling, 186

S

screensaver map, 519
second-order recurrence equation, 452
section of a fibration, 183
Segre map, 30
self-intersection number, 26
semistable element, 286
Serre duality theorem, 220
set of indeterminacy, 13, 31
sheaf cohomology, 15
Siegel–Eichler transformation, 339
σ-process, 97
simple singularity, 278
sine–Gordon map, discrete, 556
singular fiber, 93
singular fiber, Kodaira type of, 209
singular point of analytic set, 10
singular point of analytic space, 262
singular value of a mapping, 179
singularity confinement, 124
singularity confinement, geometric, 124
smooth locus of analytic space, 262
soliton map, nonlinear discrete, 550
spin chain map, discrete Heisenberg, 553
stable element, 286
steep, infinitely, 396
stein factorization theorem, 182
strata, 124
stratification, 124
submersion, 180, 124
surjectivity of period map, 301
symmetric biquadratic curve, 452
symmetric biquadratic polynomial, xvii, 452
symmetric QRT mapping, 451
symmetry morphism, 292
symmetry switch, 452

T

tangent lattice bundle, 360
tangent mapping, 74, 179
tangent space, 74
Tate height, 342
tautological line bundle, 93
tensor product of lattice with $\mathbb{Q}$, 346
tensor product of line bundles, 16
ternary cubic form, 131
third roots of unity, 154
time reversible system, 3
topological Euler number, 233

topological genus, 19
topological intersection number, 24
Torelli theorem for K3 surfaces, 301
torsion group element, 349
torsion subgroup, 24, 336
torus, complex, 230
total transform, 101
total variation of rotation, 399
translation on elliptic curve, x, 34
translational involution, 60
tree of spheres, 267
triangulation theorem of Łojasiewicz, 23
trivialization of line bundle, 15
two-vector field, exterior, 28
type of singular fiber, Kodaira, 209

U

undetermined, 13
uniformization, 188
unimodular integral bilinear form, 24
unimodular lattice, 351
universal categorical quotient, 285
unit, 12
unity, roots of, 154
unode, 278
upper half-plane, complex, 41

V

vanishing cycle, 259
vanishing order, 12
variety, algebraic, 29
n-vector field, 28
vertical switch, viii, 1
very ample line bundle, 433
virtual genus of curve, 190
volume form, 28

W

web of linear system, 107
Weierstrass data of QRT root, 457
Weierstrass model of elliptic surface, 277
Weierstrass morphism, 291
Weierstrass form of elliptic curve, 37
Weierstrass $\wp$-function, 36
weight lattice, 313

Z

Zariski topology, 30
$\mathbb{Z}$-basis of lattice, 336
Zeeman's conjecture, xviii
zeroset of biquadratic function, 64
zeroset in projective space, 29
zeroset of section of line bundle, 107
zigzag, 194